非线性方程的求根

$$x^3 - 10x^2 + 30x - 25 = 0$$

$x = 0$: $f(0) = -25 < 0$

$x = 2$: $f(2) = 8 - 40 + 60 - 25 > 0$

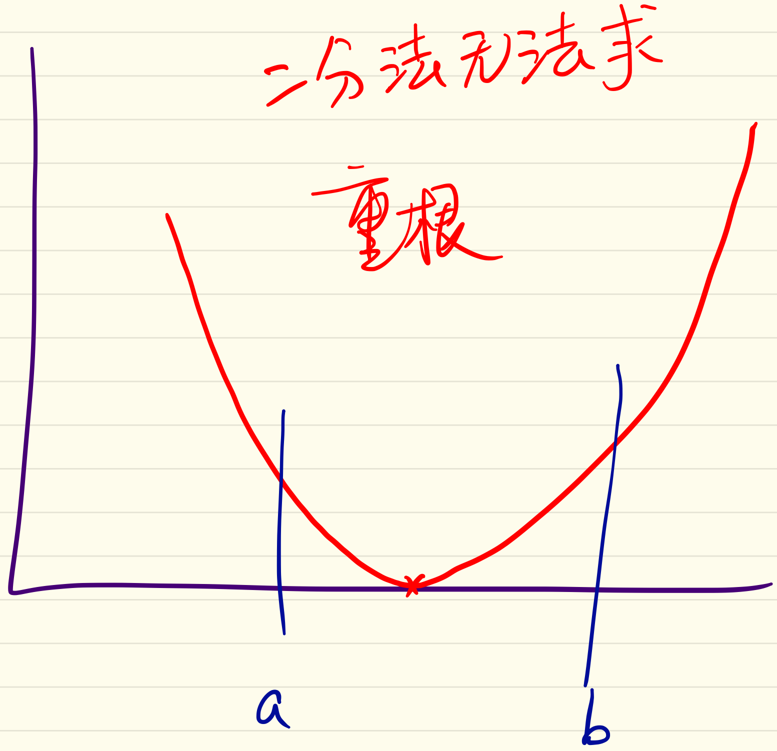
$[0, 2]$. $f(1) = 1 - 10 + 30 - 25 < 0$

