第五章 Laplace变换及其应用

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§ 5.1 Laplace变换

* Laplace变换的定义

设函数
$$\varphi(t) = \begin{cases} \varphi(t) & 0 \le t < \infty \\ 0 & t < 0 \end{cases}$$

$$\overline{\varphi}(p) = \int_0^{+\infty} \varphi(t)e^{-pt}dt$$
 称为 $\varphi(t)$ 的拉普拉斯变换

$$\varphi(t)$$
为 $\overline{\varphi}(p)$ 的原函数: $L[\varphi(t)] = \overline{\varphi}(p)$

$$\overline{\varphi}(p)$$
 为 $\varphi(t)$ 的像函数: $L^{-1}[\overline{\varphi}(p)] = \varphi(t)$



补充:Laplace变换的由来

$$\overline{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-i\omega x} dx \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \overline{f}(\omega) e^{i\omega x} d\omega$$

要求: $\int_{-\infty}^{+\infty} |f(t)| dt$ 存在. 这比较苛刻,如: t, $\sin t$, $\cos t$

设函数
$$\varphi(t) = \begin{cases} f(t) & 0 \le t < \infty \\ 0 & t < 0 \end{cases}$$
 令 $g(t) = e^{-\sigma t} \varphi(t), \sigma > 0$

$$\overline{g}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(t) e^{-i\omega t} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \varphi(t) e^{-(\sigma + i\omega)t} dt$$

$$\overline{\varphi}(p) = \int_0^{+\infty} \varphi(t) e^{-pt} dt \quad (p = \sigma + i\omega)$$

$$\varphi(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \overline{\varphi}(p) e^{pt} dp(t > 0, \sigma > \sigma_0)$$
数学物理方程



例 5.1 求
$$\varphi(t)=1$$
 的拉氏变换。

解:原函数应理解为阶跃函数 $H(t) = \begin{cases} 1 & t \ge 0 \\ 0 & t < 0 \end{cases}$

$$\overline{\varphi}(p) = \int_0^{+\infty} 1 \cdot e^{-pt} dt = \frac{1}{p} \qquad (\operatorname{Re} p > 0) \qquad L(1) = \frac{1}{p}$$

例 5.2 求 $\varphi(t) = t^n$ (n为正整数)的拉氏变换。

解:
$$\overline{\varphi}(p) = \int_0^{+\infty} t^n \cdot e^{-pt} dt = \frac{n!}{p^{n+1}}$$
 (Re $p > 0$)

$$L(t^n) = \frac{n!}{p^{n+1}}$$



例5.3 求
$$\varphi(t) = e^{st}(s$$
为实常数) 的拉氏变换。

解:
$$\overline{\varphi}(p) = \int_0^{+\infty} e^{st} \cdot e^{-pt} dt = \frac{1}{p-s}$$
 (Re $p > s$)
$$L(e^{st}) = \frac{1}{p-s}$$

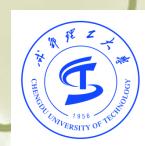
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求 $\varphi(t) = t^n e^{st} (n$ 为正整数, s为实常数)

的拉氏变换。

解:
$$\overline{\varphi}(p) = \int_0^{+\infty} t^n e^{st} \cdot e^{-pt} dt = \frac{n!}{(p-s)^{n+1}}$$
 (Re $p > s$)

$$L(t^n e^{st}) = \frac{n!}{(p-s)^{n+1}}$$



例 5.5 求 $\varphi(t) = \sin \omega t$ 的拉氏变换。

解:
$$\overline{\varphi}(p) = \int_0^{+\infty} \sin \omega t \cdot e^{-pt} dt = \frac{\omega}{p^2 + \omega^2}$$
 (Re $p > 0$)

$$L(\sin \omega t) = \frac{\omega}{p^2 + \omega^2}$$

同理可证明:

$$L(\cos \omega t) = \frac{p}{p^2 + \omega^2}$$



❖ Laplace变换的性质 $L[\varphi(t)] = \overline{\varphi}(p)$

性质1(导数性质)

$$L[\varphi^{(n)}(t)] = p^{n}\overline{\varphi}(p) - p^{n-1}\varphi(0) - p^{n-2}\varphi'(0) - \dots$$
$$-p\varphi^{(n-2)}(0) - \varphi^{(n-1)}(0)$$

$$L[\varphi'(t)] = p\overline{\varphi}(p) - \varphi(0)$$

$$L[\varphi''(t)] = p^2 \overline{\varphi}(p) - p\varphi(0) - \varphi'(0)$$



性质2 (积分性质)

$$L\left[\int_0^t \varphi(\tau)d\tau\right] = \frac{1}{p}\overline{\varphi}(p)$$

性质3 (像函数微分性质)

$$L[(-1)^n t^n \varphi(t)] = \frac{d^n \overline{\varphi}(p)}{dp^n}$$



§ 5.2 Laplace变换的反演

$$L^{-1}[\overline{\varphi}(p)] = \varphi(t)$$

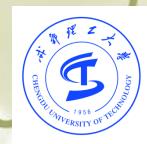
延迟定理
$$L^{-1}\left[e^{-\tau}\overline{\varphi}(p)\right] = \begin{cases} \varphi(t-\tau), (t \geq \tau) \\ 0, (t < \tau) \end{cases}$$

