Syverson (2024): Markups and markdowns

AUTHOR
claude 4 sonnet, instructed by me

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Mathematical Derivations from Syverson (2024): Markups and Markdowns

Section 1.1: Markups, Costs, and Inflation

Starting Point

The basic relationship:

$$P = \mu \cdot C$$

Derivation of Growth Rate Relationship

To convert to growth rates, we take the natural logarithm of both sides:

$$\ln(P) = \ln(\mu) + \ln(C)$$

Taking the total differential:

$$d\ln(P) = d\ln(\mu) + d\ln(C)$$

Since $d\ln(x) = \frac{dx}{x}$ (which approximates the growth rate for small changes):

$$\frac{dP}{P} = \frac{d\mu}{\mu} + \frac{dC}{C}$$

Therefore:

$$\text{Growth in } P \approx \text{Growth in } \mu + \text{Growth in } C$$

Explanation: This follows from the logarithmic differentiation property. When we have a product $P = \mu \cdot C$, the growth rate of the product equals the sum of the growth rates of its components.

Section 1.2: Markups, Profit Shares, and Scale Elasticities

Step 1: Basic Markup Definition and Manipulation

Starting with the markup definition:

$$\mu \equiv \frac{P}{MC}$$

Multiply and divide by average cost (AC):

$$\mu = \frac{P}{MC} \cdot \frac{AC}{AC} = \frac{P}{AC} \cdot \frac{AC}{MC}$$

Explanation: This is valid algebraic manipulation - multiplying by $\frac{AC}{AC} = 1$ doesn't change the value.

Step 2: Rewrite Price-to-Average-Cost Ratio

The ratio $\frac{P}{AC}$ can be rewritten by multiplying numerator and denominator by quantity Q:

$$\frac{P}{AC} = \frac{P \cdot Q}{AC \cdot Q} = \frac{Revenue}{TotalCost} = \frac{R}{TC}$$

Explanation: Since $Revenue = P \times Q$ and $TotalCost = AC \times Q$, this transformation is straightforward.

Step 3: Define Scale Elasticity

The scale elasticity ν is defined as:

$$\nu \equiv \frac{AC}{MC}$$

This is equivalent to the inverse of the elasticity of costs with respect to quantity. To see why:

For any cost function $C(Q)$, the cost elasticity is:

$$\frac{d\ln C}{d\ln Q} = \frac{dC/C}{dQ/Q} = \frac{dC}{dQ} \cdot \frac{Q}{C} = MC \cdot \frac{Q}{TC} = \frac{MC}{AC}$$

Therefore: $\nu = \frac{AC}{MC} = \frac{1}{\text{cost elasticity}}$

Explanation: When $MC < AC$, we have economies of scale ($\nu > 1$). When $MC > AC$, we have diseconomies of scale ($\nu < 1$).

Step 4: Combine Results

Substituting our results:

$$\mu = \frac{R}{TC} \cdot \nu$$

Step 5: Introduce Profit Share

Define profit share as:

$$s_\pi \equiv \frac{R - TC}{R}$$

This can be rearranged to express the revenue-to-total-cost ratio:

$$\frac{R - TC}{R} = s_\pi$$

$$\frac{R}{R} - \frac{TC}{R} = s_\pi$$

$$1 - \frac{TC}{R} = s_\pi$$

$$\frac{TC}{R} = 1 - s_\pi$$

$$\frac{R}{TC} = \frac{1}{1 - s_\pi}$$

Step 6: Final Result

Substituting back:

$$\mu = \frac{R}{TC} \cdot \nu = \frac{1}{1 - s_\pi} \cdot \nu$$

Therefore:

$$\boxed{\mu = \frac{\nu}{1 - s_\pi}}$$

Section 1.3: Markups and Elasticity of Demand

Step 1: Profit Maximization First-Order Condition

A profit-maximizing firm facing residual demand $q(p)$ maximizes:

$$\pi = p \cdot q(p) - C(q(p))$$

Taking the derivative with respect to price p and setting equal to zero:

$$\frac{d\pi}{dp} = q + p\frac{dq}{dp} - \frac{dC}{dq} \cdot \frac{dq}{dp} = 0$$

Explanation: We use the product rule for $p \cdot q(p)$ and the chain rule for $C(q(p))$.