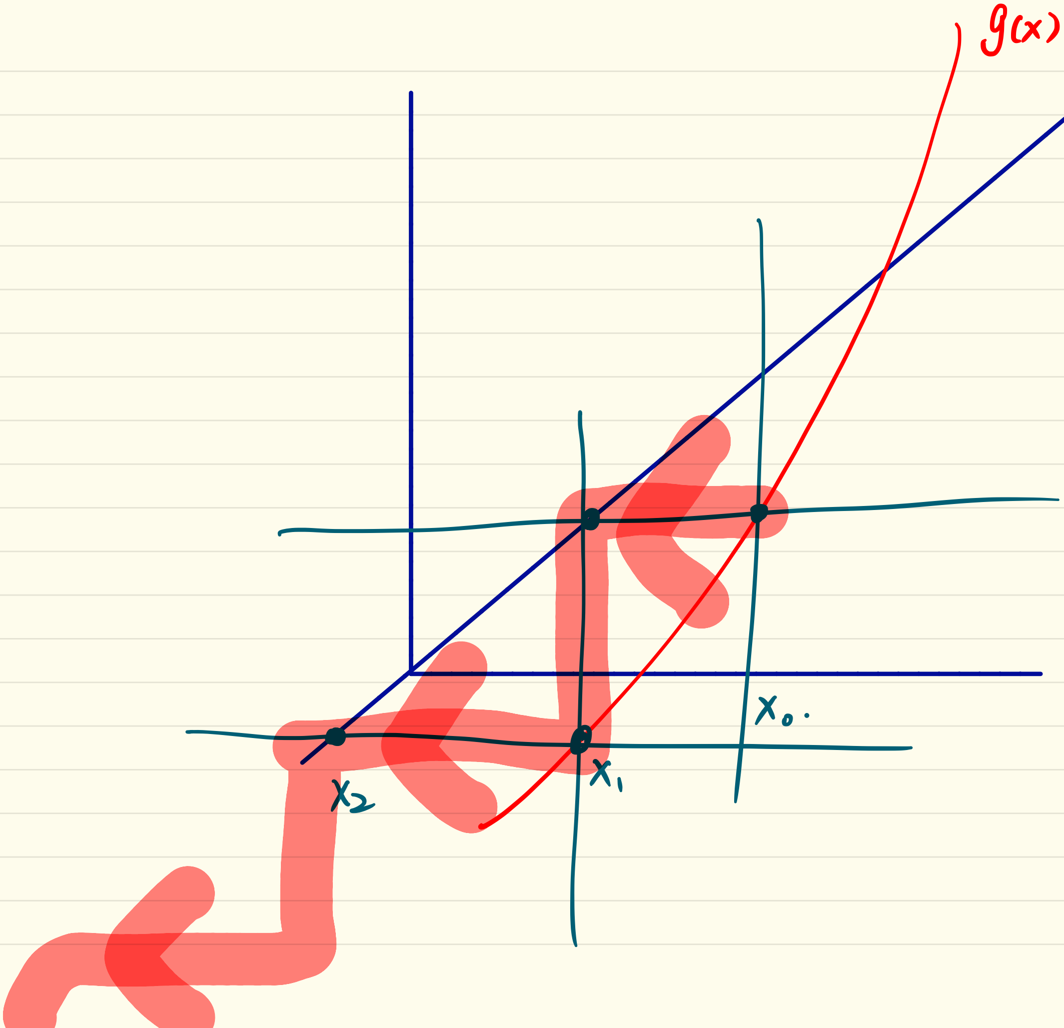
$f(0) < 0$. $f(1) < 0$ $f(2) > 0$

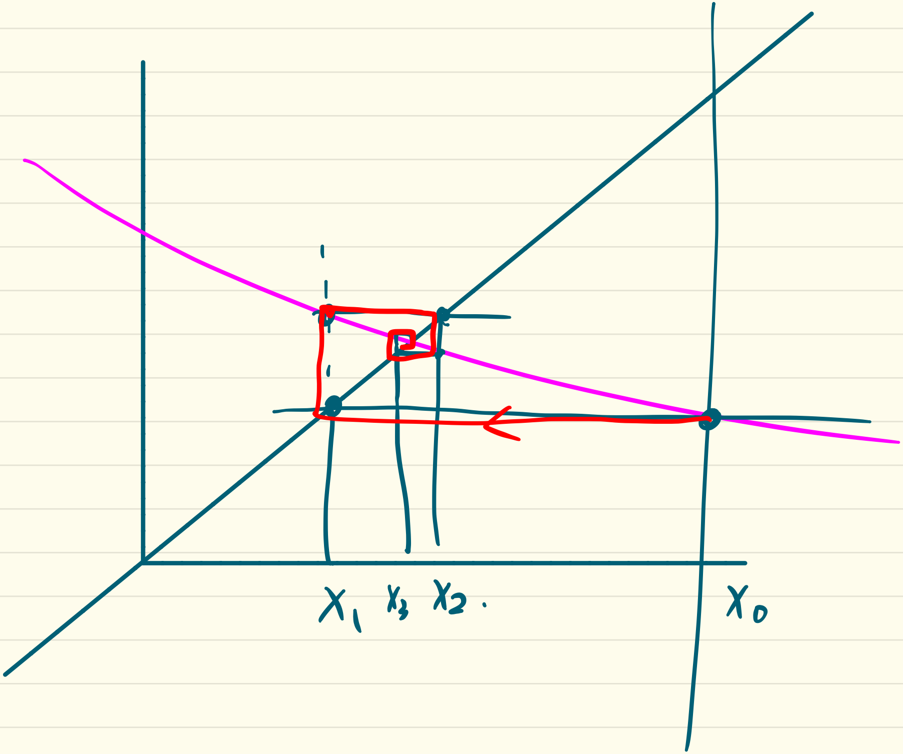
$[1, 2]$ $f(1.5) = 1.5^3 - 10 \times 1.5^2 + 30 \times 1.5 - 25 = \cdots$