证明: 接定义
$$L[\varphi(t-\tau)] = \int_{\tau}^{+\infty} \varphi(t-\tau)e^{-pt}dt$$

$$= \int_{0}^{+\infty} \varphi(t)e^{-p(t+\tau)}dt$$

$$= e^{-\tau p}\overline{\varphi}(p)$$

注意: 当t<T时, $\phi(t-T)=0$



位移定理
$$L^{-1}[\overline{\varphi}(p+\lambda)] = e^{-\lambda t}\varphi(t)$$

证明:
$$\int_0^{+\infty} e^{-\lambda t} \varphi(t) e^{-pt} dt = \int_0^{+\infty} \varphi(t) e^{-(p+\lambda)t} dt = \overline{\varphi}(p+\lambda)$$

卷积定理

若
$$L^{-1}[\overline{\varphi}_1(p)] = \varphi_1(t), L^{-1}[\overline{\varphi}_2(p)] = \varphi_2(t)$$
,则

$$L^{-1}[\overline{\varphi}_{1}(p) \cdot \overline{\varphi}_{2}(p)] = \int_{0}^{t} \varphi_{1}(\tau) \varphi_{2}(t - \tau) d\tau$$
$$= \varphi_{1}(t) * \varphi_{2}(t)$$



$$L^{-1}[\overline{\varphi}_1(p)\cdot\overline{\varphi}_2(p)] = \int_0^t \varphi_1(\tau)\varphi_2(t-\tau)d\tau$$

证明:考虑等式

$$\int_0^{+\infty} \left[\int_0^t \varphi_1(\tau) \varphi_2(t-\tau) d\tau \right] e^{-pt} dt = \int_0^{+\infty} \left[\int_{\tau}^{+\infty} \varphi_1(\tau) \varphi_2(t-\tau) e^{-pt} dt \right] d\tau$$

做积分变量代换 $t \rightarrow t' + \tau$

右边 =
$$\int_{0}^{+\infty} \varphi_{1}(\tau) d\tau \left[\int_{0}^{+\infty} \varphi_{2}(t') e^{-p(t'+\tau)} dt' \right]$$
$$= \int_{0}^{+\infty} \varphi_{1}(\tau) e^{-p\tau} d\tau \int_{0}^{+\infty} \varphi_{2}(t) e^{-pt} dt$$
$$= \overline{\varphi}_{1}(p) \cdot \overline{\varphi}_{2}(p)$$



补充 (线性定理)

$$\alpha$$
, β 为任意常数,且 $\overline{\varphi}_1(p) = L[\varphi_1(t)], \overline{\varphi}_2(p) = L[\varphi_2(t)]$

$$L[\alpha \varphi_1(t) + \beta \varphi_2(t)] = \alpha L[\varphi_1(t)] + \beta L[\varphi_2(t)]$$

补充 (相似定理)

$$L[\varphi(at)] = \frac{1}{a}\overline{\varphi}(\frac{p}{a})$$



补充求 L[sinhat], L[coshat]

【解】

$$L[\sinh at] = L[\frac{e^{at} - e^{-at}}{2}] = \frac{1}{2} \left(\frac{1}{p - a} - \frac{1}{p + a} \right) = \frac{a}{p^2 - a^2}$$

$$L[\cosh at] = L[\frac{e^{at} + e^{-at}}{2}] = \frac{1}{2} \left(\frac{1}{p-a} + \frac{1}{p+a} \right) = \frac{p}{p^2 - a^2}$$

 $(\operatorname{Re} p > \operatorname{Re} a)$



$$\frac{\omega}{(p+\lambda)^2+\omega^2}$$
和 $\frac{p+\lambda}{(p+\lambda)^2+\omega^2}$ (λ , ω 为已知常数) 的原函数。

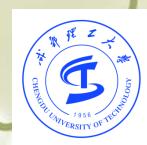
解: 由例5.5知

$$L^{-1}\left(\frac{\omega}{p^2 + \omega^2}\right) = \sin \omega t \quad (t > 0) \qquad L^{-1}\left(\frac{p}{p^2 + \omega^2}\right) = \cos \omega t$$

