Production Function Estimation with Factor-Augmenting Technology: An Application to Markups

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Production Function with Labor-Augmenting Technology

The model starts with a general form.

$$Y_{it} = F_t(K_{it}, \omega^L_{it} L_{it}, M_{it}) exp(\omega^H_{it}) exp(\varepsilon_{it}) \quad (1)$$

where $exp(\omega^L_{it})$ and $exp(\omega^H_{it})$ are labor augmenting and Hicks-neutral productivity, respectively. When $exp(\omega^L_{it})$ is not there, we collapse to a usual production function:

$$Y_{it} = F_t(K_{it}, L_{it}, M_{it}) exp(\omega^H_{it}) exp(\varepsilon_{it}). \quad (2)$$

Firms decide K_{it} in period $t-1$, while decides L_{it} and M_{it} at period t. We call K_{it} a *dynamic* input while we call L_{it} and M_{it} *static* inputs. Before deciding the inputs, firms observe $exp(\omega^H_{it})$ but observe $exp(\epsilon_{it})$ after the decision.

What's wrong with Cobb-Douglas?

When we assume Cobb-Douglas, this production function collapses to:

$$Y_{it} = K_{it}^{\beta_k} L_{it}^{\beta_l} M_{it}^{\beta_m} exp(\omega_{it}^H) exp(\omega_{it}^L) exp(\varepsilon_{it}).$$

but this implies (i) the production function is log-linear (ii) no role in labor-augmenting productivity $exp(\omega_{it}^H)$. But:

- it implies the revenue shares of flexible inputs are the same across firms (not true)
- literature has documented large heterogeneity in factor intensities, which contradicts constant elasticity
- empirical evidence suggests that the elasticity of substitution is less than one.

we can extend things with Trans-log, but it's still not consistent with the data.

Conclusion (here)

This paper discusses the identification and estimation of production function with factor-augment productivity. The paper shows:

1. Like Gandhi, Navarro, and Rivers (2020), the identification using the variation in input and output will not work.
2. Using first-order conditions will identify some relevant parameters (output elasticity, markups)
3. Even first-order conditions will not identify some parameters (factor augmenting productivity, elasticity of substitutions).

Point (2) and (3) are intuitive once explained, but hidden in Ghandi, Navarro, and Rivers (2020) which is more like a Cobb-Douglas function estimations.

Assumptions

Assumption 2.1 (*Homothetic Separability*). Suppose that:

1. The production function is of the form

$$Y_{it} = F_t(K_{it}, h_t(K_{it}, \omega_{it}^L L_{it}, M_{it}))exp(\omega_{it}^H)exp(\varepsilon_{it}). \tag{3}$$

2. $h_t(K_{it}, \cdot, \cdot)$ is homogeneous of arbitrary degree for all K
3. The firm minimizes production cost with respect to (L_{it}, M_{it}) given K_{it}, productivity shocks $(\omega_{it}^L, \omega_{it}^H)$ and input prices (pt^l, p_t^m).
4. Let $\sigma_t(K, \omega_{it}^L L_{it}, M_{it})$ be an elasticity of substitution between effective labor and materials, $\sigma_t(K, \omega_{it}^L L_{it}, M_{it})$ is always above 1, or is always below 1.

Interpretations of the assumptions

Two economic interpretations of *homothetic separability*:

1. The production is broken into stages, where h_t can be seen as an 'intermediate input' with its own production function, which is then combined with capital for production.
2. Increasing the scale of labor and materials is equivalent to increasing the scale of h_t. This means the firm decides the optimal scale of h_t rather than the optimal scale of labor and materials separately. In other words, **the optimal ratio of materials and labor is a sufficient statistic**.
3. The production cost does not involve capital, since K_{it} is a predetermined input. Cost minimization is a static problem, so it is agnostic about the firm's dynamic decisions.
4. The effective labor and materials are either substitutes or complements

Example: CES

CES will satisfy these properties:

$$Y_{it} = \left(\beta_k K_{it}^{\sigma} + \beta_l [\omega_{it}^{l} L_{it}]^{\sigma} + (1 - \beta_l - \beta_m) M_{it}^{\sigma}\right)^{\nu/\sigma} exp(\omega_{it}^{H}) exp(\epsilon_{it}).$$

with $h_t(\cdot) = \beta_l [\omega_{it}^{l} L_{it}]^{\sigma} + (1 - \beta_l - \beta_m) M_{it}^{\sigma}$. If $\sigma > 1 (< 1)$, the labor and the material are substitutes (complements).