$f(1) < 0$. $f(1.5) > 0$. $f(2) > 0$

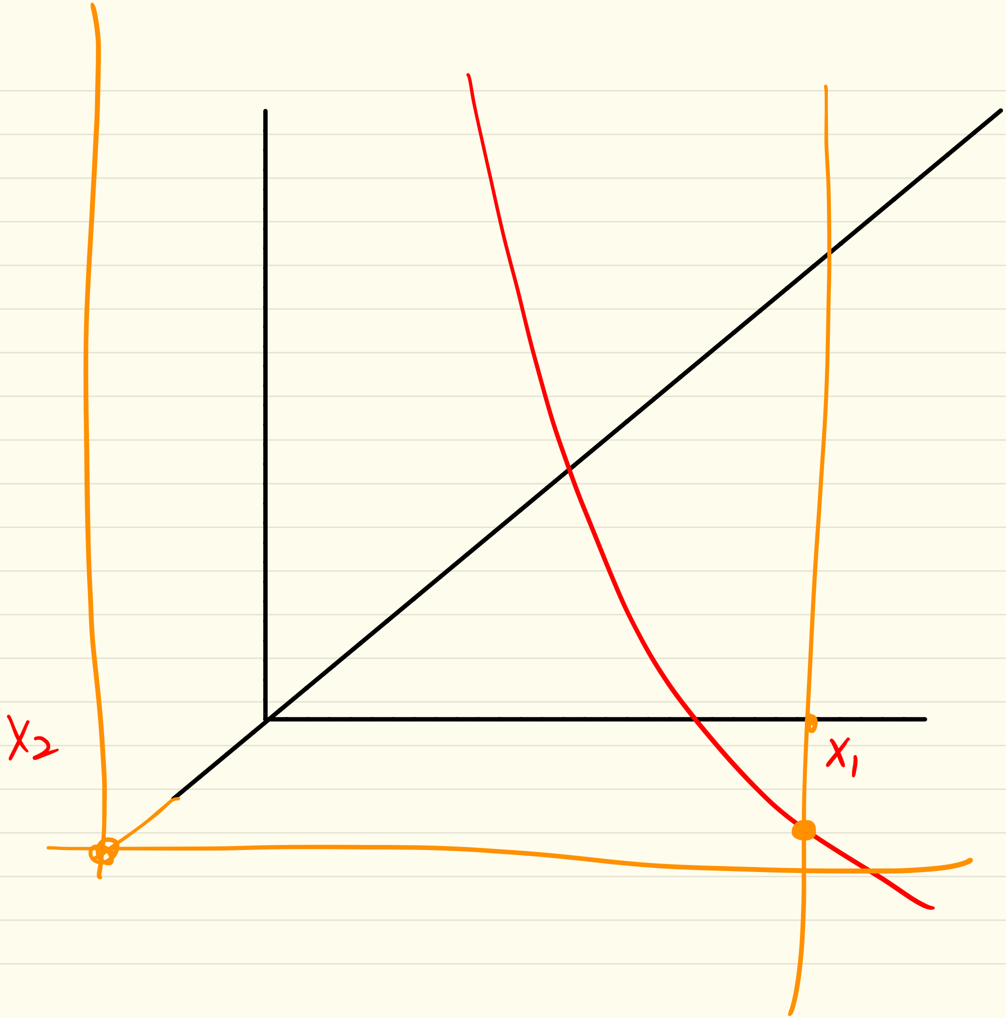
$[1, 1.5]$


二分法无法求
重根
a
b


g(x)
x_0.
x_1
x_2



x_1
x_3
x_2
x_0



x_2
x_1

$g(x) = e^{-x}$ 判断收敛性

$g'(x) = -e^{-x}$

$x \in [0,1]$. $\frac{1}{e} < |g'(x)| < 1$

牛顿法

$$f(x) = x - x^{\frac{1}{3}} - 2 \quad , \quad f(x) = 0$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$= x_k - \frac{x_k - x_k^{\frac{1}{3}} - 2}{1 - \frac{1}{3} x_k^{-\frac{2}{3}}}$$

$$= x_k - \frac{3x_k - 3x_k^{\frac{1}{3}} - 6}{3 - x_k^{-\frac{2}{3}}} = \frac{\cancel{3x_k} - x_k^{\frac{1}{3}} \cancel{- 3x_k} + 3x_k^{\frac{1}{3}} + 6}{3 - x_k^{-\frac{2}{3}}}$$

线性.