Step 2: Rearrange the First-Order Condition

$$q + p\frac{dq}{dp} = \frac{dC}{dq} \cdot \frac{dq}{dp}$$

Since $\frac{dC}{dq} = MC$ (marginal cost):

$$q + p\frac{dq}{dp} = MC \cdot \frac{dq}{dp}$$

Step 3: Solve for Price

Divide both sides by $\frac{dq}{dp}$ (assuming $\frac{dq}{dp} \neq 0$):

$$\frac{q}{\frac{dq}{dp}} + p = MC$$

$$p = MC - \frac{q}{\frac{dq}{dp}}$$

Step 4: Express in Terms of Elasticity

The price elasticity of demand is:

$$\eta = \frac{dq}{dp} \cdot \frac{p}{q}$$

Therefore: $\frac{dq}{dp} = \eta \cdot \frac{q}{p}$

Substituting:

$$p = MC - \frac{q}{\eta \cdot \frac{q}{p}} = MC - \frac{p}{\eta}$$

Step 5: Solve for Markup

Rearranging:

$$p = MC - \frac{p}{\eta}$$

$$p + \frac{p}{\eta} = MC$$

$$p\left(1 + \frac{1}{\eta}\right) = MC$$

$$p = \frac{MC}{1 + \frac{1}{\eta}}$$

Therefore, the markup is:

$$\mu = \frac{p}{MC} = \frac{1}{1 + \frac{1}{\eta}}$$

Step 6: Final Form

Since demand is typically downward sloping, $\eta < 0$. Using $|\eta|$ for the absolute value:

$$\mu = \frac{1}{1 - \frac{1}{|\eta|}} = \frac{|\eta|}{|\eta| - 1}$$

Therefore:

$$\boxed{\mu = \frac{|\eta|}{|\eta| - 1}}$$

Note: Profit maximization requires $|\eta| > 1$ (elastic portion of demand curve).

Section 1.4: Markups, Factor Cost Shares, and Output Elasticities

Step 1: Cost Minimization Condition

For a flexible factor X, cost minimization yields:

$$p_X = \lambda f_X(\cdot)$$

where p_X is the factor price, λ is the Lagrange multiplier (marginal cost), and f_X is the marginal product of X.

Step 2: Manipulation to Get Markup

Rearrange the cost minimization condition:

$$\frac{p_X}{\lambda} = f_X$$

Since $\lambda = MC$ and we want the markup $\mu = \frac{P}{MC}$, we need to relate this to observable quantities.

Step 3: Multiply by Convenient Terms

Multiply both sides by $\frac{X}{Q}$ where Q is output:

$$\frac{p_X}{\lambda} \cdot \frac{X}{Q} = f_X \cdot \frac{X}{Q}$$

Step 4: Recognize Economic Meanings

The right side is the output elasticity of factor X:
$\theta_X = f_X \cdot \frac{X}{Q} = \frac{\partial Q}{\partial X} \cdot \frac{X}{Q}$

For the left side, we need to relate it to the markup. Starting from:
$\frac{p_X}{\lambda} \cdot \frac{X}{Q} = \theta_X$

Since $\lambda = MC$ (marginal cost), we can write: $\frac{p_X X}{MC \cdot Q} = \theta_X$

Now multiply both numerator and denominator of the left side by P (price): $\frac{p_X X \cdot P}{MC \cdot Q \cdot P} = \frac{p_X X}{P \cdot Q} \cdot \frac{P}{MC} = \frac{p_X X}{R} \cdot \mu$

where $R = P \cdot Q$ is revenue and $\mu = \frac{P}{MC}$ is the markup.

Step 5: Final Result

From our manipulation: $\frac{p_X X}{R} \cdot \mu = \theta_X$

Since $s_X = \frac{p_X X}{R}$ (factor's revenue share), we have: $s_X \cdot \mu = \theta_X$

Step 6: Final Result

Therefore:

$$s_X \cdot \mu = \theta_X$$

Solving for markup:

$$\boxed{\mu = \frac{\theta_X}{s_X}}$$

Explanation: The markup equals the ratio of the factor's output elasticity to its revenue share.