Implications of the assumptions

Under the assumptions:

1. the flexible input ration, defined as $\tilde{M}_{it} = M_{it}/L_{it}$, depends on only K_{it} and ω_{it}^{L}, through an unknown function $\tilde{M}_{it} = r_{it}(K_{it}, \omega_{it}^{L})$.
2. $r_t(K_{it}, \omega_{it}^{L})$ is strictly monotone on ω_{it}^{L}.

Therefore, ω_{it}^{L} is *invertible* with respect to K_{it} and $\tilde{M}_{it}$. In CES, we have

$$logM_{it} = \sigma log(p_t^l / p_t^m) + log(\omega_{it}^{L}).$$

Additional Assumptions

Assumption 2.4 (Monotonicity). Firms' materials demand is given by:

$$M_{it} = s_t(K_{it}, \omega_{it}^L, \omega_{it}^H),$$

where $s_t(K_{it}, \omega_{it}^L, \omega_{it}^H)$ is an unknown function that is strictly increasing in ω_{it}^H. Again, this means that we can invert ω_{it}^H with respect to K_{it} and ω_{it}^L.

Invertibility

We can fully apply invertibility:

$$\omega_{it}^{L} = r_t^{-1}(K_{it}, M_{it}) \equiv \bar{r}_t(K_{it}, M_{it}) \tag{4}$$

$$\omega_{it}^{H} = s_t^{-1}(K_{it}, \omega_{it}^{L}, M_{it}) = s_t^{-1}(K_{it}, \bar{r}_t(K_{it}, M_{it}), \omega_{it}^{H}) \equiv \bar{s}_t(K_{it}, \tilde{M}_{it}, \omega_{it}^{H}), \tag{5}$$

and we use the identification strategy of Imbens and Newey (2009), who study the identification of

- *non-separable models* where a *scalar unobservable has a strictly monotone relationship with the outcome*
- the scalar observable is *independent of an instrument*.

Creating a scalar observables

The variable ω_{it}^{L} and ω_{it}^{H} is not easy to handle, since:

- We still do not know what $\bar{r}_{it}$ and $\bar{s}_{it}$ exactly look like.
- ω_{it}^{L} and ω_{it}^{H} are correlated with a lot of things (no independence).

What can we learn from ω_{it}^L?

We make use of the *Markov assumption* on productivity.

- The productivity only depends on past productivity (both ω_{it}^L and ω_{it}^H) and innovation.
- Innovation is independent of anything.

If productivity shocks are continuously distributed, we can relate labor-augmenting productivity to past productivity shocks in a following way:

$$\omega_{it}^L = g_1(\omega_{it-1}^L, \omega_{it-1}^H, u_{it}^1) \tag{6}$$

where $u_{it}^1 | \omega_{it-1}^L, \omega_{it-1}^H$ follows a uniform distribution. This is a Skorohod representation of random variables.

Under Assumptions 2.2 - 2.3, we have that $u^1_{it} \perp (K_{it}, W_{it--1})$

We can use this relationship to write an unknown function for $\tilde{M}_{it}$:

$$\begin{aligned}
\tilde{M}_{it} &= r_t(K_{it}, g_1(\omega^L_{t-1}, \omega^H_{t-1}, u^1_t)), \\
&= r_t(K_{it}, g_1(r_t(K_{it-1}, \tilde{M}_{it-1}), s_t(K_{it-1}, M_{it-1}, \tilde{M}_{it-1}), u^1_t)), \\
&= \tilde{r}_t(K_{it}, W_{it--1}, u^i_t),
\end{aligned}$$

where $W_{it} = (K_{it}, L_{it}, M_{it})$ and $\tilde{r}_{it}$ unknown.

What can we learn about u_{it}^1?

We now have the two conditions needed to derive a control variable:

- $\tilde{r}_t(K_{it}, W_{it-1}, u_{it}^1)$ is strictly monotone in u_{it}^1
- u_{it}^1 is independent of (K_{it}, W_{it-1}).

Since the distribution of u_{it}^1 is already normalized to a uniform distribution, we can identify u_{it}^1 from data as:

$$u_{it}^1 = F_{\tilde{M}_{it}|K_{it},W_{it-1}}(\tilde{M}_{it}|K_{it}, W_{it-1}) \tag{7}$$

Using this result, I can express ω_{it}^L as a function of the control variable and past inputs:

$$\omega_{it}^L = c_{1t}(W_{it-1}, u_{it}^1) \tag{8}$$

Identification of ω_{it}^H

Similarly for ω_{it}^H, we can assume *Markov*, which implies:

$$\omega_{it}^H = g_2(\omega_{it-1}^L, \omega_{it-1}^H, u_{it}^1, u_{it}^2) \tag{9}$$

where $u_{it}^2|\omega_{it-1}^L, \omega_{it-1}^H, u_{it}^1$ follows an uniform distribution. Following a similar step, we get:

$$u_{it}^2 = F_{M_{it}|K_{it}, W_{it-1}, u_{it}^1}(M_{it}|K_{it}, W_{it-1}, u_{it}^1) \tag{10}$$

and then ω_{it}^H can be written as:

$$\omega_{it}^H = c_{2t}(W_{it-1}, u_{it}^1, u_{it}^2). \tag{11}$$

Identification of structural parameters

The paper points out an identification issues:

- Showing that the production function and output elasticities are not identified from variations in inputs and output.
- Exploiting the first-order conditions of cost minimization to identify output elasticities

Very similar approach to Gandhi, Navarro and Rivers (2020).

