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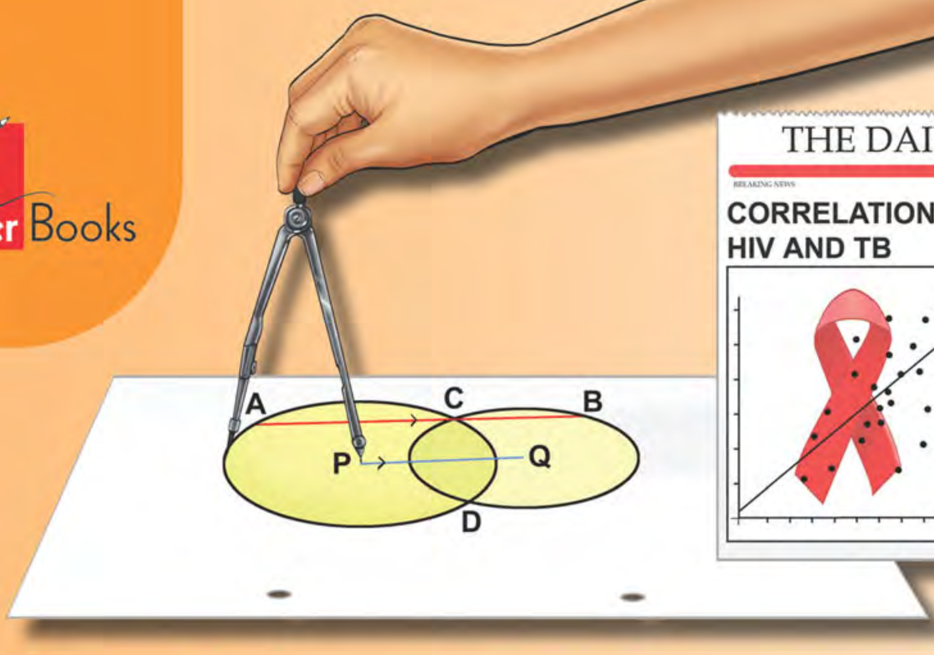

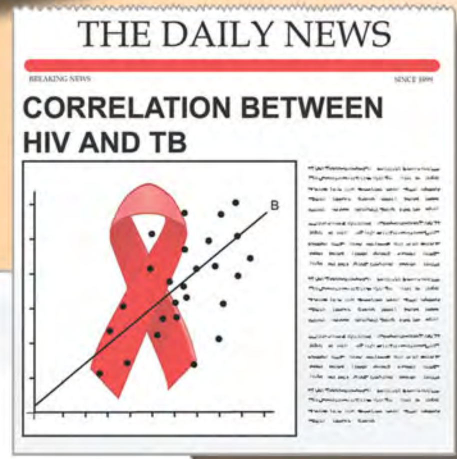

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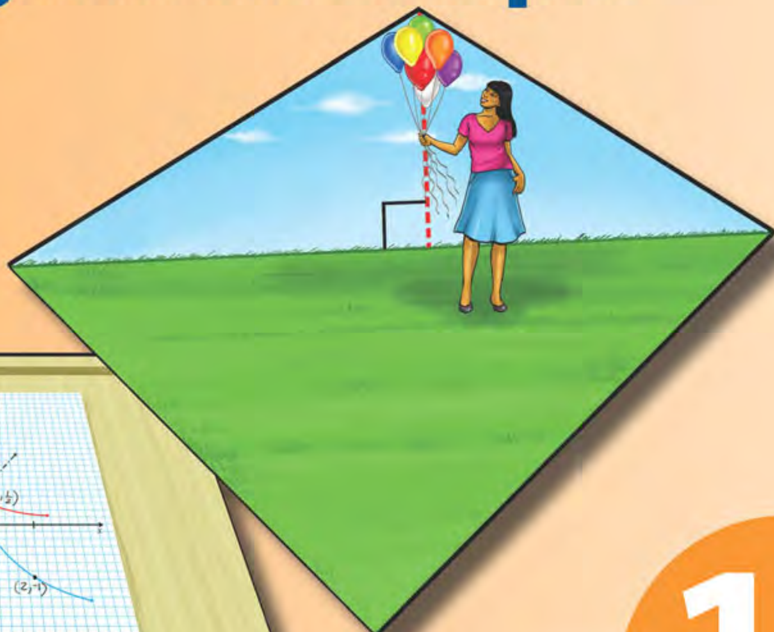

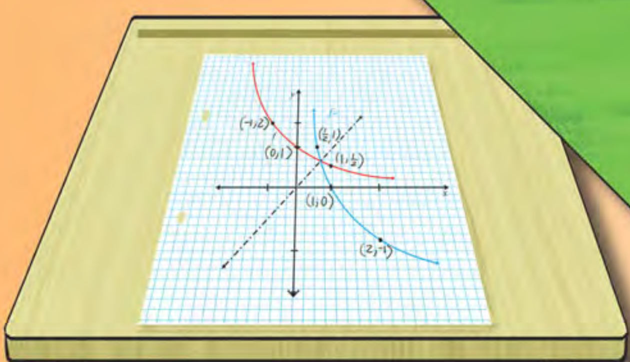

CAPS

12

Learner's Book

J Aird • L du Toit • I Harrison • C van Duyn • J van Duyn

Clever Keeping Maths Simple

Grade 12

Learner's Book

J Aird

L du Toit

I Harrison

C van Duyn

J van Duyn

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Chapter 1 Patterns, sequences and series

In this chapter you will:

- learn about arithmetic and geometric sequences
- write different series in sigma notation
- derive and use the formulae for the sum of arithmetic and geometric series.

Introduction

A **sequence** is an ordered set of numbers.

A **progression** is a sequence in which we can obtain the value of any element based on the values of the preceding elements.

The following table shows the first four terms of two different sequences. We refer to each term in a sequence using the notation T_n where n represents the position of the term. Both sequences are progressions.

	Term 1 T_1	Term 2 T_2	Term 3 T_3	Term 4 T_4
Sequence 1	7	14	21	28
Sequence 2	2	4	8	16

A **series** is the sum of the elements of a sequence. For example, if we add the values of the second sequence in the previous table, we have the series $2 + 4 + 8 + 16 + \ldots$.

Arithmetic progressions (AP)	**Geometric progressions (GP)**
An arithmetic progression (AP) is a sequence in which each term after the first term is formed by **adding** a **constant value** (d) to the preceding term.	A geometric progression (GP) is a sequence in which each term after the first term is formed by **multiplying** the preceding term by a **constant ratio** (r).

Example 1

T_1	T_2	T_3	T_4	T_5
2	5	8	11	14
	$= 2 + 3$	$= 5 + 3$	$= 8 + 3$	$= 11 + 3$

$\therefore$ This is an arithmetic sequence with $d = 3$

Example 1

T_1	T_2	T_3	T_4
2	6	18	54
	$= 2 \times 3$	$= 6 \times 3$	$= 18 \times 3$

$\therefore$ This is a geometric sequence with $r = 3$

Example 2

T_1	T_2	T_3	T_4	T_5
16	12	8	4	0
	$= 16 - 4$	$= 12 - 4$	$= 8 - 4$	$= 4 - 4$

$\therefore$ This is an arithmetic sequence with $d = -4$

Example 2

T_1	T_2	T_3	T_4
16	-8	4	-2
	$= 16 \times \left(-\frac{1}{2}\right)$	$= -8 \times \left(-\frac{1}{2}\right)$	$= 4 \times \left(-\frac{1}{2}\right)$

$\therefore$ This is a geometric sequence with $r = -\frac{1}{2}$

Example 3

If $T_n = 4 - 2n$, determine the sequence.

Solution

$T_1 = 4 - 2(1) = 4 - 2 = 2$

$T_2 = 4 - 2(2) = 4 - 4 = 0$

$T_3 = 4 - 2(3) = 4 - 6 = -2$

$\therefore$ the sequence is 2; 0; -2; …

Example 3

If $T_n = 3.2^{n-1}$, determine the sequence.

Solution

$T_1 = 3.2^{1-1} = 3.2^0 = 3.1 = 3$

$T_2 = 3.2^{2-1} = 3.2^1 = 3.2 = 6$

$T_3 = 3.2^{3-1} = 3.2^2 = 3.4 = 12$

$\therefore$ the sequence is 3; 6; 12; …

Determine whether a sequence is arithmetic or geometric

Given a sequence, we can use a formula to test if a sequence is arithmetic, geometric or neither.

Arithmetic progressions (AP)

To test whether a sequence is arithmetic, use the formula : $T_3 - T_2 = T_2 - T_1 = d$

Geometric progressions (GP)

To test whether a sequence is geometric, use the formula : $\frac{T_3}{T_2} = \frac{T_2}{T_1} = r$

Example 1

Determine whether the following sequence is arithmetic, geometric or neither.

2; 4; 6; 8; 10; …

Solution

T_1	T_2	T_3	T_4	T_5
2	4	6	8	10

$T_3 - T_2 = 6 - 4$
$= 2$

$T_2 - T_1 = 4 - 2$
$= 2$

$\therefore$ This is an arithmetic sequence with $d = 2$.

Example 1

Determine whether the following sequence is arithmetic, geometric or neither.

2; 6; 18; 54; …

Solution

T_1	T_2	T_3	T_4
2	6	18	54

$\frac{T_3}{T_2} = \frac{18}{6} = 3$ $\frac{T_2}{T_1} = \frac{6}{2} = 3$

$\therefore$ This is a geometric sequence with $r = 3$.

Exercise 1.1

1. Determine whether the following sequences are arithmetic, geometric or neither:
   - a) 5; 8; 11; 14; 17; …
   - b) 5; 10; 20; 40; 80; …
   - c) 1; 4; 9; 16; 25; …
   - d) 32; 16; 8; 4; 2; …
   - e) 6; 10; 14; 18; 22; …
   - f) 12; 7; 2; -3; -8; …
   - g) $\frac{1}{2}$; $\frac{3}{4}$; $\frac{5}{6}$; $\frac{7}{8}$; $\frac{9}{10}$; …
   - h) 1; -3; 9; -27; 81; …
   - i) 7; 4; 1; -2; -5; …
   - j) 1; 8; 27; 64; 125; …
   - k) 2; 3; $\frac{9}{2}$; $\frac{27}{4}$; $\frac{81}{8}$; …
   - l) 2; $\frac{7}{2}$; 5; $\frac{13}{2}$; 8; $\frac{19}{2}$; …
   - m) 3; -9; 27; -81; 243; …
   - n) 3; 1; $\frac{1}{3}$; $\frac{1}{9}$; $\frac{1}{27}$; …
   - o) 1; 1; 2; 3; 5; 8; 13; …
2. Given the general term of the sequence:
   - i) Determine the first five terms of the sequence.
   - ii) State whether the sequence is arithmetic, geometric or neither.
   - a) $T_n = 2n + 1$
   - b) $T_k = 3k - 2$
   - c) $T_n = 2^n$
   - d) $T_n = n^2$
   - e) $T_k = k^2 + 5$
   - f) $T_n = -5n$
   - g) $T_n = 7 - n$
   - h) $T_k = 2 \,.\, 3^k$
   - i) $T_n = \frac{n}{n+1}$
   - j) $T_n = 4 \,.\, 2^{1-n}$

The general term of a sequence

In this section, we explain how to find the general term of a sequence.

Exercise 1.2

Fill in the spaces:

Arithmetic progressions (AP)		Geometric progressions (GP)	
Given the sequence:		Given the sequence:	
2; 5; 8; 11; 14; 17; … with $d = 3$		1; 2; 4; 8; 16; 32; 64 …. with $r = 2$	
Term: general form		**Term: general form**	
$T_1 = 2$	a	$T_1 = 1$:	a
$T_2 = 5 = 2 + 3$:	$a + d$	$T_2 = 2 = 1 \times 2$:	ar
$T_3 = 8 = 2 + 3 + 3 = 2 + 2(3)$:	$a + 2d$	$T_3 = 4 = 1 \times 2 \times 2 = 1 \times 2^2$:	ar^2
$T_4 = 11 = 2 + 3 + 3 + 3$ $= 2 + 3(3)$:	____	$T_4 = 8 = 1 \times 2 \times 2 \times 2 = 1 \times 2^3$:	____
$T_5 = 14 = 2 + 3 + 3 + 3 + 3$ $= 2 + 4(3)$:	____	$T_5 = 16 = 1 \times 2 \times 2 \times 2 \times 2$ $= 1 \times 2^4$:	____
$T_6 = 17 = 2 + 3 + 3 + 3 + 3 + 3$ $= 2 + 5(3)$:	____	$T_6 = 32 = 1 \times 2 \times 2 \times 2 \times 2 \times 2$ $= 1 \times 2^5$:	____
$T_{10} =$ ______________________	____	$T_{10} =$ ______________________	____
$T_{21} =$ ______________________	____	$T_{21} =$ ______________________	____
$T_n =$ ______________________	____	$T_n =$ ______________________	____

From the previous exercise, we found that:

- the general term of an arithmetic sequence is $\boldsymbol{T_n = a + (n - 1)d}$
- the general term of a geometric sequence is $\boldsymbol{T_n = a.r^{n-1}}$

Arithmetic progressions (AP)

Example 1

Determine the 12th term of the sequence 3; 7; 11; …

Solution

We first need to determine whether the sequence is arithmetic or geometric:

$T_3 - T_2 = 11 - 7 = 4$ $\qquad$ $T_2 - T_1 = 7 - 3 = 4$

$\therefore$ the sequence is arithmetic with $d = 4$

Since $T_n = a + (n - 1)d$, the 12th term will be:

$$T_{12} = a + (12 - 1)d = a + 11d$$

We know that $a = 3$, $d = 4$ and $n = 12$:

$$T_{12} = a + 11d = 3 + 11(4) = 47$$

Example 2

Determine the nth term of the sequence 1; 6; 11; 16;…

Solution

T_1	T_2	T_3	…	T_n
a	$a + d$	$a + 2d$		$a + (n - 1)d$
1	6	11		

$a = 1$, $d = 5$ and $T_n = a + (n - 1)d$:

$T_n = a + (n - 1)d$

$T_n = 1 + (n - 1)5$

$T_n = 1 + 5n - 5$

$T_n = 5n - 4$

Geometric progressions (GP)

Example 1

Determine the eighth term of the sequence $2; 1; \frac{1}{2}; \ldots$

Solution

Determine whether the sequence is arithmetic or geometric:

$\frac{T_3}{T_2} = \frac{1}{2} \div 1 = \frac{1}{2}$ $\qquad$ $\frac{T_2}{T_1} = \frac{1}{2}$

$\therefore$ the sequence is geometric with $r = \frac{1}{2}$

Since $T_n = a.r^{n-1}$, the eighth term will be:

$$T_8 = a.r^{8-1} = ar^7$$

We know that $a = 2$, $r = \frac{1}{2}$ and $n = 8$:

$$T_8 = ar^7 = 2\left(\frac{1}{2}\right)^7 = \frac{2}{1} \times \frac{1}{128} = \frac{1}{64}$$

Example 2

Determine T_n for the sequence 3; 9; 27; …

Solution

T_1	T_2	T_3	…	T_n
a	ar	ar^2		ar^{n-1}
3	9	27		

$a = 3$, $r = 3$ and $T_n = ar^{n-1}$:

$$T_n = ar^{n-1} = 3(3)^{n-1} = 3^1 \times 3^{n-1} = 3^{1+n-1} = 3^n$$

Example 3

Determine which term is equal to -106 in the following arithmetic progression:
$8; 2; -4; \ldots$

Solution

T_1	T_2	T_3	…	T_n
a	$a + d$	$a + 2d$		$a + (n - 1)d$
8	2	-4		-106

$a = 8$, $d = -6$, $n = ?$ and $T_n = -106$:

$$\begin{aligned} T_n &= a + (n - 1)d \\ -106 &= 8 + (n - 1)(-6) \\ -106 &= 8 - 6n + 6 \\ 6n &= 14 + 106 \\ 6n &= 120 \\ \frac{6n}{6} &= \frac{120}{6} \\ n &= 20 \end{aligned}$$

$\therefore$ the 20th term is equal to -106.

Example 3

Determine which term is equal to $\frac{128}{729}$ in the following geometric progression:
$3; 2; \frac{4}{3}; \ldots$

Solution

T_1	T_2	T_3	…	T_n
a	ar	ar^2		ar^{n-1}
3	2	$\frac{4}{3}$		$\frac{128}{729}$

$a = 3$, $r = \frac{2}{3}$, $n = ?$ and $T_n = \frac{128}{729}$

$$T_n = ar^{n-1}$$

$$\frac{128}{729} = 3\left(\frac{2}{3}\right)^{n-1}$$

$$\frac{1}{3} \times \frac{128}{729} = \frac{1}{3} \times 3\left(\frac{2}{3}\right)^{n-1}$$ *Divide both sides by 3 to isolate the power*

$$\frac{128}{2\,187} = \left(\frac{2}{3}\right)^{n-1}$$

$$\left(\frac{2}{3}\right)^{n-1} = \frac{128}{2\,187}$$ *To solve, we need to write $\frac{128}{2\,187}$ as a power of $\frac{2}{3}$*

$$\left(\frac{2}{3}\right)^{n-1} = \left(\frac{2}{3}\right)^{7}$$

$\therefore n - 1 = 7$ *Equate the exponents*

$n = 8$

$\therefore$ the eighth term is equal to $\frac{128}{729}$

Exercise 1.3

1. Determine the required term in each sequence:
   - a) T_{12} of $3; 7; 11; \ldots$
   - b) T_7 of $6; 12; 24; \ldots$
   - c) T_9 of $1; 3; 9; 27; \ldots$
   - d) T_{15} of $0; -2; -4; \ldots$
   - e) T_{21} of $7; 10; 13; \ldots$
   - f) T_{13} of $-12; -8; -4; \ldots$
   - g) T_7 of $3; -12; 48; \ldots$
   - h) T_{14} of $4; -4; 4; \ldots$
   - i) T_8 of $3; 6; 9; \ldots$
   - j) T_{10} of $15; 75; 375; \ldots$
2. Determine the number of terms in each of the following sequences:
   - a) $3; 7; 11; \ldots 147$
   - b) $3; 6; 12; \ldots 96$
   - c) $-5; -8; -11; \ldots -41$
   - d) $0; -2; -4; \ldots -220$
   - e) $5; -10; 20; \ldots 5\,120$
   - f) $4; 12; 36; 108; \ldots 8\,748$
   - g) $4; \frac{7}{2}; 3; \frac{5}{2}; \ldots -3$
   - h) $4; 2; 1; \frac{1}{2}; \ldots \frac{1}{256}$
   - i) $17; \frac{17}{18}; \frac{17}{18^2}; \ldots \frac{17}{18^{10}}$
   - j) $20; 19\frac{1}{4}; 18\frac{1}{2}; \ldots -1$

A comparison of arithmetic and geometric progressions

Arithmetic progressions (AP)	Geometric progressions (GP)
Example 1 In an arithmetic sequence, $T_1 = 4$ and $T_{10} = 31$. Determine: 1. the sequence. 2. the 15th term. **Solution** 1. $T_1 = 4$ and $T_{10} = 31$ $a = 4$ $T_n = a + (n - 1)d$ $31 = 4 + (10 - 1)d$ T_{10} $31 = 4 + 9d$ *General form* $9d = 31 - 4$ $9d = 27$ $d = \frac{27}{9} = 3$ $\therefore$ the sequence is 4; 7; 10; … 2. $a = 4$ and $d = 3$ $\therefore T_{15} = a + 14d$ $= 4 + 14 \times 3$ $= 46$ The 15th term is 46.	**Example 1** In a geometric sequence, $T_1 = 4$ and $T_{10} = \frac{1}{128}$. Determine: 1. the sequence. 2. the 15th term. **Solution** 1. $T_1 = 4$ and $T_{10} = \frac{1}{128}$ $a = 4$ $ar^9 = \frac{1}{128}$ *General form* To solve for r, substitute $a = 4$ into the equation $ar^9 = \frac{1}{128}$ $4r^9 = \frac{1}{128}$ $r^9 = \frac{1}{512}$ *Divide both sides by 4* $r^9 = \left(\frac{1}{2}\right)^9$ $r = \frac{1}{2}$ $\therefore$ the sequence is 4; 2; 1; … 2. $a = 4$ $r = \frac{1}{2}$ $\therefore T_{15} = ar^{14}$ $= 4\left(\frac{1}{2}\right)^{14}$ $= \frac{1}{4\,096}$

Example 2

In an arithmetic sequence, $T_3 = -2$ and $T_8 = 23$. Determine the first term and the common difference.

Solution

$T_3 = -2$ and $T_8 = 23$

We need to solve these two equations simultaneously. We can do this using the substitution method or by elimination. Here, we use the elimination method.

$a + 2d = -2$ ① $\quad a + 7d = 23$ ②

$$\begin{array}{rll} a + 7d = & 23 & ② \\ -\ \underline{(a + 2d = -2)} & & ① \\ 5d = & 25 & ② - ① \\ d = & 5 & \end{array}$$

Therefore:

$d = 5$

$a + 2(5) = -2$ *Substitute $d = 5$ into ① to solve for a*

$a + 10 = -2$

$a = -12$

The first term is –12. The common difference is 5.

Example 2

In a geometric sequence, $T_4 = \frac{2}{3}$ and $T_6 = \frac{3}{2}$. Determine the second term.

Solution

$T_4 = \frac{2}{3}$ $\quad T_6 = \frac{3}{2}$

$ar^3 = \frac{2}{3}$ ① $\quad ar^5 = \frac{3}{2}$ ②

Divide equation ② by equation ① to eliminate a. Then, solve for r.

$$\frac{ar^5}{ar^3} = \frac{3}{2} \div \frac{2}{3}$$

$$r^2 = \frac{3}{2} \times \frac{3}{2}$$

$$r^2 = \frac{9}{4}$$

$$r = \pm\frac{3}{2}$$

Since the value of r is squared, there will be two solutions to this equation

Since there are two values of r, there will be two different sequences.

To find the value of a, substitute the value of r into equation ①:

If $r = \frac{3}{2}$:

$$a\left(\frac{3}{2}\right)^3 = \frac{2}{3}$$

$$a\left(\frac{27}{8}\right) = \frac{2}{3}$$

$$a\left(\frac{27}{8}\right) \times \frac{8}{27} = \frac{2}{3} \times \frac{8}{27}$$

$$a = \frac{2}{3} \times \frac{8}{27}$$

$$a = \frac{16}{81}$$

$$T_2 = ar$$

$$= \frac{16}{81} \times \frac{3}{2}$$

$$= \frac{8}{27}$$

If $r = -\frac{3}{2}$:

$$a\left(-\frac{3}{2}\right)^3 = \frac{2}{3}$$

$$a\left(\frac{-27}{8}\right) = \frac{2}{3}$$

$$a\left(\frac{-27}{8}\right) \times \frac{-8}{27} = \frac{2}{3} \times \frac{-8}{27}$$

$$a = \frac{2}{3} \times \frac{-8}{27}$$

$$a = \frac{-16}{81}$$

$$T_2 = ar$$

$$= \frac{-16}{81} \times \frac{3}{2}$$

$$= \frac{-8}{27}$$

Example 3

If $(x + 1); (x + 4); (2x + 4); \ldots$ is an arithmetic sequence, calculate the value of x.

Solution

$$T_3 - T_2 = T_2 - T_1$$
$$(2x + 4) - (x + 4) = (x + 4) - (x + 1)$$
$$2x + 4 - x - 4 = x + 4 - x - 1$$
$$x = 3$$

Therefore:

$T_1 = 3 + 1 = 4$

$T_2 = 3 + 4 = 7$

$T_3 = 2 \times 3 + 4 = 10$

The sequence is 4; 7; 10; …

Example 3

If $(x + 1); (x + 3); (2x + 3) \ldots$ is a geometric progression, calculate the value of x.

Solution

$$\frac{T_3}{T_2} = \frac{T_2}{T_1}$$
$$\frac{2x + 3}{x + 3} = \frac{x + 3}{x + 1}$$
$$(2x + 3)(x + 1) = (x + 3)(x + 3)$$
$$2x^2 + 5x + 3 = x^2 + 6x + 9$$
$$x^2 - x - 6 = 0$$
$$(x - 3)(x + 2) = 0$$

$x = 3$ or $x = -2$

If $x = 3$, then the sequence is 4; 6; 9; …

If $x = -2$, then the sequence is $-1; 1; -1; \ldots$

Exercise 1.4

1. Determine T_{10} of the following arithmetic sequences:
   a) $a = 4$ and $d = \frac{1}{2}$
   b) $T_2 = -10$ and $T_6 = -160$
   c) $T_5 = 8$ and $d = 3$
   d) $T_2 = 4x$ and $T_5 = 10x$
2. Determine T_{10} of the following geometric sequences:
   a) $a = 4$ and $r = \frac{1}{2}$
   b) $T_2 = -10$ and $T_6 = -160$
   c) $T_5 = \frac{4}{81}$ and $r = \frac{1}{3}$
   d) $T_2 = 4x$ and $T_5 = \frac{x^4}{8}$
3. Calculate the following terms:
   a) T_{12} of the arithmetic progression, if $T_5 = 13$ and $T_{20} = 43$.
   b) T_5 of the geometric progression, if $T_3 = -\frac{1}{4}$ and $T_8 = 8$.
   c) T_4 of the geometric progression, if $T_6 = 64$ and $T_{10} = 1\ 024$.
   d) T_7 of the arithmetic progression, if $T_5 = 12$ and $T_8 = 3$.
4. Give the first three terms of an arithmetic sequence in which 3 is the seventh term and the 12th term is -3.
5. Which term of the sequence 12; 17; 22; … is 132?
6. If $x - 2; 2x - 2; 4x + 2$ are three successive terms of a geometric sequence:
   a) Calculate the value of x.
   b) Determine the sequence.
   c) Determine the 9th term of the sequence.
   d) Which term of the sequence will be equal to 4 374?
7. The first two terms of an arithmetic sequence are m and n, respectively. Calculate the 10th term.

8. If $2x - 3$; $3x + 1$; $5x + 2$ are the first three terms of an arithmetic sequence, calculate the value of x.
9. Determine the geometric progression in which $T_4 = 224$ and the common ratio is 2.
10. Find which term of the sequence -9; -15; -21; … is -45.
11. The numbers 4; x; y form an arithmetic sequence. The numbers x; y; 18 form a geometric sequence. Calculate the values of x and y.
12. Given the sequence 2; 8; 14; 20; …
    a) Determine the 50th term.
    b) Which term will be equal to 50?
13. The following is an arithmetic sequence: $16y + 1$; $4y + 7$; $y + 4$; …
    a) Calculate the value of y.
    b) Write down the value of:
        i) the first term of the sequence
        ii) the common difference
        iii) the fifth term.
14. a) Determine T_{15} of the sequence $2 + y$; $2 + 4y$; $2 + 7y$; …
    b) Which term of the sequence is equal to $2 + 61y$?
15. A boy is repaying a debt to a friend. He pays R10 in the first week, R15 in the second week, R20 in the third week, and so on. If he finishes paying after the eighth week, how much was his last payment?