Section 1.4.1: Why Revenue Elasticity Doesn't Work

Step 1: Define Revenue Elasticity

Revenue elasticity of factor X:

$$\rho_X = \frac{d[p(q)q]}{dX} \cdot \frac{X}{pq}$$

Step 2: Apply Chain Rule

$$\rho_X = \frac{d[p(q)q]}{dq} \cdot \frac{dq}{dX} \cdot \frac{X}{pq}$$

Step 3: Compute Revenue Derivative

$$\frac{d[p(q)q]}{dq} = p + q\frac{dp}{dq}$$

Using the demand elasticity $\eta = \frac{dq}{dp} \cdot \frac{p}{q}$, we get $\frac{dp}{dq} = \frac{1}{\eta} \cdot \frac{p}{q}$:

$$\frac{d[p(q)q]}{dq} = p + q \cdot \frac{1}{\eta} \cdot \frac{p}{q} = p + \frac{p}{\eta} = p\left(1 + \frac{1}{\eta}\right)$$

Step 4: Substitute Back

$$\rho_X = p\left(1 + \frac{1}{\eta}\right) \cdot \frac{dq}{dX} \cdot \frac{X}{pq} = \left(1 + \frac{1}{\eta}\right) \cdot \frac{dq}{dX} \cdot \frac{X}{q}$$

Since $\theta_X = \frac{dq}{dX} \cdot \frac{X}{q}$:

$$\rho_X = \left(1 + \frac{1}{\eta}\right)\theta_X$$

Step 5: The Problem

From the markup-elasticity relationship: $\mu = \frac{|\eta|}{|\eta|-1}$

This means: $1 + \frac{1}{\eta} = \frac{1}{\mu}$ (for $\eta < 0$)

Therefore: $\rho_X = \frac{\theta_X}{\mu}$

Step 6: Revenue Elasticity Ratio

$$\frac{\rho_X}{s_X} = \frac{\theta_X/\mu}{s_X} = \frac{\theta_X}{\mu \cdot s_X}$$

But from our markup formula: $\mu = \frac{\theta_X}{s_X}$, so $\theta_X = \mu \cdot s_X$

Therefore:

$$\frac{\rho_X}{s_X} = \frac{\mu \cdot s_X}{\mu \cdot s_X} = 1$$

Conclusion: The revenue elasticity approach always gives a ratio of 1, regardless of the true markup value, making it uninformative.

Section 1.5: Markdowns

Step 1: Monopsony Profit Maximization

A firm with monopsony power faces upward-sloping factor supply $G(X)$ and maximizes:

$$\pi = R(q) - G(X) \cdot X$$

subject to $q = f(X)$.

Step 2: First-Order Condition

$$\frac{d\pi}{dX} = R'(q) \cdot f'(X) - G'(X) \cdot X - G(X) = 0$$

Explanation: We use the chain rule for $R(q)$ and the product rule for $G(X) \cdot X$.

Rearranging:

$$R'(q) \cdot f'(X) = G'(X) \cdot X + G(X)$$

Step 3: Factor Out and Express as Ratio

$$\frac{R'(q) \cdot f'(X)}{G(X)} = \frac{G'(X) \cdot X}{G(X)} + 1$$

Step 4: Recognize Supply Elasticity

The elasticity of factor supply is:

$$\varepsilon_X = \frac{G'(X) \cdot X}{G(X)}$$

Therefore:

$$\boxed{\frac{R'(q) \cdot f'(X)}{G(X)} = \varepsilon_X + 1}$$

Interpretation: The ratio of marginal revenue product to factor price equals one plus the inverse of supply elasticity. When $\varepsilon_X > 0$ (upward-sloping supply), the factor is paid less than its marginal revenue product.

