Note on growth and growth accounting

1. Growth and the growth rate

In this section aspects of the mathematical concept of the rate of growth used in growth models and in the empirical analysis of growth are set out.

Notation used in growth theory. Use is often made of the so-called dot notation $\dot{Y}$, to refer to the rate of change of key variables. "Y dot" is the rate of change of output and is closely related to the familiar concept of the change in output between period t and period $t + 1$, $Y_{t+1} - Y_t$, which is often abbreviated as ΔY. $\dot{Y}$ stands for the continuous time equivalent of ΔY and is defined as the time derivative of Y, that is dY/dt. To summarize:

$$\Delta Y \equiv Y_{t+1} - Y_t \qquad \text{(Rate of change, discrete time)}$$

$$\dot{Y} \equiv dY/dt. \qquad \text{(Rate of change, continuous time)}$$

Does it matter whether we work in continuous time or discrete time? Generally speaking in growth theory it is easier and requires less cumbersome notation to derive the results in continuous time.[1] The *(proportional) rate of growth* is defined in discrete time as

$$\gamma_Y = \frac{Y_{t+1} - Y_t}{Y_t} = \frac{\Delta Y}{Y} \qquad \text{(Growth rate, discrete time)}$$

and in continuous time as

$$g_Y = \frac{\dot{Y}}{Y} = \frac{dY/dt}{Y}. \qquad \text{(Growth rate, continuous time)}$$

In each case, the growth rate, e.g. 0.02, is multiplied by 100 to produce a *percentage* growth rate.

Calculating growth rates. As this chapter is about economic growth it is crucial to understand the concept of a growth rate and how to measure it using data. The simplest notion of a growth rate is the percentage change in GDP between periods (quarters or years) as defined above. For example US GDP in the year 2000 was $\$8,987$ bill. and it was $\$8,627$ bill. in the year 1999 (both measured in 1995 prices). Hence the US economy grew by 4.2% between 1999 and 2000.

Once we move beyond growth rates over very short periods, it is necessary to calculate the so-called compound growth rate. Economists interested in growth often focus on average annual growth rates over lengthy periods of time. We take an example and compare three methods.

- First, we calculate the average growth rate in Chinese GDP per capita between 1988 and 1998 using a 'long average' method: $1/10 \cdot \left(\frac{Y_{98} - Y_{88}}{Y_{88}}\right) \cdot 100 = 7.16\%$ p.a..
- Second, we calculate the annual growth rate each year of the decade and take the average: the answer is 5.7% p.a..
- Third, we calculate the annual compound growth rate using a formula: AGR (annual percentage growth rate) $= (\exp(1/10 \cdot \log(Y_{98}/Y_{88})) - 1) \cdot 100 = 5.6\%$ p.a.. Note that it is the $\ln$ function on the calculator that must be used since this is the natural logarithm function; although we use $\log$ in the text, this always refers to the natural logarithm.

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[1]From a mathematical point of view we could prove all our results in discrete time as well, and the equations we have derived in continuous time would hold as approximations to the discrete time results.

The 'long average' method is incorrect: it overstates the growth rate because it neglects the fact that the base for growth is continuously rising.[2] The average of the annual growth rates will be a reasonable approximation to the compound growth rate for low growth rates but loses accuracy as the growth rate rises. The comparison between the average of the annual growth rates and the compound growth rate is exactly the same as between simple interest (compounded annually) and compound interest (compounded continuously). The AGR is the equivalent of the APR for interest rates (the annual percentage rate of interest reported on credit card bills, for example). We need to explain some more concepts before we return to show where the formula for the annual compound growth rate (AGR) comes from.