Arithmetic and geometric means

Arithmetic mean (AM)

If a; x; b is an arithmetic progression then x is the **arithmetic mean** of a and b.

$$T_3 - T_2 = T_2 - T_1$$
$$b - x = x - a$$
$$2x = a + b$$
$$x = \frac{a+b}{2}$$

$\therefore$ the arithmetic mean is $\frac{a+b}{2}$

Geometric mean (GM)

If a; x; b is a geometric progression then x is the **geometric mean** of a and b.

$$\frac{T_3}{T_2} = \frac{T_2}{T_1}$$
$$\frac{b}{x} = \frac{x}{a} \qquad (a, x, b \neq 0)$$
$$x^2 = ab$$
$$x = +\sqrt{ab}$$

By definition, the geometric mean is $\sqrt{ab}$.

Example 1 (AM)

Calculate the arithmetic mean of 3 and 11.

Solution

$$\text{AM} = \frac{3+11}{2} = \frac{14}{2} = 7$$

Therefore, 3; 7; 11; … is an arithmetic sequence.

Example 1 (GM)

Calculate the geometric mean of 2 and 8.

Solution

$$\text{GM} = \sqrt{2 \times 8} = \sqrt{16} = 4$$

Therefore 2; 4; 8; … is a geometric sequence.

Example 2

Insert three arithmetic means between -2 and 10.

Solution

This means we need to insert three numbers between -2 and 10 such that the numbers form an arithmetic sequence.

T_1	T_2	T_3	T_4	T_5
a				$a + 4d$
-2				10

So $T_1 = -2$, $a = -2$

$$T_5 = 10$$
$$a + 4d = 10$$
$$-2 + 4d = 10$$
$$4d = 12$$
$$d = 3$$

$\therefore$ the sequence is: $-2; 1; 4; 7; 10$

Example 2

Insert two geometric means between x^3 and y^3.

Solution

This means we need to insert two numbers between x^3 and y^3 such that the numbers form a geometric sequence.

T_1	T_2	T_3	T_4
a			ar^3
x^3			y^3

So $T_1 = x^3$, $a = x^3$

$$T_4 = y^3$$
$$ar^3 = y^3$$
$$(x^3)r^3 = y^3$$
$$r^3 = \frac{y^3}{x^3}$$
$$r = \frac{y}{x}$$

$\therefore$ the sequence is $x^3; x^2y; xy^2; y^3$

Exercise 1.5

1. Determine the arithmetic mean between
   a) 4 and 18 b) 3 and 12 c) -8 and -2 d) 2,7 and 14,5
2. Determine the geometric mean of
   a) 3 and 12 b) 5 and 20 c) $\frac{1}{2}$ and 50 d) 4 and 16
3. Insert four arithmetic means between 4 and 19.
4. Insert three geometric means between $\frac{1}{4}$ and 4.

Sigma notation

The symbol $\sum$ is the Greek symbol *sigma*. We use this symbol to find the sum of a sequence.

When we add the terms in a sequence, we call this a **series**. For example, if we have a sequence 5; 8; 11; 14; 17, then the series is $5 + 8 + 11 + 14 + 17$.

Example 1

Evaluate: $\sum_{n=1}^{5} (3n + 2)$

Solution

$\sum_{n=1}^{5} (3n + 2)$ means that we need to find the sum of a sequence. To find the terms of the sequence, we first substitute $n = 1$ into the general term of $3n + 2$. Then, we substitute $n = 2$, $n = 3$, and so on, until we reach $n = 5$. So, the number below the sigma symbol tells us where to start ($n = 1$), and the number above the sigma symbol tells us where to stop ($n = 5$).

$$\begin{aligned}\text{So } \sum_{n=1}^{5} (3n + 2) &= (3 \,.\, 1 + 2) + (3 \,.\, 2 + 2) + (3 \,.\, 3 + 2) + (3 \,.\, 4 + 2) + (3 \,.\, 5 + 2) \\ &= 5 + 8 + 11 + 14 + 17 \\ &= 55\end{aligned}$$

Example 2

Evaluate: $\sum_{r=0}^{6} (2 \,.\, 3^r)$

Solution

In this case, we start with $r = 0$ and stop when $r = 6$. In other words, we begin by substituting $r = 0$ into the general term $2 \,.3^r$ and continue to $r = 6$.

$$\begin{aligned}\sum_{r=0}^{6} (2 \,.\, 3^r) &= 2 \,.\, 3^0 + 2 \,.\, 3^1 + 2 \,.\, 3^2 + 2 \,.\, 3^3 + 2 \,.\, 3^4 + 2 \,.\, 3^5 + 2 \,.\, 3^6 \\ &= 2 \,.\, 1 + 2 \,.\, 3 + 2 \,.\, 9 + 2 \,.\, 27 + 2 \,.\, 81 + 2 \,.\, 243 + 2 \,.\, 729 \\ &= 2 + 6 + 18 + 54 + 162 + 486 + 1\,458 \\ &= 2\,186\end{aligned}$$

Example 3

Evaluate: $\sum_{i=3}^{10} (-2i + 5)$

Solution

In this case, we start by substituting $i = 3$ into the general term $-2i + 5$, and continue to $i = 10$:

$$\begin{aligned}\sum_{i=3}^{10} (-2i + 5) &= (-2 \,.\, 3 + 5) + (-2 \,.\, 4 + 5) + (-2 \,.\, 5 + 5) + (-2 \,.\, 6 + 5) + \\ &\quad (-2 \,.\, 7 + 5) + (-2 \,.\, 8 + 5) + (-2 \,.\, 9 + 5) + (-2 \,.\, 10 + 5) \\ &= (-6 + 5) + (-8 + 5) + (-10 + 5) + (-12 + 5) + (-14 + 5) + \\ &\quad (-16 + 5) + (-18 + 5) + (-20 + 5) \\ &= -1 - 3 - 5 - 7 - 9 - 11 - 13 - 15 \\ &= -64\end{aligned}$$

Number of terms = top number – bottom number + 1

In Example 1, $\sum_{n=1}^{5}(3n+2)$ the number of terms is $5-1+1=5$.

In Example 2, $\sum_{r=0}^{6}(2\,.\,3^r)$ the number of terms is $6-0+1=7$.

In Example 3, $\sum_{i=3}^{10}(-2i+5)$ the number of terms is $10-3+1=8$.

So if $\sum_{n=a}^{b}T_n$ the number of terms in the series will be $\boldsymbol{b-a+1}$.

Also, note that each example used different variables (n, r and i).

Example 4

Evaluate: $\sum_{n=1}^{6}5$

Solution

We know that there are $6-1+1=6$ terms in this series. Here, the general term is a constant, so we write the series as follows:

$$\begin{aligned}\sum_{n=1}^{6}5 &= 5+5+5+5+5+5\\ &= 6\times 5\\ &= 30\end{aligned}$$

Writing a series in sigma notation

Example 1

Write the following series in sigma notation: $5+8+11+14+17$

Solution

First, we need to determine the general term, T_n, for the series.
The series is an arithmetic series with $a=5$ and $d=3$. Therefore:

$$\begin{aligned}T_n &= a+(n-1)d\\ &= 5+(n-1)3\\ &= 5+3n-3\\ &= 3n+2\end{aligned}$$

Now we write in the formula after the sigma symbol:

$$\sum_{n=\ldots}^{\ldots}(3n+2)$$

Next, we need to determine the numbers above and below the sigma symbol.
To do so, we need to solve the following equations:

The first term of the series is 5:

$$\begin{aligned}3n+2&=5\\ 3n&=3\\ n&=\mathbf{1}\end{aligned}$$

The last term of the series is 17:

$$\begin{aligned}3n+2&=17\\ 3n&=15\\ n&=\mathbf{5}\end{aligned}$$

$$\therefore \sum_{n=1}^{5}3\,n+2$$

Example 2

Write the following series in sigma notation: $1 + 4 + 9 + 16 + 25 + 36 + 49 + 64$

Solution

By inspection, we see that the series $1 + 4 + 9 + 16 + 25 + 36 + 49 + 64$ is made up of the first eight perfect squares.

$\therefore 1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 = \sum_{k=1}^{8} k^2$

Example 3

Write the following series in sigma notation: $6 + 12 + 24 + 48$

Solution

$6 + 12 + 24 + 48$ is a geometric series with $a = 6$ and $r = 2$. Therefore:

$$
\begin{aligned}
T_n &= a \,.\, r^{k-1} \\
&= 6 \,.\, 2^{k-1} && 6 = 2 \times 3 \\
&= 2 \times 3 \times 2^{k-1} \\
&= 2^1 \times 3 \times 2^{k-1} \\
&= 3 \,.\, 2^{1+k-1} \\
&= 3 \,.\, 2^k
\end{aligned}
$$

$\therefore \sum_{k=\ldots}^{\ldots} 3 \,.\, 2^k$

The first term of the series is 6:

so $3 \,.\, 2^k = 6$

$2^k = 2$ *Divide both sides of the equation by 3*

$\therefore k = \mathbf{1}$

The last term of the series is 48:

so $3 \,.\, 2^k = 48$

$2^k = 16$

$2^k = 2^4$

$k = \mathbf{4}$

$\therefore \sum_{k=1}^{4} 3 \,.\, 2^k$

Exercise 1.6

1. Evaluate the following:

   a) $\sum_{r=1}^{12} 3r$ b) $\sum_{r=0}^{10} (2r + 5)$ c) $\sum_{n=1}^{5} \left(\frac{1}{5^n}\right)$ d) $\sum_{n=3}^{10} n^2$

   e) $\sum_{i=1}^{6} \left(\frac{2}{3}\right)^i$ f) $\sum_{k=2}^{8} 3 \,.\, 2^k$ g) $\sum_{r=1}^{6} r$

2. Write the following in sigma notation:

   a) $1 + 3 + 5 + \ldots + 17$ b) $3 + 9 + 27 + \ldots + 729$

   c) $7 + 10 + 13 + \ldots + 25$ d) $64 + 32 + 16 + \ldots + \frac{1}{2}$

   e) $16 + 25 + 36 + \ldots + 100$ f) $4 + 7 + 10 + \ldots + 37$

   g) $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \ldots + \frac{10}{11}$ h) $1 + 4 + 16 + \ldots + 16\ 384$

Series

- When we add the terms of an arithmetic progression we obtain the **arithmetic series (S_n)**.
- When we add the terms of a geometric progression we obtain the **geometric series (S_n)**.

Example 1

If $T_n = 3n + 2$, determine S_5.

Solution

$$S_5 = \overset{T_1}{5} + \overset{T_2}{8} + \overset{T_3}{11} + \overset{T_4}{14} + \overset{T_5}{17} = 55$$

Also note:

$$\underbrace{\underbrace{\overset{T_1}{5} + \overset{T_2}{8} + \overset{T_3}{11} + \overset{T_4}{14}}_{S_4} + \overset{T_5}{17}}_{S_5}$$

So:

$T_5 = S_5 - S_4 = (5 + 8 + 11 + 14 + 17) - (5 + 8 + 11 + 14) = 17$

In general: $\boldsymbol{T_n = S_n - S_{n-1}}$

Example 2

Given that $S_n = 2n^2 + 3$, determine

1. the fifth term
2. T_{25}

Solution

1. $T_5 = S_5 - S_4$
   $T_5 = [2(5)^2 + 3] - [2(4)^2 + 3]$
   $T_5 = 53 - 35$
   $T_5 = 18$

2. $T_{25} = S_{25} - S_{24}$
   $T_{25} = [2(25)^2 + 3] - [2(24)^2 + 3]$
   $T_{25} = 1\,253 - 1\,155$
   $T_{25} = 98$

The sum of an arithmetic series

Find the sum of the arithmetic series $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$.

We can write this as:

$$\begin{array}{lll} S_{10} & = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 & \\ S_{10} & = 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 & \textit{Write the series backwards} \\ \hline 2S_{10} & = 11 + 11 + 11 + 11 + 11 + 11 + 11 + 11 + 11 + 11 & \textit{Add} \\ 2S_{10} & = 10 \times 11 & \textit{There are 10 terms} \\ 2S_{10} & = 110 & \\ S_{10} & = \frac{110}{2} = 55 & \end{array}$$

It can become quite tedious to add a large number of terms. Also, the more terms we have to add, the more chance there is of making a mistake. Fortunately, we can use the same technique we have just used to develop a formula to add all the terms in a general arithmetic series.

$$\begin{aligned} S_n &= T_1 + T_2 + T_3 + \ldots + T_{n-1} + T_n \\ &= (a) + (a + d) + (a + 2d) + \ldots + (a + (n - 2)d) + (a + (n - 1)d) \end{aligned}$$

Proof of the formula for the sum of an arithmetic series:

$$\begin{aligned} S_n &= a + (a + d) + (a + 2d) + \ldots + (a + (n - 3)d) + (a + (n - 2)d) + (a + (n - 1)d) \\ S_n &= (a + (n - 1)d) + (a + (n - 2)d) + (a + (n - 3)d) \ldots + (a + 2d) + (a + d) + a \\ 2S_n &= (2a + (n - 1)d) + (2a + (n - 1)d) + (2a + (n - 1)d) + \ldots + \\ &\quad (2a + (n - 1)d) + (2a + (n - 1)d) \\ 2S_n &= n(2a + (n - 1)d) \qquad \textit{Since there are n terms} \\ S_n &= \frac{n(2a + (n - 1)d)}{2} \\ &= \frac{n}{2}(2a + (n - 1)d) \end{aligned}$$

We can write the last term (l) of an arithmetic series as $l = a + (n - 1)d$. This means we can write the formula

$$S_n = \frac{n}{2}(2a + (n - 1)d) \text{ as } S_n = \frac{n}{2}(a + a + (n - 1)d)$$

Or

$$S_n = \frac{n}{2}(a + l)$$

The sum of a geometric series

Find the sum of the geometric series $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128$.
Here, $a = 1$; $r = 2$ and the number of terms is $n = 8$. The technique we use here is to multiply each term by the common ratio r. Therefore, we have:

$$\begin{aligned} S_8 &= 1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 \qquad ① \\ 2S_8 &= \quad\;\; 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 \qquad \textit{Multiply by } r = 2 \\ \hline S_8 - 2S_8 &= 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 - 256 \qquad ① - ② \\ (1 - 2)S_8 &= 1 - 256 \\ -1S_8 &= -255 \\ S_8 &= -\frac{255}{-1} = 255 \end{aligned}$$

Again, using the general form of a geometric series, we can determine a formula for the sum of n terms of a geometric series.

$$\begin{aligned} S_n &= T_1 + T_2 + T_3 + \ldots + T_{n-1} + T_n \\ &= a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1} \end{aligned}$$

Proof of the formula for the sum of a geometric series:

$$S_n = a + ar + ar^2 + \dots + ar^{n-2} + ar^{n-1} \quad ①$$

$$r \times S_n = ar + ar^2 + \dots + ar^{n-2} + ar^{n-1} + ar^n \quad ②$$

$$S_n - rS_n = a + 0 + 0 + \dots + 0 + 0 - ar^n \quad ① - ②$$

$$S_n - rS_n = a - ar^n$$

$$S_n(1 - r) = a(1 - r^n) \quad \textit{Factorise}$$

$$\frac{S_n(1 - r)}{(1 - r)} = \frac{a(1 - r^n)}{(1 - r)}$$

$$S_n = \frac{a(1 - r^n)}{(1 - r)} \quad r \neq 1$$

In our proof, we subtracted equation ② from equation ①. This formula is easier to use if $r < 1$.

We could also have subtracted equation ① from equation ②. This would give us a slightly different formula: $S_n = \frac{a(r^n - 1)}{(r - 1)}$. This formula is easier to use if $r > 1$.

The proofs for both an arithmetic series formula and a geometric series formula must be known for exam purposes.

Arithmetic series

For an arithmetic series, we use the following formulae:
$S_n = \frac{n}{2}(2a + (n - 1)d)$ and $S_n = \frac{n}{2}(a + l)$

Example 1

Determine the sum of the series $2 + 5 + 8 + \dots$ to 20 terms.

Solution

$a = 2$; $d = 3$; $n = 20$; $S_{20} = ?$

$$S_n = \frac{n}{2}\big(2a + (n - 1)d\big)$$

$$S_{20} = \frac{20}{2}(2(2) + 19(3))$$

$$= 10(61)$$

$$= 610$$

Geometric series

For a geometric series, we use the following formulae:

$S_n = \frac{a(1 - r^n)}{(1 - r)}$ if $r < 1$

$S_n = \frac{a(r^n - 1)}{(r - 1)}$ if $r > 1$

Example 1

Use a formula to determine the sum of the series $27 + 9 + 3 + \dots$ to four terms.

Solution

$a = 27$; $r = \frac{1}{3}$; $n = 4$; $S_4 = ?$

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$S_4 = \frac{27\left(1 - \left(\frac{1}{3}\right)^4\right)}{1 - \frac{1}{3}}$$

$$S_4 = \frac{27\left(1 - \frac{1}{81}\right)}{\frac{2}{3}}$$

$$S_4 = 27\left(\frac{81 - 1}{81}\right) \times \frac{3}{2}$$

$$S_4 = \frac{27}{1} \times \frac{80}{81} \times \frac{3}{2}$$

$$S_4 = 40$$

Example 2

Determine $3 + 7 + 11 + \ldots + 59$.

Solution

$a = 3$; $d = 4$; $n = ?$; $T_n = 59$; $S_n = ?$

We first need to calculate **the number of terms (n)** in this series before we can calculate the sum.

We use the formula $T_n = a + (n - 1)d$ to calculate the number of terms. We also know the last term (l) in the series, so we can substitute n into the formula $S_n = \frac{n}{2}(a + l)$ to calculate the sum.

$$T_n = a + (n - 1)d$$
$$59 = 3 + (n - 1)4$$
$$59 = 3 + 4n - 4$$
$$60 = 4n$$
$$n = 15$$

$$S_n = \frac{n}{2}(a + l)$$
$$S_{15} = \frac{15}{2}(3 + 59)$$
$$S_{15} = \frac{15}{2}(62)$$
$$S_{15} = 465$$

Example 2

Determine $8 + 4 + 2 + \ldots + \frac{1}{32}$

Solution

$a = 8$; $r = \frac{1}{2}$; $n = ?$; $T_n = \frac{1}{32}$; $S_n = ?$

We first need to calculate **the number of terms (n)** in this series before we can calculate the sum.

We use the formula $T_n = ar^{n-1}$ to calculate the number of terms.
Then, we substitute n into the formula $S_n = \frac{a(1 - r^n)}{1 - r}$ $(r < 1)$ to calculate the sum.

$$T_n = ar^{n-1}$$
$$\frac{1}{32} = 8\left(\frac{1}{2}\right)^{n-1}$$
$$\frac{1}{32} \times \frac{1}{8} = \frac{1}{8} \times 8\left(\frac{1}{2}\right)^{n-1}$$
$$\frac{1}{256} = \left(\frac{1}{2}\right)^{n-1}$$
$$\left(\frac{1}{2}\right)^{8} = \left(\frac{1}{2}\right)^{n-1}$$
$$8 = n - 1$$
$$9 = n$$

$$S_9 = \frac{8\left(1 - \left(\frac{1}{2}\right)^9\right)}{1 - \frac{1}{2}}$$
$$S_9 = \frac{8\left(1 - \frac{1}{512}\right)}{\frac{1}{2}}$$
$$S_9 = 8\left(\frac{512 - 1}{512}\right) \times \frac{2}{1}$$
$$S_9 = \frac{8}{1}\left(\frac{511}{512}\right) \times \frac{2}{1}$$
$$S_9 = \frac{511}{32}$$

Example 3

How many terms of the series $1 + 4 + 7 + \ldots$ must be added to give a sum of 145?

Example 3

How many terms of the series $5 + 10 + 20 + \ldots$ must be added to give a sum of 1 275?

Solution $a = 1; d = 3; n = ?; S_n = 145$ $S_n = \frac{n}{2}(2a + (n-1)d)$ $145 = \frac{n}{2}(2(1) + (n-1)3)$ $290 = n(2 + 3n - 3)$ *Multiply both sides by 2* $290 = n(3n - 1)$ $0 = 3n^2 - n - 290$ $0 = (3n + 29)(n - 10)$ *Factorise* $n = -\frac{29}{3}$ or $n = 10$ *Solve for n* Since the number of terms is always a natural number, $n = -\frac{29}{3}$ is not valid. In other words, n cannot be fraction or a negative number. Therefore, $n = 10$.	**Solution** $a = 5; r = 2; n = ?; S_n = 1\,275$ $S_n = \frac{a(r^n - 1)}{r - 1}$ $1\,275 = \frac{5(2^n - 1)}{2 - 1}$ $1\,275 = 5(2^n - 1)$ $255 = (2^n - 1)$ $256 = 2^n$ $2^8 = 2^n$ $\therefore n = 8$
Example 4 Determine $\sum_{n=1}^{25}(3n - 5)$ **Solution** $\sum_{n=1}^{25}(3n - 5) = -2 + 1 + 4 + \ldots + 70$ $a = -2; d = 3; n = 25; S_{25} = ?$ $S_n = \frac{n}{2}(2a + (n-1)d)$ $S_{25} = \frac{25}{2}(2(-2) + 24(3))$ $S_{25} = \frac{25}{2}(68)$ $S_{25} = 850$	**Example 4** Evaluate $\sum_{n=2}^{10}(3 \,.\, 2^{n-1})$ **Solution** $\sum_{n=2}^{10}(3 \,.\, 2^{n-1}) = 3.2^1 + 3.2^2 + 3.2^3 + \ldots + 3.2^9$ $= 6 + 12 + \ldots + 1\,536$ $a = 6; r = 2; n = 10 - 2 + 1 = 9; S_9 = ?$ $S_n = \frac{a(r^n - 1)}{r - 1}$ *Since r > 1* $S_9 = \frac{6(2^9 - 1)}{2 - 1}$ $S_9 = \frac{6(512 - 1)}{1} = 3\,066$

Exercise 1.7

1. Calculate the sum of the following:
   a) $3 + 7 + 11 + \ldots$ to 15 terms
   b) $-15 - 20 - 25 - \ldots$ to 17 terms
   c) $2 + 8 + 32 + \ldots$ to 10 terms
   d) $-3 + 6 - 12 + \ldots$ to 13 terms
   e) $-5 - 12 - 19 - \ldots$ to 11 terms
   f) $32 + 16 + 8 + \ldots$ to 10 terms
2. Determine the sum of the following series:
   a) $5 + 12 + 19 + \ldots + 54$
   b) $2 + \frac{6}{7} + \frac{18}{49} + \ldots + 2\left(\frac{3}{7}\right)^8$
   c) $-75 - 70 - 65 \ldots + 0$
   d) $7 + 10 + 13 + \ldots + 82$
   e) $9 + 3 + 1 + \ldots + \frac{1}{243}$
   f) $20 + 17 + 14 + \ldots -7$
   g) $-4 + 0 + 4 + \ldots + 92$

3. Calculate the following:

a) $\sum_{r=1}^{20} 3r$ b) $\sum_{r=1}^{15} (2r + 5)$ c) $\sum_{r=5}^{10} \left(\frac{1}{2}\right)^r$ d) $\sum_{n=3}^{10} (9 - 3n)$

e) $\sum_{k=1}^{8} 3 \,.\, 2^k$ f) $\sum_{n=0}^{6} \left(\frac{2}{3}\right)^n$ g) $\sum_{k=2}^{8} (2k - 3)$

4. How many terms of the following series must be added to give the indicated sum?

a) $3 + 7 + 11 + \dots\dots\dots = 210$ b) $3 + 12 + 48 + \dots\dots\dots = 4\,095$

c) $6 + 3 + \frac{3}{2} + \dots\dots\dots = \frac{765}{64}$ d) $7 + 4 + 1 + \dots\dots\dots = -143$

e) $-7 - 5 - 3 - \dots\dots\dots = -12$

5. Determine the sum of the first 50 even numbers.

Mixed problems

Arithmetic sequences and series

Example 1

In an arithmetic sequence, $T_2 = 5$ and $T_6 = 21$. Determine the sum of the first 20 terms of the sequence.

Solution

$T_2 = 5$ and $T_6 = 21$

$a + d = 5$ ① $a + 5d = 21$ ②

$$\begin{aligned} a + 5d &= 21 \quad ② \\ -\;(a + d) &= 5 \quad ① \\ \hline 4d &= 16 \quad ② - ① \\ d &= 4 \\ \therefore a + 4 &= 5 \quad \textit{Substitute } d = 4 \textit{ into } ① \\ a &= 1 \end{aligned}$$

$a = 1;\ d = 4;\ n = 20;\ S_{20} = ?$

$$S_n = \frac{n}{2}(2a + (n - 1)d)$$

$$S_{20} = \frac{20}{2}(2(1) + (20 - 1)(4))$$

$$S_{20} = 10(2 + 19 \times 4)$$

$$S_{20} = 780$$

Geometric sequences and series

Example 1

In a geometric sequence, $T_2 = 4$ and $T_6 = \frac{1}{4}$. Determine the sum of the first 12 terms of the sequence if $r > 0$.

Solution

$T_2 = 4$ and $T_6 = \frac{1}{4}$

$ar = 4$ ① $ar^5 = \frac{1}{4}$ ②

$\frac{ar^5}{ar} = \frac{1}{4} \div 4$ ② ÷ ①

$r^4 = \frac{1}{4} \times \frac{1}{4}$

$r^4 = \frac{1}{16}$

$r = \frac{1}{2}$ *since $r > 0$*

$a\left(\frac{1}{2}\right) = 4$ *Substitute $r = \frac{1}{2}$ into equation ①*

$a = 8$

$a = 8;\ r = \frac{1}{2};\ n = 12;\ S_{12} = ?$

$$S_n = \frac{a(1 - r^n)}{(1 - r)} \quad \text{if } r < 1$$

$$S_{12} = \frac{8\left(1 - \left(\frac{1}{2}\right)^{12}\right)}{\left(1 - \frac{1}{2}\right)}$$

$$S_{12} = \frac{8\left(1 - \frac{1}{4\,096}\right)}{\frac{1}{2}} = \frac{4\,095}{256}$$

Example 2

Determine the largest value of n such that

$$\sum_{r=1}^{n}(2r+5) < 150$$

Solution

$$\sum_{r=1}^{n}(2r+5) = 7 + 9 + 11 + \dots + (2n+5)$$

This is an arithmetic series with $a = 7$ and $d = 2$.

$$S_n = \frac{n}{2}(2(7) + (n-1)2)$$
$$= \frac{n}{2}(2n + 12)$$
$$= n^2 + 6n$$

$\therefore n^2 + 6n < 150$

$n^2 + 6n - 150 < 0$

Since this trinomial does not factorise, we use the quadratic formula to solve for n.