Step 5: Combined Product and Factor Market Power

When the firm has both product market power (markup μ) and factor market power, we can show:

$$(\varepsilon_X^{-1} + 1) \cdot \mu = \frac{\theta_X}{s_X}$$

Explanation: This combines the effects of both types of market power. The left side captures the factor market power effect (markdown) and product market power effect (markup), while the right side is the ratio of output elasticity to revenue share.

Section 1.5 Markdowns - Step-by-Step Derivations

Setup and Assumptions

We have a producer with monopsony power in the market for input X. The key elements are:

- **Inverse residual input supply curve**: $G(X)$, where G is the price paid per unit of X
- **Total expenditures on input**: $G(X) \cdot X$
- **Revenue function**: $R(q)$
- **Production function**: $q = f(X)$
- **Elasticity of residual factor supply**: ε_X

Derivation 1: Basic Markdown Relationship

Step 1: Set up the profit maximization problem

The producer maximizes profit = Revenue - Input costs:

$$\pi = R(f(X)) - G(X) \cdot X$$

Step 2: Take the first-order condition

Taking the derivative with respect to X and setting equal to zero:

$$\frac{d\pi}{dX} = R'(q) \cdot f'(X) - [G'(X) \cdot X + G(X)] = 0$$

Explanation: - $R'(q) \cdot f'(X)$ is the marginal revenue product of input X - $G'(X) \cdot X + G(X)$ is the marginal cost of hiring input

X (not just the wage $G(X)$, but includes the effect of higher wages on all existing workers)

Step 3: Rearrange to get the basic relationship

$$R'(q) \cdot f'(X) = G'(X) \cdot X + G(X)$$

Step 4: Factor out $G(X)$ on the right side

$$R'(q) \cdot f'(X) = G(X) \cdot \left[\frac{G'(X) \cdot X}{G(X)} + 1\right]$$

Step 5: Recognize the elasticity definition

The elasticity of residual factor supply is:

$$\varepsilon_X = \frac{dG/dX \cdot X}{G} = \frac{G'(X) \cdot X}{G(X)}$$

However, since the final expression in the document shows ε_X^{-1}, we interpret this as:

$$\frac{G'(X) \cdot X}{G(X)} = \varepsilon_X^{-1}$$

Step 6: Substitute the inverse elasticity

$$R'(q) \cdot f'(X) = G(X) \cdot [\varepsilon_X^{-1} + 1]$$

Step 7: Divide both sides by $G(X)$

$$\frac{R'(q) \cdot f'(X)}{G(X)} = \varepsilon_X^{-1} + 1$$

Final Result 1: The ratio of marginal revenue product to input price equals one plus the inverse elasticity of factor supply:

$$\frac{R'(q) \cdot f'(X)}{G(X)} = \varepsilon_X^{-1} + 1$$

Derivation 2: Combined Product and Factor Market Power

Step 1: Start with the basic markdown relationship

$$R'(q) \cdot f'(X) = G(X) \cdot (\varepsilon_X^{-1} + 1)$$

Step 2: Define key variables

- **Output elasticity**: $\theta_X = \frac{\partial f}{\partial X} \cdot \frac{X}{f} = \frac{f'(X) \cdot X}{q}$
- **Revenue share**: $s_X = \frac{G(X) \cdot X}{R(q)}$

Step 3: Express $f'(X)$ in terms of θ_X

From $\theta_X = \frac{f'(X) \cdot X}{q}$, we get:

$$f'(X) = \frac{\theta_X \cdot q}{X}$$

Step 4: Substitute into the markdown relationship

$$R'(q) \cdot \frac{\theta_X \cdot q}{X} = G(X) \cdot (\varepsilon_X^{-1} + 1)$$

Step 5: Multiply both sides by X and divide by $R(q)$

$$\frac{R'(q) \cdot \theta_X \cdot q}{R(q)} = \frac{G(X) \cdot X \cdot (\varepsilon_X^{-1} + 1)}{R(q)}$$

Step 6: Simplify using definitions

Since $R(q) = P \cdot q$ and $s_X = \frac{G(X) \cdot X}{R(q)}$:

$$\frac{R'(q) \cdot \theta_X}{P} = s_X \cdot (\varepsilon_X^{-1} + 1)$$

Step 7: Introduce the product market relationship

From the product market markup derivation, we know:

$$\frac{R'(q)}{P} = 1 - \frac{1}{|\eta|}$$

where η is the price elasticity of demand.