根据位移定理,

$$L^{-1}\left(\frac{\omega}{(p+\lambda)^2 + \omega^2}\right) = e^{-\lambda t} \sin \omega t \quad (t > 0)$$

$$L^{-1}(\frac{p+\lambda}{(p+\lambda)^2+\omega^2}) = e^{-\lambda t}\cos\omega t \quad (t>0)$$
 数学物理方程



解: 由例5.1知
$$L^{-1}(\frac{1}{p}) = H(t) = \begin{cases} 1 & t \ge 0 \\ 0 & t < 0 \end{cases}$$

所以
$$L^{-1}(\frac{e^{-ap}}{p}) = H(t-a)$$
 已知 $L^{-1}(\frac{1}{p+b}) = e^{-bt}$

由卷积定理

$$L^{-1}\left[\frac{e^{-ap}}{p(p+b)}\right] = H(t-a) * e^{-bt} = \frac{1}{b}\left[1 - e^{-b(t-a)}\right] \quad (t > a)$$

数学物理方程

反演公式

若
$$L^{-1}[\overline{\varphi}(p)] = \varphi(t)(\operatorname{Re} p > s_0, s_0$$
为已知实数)

那么 $\varphi(t)$ 在连续点处有:

$$\varphi(t) = \frac{1}{2\pi \mathbf{i}} \int_{a-i\infty}^{a+i\infty} \overline{\varphi}(p) e^{pt} dp(t>0, a>s_0)$$

黎曼一梅林公式



反演积分展开定理

若当p→∞ 时, $\bar{\varphi}(p)$ →0 并且在p平面中,

 $\overline{\varphi}(p)$ 只有有限个孤立奇点 $p_1,p_2,...,p_n$,那么必然存在一个实数a,使得这些奇点全部在Rep < a的半平面内,而且

$$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \overline{\varphi}(p) e^{pt} dp = \sum_{k=1}^{n} \operatorname{Res} f\left[\overline{\varphi}(p_k) e^{p_k t}\right]$$

$$\varphi(t) = \sum_{k=1}^{n} \operatorname{Resf}\left[\overline{\varphi}(p_k)e^{p_k t}\right] \qquad (t > 0)$$
 数学物理方程



例 5.8 求
$$\frac{1}{(p^2+1)^2}$$
 的原函数。

解: 函数在复平面上有两个二阶极点 $p_1=i,p_2=-i$

$$\operatorname{Re} sf[\overline{\varphi}(p_{1})e^{p_{1}t}] = \lim_{p \to i} \frac{d}{dp} \left[(p-i)^{2} \cdot \frac{1}{(p^{2}+1)^{2}} \cdot e^{pt} \right]$$
$$= \lim_{p \to i} \left[\frac{t}{(p+i)^{2}} - \frac{2}{(p+i)^{3}} \right] e^{pt} = -\frac{e^{it}}{4} (i+t)$$

$$\operatorname{Re} sf[\overline{\varphi}(p_2)e^{p_2t}] = \frac{e^{-it}}{4}(i-t)$$

$$\varphi(t) = \sum_{k=1}^{2} \operatorname{Re} sf[\overline{\varphi}(p_k)e^{p_k t}] = -\frac{e^{it}}{4}(i+t) + \frac{e^{-it}}{4}(i-t) = -\frac{te^{it}}{4}$$

数学物理方程

*§5.3 Laplace变换的应用

*本节内容不要求



本章作业

5-1; 5-2;

补充作业:解下列方程

(1)
$$\begin{cases} T_n''(t) + (n\pi a/l)^2 T_n(t) = 0\\ T_n(0) = C, T_n'(0) = D, \end{cases}$$

(2)
$$\begin{cases} T''_m(t) + (m\pi\alpha/l)^2 T_m(t) = A\sin\omega t \\ T_m(0) = 0, T'_m(0) = 0 \end{cases}$$



$$\begin{cases} T''_m(t) + (m\pi a/l)^2 T_m(t) = A \sin \omega t \\ T_m(0) = 0, T'_m(0) = 0 \end{cases}$$

解: 方程两边同时进行Laplace变换,则

$$p^{2}\overline{T}_{m}(p) + (m\pi a/l)^{2}\overline{T}_{m}(p) = A\frac{\omega}{p^{2} + \omega^{2}}$$

解得
$$\overline{T}_m(p) = \frac{A\omega}{p^2 + \omega^2} \cdot \frac{1}{p^2 + (m\pi a/l)^2}$$

$$= \frac{A\omega}{(m\pi a/l)^{2} - \omega^{2}} \left[\frac{1}{p^{2} + \omega^{2}} - \frac{1}{p^{2} + (m\pi a/l)^{2}} \right]$$

所以

$$T_{m}(t) = \frac{A}{(m\pi a/l)^{2} - \omega^{2}} \left[\sin \omega t - \frac{\omega l}{m\pi a} \frac{m\pi at}{l}\right]$$
数学物理方程