Solving the equation $n^2 + 6n - 150 = 0$, we get:

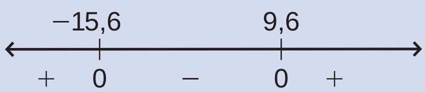
$$n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$n = -6 \pm \frac{\sqrt{36 - 4(1) \times (-150)}}{2(1)}$$

$$n = \frac{-6 \pm \sqrt{636}}{2}$$

$n = -15{,}6$ or $n = 9{,}6$

If $n^2 + 6n - 150 < 0$
using a number line, we get:



$\therefore -15{,}6 < n < 9{,}6$

$\therefore$ the largest value of n is 9.

Exercise 1.8

1. In an arithmetic sequence, $T_2 = 1$ and $T_7 = 16$. Determine the sum of the first 20 terms of the sequence.
2. Determine the value of n if:
   a) $\sum_{i=1}^{n} (3i - 5) = 3\,430$
   b) $\sum_{i=1}^{n} 3 \cdot 2^{i-1} = 12\,285$
   c) $\sum_{k=1}^{n} (2k + 7) = 1\,008$
   d) $\sum_{r=1}^{n} (4 - 3r) = -23\,125$
   e) $\sum_{k=1}^{n} \frac{1}{2}(3^{k-1}) = 1\,640$
   f) $\sum_{k=1}^{n} \frac{1}{3}(2^{k-1}) = 341$
3. In a geometric sequence, $T_3 = \frac{1}{4}$ and $T_6 = 2$. Determine the sum of the first 11 terms of the sequence.
4. In an arithmetic sequence, $T_4 = 12$ and $T_7 = 0$. Determine the number of terms if the sum of the series is 60. Why do you obtain two values of n?
5. In an arithmetic sequence the seventh term exceeds the fourth term by 15. Determine:
   a) the value of d, the common difference.
   b) the value of a, if $T_7 = 13$.
   c) the tenth term.
   d) the sum of the first 15 terms.
6. In a geometric sequence, the fifth term is four times the third term, and the second term is 4.
   If $r < 0$, determine:
   a) the value of r, the common ratio.
   b) the value of a.
   c) the tenth term.
   d) the sum of the 15 terms.
7. a) Determine the largest value of n such that:
      i) $\sum_{r=1}^{n} (3r + 4) < 120$
      ii) $\sum_{i=1}^{n} (2i - 3) < 1\,000$
   b) What is the smallest value of n such that $\sum_{k=1}^{n} (5 - 2k) < -550$?
8. The first term of a geometric series is 9. The seventh term is $\frac{64}{81}$. Determine two possible values for the sum to seven terms of the sequence.
9. The sum of the first three terms of an arithmetic series is 33. The sixth term is 12 more than the fourth term. Determine:
   a) the common difference and the first term.
   b) the tenth term.
10. The sum of the first four terms of a geometric series is 272.
    The common ratio is $\frac{3}{5}$. Calculate:
    a) the first term.
    b) the seventh term.

11. The numbers $x + 3$; $5x - 1$; $7x + 1$ are the first three terms of an arithmetic sequence. Calculate:
    a) the value of x.
    b) the sum of the first 30 terms of the sequence.
12. The numbers $4x$; $2x + 8$; $x + 10$ are the first three terms of a geometric sequence. Calculate:
    a) the value of x.
    b) the sum of the first six terms of the sequence.

Applications of arithmetic and geometric sequences and series

Arithmetic sequences and series

Example 1

A ladder has 12 rungs. The lowest rung is 800 mm long. Each succeeding rung is 40 mm shorter than the previous rung. Calculate the total length of 12 rungs.

Solution

The lowest rung is 800 mm
$\therefore$ The second rung will be
$800 - 40 = 760$ mm.
The third rung will be $760 - 40 = 720$ mm, and so on.
So the sequence is: $800 + 760 + 720 + \ldots$

$a = 800$; $d = -40$; $n = 12$; $S_{12} = ?$

$$S_n = \frac{n}{2}(2a + (n - 1)d)$$

$$S_{12} = \frac{12}{2}(2(800) + 11(-40))$$

$$S_{12} = 6(1\ 160)$$

$$S_{12} = 6\ 960$$

Geometric sequences and series

Example 1

Michelle emails a letter to three of her friends. She asks them not to break the chain. They need to each forward the email to three other friends. If this process continues, determine how many people would have received the email if it is forwarded five times. Include the first time Michelle sent the e-mail.

Solution

The sequence of the number of emails is:
$3 + 9 + 27 + \ldots$
$a = 3$; $r = 3$; $n = 5$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_5 = \frac{3(3^5 - 1)}{3 - 1}$$

$$S_5 = \frac{3(243 - 1)}{2}$$

$$S_5 = 363$$

Example 2

A man's income is R96 000 a year. Each year, his income increases by R7 200. His expenses amount to R66 000 a year, and increase by R4 200 every year. How long will it take him to save more than R180 940?

Solution

Income: 96 000; 103 200; 110 400
Expenses: 66 000; 70 200; 74 400
Savings: 30 000; 33 000; 36 000

$a = 30\ 000$; $d = 3\ 000$; $n = ?$ $S_n > 180\ 940$

$S_n = \frac{n}{2}(2a + (n-1)d)$
$\frac{n}{2}(60\ 000 + (n-1)3\ 000) > 180\ 940$
$n(60\ 000 + 3\ 000n - 3\ 000) > 361\ 880$

$3\ 000n^2 + 57\ 000n - 361\ 800 > 0$
Using the quadratic formula:

$$n = \frac{-57\ 000 \pm \sqrt{57\ 000^2 - 4(3\ 000 \times (-361\ 800))}}{2 \times 3\ 000}$$

$n = 5{,}02$ or $n = -24{,}02$ (N/A)
So it will take six years to save more than R180 940.

Example 2

A ball is dropped from a height of 12 m. The ball bounces back $\frac{2}{3}$ of the height of its previous bounce. Calculate the distance that the ball has travelled from the time it was dropped until it touches the ground for the fifth time. Round your answer to one decimal place.

Solution

$12 + 2\left(12\left(\frac{2}{3}\right) + 12\left(\frac{2}{3}\right)^2 + 12\left(\frac{2}{3}\right)^3 + 12\left(\frac{2}{3}\right)^4\right)$

$a = 12\left(\frac{2}{3}\right)$; $r = \frac{2}{3}$; $n = 4$

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$S_4 = \frac{12\left(\frac{2}{3}\right)\left(1 - \left(\frac{2}{3}\right)^4\right)}{1 - \left(\frac{2}{3}\right)}$$

$$S_4 = \frac{8\left(1 - \left(\frac{16}{81}\right)\right)}{\frac{1}{3}}$$

$$S_4 = 8\left(\frac{81 - 16}{81}\right) \times \frac{3}{1}$$

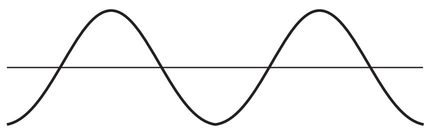
$$S_4 = \frac{8}{1} \times \frac{65}{81} \times \frac{3}{1}$$

$S_4 = 19{,}26$ m
$\therefore$ the total distance travelled is:
$12 + 2 \times (19{,}26) = 50{,}5$ m

Exercise 1.9

1. Farmer Langa starts farming with 450 sheep. He finds that his stock increases by 12% each year. How many sheep will Langa have on his farm at the end of five years?
2. Farmer Joe starts farming with a certain number of cattle. He finds that, each year, he has 30 more cattle. At the end of five years, the farmer has 2 420 cattle. How many cattle did the farmer start with?
3. Dean's granny gives him R1 on his first birthday, R2 on his second birthday, R3 on his third birthday, R4 on his fourth birthday, and so on.
   a) How much money will Dean receive on his 20th birthday?
   b) Calculate the total amount of money Dean would have received from his granny over the 20 years.

4. Sipho's granny gives him one cent on his first birthday, two cents on his second birthday, four cents on his third birthday, eight cents on his fourth birthday, and so on.
   a) How much money will Sipho receive on his 20th birthday?
   b) Calculate the total amount of money Sipho would have received from his granny over the 20 years.
5. Refer to questions 3 and 4. Would Dean or Sipho have received more money over the 20 years?
6. An athlete is training to run the Comrades Marathon. He runs 12 km on the first day and increases his distance by 2 km each day.
   a) On which day would he cover a distance of 32 km?
   b) After how many days would he have covered a total distance of 210 km?
7. The road works department is tarring a road. They set up camp at the start of the road. The workers manage to tar 0,6 km of road every day and return to their camp site at the end of every day.
   a) How far will the workers travel on the 20th day?
   b) How far will the workers have travelled in total after 20 days?
8. Kashiv saves R500 in the first month of his working career. He saves the same amount at the end of each month of the year. Each subsequent year, he manages to save 10% more than he saved the previous year. Calculate:
   a) Kashiv's total savings at the end of the first year
   b) the amount that Kashiv would be saving monthly in his sixth year
   c) the total amount that Kashiv would have saved at the end of six years.
9. A horizontal line intersects part of the sine curve at four points. It therefore divides the curve into five parts.



   a) If a second line is drawn to intersect the curve, into how many parts will the curve be divided?
   b) If ten lines are drawn to intersect the curve, how many parts will the curve be divided into?
10. A factory manufactures a product for R200,00. Each time the product is bought and sold a profit of 25% is made.
    a) If the product is bought and sold seven times, what will the price of the product be?
    b) Calculate the difference between the original price and the price after it has been sold for the seventh time.
11. Vaughan is preparing for a bicycle race. In the first week, he rides 132 km. He then increases his distance by 12 km each week.
    a) What distance did Vaughan ride in the seventh week?
    b) What was the total distance Vaughan covered after seven weeks?

12. A water tank contains 216 ℓ of water at the end of day 1. Because of a leak, the tank loses one-sixth of the previous day's contents.
    How many litres of water will there be in the tank at the end of the:
    a) second day
    b) third day
    c) seventh day?
13. Zintle decides to join a stokvel to save for her son's education when he leaves school. She joins the stokvel in January of his Grade 1 year and has to pay R300,00 monthly. The stokvel payments increase by R50,00 each year. If there are 12 people in the stokvel and Zintle is paid in December, how much will she have saved at the end of 12 years, assuming that she does not spend any of her stokvel payments?

Infinite series

Exercise 1.10

For each series:

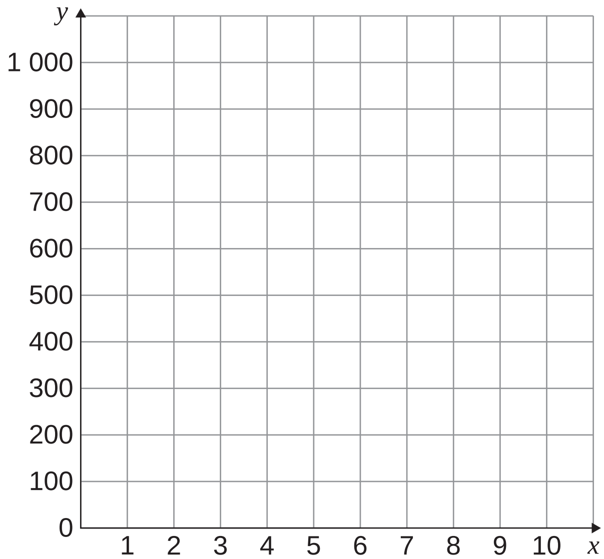
a) Complete the table.

b) Plot a graph where the *x*-axis represents the number of terms and the *y*-axis represents the sum of the terms.

1. $1 + 2 + 4 + 8 + \ldots$

S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}

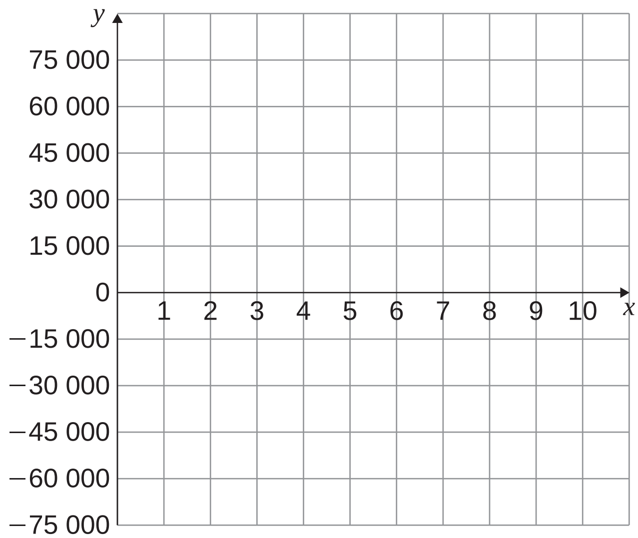
Graph for series 1:



2. $-5 + 15 - 45 \ldots$

S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}

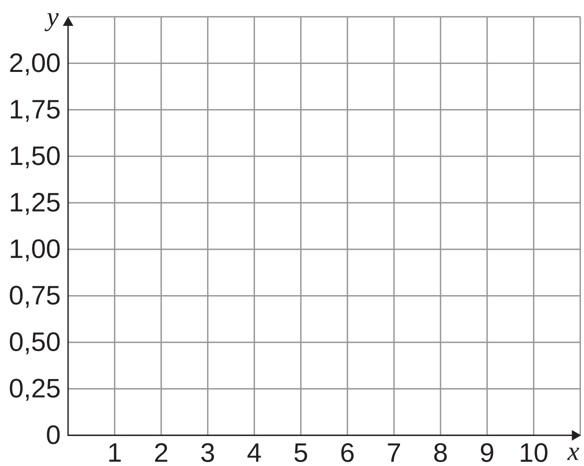
Graph for series 2:



3. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}

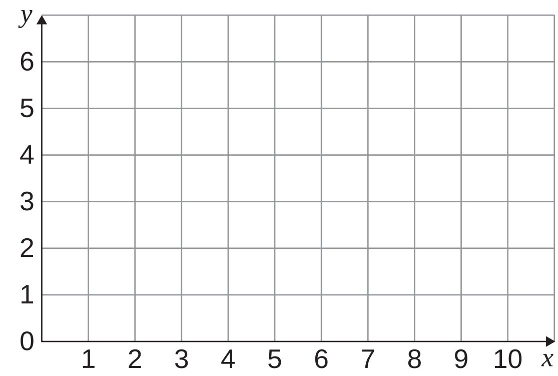
Graph for series 3:



4. $6 - 4 + \frac{8}{3} - \frac{16}{9} + \ldots$

S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}

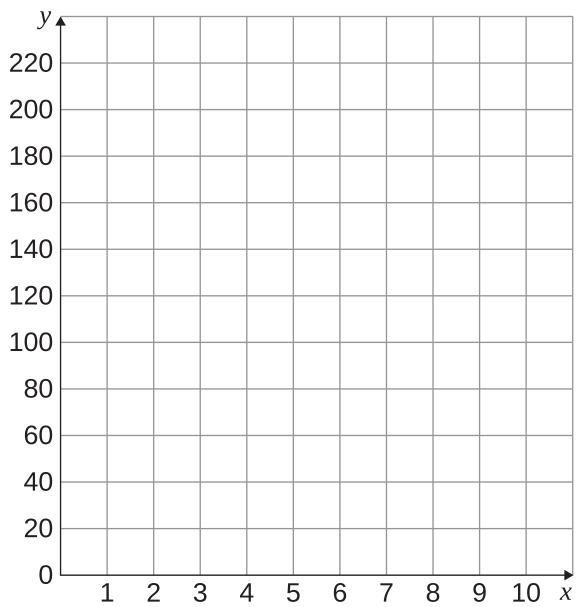
Graph for series 4:



5. $3 + 7 + 11 + \ldots$

S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}

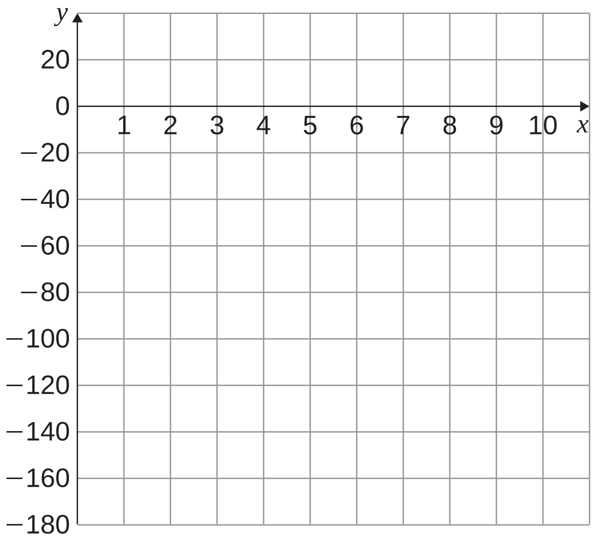
Graph for series 5:



6. $6 + 1 - 4 - \ldots$

S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}

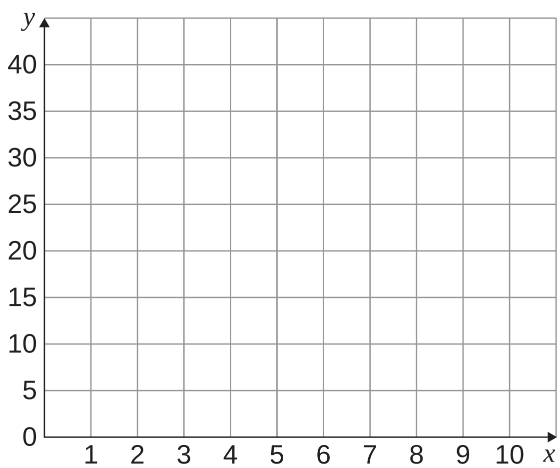
Graph for series 6:



7. $27 + 9 + 3 + \ldots$

S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}

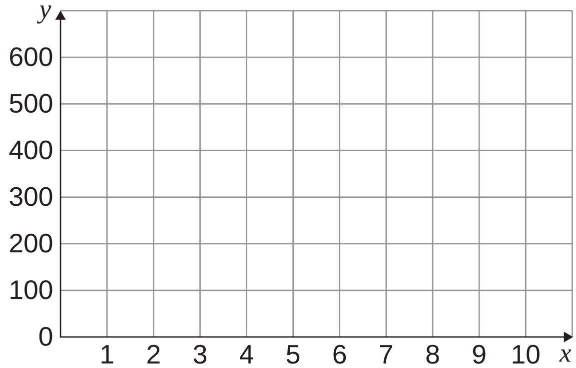
Graph for series 7:



8. $6 + 9 + \frac{27}{2} + \ldots$

S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}

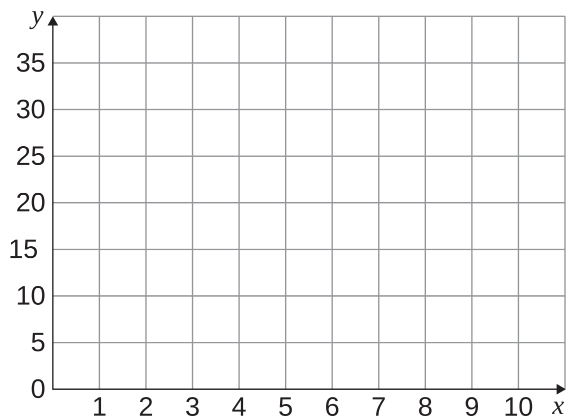
Graph for series 8:



9. $32 - 8 + 2 - \ldots$

S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}

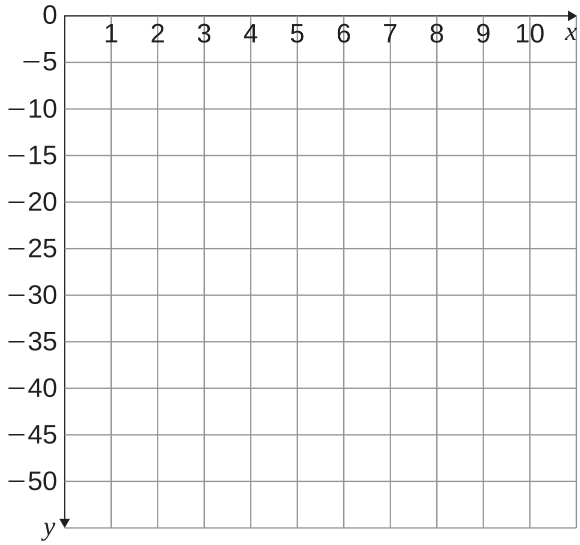
Graph for series 9:



10. $-5 - 5 - 5 - \ldots$

S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}

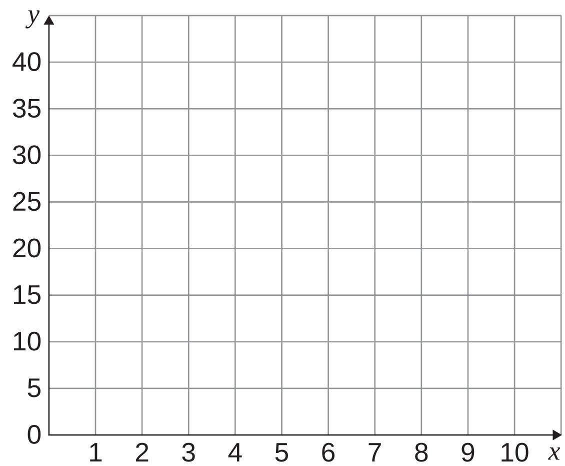
Graph for series 10:



11. $15 + 9 + \frac{27}{5} + \ldots$

S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}

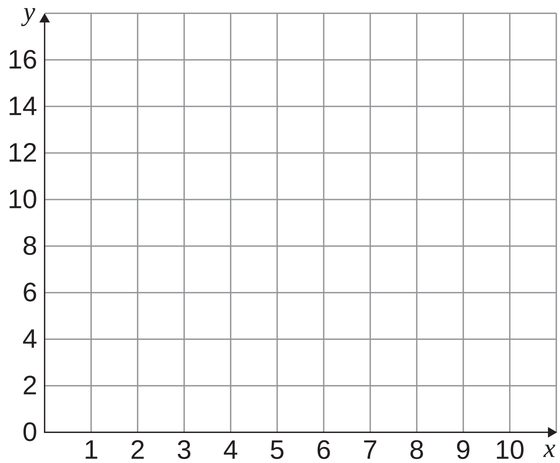
Graph for series 11:



12. $16 - 8 + 4 - \ldots$

S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}

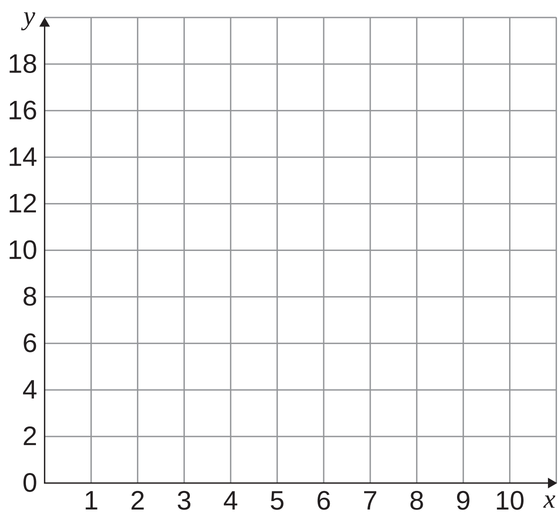
Graph for series 12:



13. $4 + 3\frac{1}{2} + 3 + \ldots$

S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}

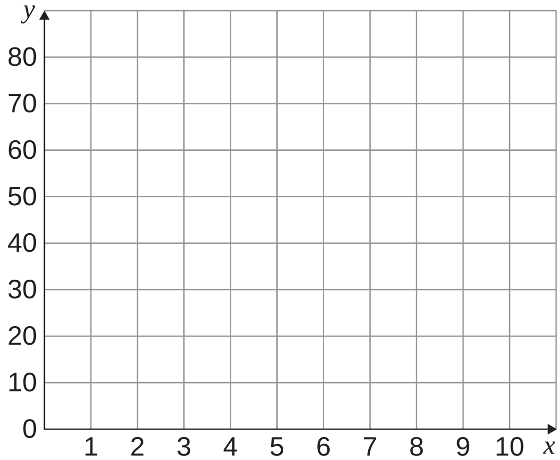
Graph for series 13:



14. $5 + 5\frac{2}{3} + 6\frac{1}{3} + \ldots$

S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}

Graph for series 14:



Until now, we have only ever added a finite number of terms. An infinite series has an infinite number of terms. In some cases, we can work out the sum of an infinite series. We write this as S_∞, which means the sum to infinity. If you could add an infinite number of terms, what do you think would be the sum of the series?

Using the tables and graphs from the previous questions, complete the following table:

Sequence	AP or GP	$d =$	$r =$	Conclusion about the S_∞
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				
11				

Sequence	AP or GP	$d =$	$r =$	Conclusion about the S_∞
12				
13				
14				

Tick the relevant block:

A series:

- **converges** if the sum approaches a particular value as we add more terms.
- **diverges** if the sum of the series becomes a very large positive or negative number as we add more terms.
- **oscillates** if, as we add more terms, the sum of the series changes between being positive and negative.

Series	Geometric series				Arithmetic series	
	$r < -1$	$-1 < r < 1$	$r = 1$	$r > 1$	$d < 0$	$d > 0$
Converges						
Diverges						
Oscillates						

Conclusion:

____

____

Sum to infinity

From the investigation, we find that only a geometric series will converge. In fact, a geometric series will only converge if $-1 < r < 1$. We can derive a formula for the sum to infinity as follows:

$$S_n = \frac{a - ar^n}{1 - r}$$

$$\therefore S_n = \frac{a}{1 - r} - \frac{ar^n}{1 - r}$$

if $-1 < r < 1$, then $r^n \to 0$ as $n \to \infty$ (Remember that the arrow $\to$ means 'tends to')

$$\therefore \frac{ar^n}{1 - r} \to 0 \text{ as } n \to \infty$$

$$\therefore S_n \to \frac{a}{1 - r} \text{ as } n \to \infty$$

$$\therefore S_\infty = \frac{a}{1 - r}$$

Example 1

Determine: $\sum_{n=1}^{\infty} 2 \cdot \left(\frac{1}{2}\right)^{n-1}$

Solution

$$\sum_{n=1}^{\infty} 2 \cdot \left(\frac{1}{2}\right)^{n-1} = 2 \cdot \left(\frac{1}{2}\right)^{0} + 2 \cdot \left(\frac{1}{2}\right)^{1} + 2 \cdot \left(\frac{1}{2}\right)^{2} + 2 \cdot \left(\frac{1}{2}\right)^{3} + \ldots$$
$$= 2 + 1 + \frac{1}{2} + \frac{1}{4} + \ldots$$

$a = 2 \qquad r = \frac{1}{2}$

$$S_{\infty} = \frac{a}{1 - r}$$
$$= \frac{2}{1 - \frac{1}{2}}$$
$$= \frac{2}{\frac{1}{2}}$$
$$= 2 \times \frac{2}{1}$$
$$= 4$$

Example 2

Use the formula for S_{∞} of a geometric series to express $0,\dot{6}$ as a common fraction.

