Production Approach Workshop@IDE-JETRO Session 3

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Session 3: Production Approach with Revenue Data (2) Estimation

- Semiparametric estimator
- Estimation code

Parametric Assumption

- Cobb-Douglas production function; AR1 TFP process; Nonparametric demand

$$\begin{aligned} y_{it} &= \theta_m m_{it} + \theta_l l_{it} + \theta_k k_{it} + \omega_{it} \\ \omega_{it} &= \rho \omega_{it-1} + \eta_{it} \\ p_{it} &= \psi_t \left(y_{it}, u_{it} \right) \end{aligned}$$

- Control function and revenue function

$$\begin{aligned} \omega_{it} &= \lambda_t \left(m_{it}, u_{it} \right) - \theta_k k_{it} - \theta_l l_{it} \\ r_{it} &= \varphi_t \left(\lambda_t \left(m_{it}, u_{it} \right), u_{it} \right) \\ &= \phi_t \left(m_{it}, u_{it} \right) \end{aligned}$$

Semiparametric Approximation

- Basis expansion of $f(X)$

$$f(X) = \sum_{s=1}^{S} h_s(X)\beta_s$$

- Examples
  - 3rd degree polynomials.
    $h_1(X) = 1; h_2(X) = X; h_3(X) = X^2; h_4(X) = X^3$
  - 3rd degree polynomials with two knots (ξ_1, ξ_2).

$$h_1(X) = 1; h_2(X) = X; h_3(X) = X^2; h_4(X) = X^3;$$
$$h_5(X) = [X - \xi_1]_+^3; h_6(X) = [X - \xi_2]_+^3$$

where $[Y]_+ = \max\{0, Y\}$.

B splines

- B(Basis) splines

$$f(X) = \sum_{s=1}^{S} B_s(X)\beta_s$$

- $B_s(X)$: B spline basis function
  - Parameters: degree, knots (number and positions)
  - S (degree of freedom)=degree+1+number of knots
  - Order= degree+1
- Hastie, Tibshirani, Friedman and Friedman (2009) *Elements of Statistical Learning* (Chapter 5)

B spline basis function

- Parameters
  - order$M$
  - K internal knots $(\xi_1 \leq \xi_2 \leq ... \leq \xi_K)$
  - $\xi_0 \leq \xi_1$ and $\xi_K \leq \xi_{K+1}$: boundary knots (usually possible data range)
- Augmented knots $\{\tau_1, \tau_2 ...., \tau_{K+2M+1}\}$ by adding M knots outside each boundary knot

$$
\begin{gathered}
\tau_1 \leq \tau_2 \leq ... \leq \tau_M = \xi_0 \\
\tau_{i+M} = \xi_i \text{ for } i = 1, ..., K \\
\xi_{K+1} \leq \tau_{M+K+1} \leq ... \leq \tau_{K+2M+1}
\end{gathered}
$$

B spline basis function

- B spline basis function of order 1

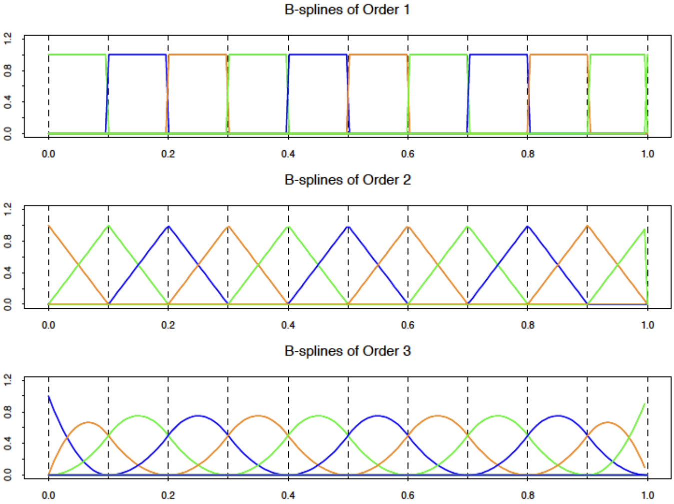
$$B_{i,1}(X) = \begin{cases} 1 & \text{if } X \in [\tau_i, \tau_{i+1}) \\ 0 & \text{otherwise} \end{cases}$$

- B spline basis function of order $m \leq M$

$$B_{i,m}(X) = \frac{X - \tau_i}{\tau_{i+m-1} - \tau_i} B_{i,m-1}(X) + \frac{\tau_{i+m} - X}{\tau_{i+m} - \tau_{i+1}} B_{i,m-1}(X)$$

