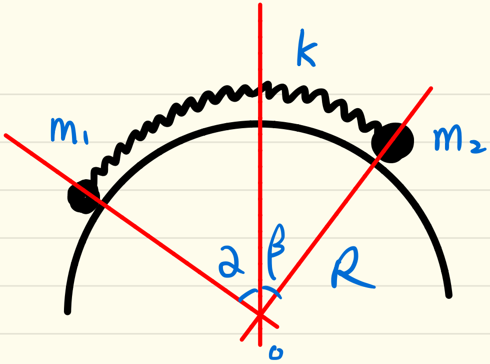


① 求广义力 Q_1 和 Q_2

广义坐标 $q_1=\alpha$，$q_2=\beta$

(1) 势：(选O点为重力零势能点.)

$$V = m_1 g \cdot R \cdot \cos q_1 + m_2 g R \cos q_2 + \frac{1}{2} k\left(R\cdot(q_1+q_2) - L\right)^2$$

广义力 $Q_1 = -\dfrac{\partial V}{\partial q_1} = m_1 g R \cdot \sin q_1 - k\left(R(q_1+q_2)-L\right)R$

$$Q_2 = -\frac{\partial V}{\partial q_2} = m_2 g R \sin q_2 - k\left(R(q_1+q_2)-L\right)R$$

平衡. 虚功原理:

$\alpha \to \alpha + \delta\alpha$

$\beta \to \beta + \delta\beta$

$\delta W = -\delta V$

$$V = m_1 g \cdot R \cdot \cos(q_1 + \delta q_1) + m_2 g R \cos(q_2 + \delta q_2) + \frac{1}{2} k (R \cdot (q_1 + \delta q_1 + q_2 + \delta q_2) - L)^2$$

做 Taylor 展开.

$$V = m_1 g R \cos q_1 - m_1 g R \sin q_1 \cdot \delta q_1 + O(\delta q_1^2) + m_2 g R \cos q_2 - m_2 g R \sin q_2 \cdot \delta q_2 + O(\delta q_2^2)$$

$$\frac{1}{2} k (R(q_1 + q_2) - L)^2 - kR \cdot \delta q_1 (L - Rq_1) - kR\, \delta q_2 (L - Rq_2)$$

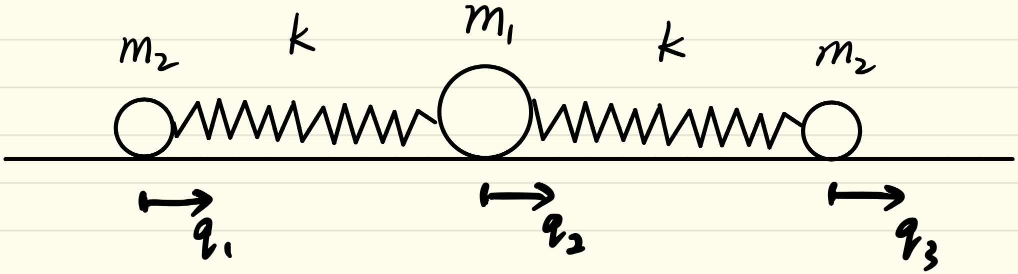
$$\delta V = \left[-m_1 g R \sin q_1 - kR\cdot(L - R q_1)\right]\delta q_1$$

$$+\left[-m_2 g R \sin q_2 - kR(L - R q_2)\right]\delta q_2 + O(\delta q_1^2) + O(\delta q_2^2)$$

$\delta W = -\delta V = 0$ 平衡条件:

$$\begin{cases} -m_1 g R \sin q_1 - kR\cdot(L - R q_1) = 0 \\ -m_2 g R \sin q_2 - kR(L - R q_2) = 0 \end{cases}$$

其实就是 $Q_1 = 0$ 和 $Q_2 = 0$



求特征频率

平衡位置，广义坐标为0.

动能：$T = \frac{1}{2} m_2 \dot{q}_1^2 + \frac{1}{2} m_1 \dot{q}_2^2 + \frac{1}{2} m_2 \dot{q}_3^2$

$$V = \frac{1}{2} k (q_2 - q_1)^2 + \frac{1}{2} k (q_3 - q_2)^2$$

拉格朗日量 $L = T - V$

代入拉格朗日方程 $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = 0$

$\alpha = 1, 2, 3$ 分别得到：

$$\begin{cases} m_2\ddot{q}_1 + k(q_1 - q_2) = 0 \\ m_1\ddot{q}_2 + \underbrace{k(q_2 - q_1) + k(q_2 - q_3)}_{k(2q_2 - q_1 - q_3)} = 0 \\ m_2\ddot{q}_2 + k(q_3 - q_2) = 0 \end{cases}$$

即

$$\begin{pmatrix} m_2 & 0 & 0 \\ 0 & m_1 & 0 \\ 0 & 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{pmatrix} + \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = (0, 0, 0)$$