Solution

$0,\dot{6} = \frac{6}{10} + \frac{6}{100} + \frac{6}{1\,000} + \ldots \qquad \therefore a = \frac{6}{10} \qquad r = \frac{1}{10}$

$$S_{\infty} = \frac{a}{1 - r}$$
$$= \frac{\frac{6}{10}}{1 - \frac{1}{10}}$$
$$= \frac{\frac{6}{10}}{\frac{9}{10}}$$
$$= \frac{6}{10} \times \frac{10}{9}$$
$$= \frac{2}{3}$$

Example 3

For which values of x will the series $(x + 1) + (x + 1)^2 + (x + 1)^3 + \ldots$ converge?

Solution

For the series $(x + 1) + (x + 1)^2 + (x + 1)^3 + \ldots\ r = x + 1$

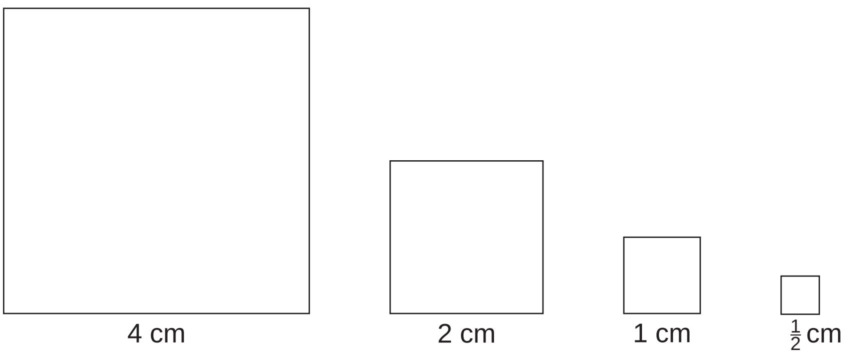
For the series to converge, $-1 < r < 1$

$\therefore -1 < x + 1 < 1$

$-2 < x < 0$

Exercise 1.11

1. Determine the sum to infinity for the following series:
   a) $27 + 9 + 3 + 1 + \ldots$
   b) $4;\ 2;\ 1;\ \frac{1}{2};\ \ldots$
   c) $16 - 4 + 1 - \frac{1}{4} + \ldots$
   d) $6 + 4 + \frac{8}{3} + \frac{16}{9} + \ldots$
   e) $-32 + 16 - 8 + 4 - \ldots$
   f) $25 + 15 + 9 + \frac{27}{5} + \ldots$
2. Evaluate the following, if possible. If not possible, give a reason why the sum to infinity cannot be found.
   a) $\sum_{n=1}^{\infty} 2 \cdot \left(\frac{1}{3}\right)^{n-1}$
   b) $\sum_{n=1}^{\infty} \left(-\frac{4}{5}\right)^{n-1}$
   c) $\sum_{n=1}^{\infty} \frac{1}{3}(2)^{n-1}$
   d) $\sum_{n=0}^{\infty} 18\left(\frac{2}{3}\right)^{n-1}$
   e) $\sum_{n=1}^{\infty} 18\left(\frac{3}{2}\right)^{n-1}$
   f) $\sum_{n=2}^{\infty} 3^{1-n}$
3. Convert each of the following to a common fraction:
   a) $0{,}\dot{8}$
   b) $2{,}\dot{3}\dot{4}$
   c) $1{,}\dot{2}\dot{5}$
4. Given the sequence $5(4^5) + 5(4^4) + 5(4^3) + \ldots$
   a) Show that the series is convergent.
   b) Calculate the sum to infinity of the series.
5. Given the geometric series $9x + 3x^2 + x^3 + \ldots$
   a) Show that $T_n = 27\left(\frac{x}{3}\right)^n$
   b) For which values of x will the series converge?
   c) Calculate the sum to infinity if $x = 2$.
6. In a sequence of squares, the sides of the first square are 4 cm long. The sides of each subsequent square are half that of the previous square.



   a) Determine the length of the side of the eighth square.
   b) Write down the series for the perimeter of the squares.
   c) Determine the sum of the perimeters of the squares if they continue infinitely.
   d) Write down the series for the areas of the squares.
   e) Determine the sum of the areas of the squares if they continue infinitely.

7. The numbers $5m - 7$; $2m + 2$; $m + 3$ are positive numbers and the first three terms of a convergent geometric series. Calculate:
   a) the value of m.
   b) the sum to infinity of the series.
8. The sum to infinity of a geometric series is $\frac{128}{3}$ and the common ratio is $\frac{1}{4}$. Calculate the first term of the series.
9. In the series $6 + 3 + \frac{3}{2} + \frac{3}{4} + \ldots$, A is the sum to infinity and B is the sum to n terms. Calculate:
   a) the value of A
   b) the value of B in terms of n
   c) the value of n for which is $A - B = \frac{3}{64}$.
10. A plant is a 100 cm tall when planted. At the end of the first year, the plant is 120 cm tall. Each year, the plant grows by half the amount of the previous year.
    a) What will the height of the plant be after six years?
    b) Show that the plant will never exceed a height of 140 cm.

Summary

	Arithmetic	Geometric
Test	$T_3 - T_2 = T_2 - T_1$	$\frac{T_3}{T_2} = \frac{T_2}{T_1}$
General form	$T_n = a + (n - 1)d$	$T_n = a \cdot r^{n-1}$
Sum	$S_n = \frac{n}{2}\left(2a + (n - 1)d\right)$ or $S_n = \frac{n}{2}(a + l)$	$S_n = \frac{a(1 - r^n)}{1 - r}; r < 1$ or $S_n = \frac{a(r^n - 1)}{r - 1}; r > 1$
Mean	$\frac{a + b}{2}$	$\sqrt{ab}$
Converges for:		$-1 < r < 1$
S_∞		$S_\infty = \frac{a}{1 - r}$

Revision exercise

1. Given the following sequence: 12; 6; 3; …
   a) Show that $T_n = 3 \, . \, 2^{3-n}$
   b) Determine: $\sum_{n=1}^{10} T_n$
2. Given the series: 16; 12; 9; $\frac{27}{4}$; …
   a) Calculate the sum of the first ten terms of the series.
   b) Determine the sum to infinity.
   c) Write the sum of the first ten terms of the series in sigma notation.
3. a) Prove that $\sum_{r=1}^{n} (3r + 4) = \frac{n(3n + 11)}{2}$
   b) Hence find the sum of the first 30 terms of the series.
   c) How many terms of the series will give a sum of 996?
4. How many terms of the series 2 + 5 + 8 + 11 + … add up to 876?
5. In an arithmetic progression, $S_6 = 20$ and $S_5 = 12$. Determine the value of T_6
6. For which values of x will the geometric series $3 + 3(x + 1) + 3(x + 1)^2 + \ldots$ be convergent?
7. The sum of the third and the seventh terms of an arithmetic series is 48. The sum of the first ten terms of the series is 265. Determine the first three terms of the series.
8.

T_1	T_2	T_3	T_4	T_5
	–4	8		32

   This sequence could be either arithmetic or geometric.
   a) Determine the nth term in each case.
   b) Hence, determine the first and fourth terms if the sequence is:
      i) arithmetic
      ii) geometric.
9. Given the sequence: $\frac{1}{2}$; 1 ; $\frac{1}{4}$; 5; $\frac{1}{8}$; 9; …
   a) Write down the next four terms if the sequence continues in the same manner.
   b) Determine the sum of the first 40 terms of the sequence.
10. Determine the nth of the sequence 4; 2; … if the sequence is
   a) arithmetic
   b) geometric.

11. The sum of the first n terms of a series is given by the formula $S_n = 3^{n-1} + 9$.
    a) Determine the sum of the first 20 terms of the series.
    b) Determine the 20th term of the series.
    c) Show that $T_n = 2 \cdot 3^{n-2}$.
12. Determine the 15th term of the arithmetic progression: $x + y$; $2x$; $3x - y$; …
13. A ball drops from a height of 16 metres and rebounds half its distance on each bounce. Calculate the total distance it will have travelled before coming to rest.
14. The sum of n terms of the arithmetic series $2 + 5 + 8$ … is equal to the sum of n terms of the arithmetic series $5 + 6\frac{1}{2} + 8 + 9\frac{1}{2} +$ … Calculate the value of n.
15. A tree is planted and the height is measured at the end of each year. The height of the tree is found to be 1 m at the end of the first year. In the second year, the tree increases in height by 15 cm. The tree increases in height each year by $\frac{4}{5}$ of the previous year's growth.
    a) Complete the table:

	Year 1	Year 2	Year 3	Year 4	Year 5
Height of tree in m	1	1,15			
Growth in cm		15	12		

    b) Determine the increase in the height of the tree at the end of the 12th year.
    c) Determine the height of the tree after 12 years.
    d) Show that the maximum height the tree will reach will be 1,75 m.
16. Since 2003, the deaths per 100 000 people at risk due to malaria in Africa have roughly followed the following pattern:

T_1	T_2	T_3	T_4	T_5
2003	2004	2005	2006	2007
126,2	123,2	119	113,6	107

    a) Determine whether this follows the pattern of
        i) a geometric sequence
        ii) an arithmetic sequence
        iii) a quadratic sequence.
    b) Determine the nth term of this sequence.
    c) What percentage reduction has taken place between 2003 and 2010?

Chapter 2 Functions and inverse functions

In this chapter you will:

- revise functions dealt with in grades 10 and 11
- define a function
- learn about the inverse of a function
- learn how to sketch the graphs of inverse functions.

In grades 10 and 11, you learnt how to draw the graphs of different functions, namely:

- the straight line: $y = ax + q$
- the parabola: $y = a(x + p)^2 + q$
- the hyperbola: $y = \frac{a}{(x + p)} + q$
- the exponential function: $y = a \cdot b^{x+p} + q$

Revising functions dealt with in Grade 10 and 11

Before sketching a graph:

- you need to identify the curve, so you need to be familiar with the general equation of each function
- draw a rough sketch of the graph
- calculate the possible x- and y-intercepts of the graph
- write down the equations of any asymptotes
- write down the equation of the axis of symmetry of a parabola and the coordinates of the turning point.

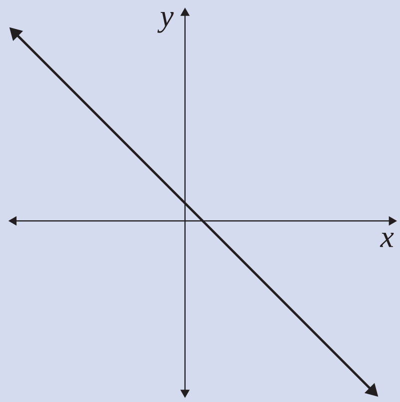
Example 1

Sketch the graph of $y = -\frac{1}{2}x + 3$

Solution

- This is the graph of a **straight line** with a negative gradient: gradient $= -\frac{1}{2}$

  Rough sketch:



- To calculate the y-intercept, we let $x = 0$ and solve for y:

  $$y = -\frac{1}{2}(0) + 3$$
  $$= 3$$

  The y-intercept is (0; 3).

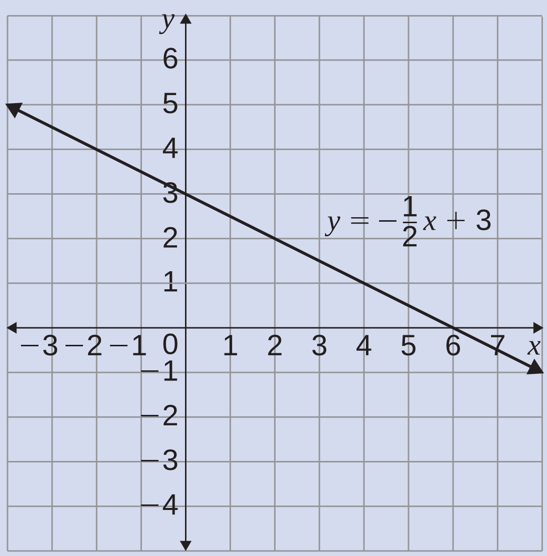
- To calculate the x-intercept, we let $y = 0$ and solve for x:

  $0 = -\frac{1}{2}x + 3$

  $0 = -x + 6$ *Multiply both sides by 2*

  $x = 6$

  The x-intercept is (6; 0).



Example 2

Sketch the graph of $y = 2(x - 3)^2 - 8$

Solution

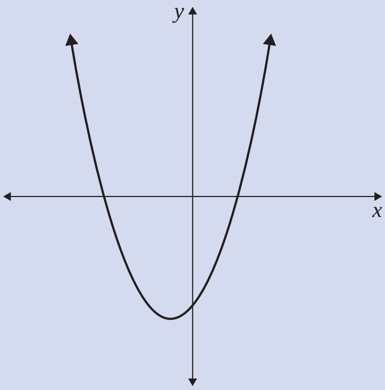
- This is the graph of a **parabola**.
- The equation of the axis of symmetry is $x = 3$.
- The turning point of the graph is $(3; -8)$.
- To calculate the y-intercept, let $x = 0$ and solve for y:

  $y = 2(0 - 3)^2 - 8$

  $y = 10$

  The y-intercept is $(0; 10)$.

- Rough sketch:



- To calculate the x-intercept, let $y = 0$ and solve for x:

$0 = 2(x - 3)^2 - 8$

$8 = 2(x - 3)^2$ *Add 8 to both sides*

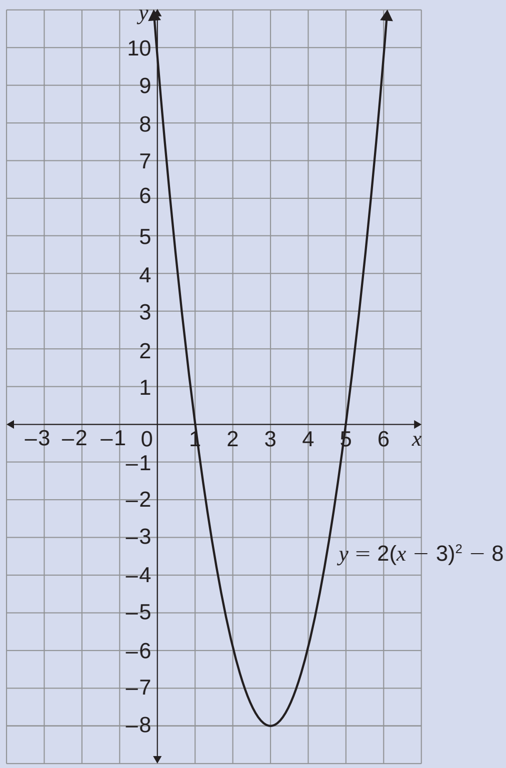
$4 = (x - 3)^2$ *Divide both sides by 2*

$\pm\sqrt{4} = (x - 3)$ *Take the square root of both sides*

$x - 3 = -2$ or $x - 3 = 2$

$x = 1$ $x = 5$

The x-intercepts are $(1; 0)$ and $(5; 0)$.



Example 3

Sketch the graph of $y = \frac{-4}{x-1} + 2$.

Solution

- This is the graph of a **hyperbola**.
- The equation of the asymptotes are $y = 2$ and $x = 1$.
- To calculate the y-intercept, we let $x = 0$ and solve for y:

  $y = \frac{-4}{0-1} + 2$

  $y = 6$

  The y-intercept is (0; 6).
- To calculate the x-intercept, we let $y = 0$ and solve for x:

  $0 = \frac{-4}{x-1} + 2$

  $\frac{4}{x-1} = 2$ *Add $\frac{4}{x-1}$ to both sides*

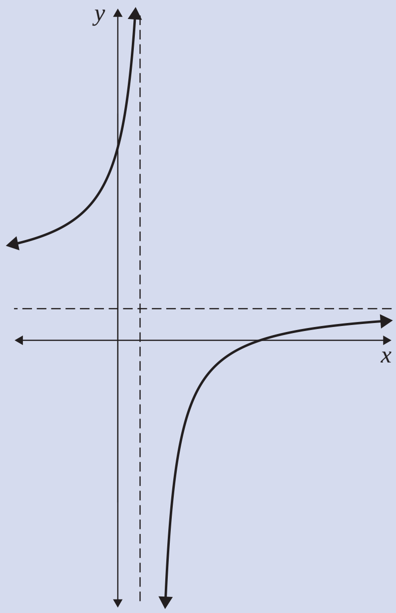
  $4 = 2(x - 1)$ *Find the LCD*

  $4 = 2x - 2$ *Multiply out the bracket*

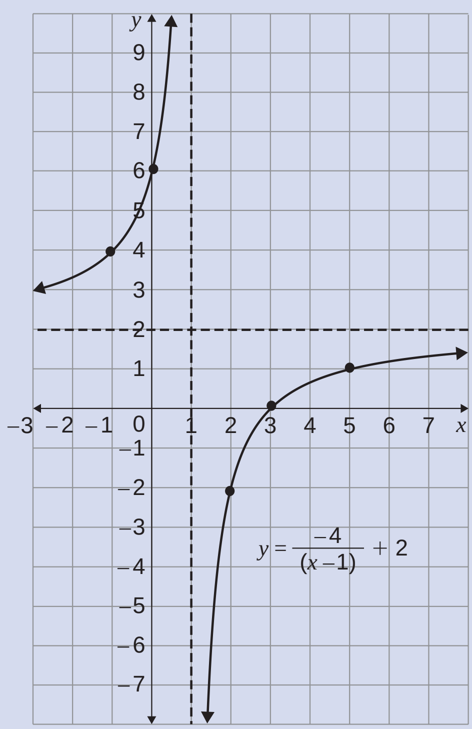
  $6 = 2x$ *Add 2 to both sides*

  $x = 3$ *Divide both sides by 2*

  The x-intercept is (3; 0).
- Rough sketch:



- Determine the coordinates of 3 other points to help you draw the graph.



Example 4

Sketch the graph of $y = -2\left(\frac{1}{2}\right)^{x+1} + 4$

Solution

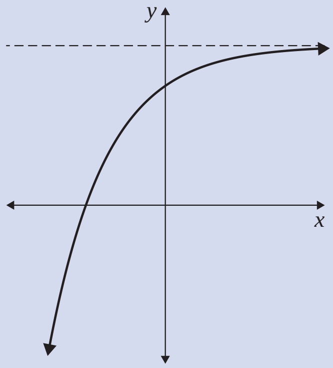
- This is the graph of an **exponential function**.
- The equation of the asymptote is $y = 4$.
- To calculate the y-intercept, we let $x = 0$ and solve for y:

  $y = -2\left(\frac{1}{2}\right)^{0+1} + 4$

  $y = 3$

  The y-intercept is (0; 3).
- To calculate the x-intercept, we let $y = 0$ and solve for x:

- Rough sketch:



$$0 = -2\left(\frac{1}{2}\right)^{x+1} + 4$$

$2\left(\frac{1}{2}\right)^{x+1} = 4$ *Add $2\left(\frac{1}{2}\right)^{x+1}$ t o both sides*

$\left(\frac{1}{2}\right)^{x+1} = 2$ *Divide both sides by 2*

$(2^{-1})^{x+1} = 2^1$ *Write $\frac{1}{2}$ as 2^{-1}*

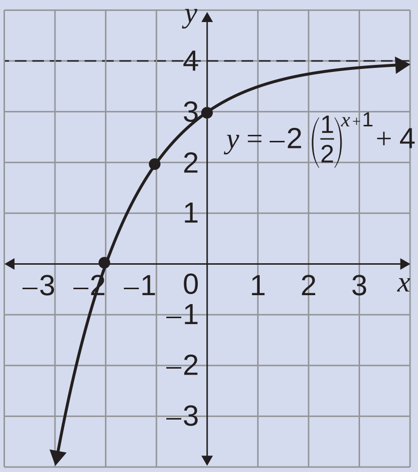
$2^{-x-1} = 2^1$ *Raising a power to a power*

$\therefore -x - 1 = 1$ *Equate the exponents*

$-x = 2$

$x = -2$ *Solve for x*

The x-intercept is (–2; 0).



Exercise 2.1

1. Sketch the graphs of the following:

a) $y = \frac{2}{x-1}$ b) $y = 2(x-1)^2$ c) $y = 2^x - 1$

d) $y = 2^{x-1}$ e) $y = 2x - 1$ f) $y = 2x^2$

g) $y = \frac{2}{x} - 1$ h) $y = \frac{x}{2} - 1$ i) $y = \frac{2}{x+2} + 1$

j) $y = -2(x+1)^2 + 2$ k) $y = x^{-1}$ l) $y = x^1$

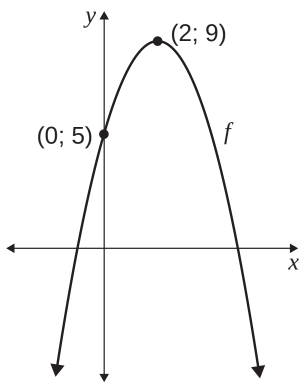
m) $y = x^2$ n) $y = 2^{-x}$ o) $y = \left(\frac{1}{2}\right)^{x+2}$

p) $y = \frac{4}{x-1} + 2$ q) $y = -1\left(\frac{1}{2}\right)^x + 4$ r) $y = \frac{-2}{x+2} - 1$

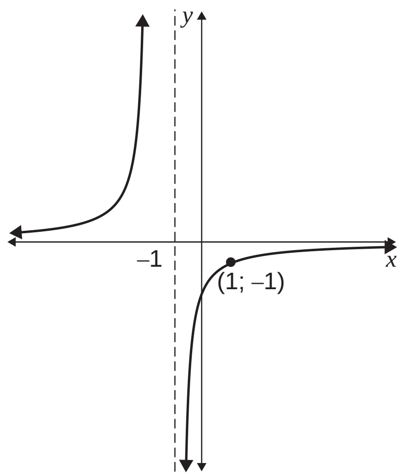
s) $y = \frac{1}{2}(x-2)^2 - 8$ t) $y = 3 \,.\, 2^{x+1} - 3$

2. Determine the equations of the following:

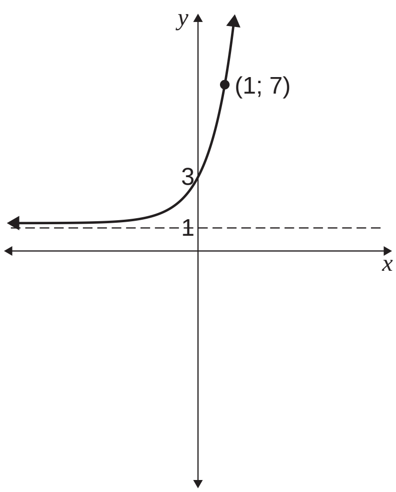
a) $f(x) = a(x+p)^2 + q$



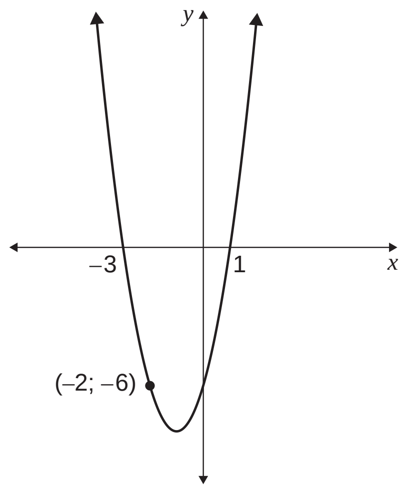
b) $g(x) = \frac{a}{x+p}$



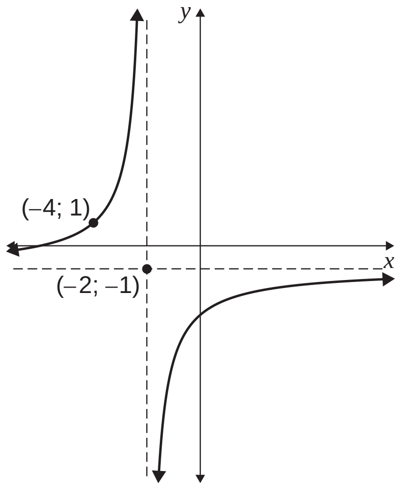
c) $h(x) = ab^x + q$



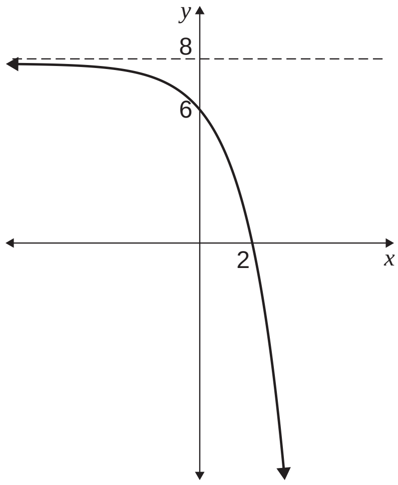
d) $p(x) = ax^2 + bx + c$



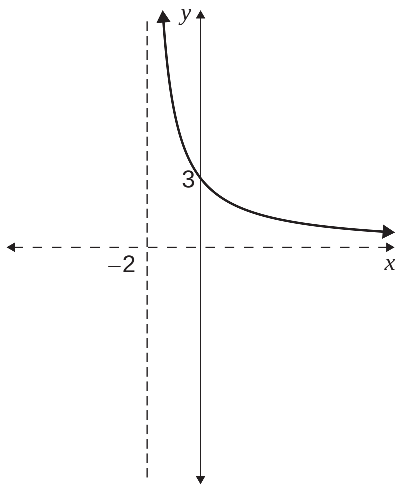
e) $g(x) = \dfrac{a}{x+p} + q$



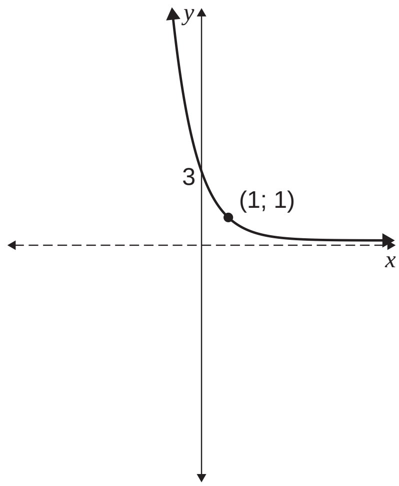
f) $h(x) = -1.b^{x+p} + q$



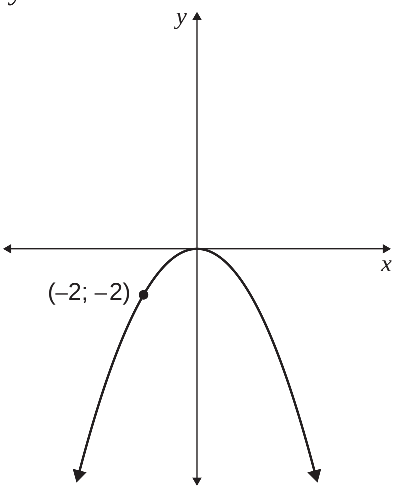
g) $y = \dfrac{a}{x+p}; y > 0$



h) $y = b^{x+p}$



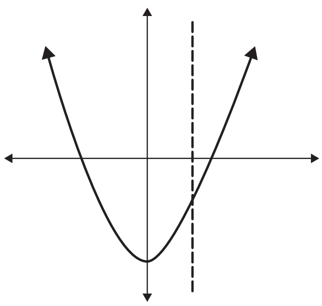
i) $y = ax^2$

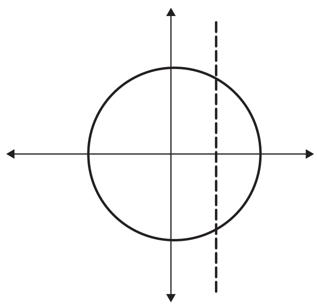


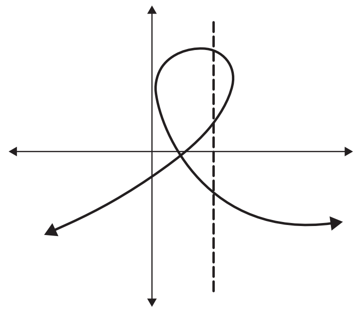
Functions

A **relation** is any relationship between two variables. A **function** is a special kind of relation in which:

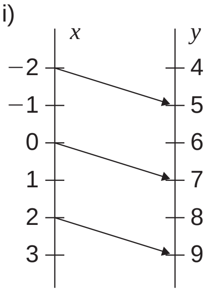
- For every *x*-value, there is at most one *y*-value. Each element of the domain is associated with **only one element** of the range. In other words, the *x*-values are never repeated in the set of ordered pairs of a function. For example:
  {(1; 2); (2; 4); (3; 6)} is a function
  {(**1**; 2); (**1**; −2); (**2**; 4); (**2**; −4)} is a NOT a function, because the *x*-coordinates are repeated.
- Any vertical line will cut the graph of a function once and only once. For example:


Function

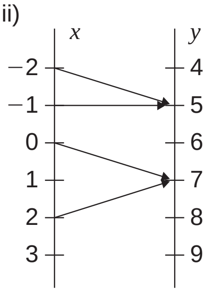

Not a function


Not a function

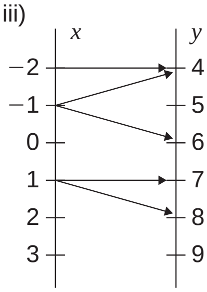
- A function has one-to-one or many-to-one **mapping**.

i)


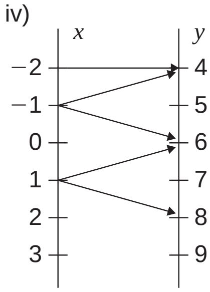
One-to-one mapping
function

ii)


Many-to-one mapping
function

iii)


One-to-many mapping
not a function

iv)


Many-to-many mapping
not a function

If we list the ordered pairs for each mapping, we have:

i) {(−2; 5); (0; 7); (2; 9)}
   No *x*-coordinate is repeated, so the relation is a function.
   Every *x*-value maps onto only one *y*-value. In other words, neither the *x*- nor the *y*-values are repeated.
   The domain is $x \in \{-2; 0; 2\}$ and the range is $y \in \{5; 7; 9\}$.

ii) {(−2; 5); (−1; 5); (0; 7); (2; 7)}
   No *x*-coordinate is repeated, so the relation is a function.
   Many *x*-values map onto more than one *y*-value. In other words, the *x*-values are not repeated, but the *y*-values are repeated.
   The domain is $x \in \{-2; -1; 0; 2\}$ and the range is $y \in \{5; 7\}$.

iii) $\{(-2; 4); (-1; 4); (-1; 6); (1; 7); (1; 8)\}$

Two x-coordinates are repeated, so the relation is not a function.

