VE477 Lab8 Report

Student Name: Zhou Haoquan

Student ID: 519370910059

General Questions

1. What is Linear Programming

Given a mathematical model where an **objective** is represented as a linear function and some **constraints** are expressed as equalities or inequalities, maximize the objective while respecting the constraints.

2. Provide examples of situations where linear programming is used in practice.

- **Business**: maximize the profit when producing various items sold at different prices.
- **Airlines**: following the regulation on what crew can do, maximize the number of flights possible given the number of crew members.
- **Exploration**: decide where to drill for oil in order to maximize profit under some restrictions like the geography constraints or policy constraints.

3. What are the standard and slack forms and how good are they to express a linear program?

The Standard Form

Find `n` real numbers `x1,...,xn` that maximize

$$\sum_{j=1}^{n} c_j x_j$$

while being subject to

$$\begin{cases} \sum_{j=1}^{n} a_{ij} x_j \le b_i \ , \ 1 \le i \le m \\ x_j \ge 0 \ , \ 1 \le j \le n \end{cases}$$

The Slack Form

Rewritten the constrains above as

$$\begin{cases} s = b_i - \sum_{j=1}^{n} aij x_j, \\ s \ge 0 \end{cases}$$

with all the `non-basic variables` on the right and all the `basic variables` on the left.

How They Perform

The Standard Form is has more mathematical meanings while the Slack Form can be easily interpreted as data structures and thus becomes more friendly to programming.

4. What algorithms exist to solve linear programs? Provide a simple but clear description of the simplex method.

There are three kinds of algorithms solving the linear programs:

1. **Ellipsoid algorithm**
2. **Interior-point methods**
3. **Simplex algorithm**

The Simplex algorithm

1. Given a slack system, first find a basic solution by setting all the non-basic variables on the right hand side to 0. Then the basic variables will have a value correspondingly.
2. Then a non-basic variable is chosen and increased as much as possible without violating any of the constraints. Then exchange the original basic variable of that tightest equation with this variable. Update all the other constraints.
3. After adjusting all the original non-basic variables, the maximum value that the basic variables can reach is the optimal solution.

5. What is duality and when could it be applied when running the simplex method?

Given a Linear Program where the objective is to be maximized, its dual has the same optimal value and is described as:

Find `m` real numbers `y1,...,ym` that minimize

$$\sum_{i=1}^{m} b_i y_i$$

while subject to

$$\begin{cases} \sum_{i=1}^{m} a_{ij} y_i \geq c_j,\ 1 \leq j \leq n \\ y_i \geq 0,\ 1 \leq i \leq m \end{cases}$$

The duality can be used as:

1. Primal form can be applied when there is more variable than constraints.
2. Dual form can be applied when there is more constraints than variables.

Toy example for the simplex method

1.

a) The standard form

First, negate the coefficients from the objective. Result in

$$\text{maximizing } 2x_1 - 3x_2$$

Since `x2` has no constraints and we require that the variables greater or equal to 0, then we write it as

$$x_2 = x_2^{'} - x_2^{''} \qquad x_2^{'}, x_2^{''} \geq 0$$

Also, we write the equation with two "less or equal" inequalities. The replace the

$$x_2^{'} \text{ and } x_2^{''} \text{ with } x_2 \text{ and } x_3$$

The problem then becomes

$$\text{maximize } 2x_1 - 3x_2 + 3x_3, \text{ subject to}$$

$$\begin{cases} x_1 + x_2 - x_3 \le 7 \\ -x_1 - x_2 + x_3 \le -7 \\ x_1 - 2x_2 + 2x_3 \le 4 \\ x_1, x_2, x_3 \ge 0 \end{cases}$$

b) The slack form

We can write the three inequalities into new equations and combine them with the objective. The slack form will be

$$\begin{cases} z = 2x_1 - 3x_2 + x_3 \\ x_4 = 7 - x_1 - x_2 + x_3 \\ x_5 = -7 + x_1 + x_2 - x_3 \\ x_6 = 4 - x_1 + 2x_2 - 2x_3 \end{cases}$$

2.

First, we choose `x1` and the first augmenting non-basic variable and augment it to `9`, the equations become

$$\begin{cases} z = 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4} \\ x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} \\ x_4 = 21 = \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4} \\ x_5 = 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2} \end{cases}$$

Then we augment `x3` and rewrite the equations as

$$\begin{cases} z = \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16} \\ x_1 = \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16} \\ x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8} \\ x_4 = \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16} \end{cases}$$

Finally we augment `x2` and rewrite the equations as

$$\begin{cases} z = 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\ x_1 = 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\ x_2 = 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\ x_4 = 18 - \frac{x_3}{2} + \frac{x_5}{2} \end{cases}$$

From the equations, we can find the optimal situation and its corresponding conditions as

$$\begin{cases} z = 28 \\ x_1 = 8 \\ x_2 = 4 \\ x_3 = 0 \\ x_4 = 18 \\ x_5 = 0 \\ x_6 = 0 \end{cases}$$