Interpolation of two B spline basis functions with order $m - 1$

B spline basis functions with 10 knots (Hastie et al., 2009)



Multivariate B splines

- Tensor-product B splines

$$
\begin{aligned}
f(X,Z) &= \sum_{s_1=1}^{S_1} \sum_{s_1=1}^{S_1} B^X_{s_1}(X) B^Z_{s_2}(Z) \beta_{s_1 s_2} \\
&= \left(B^X(X)^T \otimes B^Z(Z)^T \right) \beta \\
&\equiv B^f\,(X,Z)^T\,\beta
\end{aligned}
$$

  - $B^X(X)$: $S_1 \times 1$ vector of B spline basis functions of X
  - $B^Z(Z)$: $S_2 \times 1$ vector of B spline basis functions of X
  - $B^f\,(X,Z)$: $(S_1 S_2) \times 1$ vector

Derivatives of B splines

- Partial derivative

$$\begin{aligned}\frac{\partial f(X,Z)}{\partial X} &= \sum_{s_1=1}^{S_1}\sum_{s_1=1}^{S_1} dB^X_{s_1}(X)B^Z_{s_2}(Z)\beta_{s_1 s_2} \\ &= \left(dB^X(X)^T \otimes B^Z(Z)^T\right)\beta \\ &\equiv \partial_X B^f\,(X,Z)^T\,\beta\end{aligned}$$

  - $dB^X(X)$: $S_1 \times 1$ vector of the derivative of B spline basis functions of X

Semiparametric Estimator: Step 1

- L equal partitions of $[0,1]$ and quantile points $T \equiv \{\tau_1, ..., \tau_{L-1}\}$,e.g $\{0.01, 0.02, ..., 0.99\}$ for $L = 100$
- The moment condition

$$E\left[1\left\{r_{it} \leq \phi_t\left(m_{it}, \tau_l\right)\right\} - \tau_l | m_{it-2}\right] = 0 \text{ for all } \tau_l \in T$$

- GMM quantile regression by Firpo, Galvao, Pinto, Poirier and Sanroman (2022)
  - Kernel smoothing of $1\{r_{it} \leq \phi_t\left(m_{it}, \tau_l\right)\}$ by CDF K with bandwidth b_n

$$E\left[K\left(\frac{\phi_t\left(m_{it}, \tau_l\right) - r_{it}}{b_n}\right) - \tau_l \middle| m_{it-2}\right] = 0 \text{ for all } \tau_l \in T$$

Semiparametric Estimator: Step 1

- B splines

$$E\left[\left(K\left(\frac{B^{\phi}\left(m_{it},\tau_l\right)^T\alpha - r_{it}}{b_n}\right) - \tau_l\right)B^m\left(m_{it-2}\right)\right] = 0 \text{ for all } \tau_l \in T$$

where

$$\phi_t\left(m_{it},\tau_l\right) = \sum_{s_1}\sum_{s_2} B_{s_1}(m_{it})B_{s_2}(\tau_l) = B^{\phi}\left(m_{it},\tau_l\right)^T\alpha$$

- $B^m\left(m_{it-2}\right)$: a $S_1 \times 1$ vector of B spline basis functions of m_{it-2} as IVs
- $L - 1 \geq S_2$ is necessary for identification

Semiparametric Estimator: Step 1

- GMM by stacking all $S_1(L-1)$ equations
  - Firpo, Galvao, Pinto, Poirier and Sanroman (2022) provides the expression of the variance weighting matrix
- Impose the monotonicity of $\phi_t\left(m_{it}, \tau_l\right)$ as linear restrictions on α

$$\frac{\partial \phi_t\left(m_{it}, u_{it}\right)}{\partial m_{it}} = \partial_m B^{\phi}\left(m_{it}, \tau_l\right)^T \alpha \geq 0$$
$$\frac{\partial \phi_t\left(m_{it}, u_{it}\right)}{\partial u_{it}} = \partial_u B^{\phi}\left(m_{it}, \tau_l\right)^T \alpha \geq 0.$$

Semiparametric Estimator: Step 2

- Profile likelihood estimator by Linton, Sperlich and Van Keilegom (2008)
- Conditional CDF of m_{it} on other variables v_{it}

$$G_{m|V}\left(m_{it}|v_{it}\right) = G_{\eta}\left(\lambda_t\left(m_{it}, u_{it}\right) - Z_{it}\delta\right)$$

- Conditional density

$$g_{m|v}\left(m_{it}|v_{it}\right) = g_{\eta}(\eta_{it})\frac{\partial\lambda_t\left(m_{it}, u_{it}\right)}{\partial m_{it}}$$