Non-Identification

Take a logarithm of the production function. We have:

$$y_{it} = f_t(K_{it}, h_t(K_{it}, \omega_{it}^L L_{it}, M_{it})) + \omega_{it}^H + \epsilon_{it},$$

where f_t is $logF_t$. We assume, without loss of generality, that it is homogeneous of degree one. Using this property, I rewrite the production function as follows:

$$y_{it} = f_t(K_{it}, L_{it} h_t(K_{it}, \omega_{it}^L, \tilde{M}_{it})) + \omega_{it}^H + \epsilon_{it}.$$

We showed that $\omega_{it}^L = \bar{r}_{it}(K_{it}, \tilde{M}_{it})$, which implies:

$$y_{it} = f_t(K_{it}, L_{it} h_t(K_{it}, \bar{r}_{it}(K_{it}, \tilde{M}_{it}), \tilde{M}_{it})) + \omega_{it}^H + \epsilon_{it}.$$

This representation of the production function reveals an identification problem. Proposition 4.1. Without further restrictions, h_t cannot be identified from variations in inputs and output.

Non-Identification

We can show that output elasticities or elasticity of substitution are not identified:

$$\begin{aligned}
\theta_{it}^{K} &\equiv (f_{t1} + f_{t2}h_{t1})K_{it} \\
\theta_{it}^{L} &\equiv f_{t2}h_{t2}L_t\bar{r}_t(K_{it}, \tilde{M}_{it}) \\
\theta_{it}^{M} &\equiv f_{t2}h_{t3}M_{it}
\end{aligned}$$

where all of them depend on the derivatives of $h(\cdot)$.

Non-Identification

Instead of using $h(\cdot)$, we express the function in a reduced-form way $\bar{h}_t(K_{it}, \tilde{M}_{it})$:

$$y_{it} = f_t(K_{it}, L_{it}\bar{h}_t(K_{it}, \tilde{M}_{it})) + \omega_{it}^H + \varepsilon_{it},$$

so we acknowledge that we cannot distinguish ω_{it}^L and the optimal input mix.

Identification using Output Elasticity

So far, we showed:

- Variations in inputs and outputs are not sufficient to identify the parameters of production function (and elasticities). This is because the changes input due to productivity, ω_{it}^L, completely explains the variations in input mix ($\tilde{M}_{it}$).

So we use the cost minimization conditions to identify stuff:

- We may be able to identify the parameters, like ω_{it}^L.
- Even if we could not identify the parameters, we may be able to identify something relevant to the research question.

Identification using cost minimization

The cost minimization problem is:

$$min_{L_{it},M_{it}} p_t^l L_{it} + p_t^m M_{it}$$
$$s.t. F_t(K_{it}, \omega_{it}^L L_{it}, M_{it}) exp(\omega_{it}^H) E[exp(\epsilon_{it})|I_{it}] \geq \bar{Y}_{it}$$

where $\bar{Y}_{it}$ is planned output. The first order condition is $F_{tV}\lambda_{it} = p_t^V$ where $V \in \{M, L\}$. F_{tV} is the marginal product of V, and λ_{it} is the Large multiplier. Multiplying both sides by $V_{it}/(Y_{it}p_{it})$ implies:

$$\frac{F_{tV}(K_{it}, \omega_{it}^L L_{it}, M_{it}) V_{it}}{F_t(K_{it}, \omega_{it}^L L_{it} M_{it})} \frac{E[exp(\epsilon_{it})|I_{it}]\lambda_{it}}{exp(\epsilon_{it}) p_{it}} = \frac{V_{it} p_t^v}{Y_{it} p_{it}}.$$