1.1. Growth rates, exponential and log functions. To see that the growth of GDP in discrete time, γ_Y, is the same as the growth rate in continuous time, g_Y, when the time period is short enough, let us start from the proportional growth of GDP between two periods in time given by, t and $t + \Delta t$, where Δt is small. When we make this interval smaller and smaller, we say that we are taking the limit as Δt tends to zero:

$$\lim_{\Delta t \to 0} \gamma_Y = \lim_{\Delta t \to 0} \left\{ \frac{Y_{t+\Delta t} - Y_t}{\Delta t} \cdot \frac{1}{Y_t} \right\} \tag{1.1}$$

$$= \frac{dY}{dt} \frac{1}{Y} = \frac{\dot{Y}}{Y} = g_Y. \tag{1.2}$$

We express the proportional growth expression per unit of time and then the rules of calculus tell us that if we make this period of time very small the first term in the brackets is nothing other than the definition of the derivative of output with respect to time, $\dot{Y}$. g_Y is known as the instantaneous growth rate. Thus when we quote $g_Y = 0.02$, this means that the growth rate of GDP during the period of interest is 2%.

It is very useful to see that the growth rate can be expressed in several equivalent ways:

$$g_Y = \frac{\dot{Y}}{Y} = \frac{dY/dt}{Y} = \frac{d \log Y}{dt} \qquad \text{(Equivalent expressions, growth rate)}$$

In order to see that the last equality holds we can use the chain rule of calculus and the fact that the derivative of the log of a variable, Y, is $\frac{1}{Y}$, i.e. $\frac{d \log Y}{dY} = \frac{1}{Y}$:

$$\frac{d \log Y}{dt} = \frac{d \log Y}{dY} \frac{dY}{dt} = \frac{dY/dt}{Y}. \tag{1.3}$$

We now explore further what economists mean when they say that "output grows exponentially". Let us start with the previous example where for some economy $g_Y = 0.02$. In this case, if we know the level of output initially, Y_0, and the growth rate, then the level of output at time t comes from the equation $Y_t = Y_0 \exp(g_Y \cdot t)$, where $\exp$ is the exponential function and the growth rate g_Y is assumed to be constant. How do we know that this equation describes the evolution of output for our economy that grows at rate $g_Y = 0.02$? It is easier to see this if we do the proof in 'reverse'. That is, let us start from the equation:

$$Y_t = Y_0 \exp(g_Y \cdot t) \tag{1.4}$$

where Y_0 stands for the initial value of output. Taking logs of both the left and the right side of this we obtain:

$$\log Y_t = \log Y_0 + g_Y \cdot t \tag{1.5}$$

If we now take derivatives with respect to time of this expression we obtain

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[2]To see why the first method is incorrect, apply the 7.16% annual growth rate to the base year GDP per capita of $1816 and then for each subsequent year. The result is a level of GDP p.c. in 1998 far higher than that recorded ($3117).
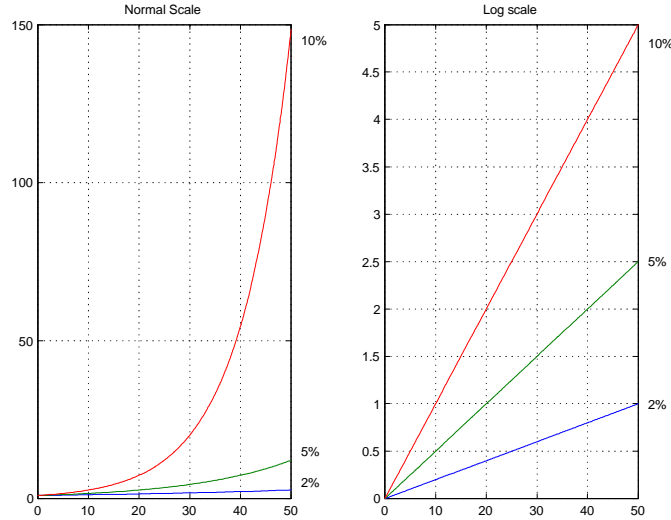
FIGURE 1. Simulation of exponential growth on a normal and logarithmic scale.

$$\frac{d \log Y_t}{dt} = \frac{d \log Y_0}{dt} + g_Y \tag{1.6}$$

$$= g_Y. \tag{1.7}$$

The initial value of output is of course fixed at all future values and hence $\frac{d \log Y_0}{dt} = 0$. Thus we have shown that $g_Y = d \log Y/dt$, as expected.