$$\begin{cases} f(x+y) = f(x) + f(y) \\ f(ax) = af(x) \end{cases}$$

或等效于:

$$f(ax+by) = af(x) + bf(y)$$

线性代数

$Ax_1 + Ax_2$

$= A(x_1 + x_2)$

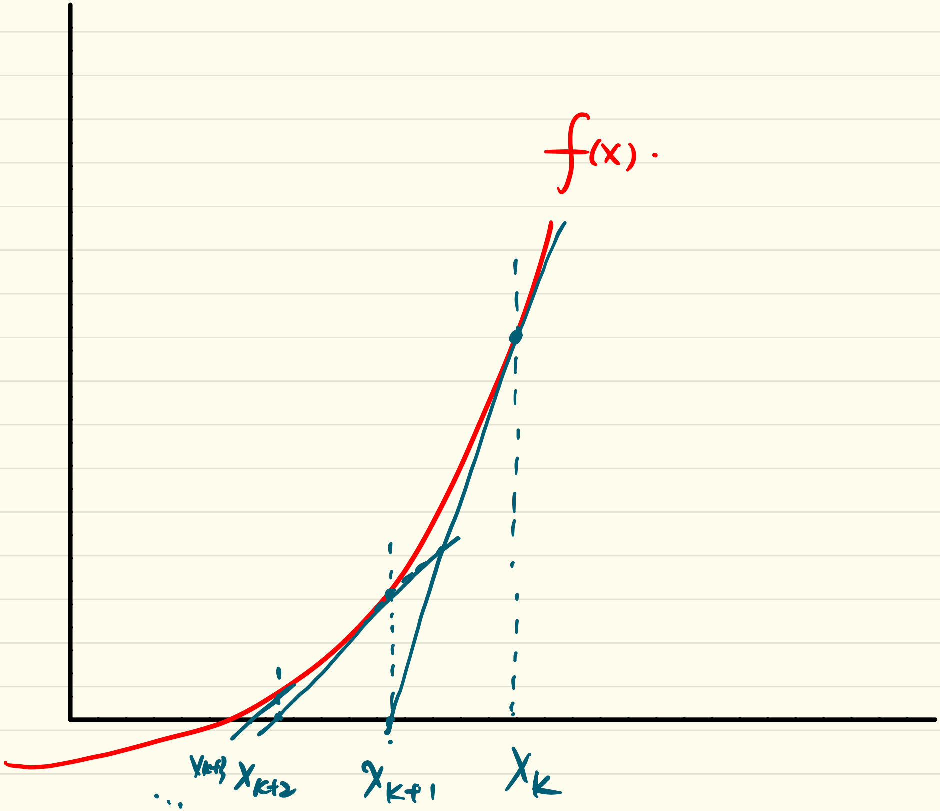
$A(ax_1) = a(Ax_1)$

微分方程

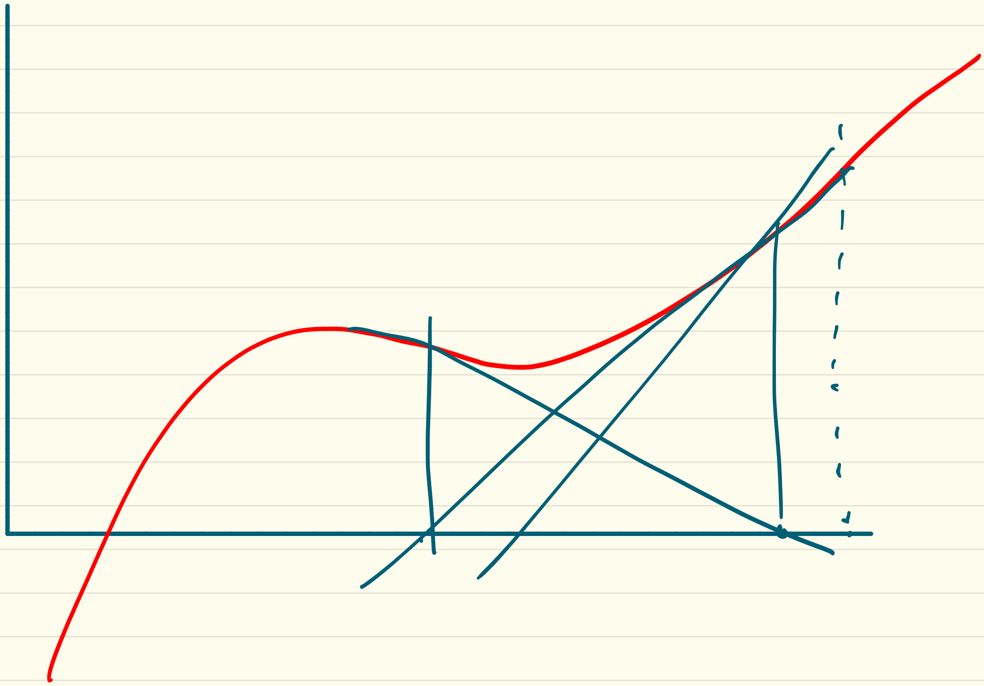
$y' = f(x)$

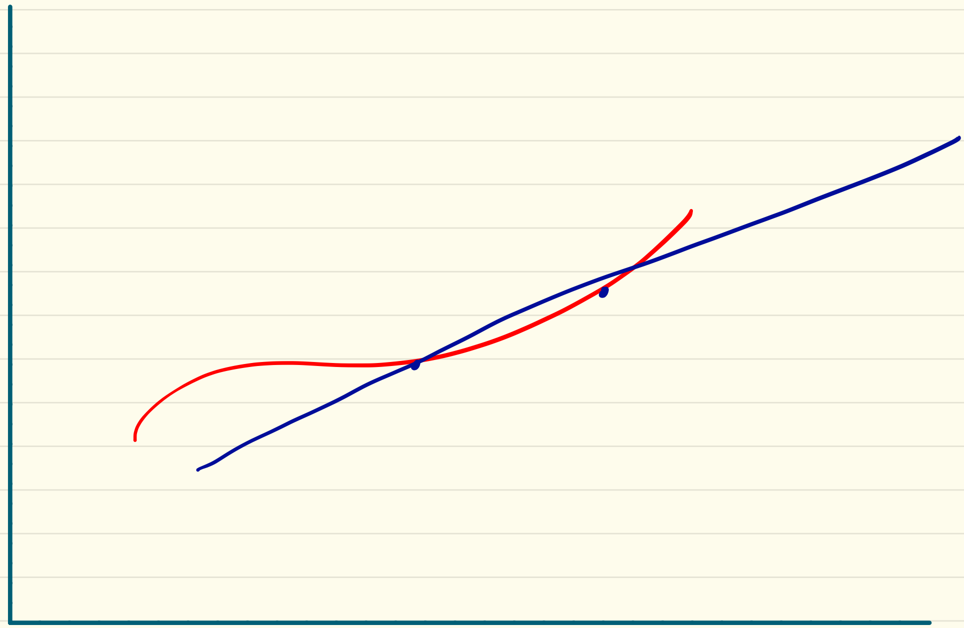
x_1, x_2 为解.

则 ax_1 也为解 $x_1 + x_2$ 也为解



f(x)
x_{k+3}
x_{k+2}
x_{k+1}
x_k
...





使用二分法求 $x = \cos x$ 在 $(0, \frac{\pi}{2})$ 上的根

$y = x - \cos x$

$f(x) = 0$

$f(x) = x - \cos x$

$(0, \frac{\pi}{2})$ $\quad x = 0, \ y < 0$

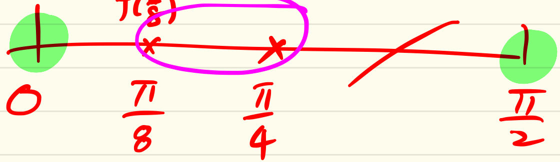
$x = \frac{\pi}{2} \quad y > 0$

$(0, \frac{\pi}{4}) \quad x = \frac{\pi}{4} \quad y > 0$

$(0, \frac{\pi}{8}) \quad x = \frac{\pi}{8}$

$f(0) < 0 \qquad f(\frac{\pi}{4}) > 0 \qquad f(\frac{\pi}{2}) > 0$

$f(\frac{\pi}{8})$



误差 $< 10^{-5}$. 需要多少次迭代.

$\frac{\pi}{4} \quad \frac{\pi}{8} \quad \frac{\pi}{16}$.