The same x-value maps onto different y-values. In other words, the x-values are repeated, but the y-values are not repeated.

The domain is $x \in \{-2; -1; 1\}$ and the range is $y \in \{4; 6; 7; 8\}$.

iv) $\{(-2; 4); (-1; 4); (-1; 6); (1; 6); (1; 8)\}$

Two x- coordinates are repeated, so the relation is not a function.

Many x-values map onto many y-values. In other words, both the x- and the y-values are repeated.

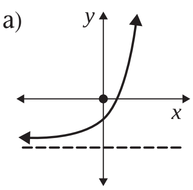
The domain is $x \in \{-2; -1; 1\}$ and the range is $y \in \{4; 6; 8\}$.

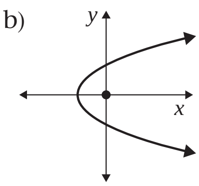
A function is **increasing** if the variables change in the same direction. In other words, as the x-values increase, the y-values increase as well. Or, as the x-values decrease, the y-values decrease as well.

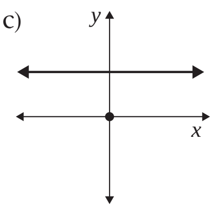
A function is **decreasing** if the variables change in different directions. In other words, as the values of x increase, the values of y decrease. Or, as the values of x decrease, the values of y increase.

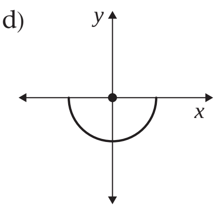
Exercise 2.2

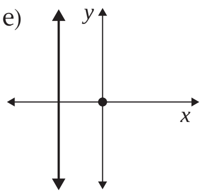
1. Determine which of the following graphs are functions.

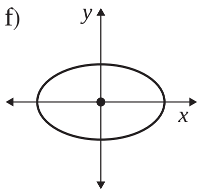
a)


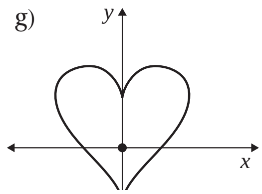
b)


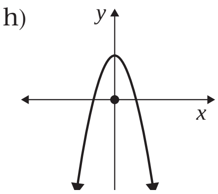
c)


d)


e)


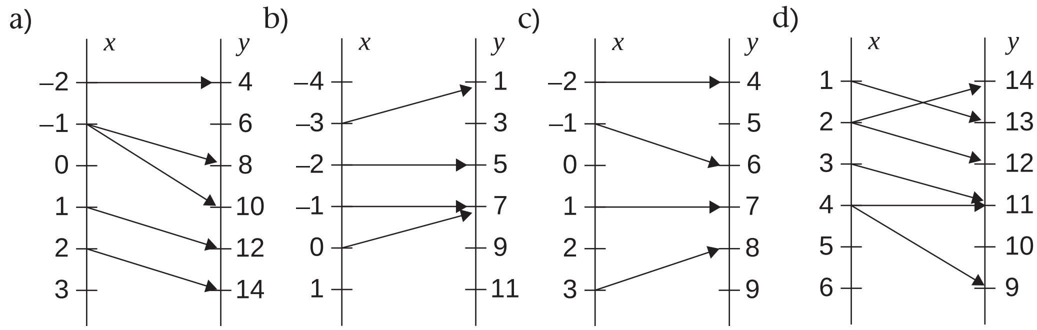
f)


g)


h)


2. $f = \{(2; 5); (3; 7); (4; 9); (5; 11)\}$
   a) Is f a function? Give a reason for your answer.
   b) Write down the domain and range of f.
   c) Determine m if $f(m) = 9$.
   d) Determine n if $f(3) = n$.
3. Given that P is not a function, determine the value(s) of x.
   $P = \{(x^2 ; 2); (x + 2; 3); (4; 5)\}$

4. Do the following mappings represent a function? Give a reason for your answer.



5. Write down the domain and range for each of the mappings in question 4.

Inverse functions

The inverse of a one-to-one function

Operation	Inverse operation
add	subtract
subtract	add
multiply	divide
divide	multiply
square	square root
square root	square

Example 1

Find the inverse of the function $f(x) = \frac{2x + 5}{3}$

Solution

If we draw a flow diagram of this function, it will look as follows:

$x \rightarrow \boxed{\text{Multiply by 2}} \xrightarrow{2x} \boxed{\text{Add 5}} \xrightarrow{2x + 5} \boxed{\text{Divide by 3}} \longrightarrow \frac{2x + 5}{3}$

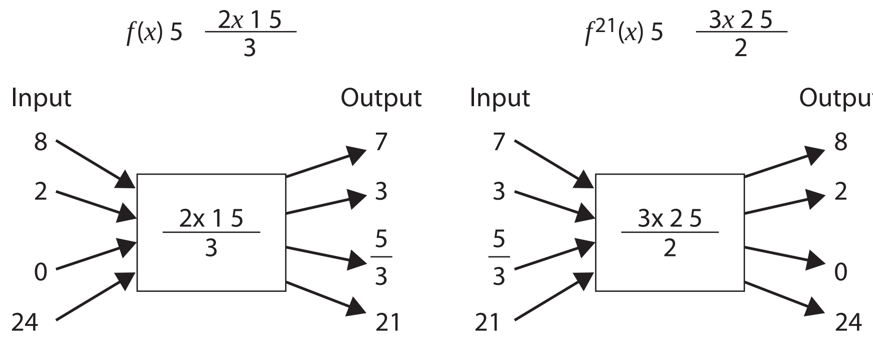
If we now replace each operation with its inverse, the flow diagram will look as follows:

$\frac{3x - 5}{2} \longleftarrow \boxed{\text{divide by 2}} \xleftarrow{3x - 5} \boxed{\text{subtract 5}} \xleftarrow{3x} \boxed{\text{multiply by 3}} \longleftarrow x$

We write the equation of the inverse function using the notation: f^{-1}

So if $f(x) = \frac{2x + 5}{3}$ then the inverse function is $f^{-1}(x) = \frac{3x - 5}{2}$

If we substitute values for x into the function, we obtain certain output values. If we now use these output values as input values to the inverse function, we get back to the original values we substituted into the function. For example:



If we list the set of ordered pairs, we have:

$f(x) = \frac{2x + 5}{3}$

$\left\{(8; 7); (2; 3); \left(0; \frac{5}{3}\right); (-4; -1)\right\}$

Domain: $x \in \{8; 2; 0; -4\}$

Range: $y \in \left\{7; 3; \frac{5}{3}; -1\right\}$

$f^{-1}(x) = \frac{3x - 5}{2}$

$\left\{(7; 8); (3; 2); \left(\frac{5}{3}; 0\right); (-1; -4)\right\}$

Domain: $x \in \left\{7; 3; \frac{5}{3}; -1\right\}$

Range: $y \in \{8; 2; 0; -4\}$

The domain of the function becomes the range of the inverse. The range of the function becomes the domain of the inverse.

Example 2

If $f(x) = 3x + 6$:

1. Write the equation of the inverse in the form f^{-1}.
2. Sketch the graphs of f and f^{-1} on the same system of axes, along with the line $y = x$.

Solution

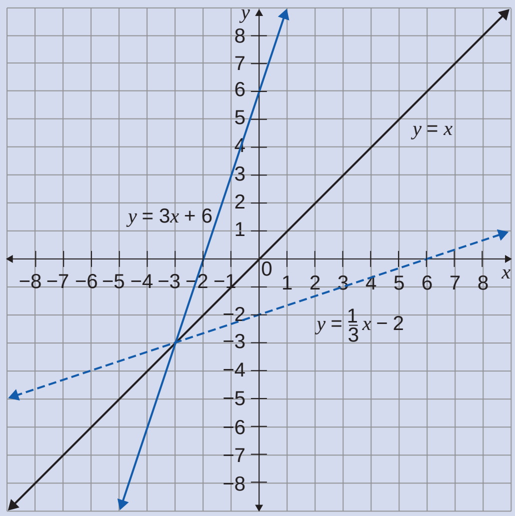
1. If $f(x) = 3x + 6$, we can determine the equation of the inverse by interchanging x and y in the original function.

   We can write $f(x) = 3x + 6$ as $y = 3x + 6$. Then, after interchanging x and y, we have the inverse function: $x = 3y + 6$. The last step is to make y the subject of the equation.

   $$x - 6 = 3y$$
   $$y = \frac{x - 6}{3}$$
   $$y = \frac{1}{3}x - 2$$

   So $f^{-1}(x) = \frac{1}{3}x - 2$

2. In the graph:
   - The graph and its inverse are symmetrical about the line $y = x$.
   - If the graphs intersect, they intersect on the line $y = x$.
   - If we interchange the x- and y-values of the coordinates of f, we can draw the graph of f^{-1}. So if the points (0; 6) and (−2; 0) satisfy the equation of f, then (6; 0) and (0; −2) satisfy the equation of f^{-1}.



- The graph and its inverse are always symmetrical about the line $y = x$.
- The inverse 'reverses' the operations of f.
- We denote the inverse of f as f^{-1}.
- The domain of f becomes the range of f^{-1}.
- The range of f becomes the domain of f^{-1}.
- To determine the inverse of any relation, interchange x and y.

Exercise 2.3

1. Complete this table: We have completed the first line as an example.

	FUNCTION	INVERSE	Interchange x and y	Is the inverse a function?
	Double it $y = 2x$	Halve it $y = \frac{x}{2}$	$x = 2y$ $\frac{x}{2} = y$ $\therefore y = \frac{x}{2}$	Yes
a)	Square it $y = x^2$			
b)	Add 2 $y = x + 2$			
c)	Subtract 4 $y = x - 4$			
d)	Divide by 5 $y = \frac{x}{5}$			
e)	Multiply by 6 $y = 6x$			

2. a) If $f(x) = 2x - 4$, complete the flow diagram.

$$x \longrightarrow \boxed{\text{Multiply by 2}} \xrightarrow{2x} \boxed{\text{Subtract 4}} \longrightarrow 2x - 4$$

$$\text{Inverse: } x \longrightarrow \boxed{\dots\dots\dots\dots} \xrightarrow{\dots\dots\dots} \boxed{\dots\dots\dots\dots} \longrightarrow \dots\dots\dots\dots$$

b) Draw f and its inverse on the same set of axes, as well as the graph of $y = x$.

3. For each of the following,
   - i) Write the equation of the inverse in the form f^{-1}.
   - ii) Sketch the graphs of f and f^{-1} on the same system of axes, as well as the line $y = x$.

a) $f(x) = 3x + 2$ b) $f(x) = \frac{1}{2}x - 6$ c) $f(x) = -2x + 4$

d) $f(x) = -\frac{1}{3}x + 1$ e) $f(x) = \frac{5}{2}x - 5$ f) $f(x) = -\frac{2}{3}x + 6$

4. $g = \{(-4; 3); (-3; 4); (0; 0); (3; -4); (4; -3)\}$
   - a) Is g a function? Give a reason for your answer.
   - b) Write down the domain and range of g.
   - c) Determine p if $g(p) = 4$.
   - d) Determine q if $g(-4) = q$.
   - e) Write g^{-1} in the form $g^{-1} = \{(\dots; \dots); \dots\}$
   - f) Is g^{-1} a function? Give a reason for your answer.
   - g) Write down the domain and range of g^{-1}.

The inverse of a many-to-one function

Example 1

1. Sketch the graph of $y = 2x^2$ and its inverse on the same set of axes.
2. Write the equation of the inverse in the form $y = \dots$
3. Write down the domain and the range of the relation and its inverse.

Solution

The characteristics of the graph $y = 2x^2$ are given below:
Axis of symmetry: $x = 0$
Turning point: $(0; 0)$
y-intercept: $(0; 0)$
x-intercepts: $(0; 0)$

If we calculate the coordinates of two other points which lie on the graph of $y = 2x^2$:
Let $x = 1$, then $y = 2(1)^2 = 2$. So $(1; 2)$ lies on the graph.
Also if $x = -2$, $y = 2(-2)^2 = 8$. So $(-2; 8)$ also lies on the graph.

Using the characteristics of $y = 2x^2$, we can determine the characteristics of its inverse.

	$y = 2x^2$	Inverse
Axis of symmetry	$x = 0$	$y = 0$
Turning point	(0; 0)	(0; 0)
***y*-intercept(s)**	(0; 0)	(0; 0)
***x*-intercept(s)**	(0; 0)	(0; 0)
	(1; 2) lies on the graph	(2; 1) lies on the inverse
	(−2; 8) lies on the graph	(8; −2) lies on the inverse

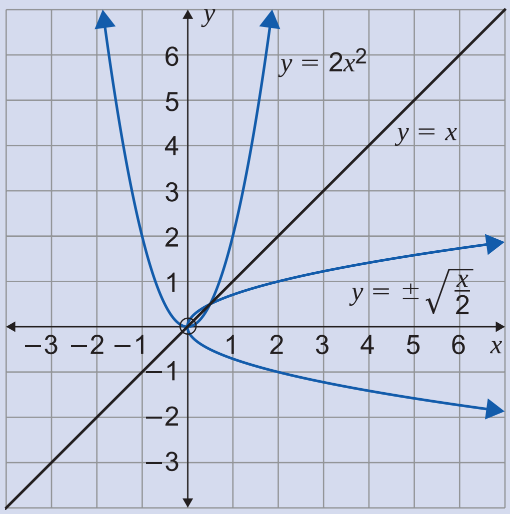
To determine the equation of the inverse in the form $y = \ldots$, we interchange x and y. So $y = 2x^2$ becomes:

$$x = 2y^2$$

$$\frac{x}{2} = y^2 \quad \text{(dividing both sides by 2)}$$

$$\pm\sqrt{\frac{x}{2}} = y \quad \text{(taking } \pm \text{ the square root on both sides)}$$

$$y = \pm\sqrt{\frac{x}{2}}$$



When dealing with one-to-one relations, both the relation and its inverse are functions.
When dealing with many-to-one relations, the relation is a function, but its inverse is not a function. However, we can restrict the domain of the relation so that its inverse will be a function.

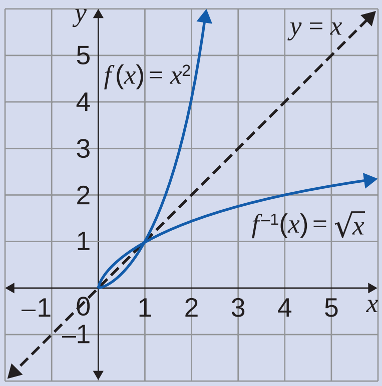
Example 2

If $f(x) = x^2$; $x \geq 0$, sketch the graphs of f and f^{-1} on the same set of axes.

Solution

	$y = f(x) = x^2$; $x \geq 0$	Inverse
Equation		$x = y^2$; $y \geq 0$ $\sqrt{x} = y$ *We only take the positive square root since* $y \geq 0$ $y = \sqrt{x}$ $y \geq 0$
Axis of symmetry:	$x = 0$	$y = 0$
Turning point:	(0; 0)	(0; 0)
y-intercept(s):	(0; 0)	(0; 0)
x-intercept(s):	(0; 0)	(0; 0)
Another point which lies on the graph	(2; 4)	(4; 2)

If a graph and its inverse intersect, they will intersect on the line $y = x$.



The inverse of an exponential function

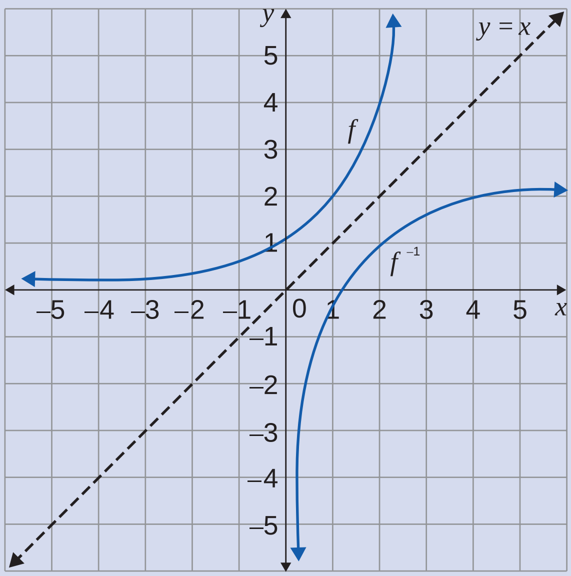
Example

1. If $f(x) = 2^x$; sketch the graphs of f and f^{-1} on the same set of axes.
2. Write down the domain and range of f and f^{-1}.

Solution

1. At this stage, you have not learnt how to write the inverse of $f(x) = 2^x$ in the form $f^{-1}(x) = \ldots$ You will learn how to do so in Chapter 3 when we discuss logarithms. For now we will accept the fact that the equation of the inverse of $f(x) = 2^x$ is written as $f^{-1}(x) = \log_2 x$. To draw the graph of the inverse of $f(x) = 2^x$, we use the same technique as before. In other words, we interchange x- and y-values. So if the point $(x; y)$ lies on f, then the point $(y; x)$ lies on the graph of f^{-1}.

	$f(x) = 2^x$	Inverse
Asymptote	$y = 0$	$x = 0$
y-intercept:	(0; 1)	none
x-intercept:	none	(1; 0)
Two other points on the graph	(1; 2) (2; 4)	(2; 1) (4; 2)



2.

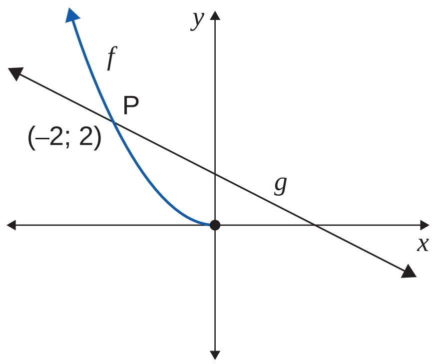
	Domain	Range
f	$x \in \mathbb{R}$	$y \in \mathbb{R}; y > 0$
f^{-1}	$x \in \mathbb{R}; x > 0$	$y \in \mathbb{R}$

Equation of the function	Equation of the inverse
$y = 2^x$	$y = \log_2 x$
$y = \left(\frac{1}{2}\right)^x$	$y = \log_{\frac{1}{2}} x$
$y = 3^x$	$y = \log_3 x$
$y = \left(\frac{1}{5}\right)^x$	$y = \log_{\frac{1}{5}} x$

Exercise 2.4

1. For each of the following:
   i) Determine the equation of the inverse.
   ii) Draw the graph of the relation, its inverse, and the line $y = x$.
   iii) Write down the domain and range for both the relation and its inverse.
   a) $f(x) = -\frac{3}{2}x^2$ b) $g(x) = -\frac{3}{2}x$
   c) $h(x) = 3x$ d) $p(x) = 3x^2$

2. The diagram represents the graphs of $y = f(x) = \frac{1}{2}x^2; x \leq 0$ and $y = g(x) = -\frac{1}{2}x + 1$. A point of intersection is P(–2; 2).



   a) Calculate the points where g intercepts the axes.
   b) Write down the equations of the inverse of f and g in the form $f^{-1}(x) = \ldots$ and $g^{-1}(x) = \ldots$
   c) On the same system of axes, sketch the graphs of f^{-1} and g^{-1}, clearly showing all intercepts with the axes.

d) Write down the domain and range of f.

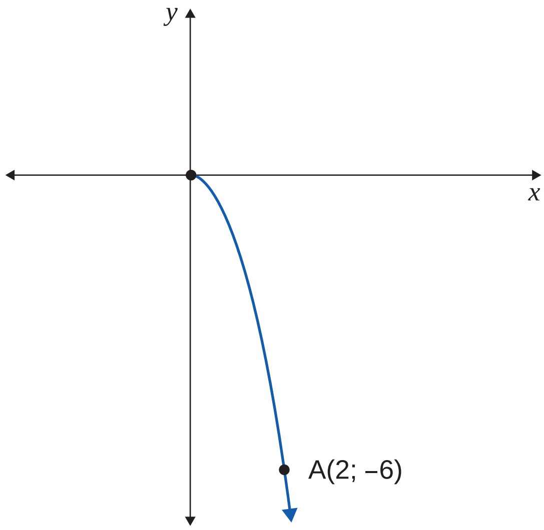
e) Write down the equation of the line about which f and f^{-1} are symmetrical.

3. a) If $f(x) = -2x^2;\ x \leq 0$, sketch the graph f and f^{-1} on the same set of axes.

   b) Write the equation of f^{-1} in the form $f^{-1}(x) = \ldots$

   c) Write down the domain and range of f and f^{-1}.

4. The diagram represents the graph of $y = f(x) = ax^2;\ x \geq 0$. Point A$(2; -6)$ lies on f.



   a) Calculate the value of a.

   b) Write down the equations of the inverse of f in the form $f^{-1}(x) = \ldots$

   c) Sketch the graph of f^{-1}, clearly showing the coordinates of at least two points on the curve.

   d) Write down the domain and range of f^{-1}.

5. Match the function with its inverse:

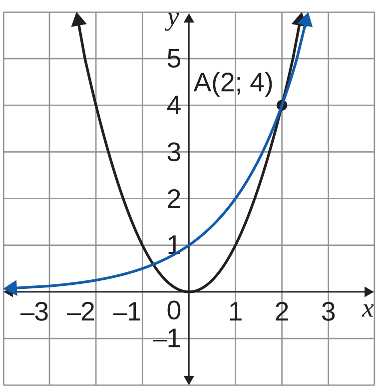
a)	$y = 2x - 4$	A	$y = \frac{x}{2} + 2$
b)	$y = \pm\sqrt{2x}$	B	$y = \frac{3}{x - 1}$
c)	$y = \frac{3}{x} + 1$	C	$y = -\sqrt{\frac{x}{2}};\ y < 0$
d)	$y = 2x^2;\ x < 0$	D	$y = \frac{1}{2}x^2$

Revision exercise

1. The following points satisfy the function *f*: (–3; 5), (–1; 7), (0; 8) and (4; 12). Write down the coordinates of four points that satisfy the inverse of *f*.
2. The inverse of the function *f* is given by the equation $f^{-1}(x) = \frac{x}{3} + 4$. Determine the equation of $f(x)$.
3. $g = \{(-4; 3); (-3; 2); (0; 1); (3; 2); (4 ; 3)\}$
   a) Is g a function? Give a reason for your answer.
   b) Write down the domain and range of g.
   c) Determine p if $g(p) = 2$.
   d) Determine q if $g(-4) = q$.
   e) Write g^{-1} in the form $g^{-1} = \{(\ldots;..);\ldots\}$
   f) Is g^{-1} a function? Give a reason for your answer.
   g) Write down the domain and range of g^{-1}.
4. $A\left(1; \frac{1}{2}\right)$ is a point on the curve of the function $f: x \rightarrow b^x$.
   a) Determine the value of b.
   b) Sketch f and f^{-1} on the same set of axes.
   c) Write down the domain and range of f and f^{-1}.
5. If $f(x) = \frac{4x + 16}{2}$, match the following:

a)	$f^{-1}(x)$	A	$y = \frac{1}{2x + 8}$
b)	$f\left(\frac{1}{x}\right)$	B	$y = \frac{2}{x} + 8$
c)	$\frac{1}{f(x)}$	C	$y = -2x + 8$
d)	$f(-x)$	D	$y = \frac{1}{2}x - 4$

6. The following graphs represent the functions $y = f(x) = x^2$ and $y = g(x) = 2^x$. One of the points of intersection of f and g is A(2; 4).