An understanding of the relationship between exponential growth and logs is very useful for any analysis of growth.[3] As the plots in the left hand panel of Fig.1 show, once we have the initial value for GDP and the growth rate, we can read off the level of output at any subsequent date. Growth paths for GDP are shown for a 2%, 5% and 10% growth rate. We also note that the equation for $\log Y_t$ is linear, with the slope of the line equal to the growth rate. The growth paths for GDP are shown in the right hand panel using the log scale: the growth paths are straight lines.

A second reason for understanding the relationship between logs and growth rates is that this relationship lies behind some very useful rules for handling growth rates. The following rules are used frequently:

(1) if $Y = X^\alpha$ then $\log Y = \alpha \log X$. This is useful because it allows us to go from levels to growth rates. For example, if we begin with the widely used production function, the Cobb Douglas,

$$Y = K^\alpha L^{1-\alpha} \qquad \text{(Cobb Douglas production function)}$$

where $\alpha < 1$, and first take logs:

$$\log Y = \alpha \log K + (1 - \alpha) \log L$$

then differentiate with respect to time and use the fact that $\frac{d \log Y}{dt} = \frac{\dot{Y}}{Y}$ we get:

$$\frac{d \log Y}{dt} = \alpha \frac{d \log K}{dt} + (1 - \alpha) \frac{d \log L}{dt}$$

$$\frac{\dot{Y}}{Y} = \alpha \frac{\dot{K}}{K} + (1 - \alpha) \frac{\dot{L}}{L}.$$

We can see from this that the growth rate of output is a weighted average of the growth rates of capital and labour inputs.

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[3]For a concise and insightful discussion of the exponential and logarithmic functions, see Chapter 9 in M. Pemberton and N. Rau Mathematics for Economists. Manchester University Press, 2001.

(2) the growth rate of the ratio of two variables is equal to the difference between the two growth rates: $g_{\frac{X}{Y}} = g_X - g_Y$. This rule is used often. For example, it says that the growth rate of output per worker $(\frac{\dot{Y}}{L})$ is equal to the growth rate of output $(\frac{\dot{Y}}{Y})$ minus the growth rate of workers $(\frac{\dot{L}}{L})$. To show this, the same technique is used as for (1): $\log(\frac{X}{Y}) = \log X - \log Y$ and $\frac{d \log(\frac{X}{Y})}{dt} = \frac{d \log X}{dt} - \frac{d \log Y}{dt}$; hence $g_{\frac{X}{Y}} = g_X - g_Y$.

(3) the growth rate of the product of two variables is equal to the sum of their growth rates: $g_{XY} = g_X + g_Y$. This follows the same logic as for 2.

1.2. Useful definitions for growth theory. In growth models in continuous time, the concept of the growth rate is $g_Y = \frac{\dot{Y}}{Y} = \frac{d \log Y}{dt}$, which we can express also using $Y_t = Y_0 \exp(g_Y \cdot t)$ and equivalently as $\log Y_t = \log Y_0 + g_Y \cdot t$. As noted above, the growth rate g_Y is the *instantaneous* growth rate. This may be calculated from data as follows:

$$
\begin{aligned}
\log Y_t &= \log Y_0 + g_Y \cdot t \\
g_Y \cdot t &= \log Y_t - \log Y_0 \\
g_Y &= \frac{1}{t} \left(\log Y_t - \log Y_0 \right),
\end{aligned}
$$

where t is the number of years, Y_0 is the base year level of output and Y_t the final year level. g_Y is multiplied by 100 to get the percentage growth rate. This is often referred to as the 'log difference' method of calculating the growth rate. When referring to the growth of output, economists typically mean the annual growth rate (the discrete time concept), rather than the instantaneous growth rate (the continuous time concept). Similarly, when discussing interest rates, it is the annual percentage rate (APR) rather than the instantaneous rate that is typically calculated. The annual percentage growth rate (or annual compound growth rate as it is sometimes called) is $(\exp g_Y - 1)$ expressed as a percentage and this is the formula for calculating the compound growth rate provided above:

$$
\begin{aligned}
AGR &= (\exp g_Y - 1) \cdot 100 \\
&= \left[\exp \left(\frac{1}{t} \left(\log Y_t - \log Y_0 \right) \right) - 1 \right] \cdot 100.
\end{aligned}
$$