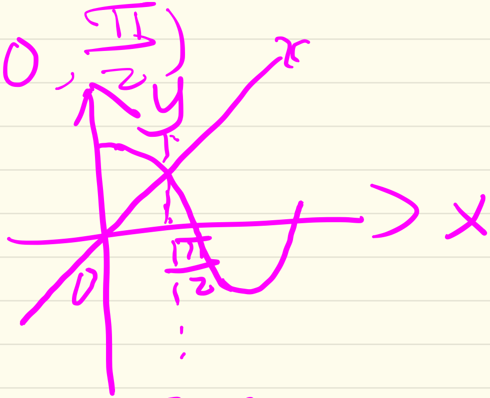
$e = \frac{\pi}{2} \cdot (\frac{1}{2})^{n} < 10^{-5}$

那简单迭代法和牛顿法呢，估计误差在 10^{-5} 之内，需多少次迭代。

$x = \cos x$

$x_{k+1} = \cos x_k$

$x \in (0, \frac{\pi}{2})$



$x_0 = \frac{\pi}{4}$

$x_1 = \frac{\sqrt{2}}{2}$

$x_2 = \cos\frac{\sqrt{2}}{2}$

误差 $< 10^{-5}$，需多少次迭代？

$L \approx \frac{\sqrt{2}}{2}$

$$\frac{\left(\frac{\sqrt{2}}{2}\right)^n}{1-\frac{\sqrt{2}}{2}} \cdot \left|\frac{\pi}{4} - \frac{\sqrt{2}}{2}\right| < 10^{-5}$$

牛顿法 $x=\cos x$

$f(x)=x-\cos x$

$f'(x)=1+\sin x$

$x_0=\frac{\pi}{4}$

$$x_{k+1}=x_k-\frac{f(x_k)}{f'(x_k)}$$

$x_1=\frac{\quad}{1+\frac{\sqrt{2}}{2}}=a$

$$=x_k-\frac{x_k-\cos x_k}{1+\sin x_k}$$

$x_2=$

$$=\frac{x_k\sin x_k+\cos x_k}{1+\sin x_k}$$

$$f(x^*) = f(x_k) + (x^* - x_k) f'(x_k)$$

$$= 0$$

$$g(x) = x - \frac{x - \cos x}{1 + \sin x}$$

L : $\quad |g(x) - g(y)| \le L\,|x - y|$.

非线性方程组.

$$\begin{cases} x_1 + \cos x_2 = 0 \\ x_1^2 - x_2 = 0 \end{cases}$$

Jacobi 矩阵 $\begin{pmatrix} 1 & -\sin x_2 \\ 2x_1 & -1 \end{pmatrix}$

初始值 $X_0 = \begin{pmatrix} x_1 = 0 \\ x_2 = 0 \end{pmatrix}$

$$X_1 = X_0 - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$x_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

对角矩阵的逆

$$A = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}$$

$$A \cdot B = 1 \qquad B = \begin{pmatrix} b_1 & & & \\ & b_2 & & \\ & & \ddots & \\ & & & b_n \end{pmatrix}$$

$$B = A^{-1}$$

$$A \cdot B = \begin{pmatrix} a_1 b_1 & & & \\ & a_2 b_2 & & \\ & & \ddots & \\ & & & a_n b_n \end{pmatrix}$$

$a_1 b_1 = 1$

$a_2 b_2 = 1$

$\vdots$

$a_n b_n = 1.$

二分法

$$|x^* - x^k| \approx \frac{\pi}{2}\left(\frac{1}{2}\right)^k$$

$$|x^* - x^{k+1}| \approx \frac{\pi}{2}\left(\frac{1}{2}\right)^{k+1}$$

$$\lim_{k\to\infty} \frac{|x^* - x^{k+1}|}{|x^* - x^k|^p} = C \qquad \lim_{k\to\infty} \frac{\left(\frac{1}{2}\right)^{k+1}}{\left(\frac{1}{2}\right)^{pk}} = C$$

牛顿法 $f(x)=0$

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$|g'(x^*)| = \cancel{1 - f'(x^*)\frac{1}{f'(x^*)}} + f(x^*)\frac{1}{f''(x^*)}$$

$$= 0$$

x^*	star
$\tilde{x}$	tilde
$\hat{x}$	hat
$\dot{x}$	dot
$\bar{x}$	bar
$\underline{x}$	underline

$$\frac{x^* - x_{k+1}}{(x^* - x_k)^2} = \frac{x^* - x_{k+2}}{(x^* - x_{k+1})^2}$$

$$x^* =$$

$$\Bigg\{$$