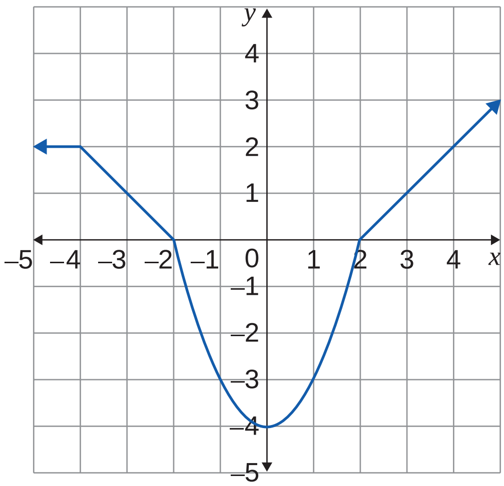
a) Write down the domain and range of f and g.

b) Draw the inverses of f and g on the same set of axes. Clearly show the coordinates of any intercepts with the axes, as well as the coordinates of two other points on each graph.

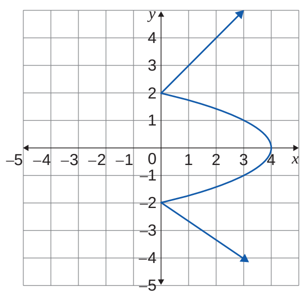
c) Write down the equation of the asymptote of g^{-1}.

d) How can the domain of f be restricted so that the inverse of f will also be a function? Hence, write down the equation of the inverse of f.

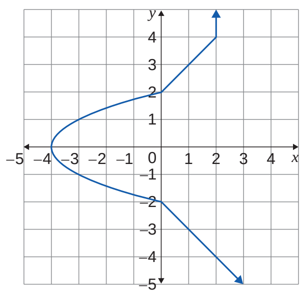
7. Which of A, B or C represent the inverse of the following relation?



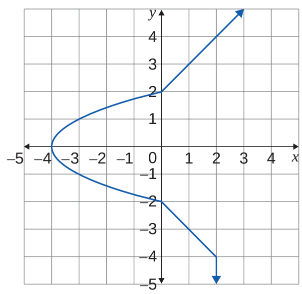
A



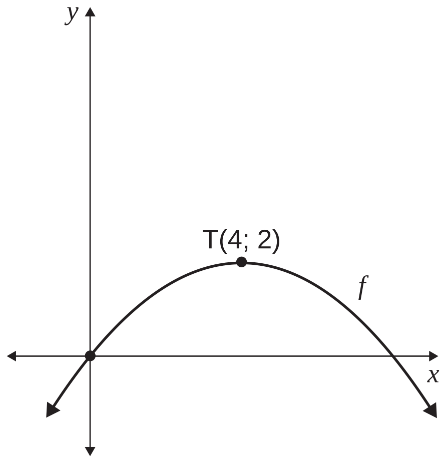
B



C



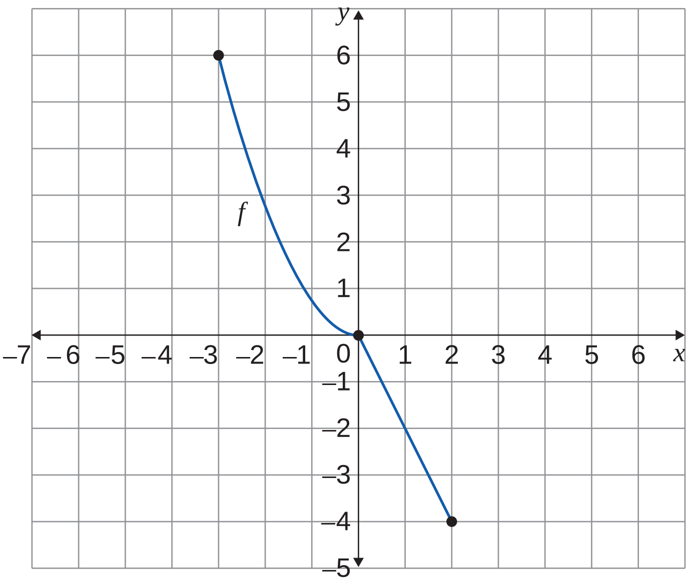
8. The following diagram represents the graph of $f(x) = ax^2 + bx + c$. The graph has a turning point of T(4; 2).



a) Determine the equation of f.

b) Draw the graph of f^{-1}, indicating at least three points on the curve.

9. This graph represents the function f.



a) Write down the domain and range of f.

b) Draw the graph of f^{-1}, the inverse of f.

c) Write down the domain and range of f^{-1}.

d) Is the inverse of f a function? Give a reason for your answer.

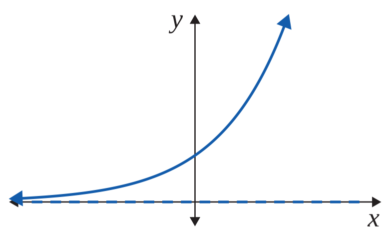
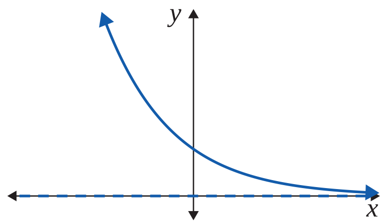
Chapter 3 Exponential and logarithmic functions

In this chapter you will:

- revise the exponential function and the exponential laws
- revise the graph defined by $y = b^x$, where $b > 0$ and $b \neq 1$
- learn the definition of a logarithm: $y = \log_b x \Leftrightarrow x = b^y$, where $b > 0$ and $b \neq 1$
- learn how to graph the function $y = \log_b x$ for both $0 < b < 1$ and $b > 1$.

Exponential functions

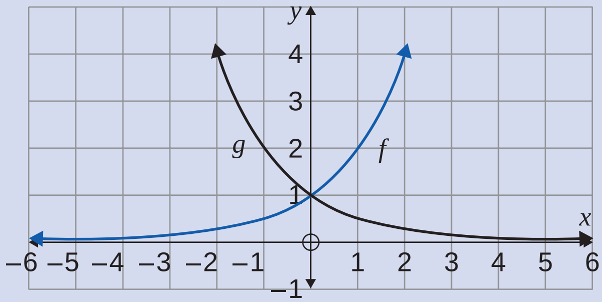
In this first section, we revise what you have learnt about graphs of the form $y = a^x$.

	Exponential functions	
	$\boldsymbol{y = a^x}$	
	$a > 1$	$0 < a < 1$
		
y-intercept	(0; 1)	(0; 1)
x-intercept	none	none
Increasing/ decreasing function	Increasing function	Decreasing function
Asymptote	$y = 0$	$y = 0$
Domain	$x \in \mathbb{R}$	$x \in \mathbb{R}$
Range	$y \in \mathbb{R}; y > 0$	$y \in \mathbb{R}; y > 0$

Example

On the same set of axes, sketch the graphs of $y = f(x) = 2^x$ and $y = g(x) = \left(\frac{1}{2}\right)^x$

Solution



Note that the graph of g is a reflection of f about the y-axis (or the line $x = 0$).

If a graph is reflected about the line $x = 0$, we obtain the equation of the reflected graph by replacing x with $-x$. For example:

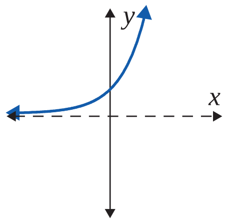
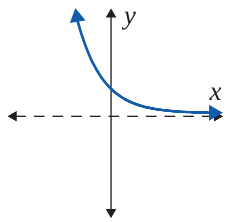
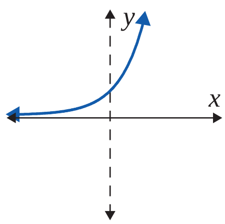
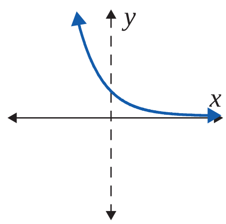
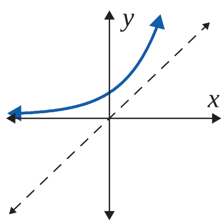
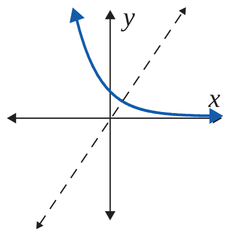
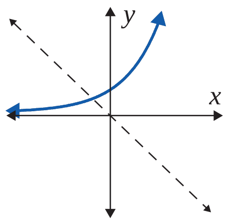
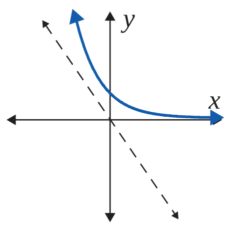
$$\begin{aligned} g(x) &= f(-x) \\ &= 2^{-x} \\ &= (2^{-1})^x \\ &= \left(\tfrac{1}{2}\right)^x \end{aligned}$$

Here is a summary of the rules to use when a graph is reflected about the x-axis, y-axis, the line $y = x$ and the line $y = -x$. You studied these rules in Grade 9.

		Coordinates change as follows
Reflections	about the x-axis	$(x; y) \rightarrow (x; -y)$
	about the y-axis	$(x; y) \rightarrow (-x ;y)$
	about the line $y = x$	$(x; y) \rightarrow (y; x)$
	about the line $y = -x$	$(x; y) \rightarrow (-y; -x)$

Exercise 3.1

1. Copy the table. In each case, reflect the graph about the given line.

Reflect the graph about the line:	$y = 2^x$	$y = \left(\frac{1}{2}\right)^x$
$y = 0$		
$x = 0$		
$y = x$		
$y = -x$		

2. Determine the equation of each of the reflections in Question 1.

	$y = 2^x$	$y = \left(\frac{1}{2}\right)^x$
Reflection about $y = 0$		
Reflection about $x = 0$		
Reflection about $y = x$		
Reflection about $y = -x$		

In Chapter 2, we pointed out that, when reflecting a graph about the line $y = x$, we simply interchange x and y in the equation of the function. We then make y the subject of the new equation. For example, the reflection of $y = 2^x$ about the line $y = x$ is $x = 2^y$. Until now, you have not been able to write this in the form $y = \ldots$. To do so, we need logarithms. The equation of the inverse of the graph $y = 2^x$ is written as $y = \log_2 x$.

Logarithmic functions

Definition: If $a^b = c$ then $b = \log_a c$

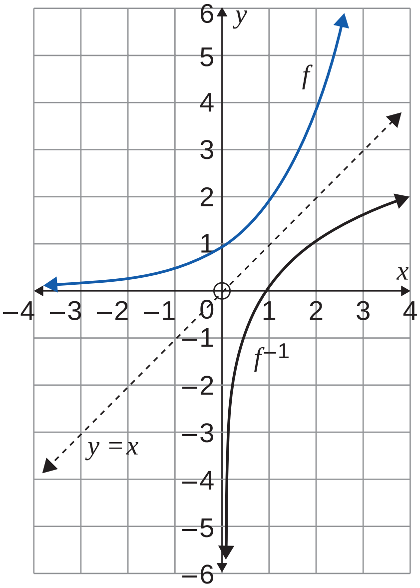
So if $x = a^y$ then $y = \log_a x$ if $a > 0$ and $a \neq 1$.

For example, if $x = 2^y$ then $y = \log_2 x$.

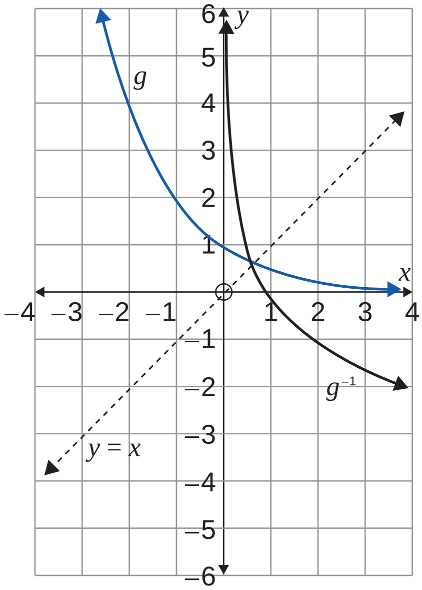
The **inverse function** of $y = a^x$ is $y = \log_a x$.

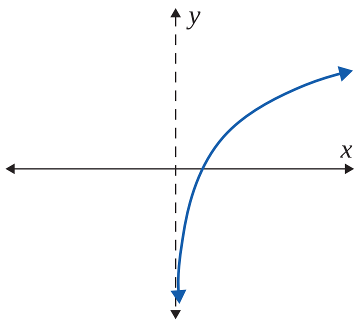
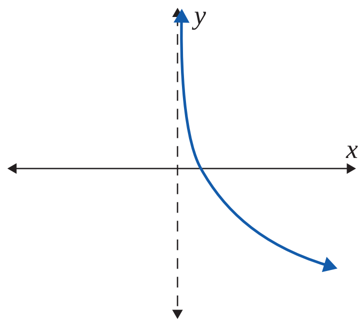
Therefore, if $f(x) = a^x$ then $f^{-1}(x) = \log_a x$ is the inverse function. For example:

- If $f(x) = 2^x$ then $f^{-1}(x) = \log_2 x$



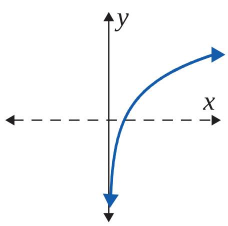
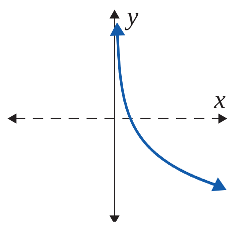
- If $g = \left(\frac{1}{2}\right)^x$ then $g^{-1} = \log_{\frac{1}{2}} x$

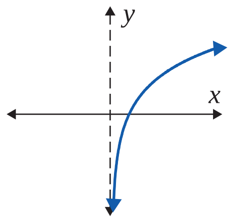
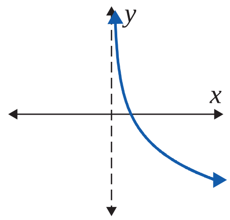
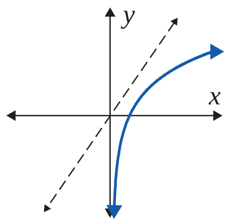
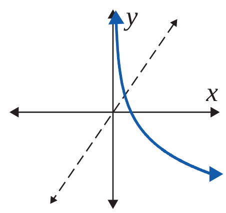
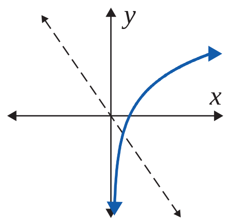
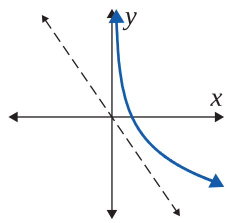


	Logarithmic functions	
	$y = \log_a x$	
	$a > 1$	$0 < a < 1$
		
y-intercept	none	none
x-intercept	(1; 0)	(1; 0)
Increasing/ decreasing function	Increasing function	Decreasing function
Asymptote	$x = 0$	$x = 0$
Domain	$x \in \mathbb{R};\ x > 0$	$x \in \mathbb{R};\ x > 0$
Range	$y \in \mathbb{R}$	$y \in \mathbb{R}$

Exercise 3.2

1. Copy the table. In each case, reflect the graph about the given line.

Reflect the graph about the line:	$y = \log_2 x$	$y = \log_{\frac{1}{2}} x$
$y = 0$		

Reflect the graph about the line:	$y = \log_2 x$	$y = \log_{\frac{1}{2}} x$
$x = 0$		
$y = x$		
$y = -x$		

2. Determine the equation of each of the reflections in Question 1.

	$y = \log_2 x$	$y = \log_{\frac{1}{2}} x$
Reflection about $y = 0$		
Reflection about $x = 0$		
Reflection about $y = x$		
Reflection about $y = -x$		

Definition of a logarithm

The logarithm of a number is the exponent to which a base must be raised to produce the number.

Definition: If $a^b = c$ then $b = \log_a c$
We read this as 'log c to the base a'.

Example 1

$\log_3 9 = 2$: log 9 to the base 3 is 2.

In other words, we need to raise 3 to the power of 2 to get 9.

$\log_3 9 = 2$ *Log form*

$3^2 = 9$ *Exponential form*

Example 2

Write $\log_5 125 = 3$ in exponential form.

Solution

$$\log_5 125 = 3$$

$$5^3 = 125$$

Example 3

Write $3^4 = 81$ in log form.

Solution

$$3^4 = 81$$

$$\log_3 81 = 4$$

Example 4

Evaluate $\log_2 32$.

Solution

Let $\log_2 32 = x$

$2^x = 32$ *Exponential form*

$2^x = 2^5$ *Writing 32 as 2^5*

$$\therefore x = 5$$

$$\therefore \log_2 32 = 5$$

Example 5

Evaluate $\log_{10} \frac{1}{100}$.

Solution

Let $\log_{10} \frac{1}{100} = x$

$10^x = \frac{1}{100}$ *Exponential form*

$$10^x = \frac{1}{10^2}$$

$$10^x = 10^{-2}$$

$$\therefore x = -2$$

$$\therefore \log_{10} \frac{1}{100} = -2$$

Exercise 3.3

1. Evaluate the following:

a) $\log_2 8$ b) $\log_3 9$ c) $\log_{10} 100$ d) $\log_4 64$
e) $\log_3 81$ f) $\log_3 \frac{1}{27}$ g) $\log_5 25$ h) $\log_2 \frac{1}{2}$
i) $\log_4 16$ j) $\log_5 5$ k) $\log_3 \sqrt{9}$ l) $\log_{10} 10\ 000$

2. Complete the table:

	$a^b = c$	$b = \log_a c$
a)		$\log_3 9 = 2$
b)	$2^4 = 16$	
c)	$10^2 = 100$	
d)		$\log_2 \frac{1}{4} = -2$
e)		$\log_5 125 = 3$
f)	$4^3 = 64$	
g)	$3^5 = 243$	
h)		$\log_{10} 1 = 0$
i)	$16^{\frac{1}{2}} = 4$	
j)	$7^{-2} = \frac{1}{49}$	
k)		$\log_9 27 = \frac{3}{2}$
l)	$10^3 = 1\ 000$	

3. Solve for x:

a) $\log_2 16 = x$ b) $\log_3 x = 81$ c) $\log_x 100 = 2$
d) $\log_4 x = 3$ e) $\log_x 32 = 5$ f) $\log_3 x = -3$
g) $\log_{10} \frac{1}{100} = x$ h) $\log_4 \sqrt{32} = x$ i) $\log_2 x = 16$
j) $\log_{\frac{1}{2}} 4 = x$ k) $\log_{\frac{1}{10}} x = -2$ l) $\log_5 \sqrt{125} = x$

Logarithmic laws

(This section is not examinable.)

In previous grades, you learnt the exponential laws and definitions.

Exponential laws and definitions

In this chapter, we work in the real number system.

(There are complex restrictions to these five laws, which we will not discuss in this chapter.)

Exponents				
Laws			**Definitions**	
1.	$a^m \times a^n = a^{m+n}$		1.	$a^0 = 1;\ a \neq 0$
2.	$a^m \div a^n = a^{m-n}$		2.	$a^{-n} = \frac{1}{a^n}$ or $\frac{1}{a^{-n}} = a^n$
3.	$(a^m)^n = a^{m \times n}$		3.	$\sqrt[n]{a^m} = a^{\frac{m}{n}}$
4.	$(ab)^n = a^n b^n$			
5.	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$			

Logarithms	
Laws	
1.	$\log_a m + \log_a n = \log_a mn$
2.	$\log_a m - \log_a n = \log_a \frac{m}{n}$
3.	$\log_a x^m = m \log_a x$
4.	$\log_a x = \frac{\log_b x}{\log_b a}$

You must be able to apply these laws:

- In general, $\log_a a = 1$
- $\log_a 1 = 0$, since $a^0 = 1$
- $\log_1 x$ has no meaning
- $\log x = \log_{10} x$. If a logarithm is written without a base, we assume the base is 10.
- $\log_a x$ has no meaning if $a \leq 0$. The base of a logarithm must be positive.
- $\log_a a^x = x$

Examples

Evaluate the following:

1. $\log_3 3$

Solution

$\log_3 3 = 1$

2. $\log_2 8$

Solution

$$\begin{aligned} \log_2 8 &= \log_2 2^3 && \textit{8 = 2}^3 \\ &= 3\log_2 2 && \textit{Use Law 3} \\ &= 3(1) && \textit{Since } \log_2 2 = 1 \\ &= 3 \end{aligned}$$

3. $\log 4 + \log 25$

Solution

$$\begin{aligned} \log 4 + \log 25 &= \log(4 \times 25) \quad \textit{Use Law 1} \\ &= \log 100 \\ &= \log_{10} 10^2 \\ &= 2\log_{10} 10 \\ &= 2(1) \\ &= 2 \end{aligned}$$

4. $\log_3 81 - \log_3 3$

Solution

$$\begin{aligned} \log_3 81 - \log_3 3 &= \log_3\left(\frac{81}{3}\right) \quad \textit{Use Law 2} \\ &= \log_3 27 \\ &= \log_3 3^3 \\ &= 3\log_3 3 \\ &= 3(1) \\ &= 3 \end{aligned}$$

5. $\log 2 + \log 3 - \log 12 - \log 4 + \log 8$

Solution

$\log 2 + \log 3 - \log 12 - \log 4 + \log 8$

$= \log\left(\frac{2}{1} \times \frac{3}{1} \times \frac{1}{12} \times \frac{1}{4} \times \frac{8}{1}\right)$

$= \log\left(\frac{48}{48}\right)$

$= \log 1$

$= 0$

6. $\log_8 1\,024$

Solution

$$\begin{aligned} \log_8 1\,024 &= \frac{\log_2 1\,024}{\log_2 8} \\ &= \frac{\log_2 2^{10}}{\log_2 2^3} \\ &= \frac{10\log_2 2}{3\log_2 2} \\ &= \frac{10}{3} \end{aligned}$$

7. $3\log 2 + 3\log 5$

Solution

$$\begin{aligned} 3\log 2 + 3\log 5 &= \log 2^3 + \log 5^3 \\ &= \log 8 + \log 125 \\ &= \log(8 \times 125) \\ &= \log 1\,000 \\ &= \log_{10} 10^3 \\ &= 3\log_{10} 10 \\ &= 3 \end{aligned}$$

8. $\log_7 8 \,.\, \log_4 49$

Solution

$\log_7 8 \,.\, \log_4 49$

$= \frac{\log 8}{\log 7} \cdot \frac{\log 49}{\log 4}$ *Change the base to base 10, using Law 4*

$= \frac{\log 2^3}{\log 7} \cdot \frac{\log 7^2}{\log 2^2}$

$= \frac{3\log 2}{\log 7} \cdot \frac{2\log 7}{2\log 2}$ *Use Law 3*

$= 3$

Exercise 3.4

1. Evaluate the following using the log laws:

a) $\log_2 8 + \log_2 4$ b) $\log_5 75 - \log_5 3$ c) $\log_3 27$

d) $\log 2 + \log 5$ e) $\log 4 + \log 25$ f) $\log_7 98 - \log_7 2$

g) $\log 18 + \log 5 - 2\log 3$ h) $\log_3 27 - \log_3 \frac{1}{3}$ i) $-\log 50 - \log 2$

2. Evaluate the following:

a) $\log_2 64 + \log_2 4$ b) $\log_2 64 - \log_2 4$ c) $\log_2 64 \times \log_2 4$

d) $\log_2 64 \div \log_2 4$ e) $\log_2 (64 + 4)$ f) $\log_2 (64 - 4)$

g) $\log_2 (64 \times 4)$ h) $\log_2 (64 \div 4)$ i) $\log_2 64^2$

j) $(\log_2 64)^2$ k) $\log_2 64^{-2}$ l) $\log_2 64^{\frac{1}{2}}$

m) $\log_{\frac{1}{2}} 16$ n) $\log_2 9 \,.\, \log_8 3$ o) $\log_5 4 \,.\, \log_{\frac{1}{2}} 25$

3. Evaluate the following:

a) $\dfrac{\log_3 81}{\log_3 27}$ b) $\log_3 \left(\dfrac{81}{27}\right)$ c) $\dfrac{\log_3 27 + \log_3 3}{\log_3 27 - \log_3 3}$

d) $\dfrac{\log_3 27 \times \log_3 3}{\log_3 27 \div \log_3 3}$ e) $\dfrac{\log_2 81 + \log_3 16}{\log_2 3 - \log_3 2}$ f) $\dfrac{\log(81 \div 27)}{\log 81 \div \log 27}$

Equations involving logarithms

When solving an equation that includes logarithms, you need to remember that:

- the base of a logarithm cannot be negative
- we cannot find the logarithm of a negative number.

For example, if $\log_a x = P$; $a > 0$, $a \neq 1$ and $x > 0$.

This means that you need to check the solutions to the logarithmic equations to make sure they do not break either of these rules. We can solve logarithmic equations by

- using the definition of a logarithm:
  If $\log_a x = P$ (log form) then $a^P = x$ (exponential form)
- taking the logarithm of each side of the equation:
  If $x = y$ then $\log_a x = \log_a y$

Example 1

Solve for x: $2\log_x 64 = 6$

Solution

$2\log_x 64 = 6$

$\log_x 64 = 3$ *Divide both sides by 2*

$x^3 = 64$ *Exponential form*

$x^3 = 4^3$

$x = 4$

Example 2

Solve for x: $(0{,}3)^x = 5$, correct to two decimal places.

Solution

$(0{,}3)^x = 5$	
$\log (0{,}3)^x = \log 5$	*Take the log of both sides of the equation*
$x \log (0{,}3) = \log 5$	*Use Law 3*
$x = \dfrac{\log 5}{\log (0{,}3)}$	*Divide both sides by log 0,3*
$x = -1{,}34$	

Example 3

Solve for x: $2 \log x = \log (x + 6)$

Solution

$2 \log x = \log (x + 6)$	
$\log x^2 = \log (x + 6)$	*Use Law 3*
$\therefore x^2 = x + 6$	
$x^2 - x - 6 = 0$	*Write in the form* $ax^2 + bx + c = 0$
$(x - 3)(x + 2) = 0$	*Factorise*
$x = 3$ or $x = -2$	

Check whether the solutions are valid. Substitute the solutions into the **original** equation:
If $x = 3$ then:
LHS $= 2 \log 3 = \log 3^2 = \log 9$
RHS $= \log (3 + 6) = \log 9$
Therefore, LHS = RHS, and the solution is valid.

If $x = -2$:
LHS $= 2 \log (-2)$, which is undefined.
Therefore, $x = -2$ is not a valid solution. The only solution is $x = 3$.