The relationship between the different concepts of the growth rate is clarified by noting that when we consider growth for one year, the familiar proportional rate of growth is G (multiplied by 100):

$$
\begin{aligned}
Y_1 &= Y_0(1 + G) \\
G &= \frac{Y_1 - Y_0}{Y_0} = \frac{\Delta Y}{Y}
\end{aligned}
$$

Extending to many periods, we have

$$
\begin{aligned}
Y_t &= Y_0(1 + G)^t \\
\log \frac{Y_t}{Y_0} &= t \log(1 + G) \\
\exp \left(\frac{1}{t} \left(\log Y_t - \log Y_o \right) \right) &= 1 + G \\
G &= \exp \left(\frac{1}{t} \left(\log Y_t - \log Y_o \right) \right) - 1 \\
&= \exp g_y - 1.
\end{aligned}
$$

This illustrates that the AGR is the multi-period analogue to the one-period proportional growth rate.

1.3. Summary.

- When using data to measure economic growth, the average of the annual percentage growth rates is a close approximation to the true annual (compound) percentage growth rate (AGR) when the growth rate is low (less than 5% p.a.). In the earlier example of Chinese growth, the average of the annual percentage growth rates is 5.58% p.a.; and the AGR is 5.55% p.a..
- Taking the difference between the natural logarithms of output at the start of the period and the end of the period and dividing by the length of the period gives the instantaneous growth rate, g_Y. When multiplied by 100, this is the percentage growth rate in continuous time. To get the AGR, we use the fact that $AGR = (\exp g_Y - 1) \cdot 100$. This is the percentage growth rate in discrete time. When referring to data, economists normally use discrete time concepts and therefore refer to the AGR; however growth theory is normally expressed in continuous time, with reference to g_Y. When growth rates are low, these are close to each other. In the example of Chinese growth, the $AGR = 5.55\%$ p.a. and $g_y \cdot 100 = 5.40\%$ p.a..
- Growth theory makes heavy use of the following equivalent expressions for the (instantaneous) growth rate: $g_Y = \frac{\dot{Y}}{Y} = \frac{dY/dt}{Y} = \frac{d\log Y}{dt}$; $Y_t = Y_0 \exp(g_Y \cdot t)$ and $\log Y_t = \log Y_0 + g_Y \cdot t$.
- Frequent use is made of the procedure of 'taking logs and differentiating' to go from expressions in levels to growth rates, and of the rules $g_{\frac{X}{Y}} = g_X - g_Y$ and $g_{XY} = g_X + g_Y$.

2. Growth accounting

The idea of growth account is to account for the contribution to the growth of output made by the growth of factor inputs (capital and labour) and to associate any growth unaccounted for to 'technological progress'. Solow referred to this residual as total factor productivity growth. Total factor productivity growth captures the impact of intangible aspects of human progress that allow both labour and capital to increase their productivity. Thinking of the period 1990 to 2000, what has made the US economy more productive and has increased welfare was not simply the fact that firms have invested and bought computers, but rather that the information revolution has allowed new machines based on computer technology to be more productive and workers who use computers to be more efficient.

Solow's method of calculating total factor productivity growth is known as (*Solow*) *growth accounting*. To see how this works we can start from a production function defined in terms of capital, labour and an index of the level of technological progress given by λ, which is a function of time:

$$Y = F(K, L, \lambda). \tag{2.1}$$

Let us now take logs of the production function and differentiate with respect to time to obtain (using the chain rule as shown in the previous section):

$$\dot{Y} = F_K \dot{K} + F_L \dot{L} + F_\lambda \dot{\lambda}, \tag{2.2}$$

where F_K is $\frac{\partial F(K,L,\lambda)}{\partial K}$, F_L is $\frac{\partial F(K,L,\lambda)}{\partial L}$ and F_λ is $\frac{\partial F(K,L,\lambda)}{\partial \lambda}$. Applying exactly the same technique as used above in deriving the expression for the growth rate of output, we obtain:

$$g_Y = \frac{F_K K}{Y} g_K + \frac{F_L L}{Y} g_L + \frac{F_\lambda \lambda}{Y} g_\lambda. \tag{2.3}$$