Where the first part of the left hand side is the output elasticity θ_{it}^V, and the right hand side is the revenue share of inputs (α_{it}^V). Taking ratio for M and L implies:

$$\theta_{it}^M / \theta_{it}^L = \alpha_{it}^M / \alpha_{it}^L.$$

Identification using cost minimization

Now, we want to identify the sum of material and labor elasticity. Recall that

$$\theta_{it}^{L} \equiv f_{t2} h_{t2} L_t \bar{r}_t(K_{it}, \tilde{M}_{it})$$
$$\theta_{it}^{M} \equiv f_{t2} h_{t3} M_{it}$$

so we need the knowledge of $h(\cdot)$ to estimate them separately. However, we can identify $\theta_{it}^{L} + \theta_{it}^{M}$ only by knowing $\bar{h}(\cdot)$ and f_t:

$$\theta_{it}^{V} = \theta_{it}^{M} + \theta_{it}^{L} = f_{t2}(K_{it}, L_{it}\bar{h}_t(K_{it}, \tilde{M}_{it}))L_{it}\bar{h}_t(K_{it}, \tilde{M}_{it}).$$

We can recover θ_{it}^{M} and θ_{it}^{L} by:

$$\theta_{it}^{L} = \theta_{it}^{V} \alpha_{it}^{L} / \alpha_{it}^{V}$$
$$\theta_{it}^{M} = \theta_{it}^{V} \alpha_{it}^{M} / \alpha_{it}^{V},$$

where $\alpha_{it}^{V} = \alpha_{it}^{M} + \alpha_{it}^{L}$.

Other identification result

Without additional assumptions, we cannot identify:

- Labor augmenting productivity ω_{it}^{L}
- Capital elasticity θ_{it}^{K}.

But possible with a reasonable (functional form) assumption.

Estimating $\bar{h}_t$ and f_t

We start with a slightly simplified function

$$y_{it} = f_t(K_{it}, h_t(\omega^L_{it} L_{it}, M_{it})) + \omega^H_{it} + \epsilon_{it}$$

where now h_t does not depend on K_{it}. The reduced form expression is

$$y_{it} = f_t(K_{it}, L_{it}\bar{h}_t(\tilde{M}_{it})) + \omega^H_{it} + \epsilon_{it},$$

now we only have to deal with ω^H_{it}! We bring back the u^1_{it}, u^2_{it} notation here:

$$y_{it} = f_t(K_{it}, L_{it}\bar{h}_t(\tilde{M}_{it})) + c_{2t}(W_{it-1}, u^1_{it}, u^2_{it}) + \epsilon_{it}|,$$

with $E[\epsilon_{it}|W_{it}, W_{it-1}, u^1_{it}, u^2_{it}]$ where $W_{it} = (K_{it}, L_{it}, M_{it})$ and u^1_{it} and u^2_{it} can be recovered by inversion.

Estimation procedure

We approximate $\bar{h}_t$ and f_t using polynomials. Specifically, we have:

$$\begin{aligned} log(\hat{\bar{h}}_t(\tilde{M}_{il})) &= a_{1t} + a_{2t}\tilde{m}_{it} + a_{3t}\tilde{m}_{it}^2 + a_{4t}\tilde{m}_{it}^3 \\ \hat{f}_t(K_{it}, \nu_{it}) &= b_{1t} + b_{2t}k_{it} + b_{3t}k_{it}^2 + b_{4t}k_{it}^3 \\ &\quad + b_{5t}\nu_{it} + b_{6t}\nu_{it}^2 + b_{7t}\nu_{it}^3 + b_{8t}k_{it}^2\nu_{it} + b_{9t}k_{it}\nu_{it}^2 + b_{10t}k_{it}\nu_{it} \end{aligned}$$

where $\nu_{it} = log(L_{it}\hat{\bar{h}}_t(\tilde{M}_{it}))$. The function to minimize is

$$J(\hat{f}_t, \hat{\bar{h}}_t, \hat{c}_{2t}) = \frac{1}{TN}\sum_{i,t}\hat{\epsilon}_{1it}^2.$$

Recall that:

$$y_{it} = f_t(K_{it}, L_{it}\bar{h}_t(\tilde{M}_{it})) + c_{2t}(W_{it-1}, u_{it}^1, u_{it}^2) + \epsilon_{it}.$$

Estiating Mark-ups

Markups can be estimated as:

$$\mu_{it} = \frac{\theta_{it}^{V}}{\alpha_{it}^{V}}$$

and this is consistent for both L and M:

$$\mu_{it}^{L} = \frac{\theta_{iL}^{L}}{\alpha_{it}^{L}} = \frac{\theta_{it}^{M}}{\alpha_{it}^{M}} = \mu_{it}^{M}$$

which nicely aligns with the criticism by Raval (2024).

Conclusion

This paper discusses the identification and estimation of production function with factor-augment productivity. The paper shows:

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- Using first-order conditions will identify some relevant parameters (output elasticity, markups)
- Even first-order conditions will not identify some parameters (factor augmenting productivity, elasticity of substitutions).

The End