Exercise 3.5

1. Solve for x:
   a) $\log_2 x = 3$
   b) $\log_3 81 = x$
   c) $\log_x 49 = 2$
   d) $\log_2 x = -1$
   e) $\log_4 x = \frac{1}{2}$
   f) $2 \log_3 27 = x$
   g) $\log_2 \frac{1}{32} = 2x$
   h) $2 \log x = \log \frac{1}{9}$
   i) $2 \log_2 x = \log_2 16$
2. Solve for x:
   a) $\log_3 (x + 2) = 0$
   b) $\log_3 x + 2 = 0$
   c) $\log_2 (2x + 1) = 3$
   d) $\log (x + 2) + \log (x - 1) = 1$
   e) $\log_2 (x + 2) + \log_2 (x - 3) = \log_2 6$
   f) $\log_3 (x - 3) + \log_3 (x + 5) = 2$
   g) $\log_2 (3x + 1) - \log_2 (x - 3) = 3$
   h) $\log (2x - 1) + \log (x - 2) = \log 5$

3. Solve for x, writing answers correct to one decimal place where necessary:

a) $2^x = 3$	b) $5^x = 12$	c) $5^x = 0{,}75$
d) $(1{,}08)^x = 2$	e) $(1{,}095)^x = 300$	f) $(1{,}105)^x = 45{,}76$
g) $(1{,}12)^{2x} = 32{,}64$	h) $3(1{,}105)^x = 45$	i) $200(1{,}11)^x = 500$
j) $P(1{,}08)^x = 2P$	k) $2(5)^x = 1{,}6$	l) $3{,}4(1{,}07)^{2x-1} = 5{,}2$

4. The number of organisms in a culture, C, after n days is given by the formula $C = ax^n$, where a and x are positive real numbers.
   a) The culture began with 1 000 organisms. After four days, there were 64 000 organisms. Calculate the values of a and x.
   b) How many days would it take for the number of organisms to grow to 65 536 000?

Exponential and logarithmic functions

You were introduced to the graph of $y = \log_a x$ at the beginning of this chapter. In this section, we explore these graphs further.

Example 1

1. If the graph of $y = f(x) = \log_a x$ passes through the point $\left(\frac{1}{2}; -1\right)$, show that $a = 2$.
2. Write down the equation of f^{-1}.
3. Draw the graph of f and f^{-1} on the same set of axes.
4. Is f an increasing or decreasing function?
5. Write down the equation of the asymptote of f.
6. On the same set of axes, draw the graph of g if g is the reflection of f about the x-axis.
7. Write the equation of g in the form $g(x) = \ldots$.

Solution

1. Substitute the point $\left(\frac{1}{2}; -1\right)$ into the equation $y = \log_a x$ and solve for a.

$-1 = \log_a \frac{1}{2}$

$a^{-1} = \frac{1}{2}$

$a^{-1} = 2^{-1}$

$\therefore a = 2$

2. You learnt in Chapter 2 that the exponential graph is the inverse of the logarithmic graph. Therefore, $f^{-1}(x) = 2^x$. We could also determine the equation of f^{-1} by interchanging x and y in the equation $y = \log_a x$. Then, $x = \log_a y$.

$y = a^x$

$\therefore f^{-1}(x) = 2^x$

3. $y = f(x) = \log_2 x$

From the summary earlier in the chapter, we know that f cuts the x-axis, but not the y-axis. Therefore, to calculate the x-intercept, we let $y = 0$:

$$0 = \log_2 x$$
$$2^0 = x$$
$$x = 1$$

Therefore, the x-intercept is $(1; 0)$.

We can determine two other points that lie on the graph $y = f(x) = \log_2 x$.

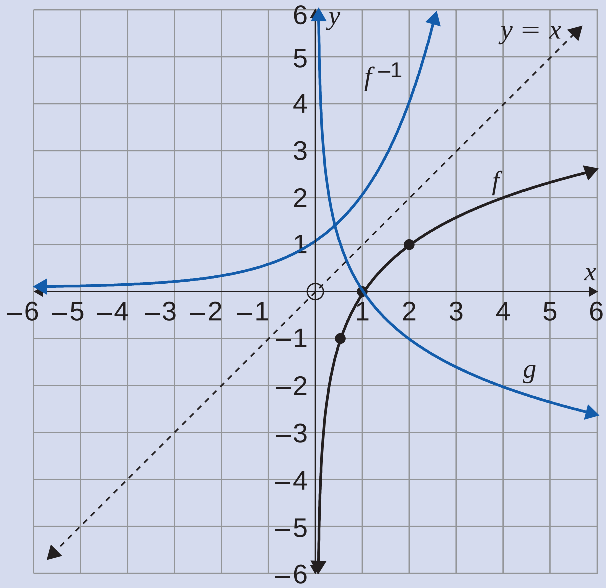
If $x = 2$ then $y = \log_2 2 = 1$ So $(2; 1)$ lies on f.

If $x = \frac{1}{2}$ then $y = \log_2 \frac{1}{2} = \log_2 2^{-1} = -1$ So $\left(\frac{1}{2}; -1\right)$ lies on f.

4. The function f is an increasing function.

5. The y-axis is an asymptote to f and has an equation of $x = 0$.

3 and 6.

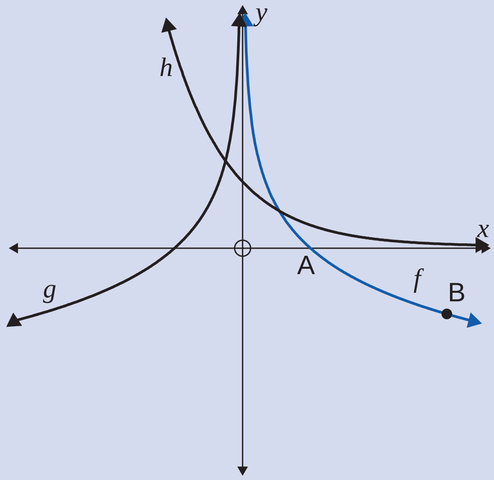


7. If f is reflected about the x-axis then $(x; y) \rightarrow (x; -y)$. In this case, the equation

$$y = \log_2 x \text{ becomes:}$$
$$-y = \log_2 x$$
$$y = -\log_2 x \text{ or } 2^{-y} = x$$
$$\left(\frac{1}{2}\right)^y = x$$
$$y = \log_{\frac{1}{2}} x$$
$$\therefore y = g(x) = -\log_2 x = \log_{\frac{1}{2}} x$$

Example 2

The sketch represents the graph of $y = f(x) = \log_{\frac{1}{3}} x$.



1. Write down the coordinates of A.
2. If B is the point (3; p), determine the value of p.
3. Determine the equation of g, the reflection of f about the y-axis.
4. Determine the equation of h, if $h(x) = f^{-1}(x)$.
5. Write down the domain and range of f.

Solution

1. A(1; 0): this is the x-intercept of f.
2. Substitute (3; p) into the equation $y = \log_{\frac{1}{3}} x$ and solve for p.

$$p = \log_{\frac{1}{3}} 3$$

$$p = \frac{\log 3}{\log \frac{1}{3}}$$

$$p = \frac{\log 3}{\log 3^{-1}}$$

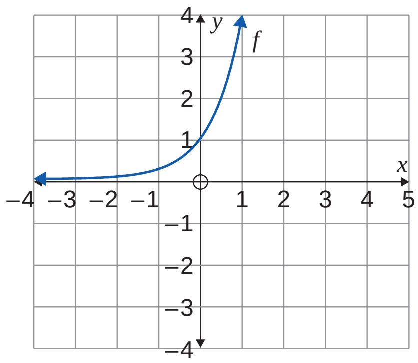
$$p = \frac{\log 3}{-1 \log 3}$$

$$p = -1$$

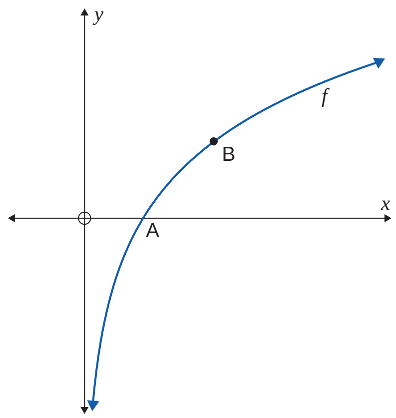
3. If a graph is reflected about the y-axis, then $(x; y) \rightarrow (-x; y)$.
   So $g(x) = \log_{\frac{1}{3}}(-x)$.
4. The inverse of a log graph is an exponential graph. Therefore, $h(x) = f^{-1}(x) = \left(\frac{1}{3}\right)^x$
5. Domain of f: $x > 0$ Range of f: $y \in \mathbb{R}$

Exercise 3.6

1. $P\left(1; \frac{1}{2}\right)$ is a point on the graph of $y = f(x) = a^x$.
   a) Calculate the value of a.
   b) Write down the equation of f^{-1}.
   c) Write down the equation of g, if g is the reflection of f about the x-axis.
   d) Draw the graphs of f, f^{-1} and g on the same set of axes. Clearly show any intercepts with the axes.
2. The following graph is that of $f(x) = 4^x$.

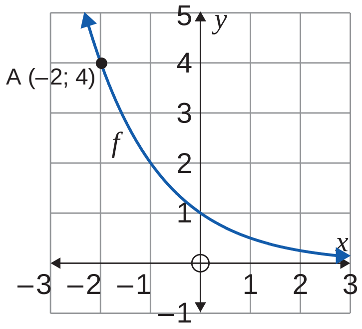


   a) Write the equation of f^{-1} in the form $y = \ldots$.
   b) Calculate the coordinates of the y-intercept of g if $g(x) = f(x + 2)$.
   c) Sketch the graphs of f^{-1} and g on the same set of axes.
   d) Use your graph to solve for x if $f(x + 2) \leq 1$.
3. The diagram represents the graph of $y = f(x) = \log_a x$. The point A is the x-intercept of f and B(2,25; 2) is a point on f.



   Determine:
   a) the equation of f
   b) the equation of g, the reflection of f about the y-axis
   c) the equation of h, the reflection of f about the line $y = x$
   d) the value of x if $\log_a x = -1$
   e) the domain of f.

4. The point A(−2; 4) lies on the graph of $f(x) = a^x$.



a) Determine the value of a.

b) Write down the domain and range of f.

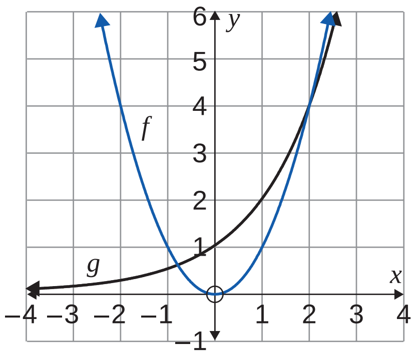
c) Write down the equation of f^{-1} in the form $y = \ldots.$

d) Write down the equation of g, if g is the reflection of f about the y-axis.

e) Sketch the graphs of f^{-1} and g on the same set of axes.

f) Use a calculator to determine the value of x, correct to two decimal places, if $g(x) = 5$.

5. The diagram represents the graphs of $y = f(x) = x^2$ and $y = g(x) = 2^x$.



a) Write down the coordinates of a point of intersection of f and g.

b) Write down the equations of f^{-1} and g^{-1}.

c) Sketch the graphs of f^{-1} and g^{-1} on the same set of axes.

d) Are the graphs of f^{-1} and g^{-1} functions? If not, restrict the domain of f or g so that the graph represents a function.

6. a) Determine the equation of the graph of $y = \log_a x$ if it passes through the point:

i) (8; 3)

ii) (3; −1)

iii) $\left(\frac{1}{2}; 1\right)$

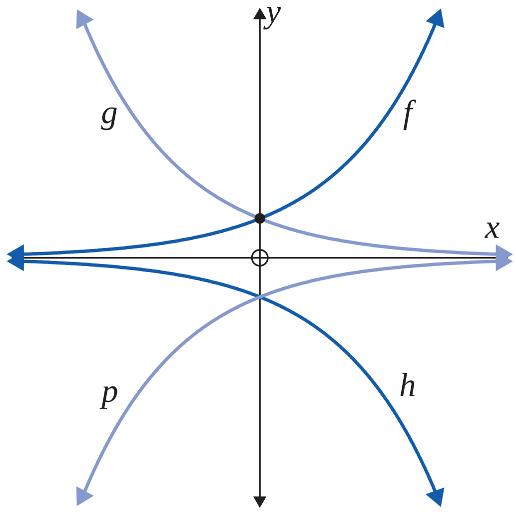
iv) $\left(\frac{1}{16}; -2\right)$

v) $\left(\frac{27}{8}; 3\right)$

b) In each case, state whether the function is an increasing or decreasing function.

Revision exercise

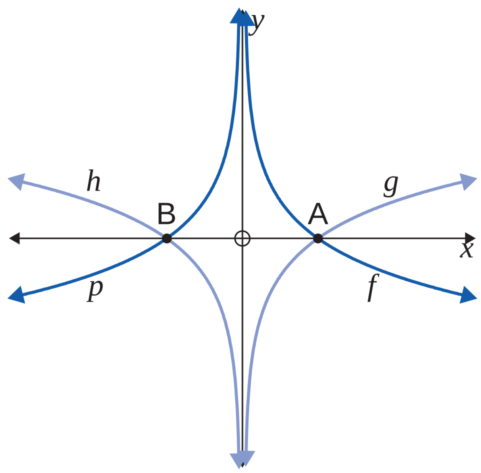
1. Given that $f(x) = \left(\frac{3}{2}\right)^x$



   a) Determine the equations of $g(x)$, $h(x)$ and $p(x)$ if
      i) g is the reflection of f about the y-axis.
      ii) h is the reflection of f about the x-axis.
      iii) p is the reflection of g about the x-axis.
   b) Write down the equations of f^{-1} and g^{-1}.
   c) Sketch the graphs of f^{-1} and g^{-1} on the same set of axes.

2. Given the function $f(x) = 2^x$.
   a) Write down the equation of $g(x) = f(x) - 4$.
   b) Calculate the coordinates of any intercepts of g with the axes.
   c) Write down the equation of the asymptote of g.
   d) Write down the equation of $f^{-1}(x)$ in the form $y = \ldots$
   e) Calculate the coordinates of any intercepts of $f^{-1}(x)$ with the axes.
   f) Write down the equation of the asymptote of $f^{-1}(x)$.
   g) Sketch the graphs of g and h on the same set of axes.
   h) Solve for x, correct to three decimal places, if $g(x) = 3$.

3. Given that $f(x) = \log_4 x$:



a) Determine the equations of $g(x)$, $h(x)$ and $p(x)$ if

i) g is the reflection of f about the x-axis

ii) p is the reflection of f about the y-axis

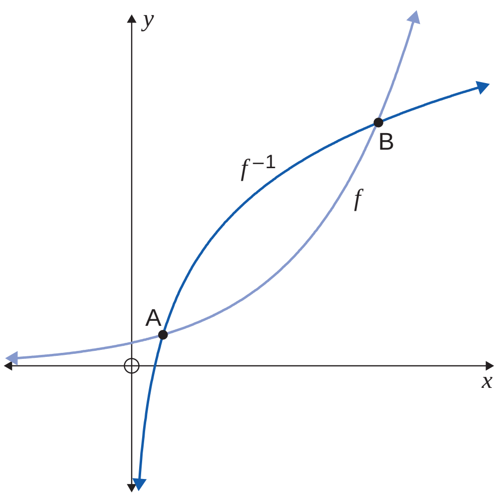
iii) h is the reflection of g about the y-axis.

b) Write down the coordinates of A and B.

c) Write down the equations of f^{-1} and g^{-1}.

d) Sketch the graphs of f^{-1} and g^{-1} on the same set of axes.

4. The diagram represents the graphs of $y = f(x) = \left(\frac{5}{4}\right)^x$ and its inverse, which intersect at A(1,35; 1,35) and B(10,57; 10,57).



a) Write down the equation of $f^{-1}(x)$ in the form $f^{-1}(x) = \ldots$.

b) Write down the equation of the asymptote of f^{-1}.

c) About which line are the graphs symmetrical?

d) Determine the equation of g, if g is the graph of f^{-1} reflected about the y-axis.

e) Determine the equation of h, if h is the graph of f shifted two units up.

f) Determine the equation of p, if p is the graph of f shifted two units to the left.

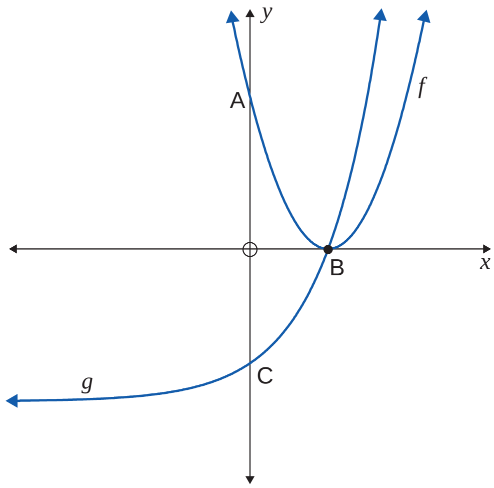
g) Determine the equation of q, if q is the graph of f reflected about the y-axis.

h) Use the graph to determine for which value(s) of x:

i) $f(x) \geq 1$ ii) $f(x) = f^{-1}(x)$

iii) $f^{-1}(x) \geq f(x)$ iv) $f^{-1}(x) \leq 1{,}35$

5. The diagram represents the graphs of $y = f(x) = (x - 2)^2$ and $y = g(x) = 2^x - 4$. Points A and C are the y-intercepts of f and g, respectively. Point B is the x-intercept of f and g, and is a point of intersection of f and g.



a) Calculate the coordinates of A, B and C.

b) Write down the equation of the asymptote of g.

c) Determine the equation of h if $h(x) = f(2x) + 4$.

d) Write down the equation of $h^{-1}(x)$ in the form $y = \ldots.$

e) Evaluate $h^{-1}\left(\frac{1}{16}\right)$.

f) Calculate the value of x, correct to two decimal places, if:

i) $h^{-1}(x) = 6$ ii) $h(x) = 6$

Chapter 4 Finance, growth and decay

In this chapter you will:

- solve problems involving present value and future value annuities
- use logarithms to calculate the value of n, the time period, in the equations $A = P(1 + i)^n$ and $A = P(1 - i)^n$
- critically analyse investment and loan options, and decide on the best available option.

Grade 11 summary

For compound growth, $A = P(1 + i)^n$

- P is the amount borrowed or invested and is called the principal amount
- i is the interest rate per **compounding period**
- n is the number of **compounding periods**
- A is the accumulated amount, which includes P and the interest.

Depreciation

Fixed assets lose value over time as a result of use or age. We refer to this loss of value as **depreciation**. We either calculate depreciation using the straight-line method or the reducing-balance method, depending on the type of asset.

- **Straight-line depreciation** is calculated as a percentage of the original value of the asset and is the same each year. The value of the asset reduces to zero over time.
- **Reducing-balance depreciation** is calculated as a percentage of the previous year's value. In this case, the depreciation changes each year, and decreases as the asset loses value. With reducing-balance depreciation, the asset always retains some value.
- The **book value** is the value of an asset after taking depreciation into account.
- The **scrap value** is the value of an asset at the end of its useful life.
- A **sinking fund** is a fund set up to replace an asset when it comes to the end of its useful life.

Items that have very little resale value are depreciated on a straight-line basis. These items include computers, printers, electronic equipment and furniture. Expensive items that have a book value after a number of years, and so can be resold, are normally depreciated on a reducing balance. These items would include heavy machinery, trucks, aeroplanes and ships.

Calculating depreciation

The formulae we use to calculate depreciation are as follows:
Straight line depreciation: $A = P(1 - i \times n)$
Reducing balance depreciation: $A = P(1 - i)^n$
where:

- A is the value of the asset after depreciation
- P is the original value of the asset
- n is the number of years after the asset was purchased
- i is the per annum rate of depreciation of the asset.

Example

The value of a motor car depreciates by 15% per year on the reducing balance method. The car is currently worth R70 000. Calculate the value of the car after four years.

Solution

$A = P(1 - i)^n$
$= 70\,000(1 - 0{,}15)^4$
$= \text{R}36\,540{,}44$

Nominal and effective interest rates

A nominal interest rate is one in which the quoted interest rate is different from the compounding period. Examples of nominal interest rates are:

- 12% **per annum** (**p.a.**) compounded **quarterly**
- 9% per **annum** compounded **monthly**

It is common practice to quote interest rates per annum, but the compounding period can be quarterly, monthly or daily. We call these interest rates nominal interest rates.

An effective interest rate is one in which the quoted interest rate is the same as the compounding period. Examples of effective interest rates are:

- 12% per **annum** compounded **annually**
- 1% per **month** compounded **monthly**
- 2,5% per **quarter** compounded **quarterly**

In financial mathematics, we often change the nominal interest rates to effective interest rates. So if the annual interest rate is compounded *monthly*, we change this to a *monthly* interest rate. For example, if the rate is quoted as 12% p.a., compounded monthly, we change this to $i = \frac{0{,}12}{12} = 0{,}01 = 1\%$ per month, compounded monthly. This is now an effective interest rate.

The formula to convert nominal interest rates to effective annual interest rates is:

$$1 + i_e = \left(1 + \frac{i_n}{m}\right)^m$$

where:

i_e is the effective annual interest rate

i_n is the nominal interest rate

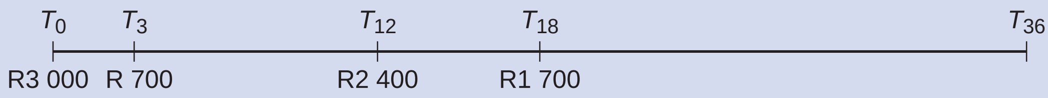
m is the number of compounding periods in one year

Example 1

Lindelani opens a savings account with an initial deposit of R3 000. Three months later, she makes a second deposit of R700. One year after the account was opened, she makes a third deposit of R2 400, followed by a fourth deposit R1 700 six months after the third deposit. The bank pays interest of 6% p.a., compounded monthly. Calculate how much she will have in her account at the end of three years.

Solution

Interest is compounded monthly, so our timeline is drawn in months. In other words, T_{12} on the timeline is the end of one year, or 12 months, and T_{36} represents the end of three years.



The interest rate is 6% p.a., compounded monthly, which is a nominal interest rate. We need to change this to an effective interest rate per month, compounded monthly: $\frac{0,06}{12}$. This means our formula looks as follows:

$$A = 3\,000\left(1 + \frac{0,06}{12}\right)^{36} + 700\left(1 + \frac{0,06}{12}\right)^{33} + 2\,400\left(1 + \frac{0,06}{12}\right)^{24} + 1\ 700\left(1 + \frac{0,06}{12}\right)^{18}$$

$$\therefore A = \text{R}8\ 980{,}14$$

Example 2

Andrew plans to save R20 000 for a deposit on a new car. He decides to use part of his annual bonus, and makes three equal annual deposits into a savings account at the beginning of each year. Calculate how much money he needs to deposit to save R20 000 after three years. Interest on the savings account is 8% p.a., compounded quarterly.

Solution

Here, T_0 represents the beginning of the first year on the timeline. T_1 is the end of the first year, but also the beginning of the second year. T_2 on the timeline is the beginning of the third year.

The interest rate is 8% p.a. compounded quarterly, which is a nominal rate. Interest is compounded quarterly (four times a year), so our effective rate is $\frac{0{,}08}{4}$

- The first deposit is in the savings account for three years. There are $3 \times 4 = 12$ quarterly compounding periods in three years.
- The second deposit is in the savings account for two years. There are eight quarterly periods.
- The last deposit has interest compounded four times in one year.

Therefore:

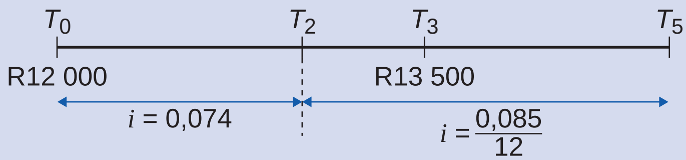
$$x\left(1 + \frac{0{,}08}{4}\right)^{12} + x\left(1 + \frac{0{,}08}{4}\right)^{8} + x\left(1 + \frac{0{,}08}{4}\right)^{4} = 20\ 000$$
$$x\left[\left(1 + \frac{0{,}08}{4}\right)^{12} + \left(1 + \frac{0{,}08}{4}\right)^{8} + \left(1 + \frac{0{,}08}{4}\right)^{4}\right] = 20\ 000$$
$$x = \text{R}6\ 320{,}06$$

So Andrew needs to deposit R6 320,06 each year to have R20 000 after three years.

Example 3

A savings account is opened with a deposit of R12 000. Three years later, a second deposit of R13 500 is made into the savings account. The interest for the first two years is 7,4% p.a., compounded annually. The interest rate then increases to 8,5% p.a., compounded monthly. Calculate the total amount in the savings account after five years.

Solution



There is a change in interest rate, which will affect the first deposit after two years. The interest paid on the first deposit is 7,4% p.a., compounded annually. For the remaining three years, the interest rate is 8,5% p.a., compounded monthly.

The interest on the second deposit is 8,5% p.a., compounded monthly for two years.

$$A = 12\ 000(1 + 0{,}074)^2\left(1 + \frac{0{,}085}{12}\right)^{36} + 13\ 500\left(1 + \frac{0{,}085}{12}\right)^{24}$$
$$A = \text{R}33\ 838{,}18$$

Exercise 4.1

1. A savings account is opened with an initial deposit of R5 000. One year later, a further R2 200 is added to the account. Six months after the second deposit, R1 000 is withdrawn from the account. The interest on the savings is 9% p.a., compounded monthly. Calculate the balance in the savings account at the end of two years.
2. Sisekelo plans to save R80 000 in four years, as a deposit on a house. He deposits equal amounts of money at the beginning of each year for four years. Interest on the savings is 8% p.a., compounded monthly. Calculate how much money he will need to deposit each year to achieve his objective.
3. Nelson deposits R3 500 into a savings account. Three years later, he adds R5 700 to the account. The interest for the first two years is 7% p.a., compounded quarterly. The interest for the last three years is 8% p.a., compounded monthly. Calculate the balance in the savings account at the end of five years.
4. A savings account is opened with an initial deposit of R8 000. Two further equal amounts of R2 500 are added to the account. The first deposit is made one year after the savings account was opened and second amount six months later. Calculate the total amount in the savings account at the end of three years if the interest rate is 7,6% p.a., compounded monthly.
5. Mr Modise makes three equal deposits of R9 000 into a savings account. He makes the first deposit when he opens the savings account. The second and third deposits are after two and four years, respectively. Interest is 6,3% p.a., compounded quarterly, for the first two years, and 7,8% p.a., compounded monthly, for the last three years. Calculate the accumulated amount in the savings account at the end of five years.
6. Jacob invests R20 000 in a pension fund on his 30th birthday. He adds another R30 000 to the fund on his 40th birthday. When he turns 50, he deposits a further R40 000. The expected growth on the investment in the pension fund is 11% p.a., compounded monthly. Calculate the amount he has saved for his pension when he turns 60.
7. Gillian deposits R1 000 in a savings account at the beginning of each year for four years. The interest paid on the savings is 8% p.a., compounded monthly, for the first two years. The rate then increases to 9% p.a., compounded quarterly, for the next three years. Calculate the total amount in the savings account at the end of five years.
8. Thapelo deposits R2 000 into a savings account at the beginning of each year for four years. Calculate how much money he will have accumulated in his savings at the end of four years if the interest paid on the savings is 6% p.a., compounded quarterly.
9. A company has computer systems to the value of R1 000 000. If the value of this equipment depreciates at 40% per annum on a reducing balance, calculate how much it is worth after 3 years?
10. The rhino population of South Africa is depreciating on a reducing balance at a rate of 12% p.a. If there are now 3 200 rhino left, how many will there be in 5 years' time?