- Log conditional likelihood

$$\sum_{i=1}^{N} \ln g_{m|v}\left(m_{it}|v_{it}\right) = \sum_{i=1}^{N}\left[\ln g_{\eta}(\eta_{it}) + \ln \frac{\partial\lambda_t\left(m_{it}, u_{it}\right)}{\partial m_{it}}\right]$$

Semiparametric Estimator: Step 2

- B splines

$$\lambda_t\left(m_{it}, u_{it}; \beta\right) = B^{\lambda}\left(m_{it}, u_{it}\right)^T \beta$$
$$\lambda_{t-1}\left(m_{it-1}, u_{it-1}\right) = B^{\lambda}\left(m_{it-1}, u_{it-1}\right)^T \gamma$$

- Transformation model

$$\begin{aligned}\lambda_t\left(m_{it}, u_{it}; \beta\right) = {} & \theta_k k_{it} + \theta_l l_{it} - \rho\theta_k k_{it-1} - \rho\theta_l l_{it-1} \\ & + B^{\lambda}\left(m_{it-1}, u_{it-1}\right)^T \gamma + \eta_{it}\end{aligned} \tag{1}$$

- For given β, the regression of (1) obtains $\eta_{it}(\beta)$

Semiparametric Estimator: Step 2

- Kernel density estimator

$$\hat{g}_\eta(\eta_{it};\beta) = \frac{1}{nc_n}\sum_{i=1}^{n} k\left(\frac{\eta_{it}(\beta)-\eta_{jt}(\beta)}{c_n}\right)$$

where k is a kernel density (e.g., Gaussian) with bandwidth c_n

- The PL estimator

$$\begin{aligned}\hat{\beta} &\in \arg\max_{\beta} \sum_{i=1}^{N} \ln \hat{g}_{m|v;\beta}\left(m_{it}|v_{it}\right) \\ &= \sum_{i=1}^{N}\left[\ln \hat{g}_\eta(\eta_{it};\beta) + \ln \partial_m B^\lambda\left(m_{it},u_{it}\right)^T\beta\right]\end{aligned}$$

- We must impose the location and scale normalization

$$\lambda_t(m_1^*,u^*) = B^\lambda\left(m_1^*,u^*\right)^T\beta = 1$$
$$\lambda_t(m_0^*,u^*) = B^\lambda\left(m_0^*,u^*\right)^T\beta = 0.$$

Semiparametric Estimator: Step 3

- Let $\hat{\lambda}_{it} = B^{\lambda}(m_{it}, u_{it})^T \hat{\beta}$ and estimate

$$\hat{\lambda}_{it} = \delta_1 k_{it} + \delta_2 l_{it} + \delta_3 k_{it-1} + \delta_4 l_{it-1} + B^{\lambda}(m_{it-1}, u_{it-1})^T \delta_5 + \eta_{it}$$

  where $\hat{\theta}_k = \hat{\delta}_1$, $\hat{\theta}_l = \hat{\delta}_2$ and $\hat{\rho} = \left(\hat{\delta}_3/\hat{\delta}_1 + \hat{\delta}_4/\hat{\delta}_2\right)/2$.

- Material elasticity

$$\hat{\theta}_m = \text{median}\left\{\frac{\partial \hat{f}_t}{\partial m_{it}}\right\}$$

$$\text{where} \frac{\partial \hat{f}_t}{\partial m_{it}} = \frac{\partial \lambda_t}{\partial m_t}\left(\frac{\partial \phi_t}{\partial m_t} - \alpha_{it}^M\right)^{-1} \alpha_{it}^M$$

Semiparametric Estimator: Step 3

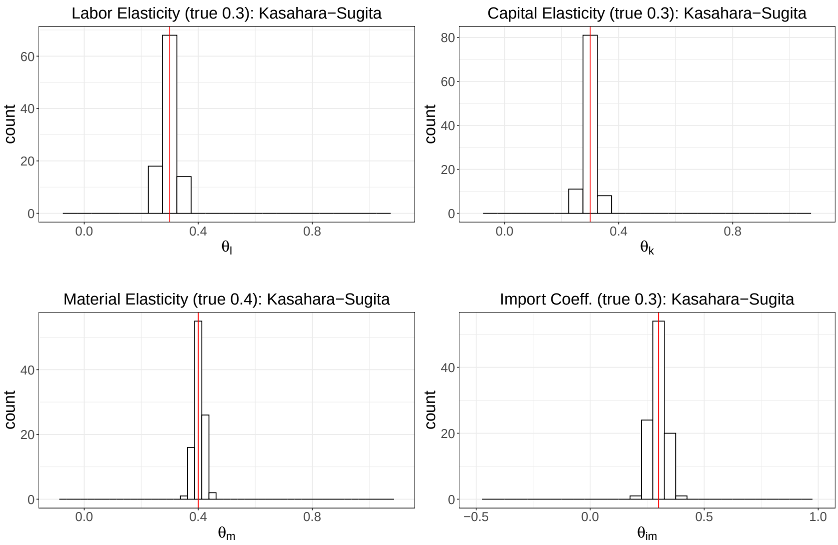
- Fix scale normalization by constant returns to scale: $b = \hat{\theta}_m + \hat{\theta}_k + \hat{\theta}_l$
- Elasticities, TFP