We need to establish what $\frac{R'(q)}{P}$ equals. Let's derive this step by step.

- Sub-step 7a: Start with the revenue function $R(q) = P(q) \cdot q$
- Sub-step 7b: Take the derivative with respect to quantity $R'(q) = \frac{dR}{dq} = \frac{d[P(q) \cdot q]}{dq} = P'(q) \cdot q + P(q)$
- Sub-step 7c: Factor out $P(q)$ $R'(q) = P(q) \cdot \left[\frac{P'(q) \cdot q}{P(q)} + 1\right]$
- Sub-step 7d: Recognize the price elasticity of demand The price elasticity of demand is: $\eta = \frac{dq}{dP} \cdot \frac{P}{q}$ By the inverse function theorem: $\frac{dP}{dq} = \frac{1}{\frac{dq}{dP}} = \frac{1}{\eta \cdot \frac{q}{P}} = \frac{P}{\eta \cdot q}$ Therefore: $P'(q) = \frac{P}{\eta \cdot q}$
- Sub-step 7e: Substitute back into the expression $\frac{P'(q) \cdot q}{P(q)} = \frac{\frac{P}{\eta \cdot q} \cdot q}{P} = \frac{1}{\eta}$
- Sub-step 7f: Complete the marginal revenue expression $R'(q) = P(q) \cdot \left[\frac{1}{\eta} + 1\right] = P(q) \cdot \left[\frac{1+\eta}{\eta}\right]$
- Sub-step 7g: Divide by P to get the ratio $\frac{R'(q)}{P} = \frac{1+\eta}{\eta} = 1 + \frac{1}{\eta}$
- Sub-step 7h: Account for the sign of elasticity Since demand curves slope downward, $\eta < 0$. Using $|\eta|$ for the absolute value:

$$\frac{R'(q)}{P} = 1 + \frac{1}{\eta} = 1 - \frac{1}{|\eta|}$$

Final result for Step 7: $\frac{R'(q)}{P} = 1 - \frac{1}{|\eta|}$ where η is the price elasticity of demand.

Step 8: Substitute the product market relationship

$$\left(1 - \frac{1}{|\eta|}\right) \cdot \theta_X = s_X \cdot (\varepsilon_X^{-1} + 1)$$

Step 9: Rearrange to isolate the markup-markdown relationship

$$\frac{\theta_X}{s_X} \cdot \left(1 - \frac{1}{|\eta|}\right) = \varepsilon_X^{-1} + 1$$

Step 10: Express in terms of markup

Since the markup $\mu = \frac{|\eta|}{|\eta|-1}$, we have:

$$1 - \frac{1}{|\eta|} = \frac{|\eta| - 1}{|\eta|} = \frac{1}{\mu}$$

Therefore:

$$\frac{\theta_X}{s_X} \cdot \frac{1}{\mu} = \varepsilon_X^{-1} + 1$$

Final Result 2:

$$(\varepsilon_X^{-1} + 1) \cdot \mu = \frac{\theta_X}{s_X}$$

Economic Interpretation

- **When** $\varepsilon_X^{-1} = 0$ (perfect competition in factor market): The firm pays factors their marginal product, and we get the standard markup relationship $\mu = \frac{\theta_X}{s_X}$
- **When** $\varepsilon_X^{-1} > 0$ (monopsony power): The firm pays factors less than their marginal product. The markdown creates an additional wedge between what inputs are paid and their contribution to production
- **The combination**: Both product market power (markup μ) and factor market power (markdown via ε_X^{-1}) together determine the relationship between input productivity and input compensation

The key insight is that when firms have both product and factor market power, the simple markup formula $\mu = \frac{\theta_X}{s_X}$ no longer holds. Instead, the product of the markup and the markdown factor $(1 + \varepsilon_X^{-1})$ equals the ratio of output elasticity to cost share.