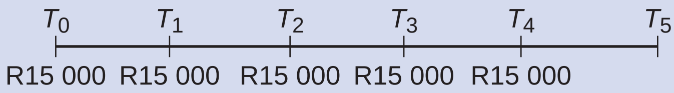
Future value annuities

An annuity is any terminating stream of fixed payments over a specified period of time subject to a rate of interest. An annuity is a series of regular payments towards a goal such as a retirement fund or the repayment of a bond. In a future value annuity capital is accumulated by means of regular equal payments into a savings account or an investment fund where compound interest is paid on the money accumulated in the fund.

Example 1

R15 000 is invested into a fund at the beginning of each year for five years. Calculate the total amount accumulated in the fund at the end of five years if interest is compounded monthly at 9% p.a.

Solution



This time, we start on the right-hand side of the timeline. The last deposit was R15 000 at time T_4, at the end of four years. This was also the beginning of the fifth year. This deposit accumulates 12 months interest. The deposit made at T_3, which is the beginning of the fourth year accumulates 24 months interest. Continuing in this way, the accumulated amount is:

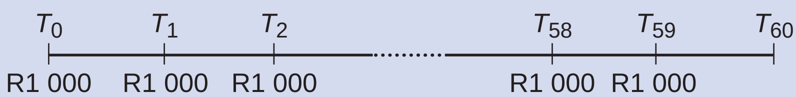
$$15\ 000\left(1+\frac{0{,}09}{12}\right)^{12}+15\ 000\left(1+\frac{0{,}09}{12}\right)^{24}+15\ 000\left(1+\frac{0{,}09}{12}\right)^{36}$$
$$+\,15\ 000\left(1+\frac{0{,}09}{12}\right)^{48}+15\ 000\left(1+\frac{0{,}09}{12}\right)^{60}=\text{R}98\ 939{,}28$$

We do not normally grow capital by random payments once a year. Saving requires a more structured and disciplined approach. So for most people, this means regular payments every month. It is more practical to invest money every month rather than at the beginning of each year.

Example 2

An amount of R1 000 is invested in a fund at the beginning of each month for five years. Calculate the total amount accumulated in the fund at the end of five years if the interest is compounded monthly at 9% p.a.

Solution



The money is invested every month for five years, so there are 60 deposits into the fund. We start with the deposits on the right-hand side of the timeline. The last deposit is made at T_{59} which is the beginning of the 60th month. This deposit will only accumulate interest for one month. The deposit at T_{58} accumulates two months interest, and so on.

$$1\ \ 000\left(1+\frac{0{,}09}{12}\right)^{1}+1\ \ 000\left(1+\frac{0{,}09}{12}\right)^{2}+1\ \ 000\left(1+\frac{0{,}09}{12}\right)^{3}+\ldots 60\text{ deposits}$$

It would take a long time to add all these amounts together, even with the help of a calculator. However, this is a geometric series. That means we can use a formula to add the terms.

The formula for the sum of a geometric series is:

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

where

$$a = 1\ 000\left(1 + \frac{0{,}09}{12}\right)^1$$

$$r = \left(1 + \frac{0{,}09}{12}\right)$$

$$n = 60$$

Therefore:

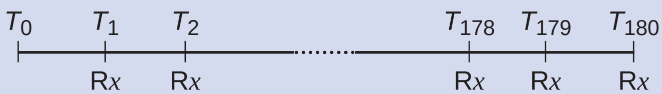
$$S_n = \frac{1\ 000\left(1 + \frac{0{,}09}{12}\right)\left[\left(1 + \frac{0{,}09}{12}\right)^{60} - 1\right]}{\left(1 + \frac{0{,}09}{12}\right) - 1}$$

$$= \text{R}75\ 989{,}82$$

Example 3

Hanyani plans to save R1 million in 15 years. He invests money in a pension fund by making equal deposits at the end of each month for 15 years. He is expecting growth in the capital of 14% p.a., compounded monthly. Calculate how much money he needs to pay each month to achieve his goal of becoming a millionaire.

Solution



In this question, there are 180 deposits, because the deposits are made monthly for 15 years. We start with the right-hand side of the timeline. In this case, the last deposit is made at the end of 15 years and so no interest is accumulated on the last deposit. The deposit at T_{179} will accumulate interest for one month. The deposit at T_{178} will accumulate interest for two months, and so on. If we add all the deposits, we have:

$$x + x\left(1 + \frac{0{,}14}{12}\right)^1 + x\left(1 + \frac{0{,}14}{12}\right)^2 + x\left(1 + \frac{0{,}14}{12}\right)^3 + \ldots \text{ 180 terms}$$

We once again have a geometric series. In this case:

$$a = x$$

$$r = \left(1 + \frac{0{,}14}{12}\right)$$

$$n = 180$$

$$S_n = \text{R}1\ 000\ 000$$

Substituting these values into the formula for a geometric series, we have:

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$1\ 000\ 000 = \frac{x\left[\left(1 + \frac{0{,}14}{12}\right)^{180} - 1\right]}{\left(1 + \frac{0{,}14}{12}\right) - 1}$$

$$x = \text{R}1\ 605{,}75$$

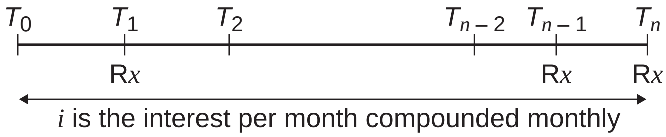
Exercise 4.2

1. James deposits R500 into a savings account at the beginning of each month for ten years. Calculate how much money will have accumulated in the savings account at the end of ten years if the interest paid on the savings is 8% p.a., compounded monthly.
2. Lindelani has R400 deducted from her salary every month towards her pension. Calculate how much money will have accumulated in her pension fund at the end of 12 years if the money grows at 11% p.a., compounded monthly.
3. Ipeleng wants to buy a new car, but needs to save R20 000 for a deposit. She opens a savings account that pays interest of 9,4 % p.a., compounded monthly. Calculate how much she must deposit at the end of each month to have enough for her deposit after two years.
4. Cathi opens a savings account. She deposits R1 000 every month, starting immediately. Calculate how much money she will have saved at the end of five years, one month after her 60th payment, if the interest paid on the savings is 7% p.a., compounded monthly.
5. A company sets up a fund that it will use to extend the business. They make quarterly deposits of R280 000, starting in three months' time. The last deposit is made at the end of six years. How much money is in the fund immediately after the 24th deposit?
6. Miriam opens a savings account. She deposits R20 000 at the beginning of January 2014. She then adds to the savings by depositing R700 at the end of every month, starting at the end of January 2014. Interest on the savings account is 6,8% p.a., compounded monthly. Calculate how much money will have accumulated in the savings account at the end of December 2015, immediately after her last deposit.
7. An amount of R25 000 is deposited into an investment every six months for seven years, starting immediately. The investment is expected to grow at 9% p.a., compounded semi-annually. Calculate the value of the investment at the end of seven years, assuming there were 14 deposits.
8. Zinhle saves money for a deposit on a car. She deposits R400 at the end of each month for three years. Calculate how much money she will have saved after her last deposit if the interest is 8,1% p.a., compounded monthly.

So far, all the problems we have worked through have used the formula for the sum of the geometric series. In the next section, we see how to find a formula specifically for the future value of an annuity.

The formula for future value annuities

In an annuity, an amount of Rx is paid each month into an investment for n months, starting in one months' time. The interest rate per month, compounded monthly, is i. We now derive a general formula for the total amount accumulated in the investment at the end of n months.



Let F be the total amount accumulated at the end of n months *immediately after* the nth payment, together with all the compound interest.

$F = x + x(1 + i) + x(1 + i)^2 + x(1 + i)^3 + \ldots n$ terms

We can use the formula for a geometric series to calculate the sum of this series:

$S_n = \frac{a(r^n - 1)}{(r - 1)}$ $\qquad a = x$ and $r = (1 + i)$

$\therefore F = \frac{x((1 + i)^n - 1)}{(1 + i) - 1} = \frac{x((1 + i)^n - 1)}{i}$

We can use the same formula for any future value annuity where the payments are made at regular intervals. The payments can be over any period with interest compounded at the end of each period. In general:

$F = \frac{x((1 + i)^n - 1)}{i}$

where:

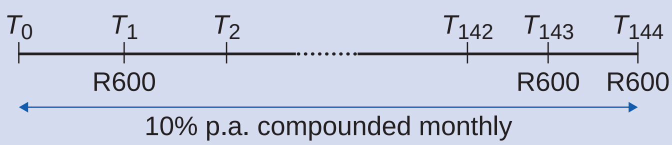
F is the total amount accumulated after n periods, immediately after the last payment
x is the amount deposited at regular intervals at the end of each period
i is the interest rate per period, compounded at the end of each period
n is the number of periods of the investment.

We can use this formula for any future value annuity. However, the payments must start at the end of the first period, and the last payment must be made at the end of n periods.

Example 1

Ayanda has R600 deducted from her salary at the end of every month for her pension. Calculate how much money she will accumulate in her fund at the end of 12 years if the growth is 10 % p.a., compounded monthly.

Solution



$$F = \frac{x\left((1+i)^n - 1\right)}{i}$$

$x = \text{R}600$ *The amount paid into the annuity at the end of every month*

$i = \frac{0{,}10}{12}$ *Interest rate per month, compounded monthly*

$n = 144$ *The investment in this annuity takes place every month for 12 years*

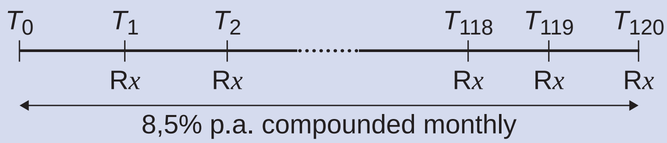
$$\therefore F = \frac{600\left(\left(1 + \frac{0{,}10}{12}\right)^{144} - 1\right)}{\frac{0{,}10}{12}}$$

$$\therefore F = \text{R}165\ 852{,}73$$

Example 2

Nolwazi wants to save R800 000 in ten years. She opens a savings account and pays money into the account at the end of every month. The interest paid on the account is 8,5% p.a., compounded monthly. Calculate how much she must deposit each month so that she will have R800 000 after her last payment.

Solution



$$F = \frac{x\left((1+i)^n - 1\right)}{i}$$

$x = ?$ *We need to calculate the amount paid into the annuity at the end of every month*

$i = \frac{0{,}085}{12}$ *The interest rate per month, compounded monthly*

$n = 120$ *The number of payments into the savings account*

$F = \text{R}800\ 000$

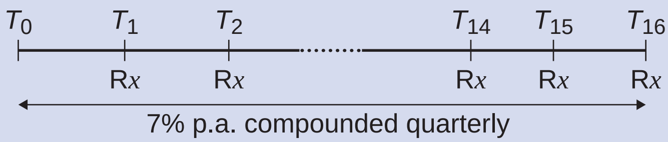
$$800\ 000 = \frac{x\left(\left(1 + \frac{0{,}085}{12}\right)^{120} - 1\right)}{\frac{0{,}085}{12}}$$

$$\therefore x = \text{R}4\ 252{,}19$$

Example 3

Mbali owns her own business. She plans to save R1 million to expand her business in four years' time. She decides to make equal *quarterly* deposits into a savings account, starting in three months' time. She will make the last deposit at the end of four years. The interest paid on the savings is 7% p.a., compounded quarterly. Calculate how much she must deposit so that she will have sufficient funds in four years' time.

Solution



$F = \frac{x((1+i)^n - 1)}{i}$

$x = ?$

$i = \frac{0,07}{4}$

$n = 16$

$F = \text{R}1\ 000\ 000$

The investment into this annuity takes place every three months for four years.

$$1\ 000\ 000 = \frac{x\left[\left(1 + \frac{0,07}{4}\right)^{16} - 1\right]}{\frac{0,07}{4}}$$

$$\therefore x = \text{R}54\ 699,58$$

Comparing investments

Whenever you decide to start investing money, you will have many choices of investment options. You can use the formula for the future value of an investment to compare these options. In this way, you can decide which offers the best return.

Example 4

You have just started your first job, and so you decide to save part of what you earn each month. You visit your bank, who offers you three savings options:

Plan	Payment	Interest
Savings plan 1:	R100 per month	8% per annum compounded monthly
Savings plan 2:	R600 every six months	9% per annum compounded semi-annually
Savings plan 3:	R1 200 per year	8,5% per annum, compounded annually

Which saving plan offers the best return over the next ten years?

Solution

We need to use the future value formula to calculate the return for each plan:

Savings plan 1:

$$F = \frac{x((1+i)^n - 1)}{i}$$

$$= \frac{100\left(\left(1 + \frac{0,08}{12}\right)^{120} - 1\right)}{\frac{0,08}{12}}$$

$$= \text{R}18\ 294,60$$

Savings plan 2:

$$F = \frac{x((1+i)^n - 1)}{i}$$

$$= \frac{600\left(\left(1 + \frac{0,09}{2}\right)^{20} - 1\right)}{\frac{0,09}{2}}$$

$$= \text{R}18\ 822,85$$

Savings plan 3:

$$F = \frac{x\left((1+i)^n - 1\right)}{i}$$

$$= \frac{1200((1+0{,}085)^{10} - 1)}{0{,}085}$$

$$= \text{R}17\ 802{,}12$$

Therefore, Savings plan 2 offers the best return over the next ten years.

Exercise 4.3

1. At the end of each month, Simphiwe deposits R400 into a savings account. Interest is calculated at 6% p.a., compounded monthly. Calculate how much money she will have saved at the end of eight years.
2. Wandile plans to save R120 000 in three years for a deposit on a house. He deposits Rx into a savings account at the end of each month, with the last deposit made at the end of three years. Interest is calculated at 7,4% p.a., compounded monthly. Calculate x.
3. Vusi has R700 deducted from his salary each month for his pension. Calculate how much money he will have accumulated for his pension at the end of 12 years if the growth of money in the pension fund is 14% p.a., compounded monthly.
4. Nomusa has her own transport company and plans to replace one of her trucks in five years' time. The new truck is expected to cost R1,4 million. She wants to save enough money so that she can pay cash for the truck. She opens a savings account and deposits equal amounts every quarter starting at the end of the first quarter. Interest is calculated at 8,2% p.a., compounded quarterly. Calculate her quarterly deposits so that she will have enough money to buy the new truck. The final payment will be made at the end of five years.
5. John deposits R500 at the end of each month in a savings account for four years. Interest is calculated at 6,7% p.a., compounded monthly. Calculate how much money he will have accumulated in the savings account immediately after the 48th payment.
6. Lerato starts working in January 2014. At the end of each month, she deposits R800 of her salary into a savings account. How much money will she have saved by the end of 2016 if the bank pays 8% interest p.a., compounded monthly?
7. Thato has a business and plans to expand his business in four years' time. He opens a savings account and deposits R40 000 into the account every six months, starting in six months' time. He makes the last payment at the end of four years. Interest is calculated at 9,5% p.a., compounded semi-annually.
   a) Calculate how much money he will have accumulated in the savings account at the end of four years.
   b) Calculate how much money he would have saved at the end of four years if he had started his first deposit immediately. Assume he made eight deposits into the savings account.

8. Mrs Naidoo buys R30 000 of unit trusts every three months, starting at the end of March 2014. Calculate the value of the unit trusts at the end of 2018 if the investment grows at 12% p.a., compounded quarterly. (Unit trusts are an investment in a collection of shares on the stock market.)

Loans and loan repayments

In the previous section, we worked out the future value of a series of payments into a savings account. In this section, we look at loans. If you take out a loan from the bank, the bank will expect you to pay it back, usually as a series of fixed monthly payments. When we save money in a bank account, the bank pays us interest. Naturally, if we borrow money from the bank, we have to pay them interest.

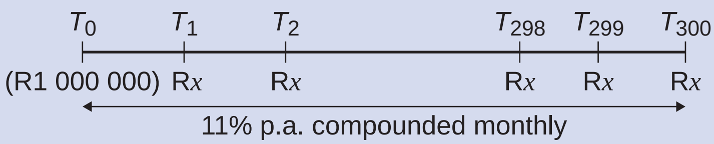
Example 1

Nkululeko buys a house and takes out a loan of R1 000 000. He repays the loan over 25 years. The interest on the loan is 11% p.a., compounded monthly.

1. Determine the monthly repayments on the loan. Assume that the first payment starts one month after drawing the loan.
2. Calculate the total amount of money required to repay the loan.

Solution

1.



The monthly repayments must equal the value of the loan plus any interest that we owe over the full period of the loan. Therefore:

Loan plus interest = Repayments plus interest at T_{300}

$$1\ 000\ 000\left(1+\frac{0,11}{12}\right)^{300} = x + x\left(1+\frac{0,11}{12}\right) + x\left(1+\frac{0,11}{12}\right)^2 + x\left(1+\frac{0,11}{12}\right)^3 \ldots 300 \text{ terms}$$

The right-hand side of the equation forms a geometric series, so we can use our formula to add up the 300 terms.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$
$$= 1\ 000\ 000\left(1+\frac{0,11}{12}\right)^{300}$$

The first value in the series is x. Therefore $a = x$. The number of terms on the right-hand side is 300. Therefore, $n = 300$. The common ratio is $r = \left(1+\frac{0,11}{12}\right)$

$$1\ 000\ 000\left(1+\frac{0,11}{12}\right)^{300} = x\frac{\left\{\left(1+\frac{0,11}{12}\right)^{300} - 1\right\}}{\left(1+\frac{0,11}{12}\right) - 1}$$

$$x = \text{R}9\ 801,13$$

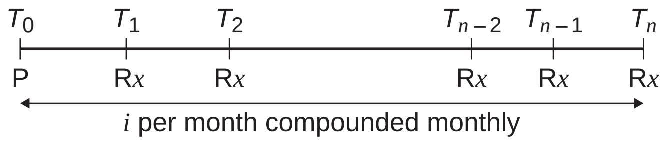
2. The total amount paid $= 300 \times$ R9 801,13 = R2 940 339,23

In general, loan repayments start at the end of the first period after the loan was drawn. For example, in the previous example, the loan repayments started at T_1, which is the end of the first month.

The present value formula for annuities

We can work out a general formula for the present value of an annuity. We prove the formula using a similar method to the previous example.

A loan of P rand is taken out from a bank. The loan is repaid in n equal payments of x rand at the end of every month, starting one month after the loan was drawn. The interest paid on the outstanding balance is i per month, compounded monthly.



The value of the loan plus the interest to T_n must equal the total of the repayments plus interest to T_n. In this way, the loan then balances with the repayments at T_n.
Therefore:
Loan plus interest = Repayments plus interest at T_n
$P(1+i)^n = x + x(1+i) + x(1+i)^2 + x(1+i)^3 + \ldots n$ equal payments
The right-hand side is a geometric series. So we can use the formula for the sum of a geometric series to add up the n terms.
$S_n = \dfrac{a(r^n - 1)}{r-1}$
The sum of the series on the right-hand side must be $S_n = P(1+i)^n$
The first value in the series is x. Therefore $a = x$
The number of terms on the right-hand side is n
The common ratio $r = (1+i)$

$$\begin{aligned} P(1+i)^n &= \frac{x\left((1+i)^n - 1\right)}{(1+i)-1} \\ &= \frac{x\left((1+i)^n - 1\right)}{i} \end{aligned}$$

Multiply both sides of the equation by $(1+i)^{-n}$:

$$\begin{aligned} P(1+i)^n \times (1+i)^{-n} &= \frac{x\left((1+i)^n - 1\right)}{i} \times \frac{(1+i)^{-n}}{1} \\ P(1+i)^0 &= \frac{x\left(1(1+i)^n(1+i)^{-n} - (1+i)^{-n}\right)}{i} \\ P &= \frac{x\left(1(1+i)^0 - (1+i)^{-n}\right)}{i} \\ P &= \frac{x\left(1 - (1+i)^{-n}\right)}{i} \end{aligned}$$

This formula $P = \dfrac{x\left(1-(1+i)^{-n}\right)}{i}$ is known as the present value formula for annuities. We can use it to solve most problems with loans and loan repayments. We derived the formula based on making monthly payments and an interest rate of i. We can use this formula for payments over any period.

In general:

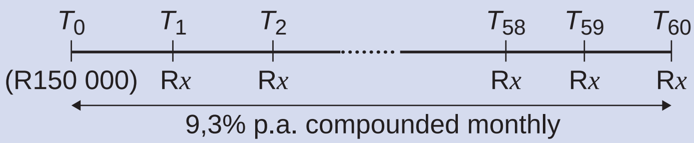
- x is the amount paid at any regular period
- i is the interest rate per period, compounded at the end of every period
- n is the number of payments

Example 1

Phumzile takes out a loan of R150 000 to buy a new car. The loan is repaid over five years. The interest paid on the amounts outstanding is 9,3% p.a., compounded monthly.

Calculate the monthly repayments.

Solution



$$P = \frac{x\left[1 - (1 + i)^{-n}\right]}{i}$$

$$150\ 000 = \frac{x\left[1 - \left(1 + \frac{0{,}093}{12}\right)^{-60}\right]}{\frac{0{,}093}{12}}$$

$\therefore x = \text{R}3\ 135{,}64$

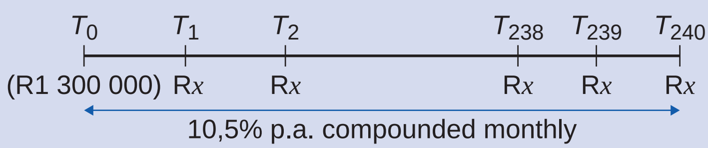
Example 2

Bonginkosi takes out a loan of R1 300 000 to buy a house. The loan is repaid over 20 years and interest charged on the loan at 10,5% p.a., compounded monthly.

1. Calculate his monthly repayments.
2. How much money has Bonginkosi paid on the loan at the end of eight years?
3. Calculate the balance outstanding on the loan after eight years, immediately after Bonginkosi's 96th payment.
4. How much of the capital has he repaid at the end of eight years?
5. How much of the money he has paid after eight years was interest?

Solution

1.



$$P = \frac{x\left[1-(1+i)^{-n}\right]}{i}$$

$$1\ 300\ 000 = \frac{x\left[1-\left(1+\frac{0{,}105}{12}\right)^{-240}\right]}{\frac{0{,}105}{12}}$$

$\therefore x = \text{R}12\ 978{,}94$

2. He paid R12 978,94 each month for eight years. Therefore, he paid R12 978,94 × 96 = R1 245 978,01
   After eight years, he has almost paid back the amount he originally borrowed. However, he still has to pay R12 978,94 each month for another 12 years because of the interest that has accumulated.
3. We use the same present value formula to calculate the balance outstanding at the end of eight years. We know how much he has to pay each month for the remaining 12 years. To find the balance on the loan, we do not need to know how much he has paid, nor do we need to know the original loan amount. All we need to know is the amount of the monthly payment and the number of payments remaining.

$$P = \frac{x\left[1-(1+i)^{-n}\right]}{i}$$

$x = \text{R}12\ 978{,}94$ *He must still pay R12 978,94 each month for the next 12 years*

$n = 12 \times 12 = 144$ *There are 12 years, and 12 monthly payments in each year*

$$P = \frac{12\ 978{,}94\left[1-\left(1+\frac{0{,}105}{12}\right)^{-144}\right]}{\frac{0{,}105}{12}}$$

$$= \text{R}1\ 060\ 249{,}05$$

It is important to realise that, although Bonginkosi has paid R1 245 978,24 in the first eight years, the balance on the loan is still R1 060 249,05. Most of the money that he paid in the first eight years was interest.

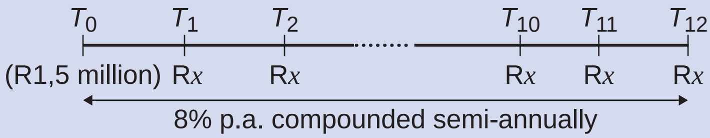
4. The actual amount paid off on the loan in the first eight years is:
   1 300 000 – 1 060 249,05 = R239 750,95
5. The amount of interest paid is:
   1 245 978,24 – 239 750,95 = R1 006 227,29

Example 3

Cebile starts her own business and takes out a loan of R1,5 million. She repays the loan through equal semi-annual payments over a period of six years. The repayments start six months after the loan was drawn. The interest on the outstanding balance of the loan is 8% p.a., compounded semi-annually.

1. Calculate the semi-annual payments.
2. Calculate the balance on the loan at the end of three years.

Solution



1.

$$P = \frac{x\left[1-(1+i)^{-n}\right]}{i}$$

$$1\ 500\ 000 = \frac{x\left[1-\left(1+\frac{0{,}08}{2}\right)^{-12}\right]}{\frac{0{,}08}{2}}$$

$$\therefore x = \text{R}159\ 828{,}26$$

2. To find the outstanding balance on the loan, we need to find the present value of all the outstanding payments.

The semi-annual payments are: $x = \text{R}159\ 828{,}26$

The number of outstanding payments: $n = 6$

$$P = \frac{x\left[1-(1+i)^{-n}\right]}{i}$$

$$= \frac{159\ 828{,}26\left[1-\left(1+\frac{0{,}08}{2}\right)^{-6}\right]}{\frac{0{,}08}{2}}$$

$$= \text{R}837\ 841{,}61$$

Exercise 4.4

1. A loan of R140 000 is repaid over six years through equal monthly payments. The interest charged on the outstanding balance of the loan is 11,4% p.a., compounded monthly. Calculate the monthly repayments on the loan.
2. Zenzele buys a house and takes out a loan of R1,2 million. The loan is repaid over 25 years. Calculate:
   a) the monthly instalment if the interest charged on the loan is 10% p.a., compounded monthly.
   b) the outstanding balance on the loan at the end of ten years.
3. Erin buys a new car and takes out a loan of R180 000. She repays the loan over five years. The bank charges interest on the reducing balance at 12% p.a., compounded monthly.
   a) Calculate her monthly repayments on the loan.
   b) Erin wants to sell her car at the end of three years. Calculate how much money she still owes on the car.
4. Nkosinathi buys a house for R1,8 million. He pays a 20% deposit and takes out a loan for the balance, which he repays over 25 years. The interest charged on the loan is 9,6% p.a., compounded monthly. Calculate:
   a) the loan amount
   b) his monthly repayments