$$\tilde{\theta}_q = \frac{\hat{\theta}_q}{b} \text{ for } q = m, k, l$$
$$\hat{\omega}_{it} = \frac{\hat{\lambda}_{it}}{b} - \tilde{\theta}_k k_{it} - \tilde{\theta}_l l_{it}$$

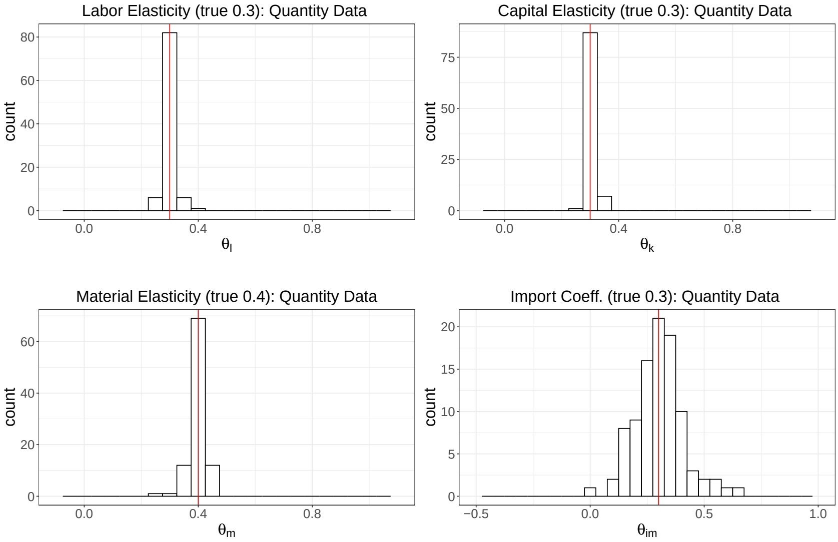
- Output, price and markup

$$\hat{y}_{it} = \tilde{\theta}_m m_{it} + \tilde{\theta}_k k_{it} + \tilde{\theta}_l l_{it} + \hat{\omega}_{it}$$
$$p_{it} = r_{it} - \hat{y}_{it}$$
$$\mu_{it} = \frac{\theta_m}{\alpha_{it}^M}$$