The growth of output is equal to a function of the growth rates of capital, labour and the technology factor. It is possible to simplify this if we make further assumptions about the market environment. This will enable it to be used to get an estimate of the final term, which is called total factor productivity growth, from data. We assume that labour and capital are traded in competitive markets and are paid their respective marginal products. This means that the marginal product of labour, $F_L(K, L, \lambda) = w$, where w is the real wage, and that the marginal product of capital,

	1948-2001	1948-1973	1974-1995	1996-2001
Total GDP Growth	2.5	3.3	1.5	2.5
- due to capital	0.9	0.9	0.7	1.2
- due to labour	0.2	0.2	0.2	0.4
Solow residual	1.3	2.1	0.6	0.9

TABLE 1. Solow Growth Accounting for the United States, 1948-2001. Source: BLS

$F_K(K, L, \lambda) = R$, where R is the rental cost of capital in the economy. Additionally, we define the *Solow residual* by $SR = \frac{F_\lambda \lambda}{Y} g_\lambda$. We can now rewrite the expression above as:

$$g_Y = \frac{RK}{Y} g_K + \frac{wL}{Y} g_L + SR. \tag{2.4}$$

Since $\sigma_K = (RK)/Y$ corresponds to the share of total income spent by the economy on payments to capital it is often called the *capital* or *profit share*. Similarly, $\sigma_L = (wL)/Y$ corresponds to the share of total income spent by the economy on payments to labour and is called the *labour share*. Hence, we have a compact expression for the Solow residual, which is also called *total factor productivity growth* as

$$TFP \text{ growth} = SR = g_Y - \sigma_K g_K - \sigma_L g_L,$$

(Total factor productivity (TFP) growth: the Solow Residual)

The Solow residual is the difference between output growth and a weighted sum of factor growths with weights given by the factor shares, i.e. it is the growth that is not attributed to the growth of either labour or capital inputs. If we accept the market assumptions we have used to derive the formula above we now have an expression that can be used to estimate the Solow residual (i.e. TFP growth) from macroeconomic data. When the production function is constant returns to scale, the sum of the labour and capital shares is one and we can simplify equation (Total factor productivity (TFP) growth: the Solow Residual) to

$$TFP \text{ growth} = SR = g_Y - \sigma_K g_K - (1 - \sigma_K) g_L.$$

In empirical analysis, it is often illuminating to work in per capita terms and we can rearrange this equation as follows:

$$\begin{aligned} TFP \text{ growth} &= (g_Y - g_L) - \sigma_K g_K + \sigma_K g_L \\ &= g_y - \sigma_K g_k, \end{aligned}$$

where $y \equiv Y/L$ and $k \equiv K/L$. We can also turn the equation around and decompose the growth of productivity into the contribution from the growth of capital intensity and the contribution of TFP growth:

$$g_y = \sigma_K g_k + TFP \text{ growth}.$$

Growth accounting exercises may also seek to separately identify the contribution to the growth of output per worker from improvements in labour quality, e.g. as measured by increased average years of education. This will lead to a further component in the above equation:

$$g_y = \sigma_K g_k + (1 - \sigma_K) q + TFP \text{ growth},$$

where q is the growth rate of the quality of labour inputs (per worker). This is weighted by labour's share in output, $1 - \sigma_K$.

2.1. Example. In Table 1 we present an analysis of growth accounting as reported by the Bureau of Labor Statistics in the United States. It is interesting to note that according to growth accounting, the so-called New Economy phase of US growth in the late 1990s is associated with a higher contribution from factor input growth, especially capital, *and* faster TFP growth. TFP growth in the late 1990s is, however, not as fast as in the 1950s and 1960s.