有通解. $q_1 = A_1 \cdot e^{\lambda t}$; $q_2 = A_2 e^{\lambda t}$ 和 $q_3 = A_3 e^{\lambda t}$

$$\begin{pmatrix} m_2 & 0 & 0 \\ 0 & m_1 & 0 \\ 0 & 0 & m_2 \end{pmatrix}\begin{pmatrix} \lambda^2 q_1 \\ \lambda^2 q_2 \\ \lambda^2 q_3 \end{pmatrix} + \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = (0, 0, 0)$$

$$\begin{pmatrix} \lambda^2 m_2 & 0 & 0 \\ 0 & \lambda^2 m_1 & 0 \\ 0 & 0 & \lambda^2 m_2 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} + \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = (0, 0, 0)$$

$$\begin{pmatrix} \lambda^2 m_2 + k & -k & 0 \\ -k & \lambda^2 m_1 + 2k & -k \\ 0 & -k & \lambda^2 m_2 + k \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = (0, 0, 0)$$

有非零的 q_1, q_2, q_3 满足上方程组的条件

$$\begin{vmatrix} \lambda^2 m_2 + k & -k & 0 \\ -k & \lambda^2 m_1 + 2k & -k \\ 0 & -k & \lambda^2 m_2 + k \end{vmatrix} = 0$$

$$(\lambda^2 m_2+k)(\lambda^2 m_1+2k)(\lambda^2 m_2+k)-k^2(\lambda^2 m_2+k)$$
$$-k^2(\lambda^2 m_2+k)=0$$

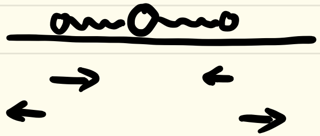
即. $$m_2^2 m_1\lambda^6+2k\left[m_1 m_2+m_2^2\right]\lambda^4+k^2(m_1+2m_2)\lambda^2=0$$

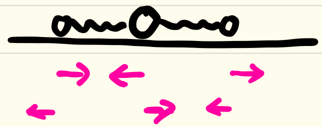
$$\lambda^2\left[m_1 m_2^2\lambda^4+2k(m_1m_2+m_2^2)\lambda^2+k^2(m_1+2m_2)\right]=0$$

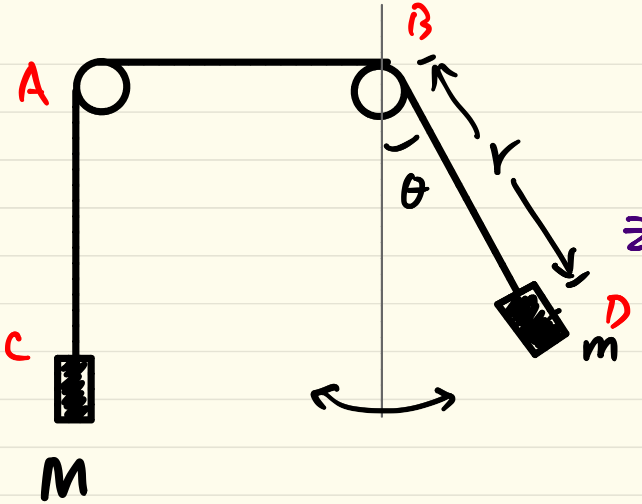
$$\lambda^2\left[\lambda^4+\left(\frac{2k}{m_2}+\frac{2k}{m_1}\right)\lambda^2+\frac{k^2}{m_2}\left(\frac{1}{m_2}+\frac{2}{m_1}\right)\right]=0$$

$$\lambda^2\left[\left(\lambda^2+\frac{k}{m_2}\right)\left(\lambda^2+k\left(\frac{1}{m_2}+\frac{2}{m_1}\right)\right)\right]=0$$

对应解 $\underbrace{\lambda^2_{1,2}=0}_{\text{对应静止.}}$; $\lambda_{3,4}=\pm i\sqrt{\frac{k}{m_2}}$ 和 $\lambda_{5,6}=\pm i\sqrt{k\left(\frac{1}{m_2}+\frac{2}{m_1}\right)}$







哈密顿量. 对保守力系统

$$H = T + V$$

动能 $T = \frac{1}{2} m \left(V_{切}^2 + V_{径}^2 \right) + \frac{1}{2} M V_{左}^2$

$V_{切} = \dot{\theta} \cdot r \ ; \quad V_{径} = \dot{r}$

$V_{左} = \dot{r}$

即 $T = \frac{1}{2} m \left(\dot{\theta}^2 r^2 + \dot{r}^2 \right) + \frac{1}{2} M \dot{r}^2$

L为AC段+BD段长度.

势能. $V = -mg\, r \cdot \cos\theta - Mg(L - r)$

$$L = T - V$$

$$p = \frac{\partial L}{\partial \dot{q}}$$

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q}$$