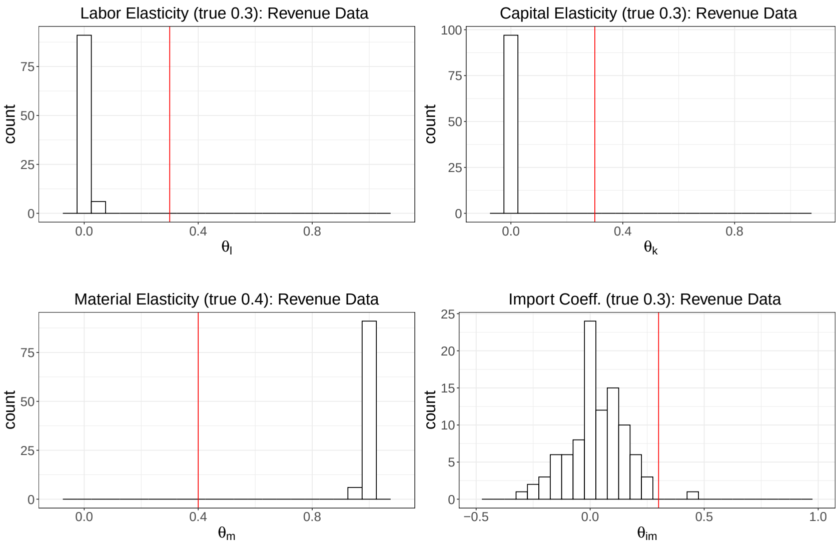
Our Estimator with Revenue Data



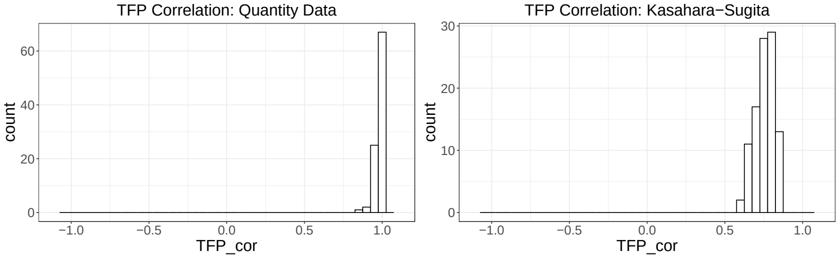
ACF with Quantity Data



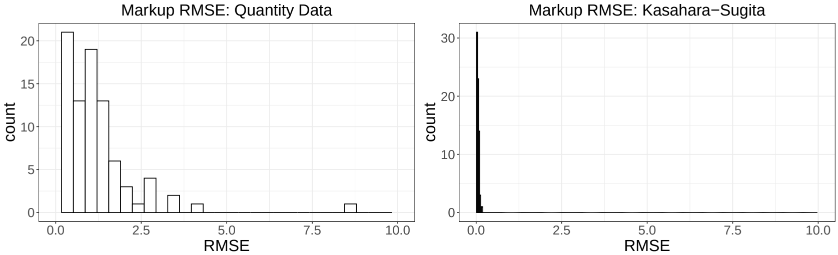
ACF with Revenue Data



TFP Correlations: ACF (Quantity) vs Our Estimator



Markup RMSE: ACF (Quantity) vs Our Estimator



Extensions

- Standard errors
- More flexible production function
- Parametric demand function
- Pooling multiple periods

More flexible production function

- *m*-separable production function

$$y_{it} = f_1(m_{it}) + f_2(k_{it}, l_{it}) + \omega_{it}$$

- Control function

$$\omega_{it} = \lambda_t(m_{it}, u_{it}) - f_2(k_{it}, l_{it})$$

- 1st step revenue

$$\begin{aligned} \varphi_t(y_{it}, u_{it}) &= \varphi_t(\theta_m m_{it} + \lambda_t(m_{it}, u_{it}), u_{it}) \\ &= \phi_t(m_{it}, u_{it}). \end{aligned}$$

Pooling multiple periods

- If we have $T > 4$ period data, we may want to pool them to estimate time-invariant elasticities more precisely.

- We need to adjust the difference in location and scale parameters across time.
- 1. Estimate b_1/b_t; e.g., $b_1/b_t = \sigma(\eta_t)/\sigma(\eta_1)$
- 2. Convert $\tilde{k}_{it} = (b_1/b_t)\, k_{it}$, $\tilde{l}_{it} = (b_1/b_t)\, l_{it}$, and so on.

Pooling multiple periods

- 3. Estimate $(\hat{\theta}_k, \hat{\theta}_l)$ with time fixed effects κ_t that adjusts location parameters:

$$\lambda_t\left(m_{it}, u_{it}; \hat{\beta}\right) = \kappa_t + \theta_k \tilde{k}_{it} + \theta_l \tilde{l}_{it} - \rho\theta_k \tilde{k}_{it-1} - \rho\theta_l \tilde{l}_{it-1} + \tilde{B}^{\lambda}\left(m_{it-1}, u_{it-1}\right)^T \gamma + \eta_{it}$$

- 4. Let $\partial \hat{f}_t / \partial m_{it}$ be an estimate from 4 period data

$$\hat{\theta}_m = \text{median}\left\{\frac{b_1}{b_t} \frac{\partial \hat{f}_t}{\partial m_{it}}\right\}$$

- 5. Identify scale parameter

$$b_1 = \hat{\theta}_m + \hat{\theta}_k + \hat{\theta}_l$$

Firpo, Sergio, Antonio F Galvao, Cristine Pinto, Alexandre Poirier, and Graciela Sanroman, "GMM quantile regression," *Journal of Econometrics*, 2022, *230* (2), 432–452.

Hastie, Trevor, Robert Tibshirani, Jerome H Friedman, and Jerome H Friedman, *The elements of statistical learning: data mining, inference, and prediction*, Vol. 2, Springer, 2009.

Linton, Oliver, Stefan Sperlich, and Ingrid Van Keilegom, "Estimation of a semiparametric transformation model," *The Annals of Statistics*, 2008, *36* (2), 686–718.