Implementation of Quantum Information Processing
ECE676/PHYS768/QIC750 Winter 2024
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Problem Set 1

Due: 2024 Jan. 26, Friday, 11:59 pm, Learn

1 Linear Algebra

There are two observables $\hat{A}$ and $\hat{B}$, whose matrices are written in a certain basis:

$$\hat{A} = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}, \qquad \hat{B} = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}, \tag{1}$$

where a and b are both real.

(1) Does $\hat{B}$ exhibit a degenerate spectrum?

(2) What is $[\hat{A}, \hat{B}]$?

(3) Find a new set of basis states which are simultaneous eigenstates of $\hat{A}$ and $\hat{B}$. Specify the eigenvalues of $\hat{A}$ and $\hat{B}$ for each of the three eigenstates. Does your specification of the eigenvalues uniquely characterize each eigenstate?

2 Pauli Operators

(1) Prove the following equality:

$$[\hat{\sigma}_i, \hat{\sigma}_j] = 2i\epsilon_{ijk}\hat{\sigma}_k \quad , \quad \{\sigma_i, \sigma_j\} = 2\delta_{ij} \tag{2}$$

where ϵ_{ijk} is the Levi-Civita tensor and $\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$ is known as the anticommutator.

(2) Prove the following vector identity involving the Pauli matrix vector, $\vec{\hat{\sigma}}$:

$$(\vec{\hat{\sigma}} \cdot \vec{a})(\vec{\hat{\sigma}} \cdot \vec{b}) = \vec{a} \cdot \vec{b} + i(\vec{\hat{\sigma}} \cdot (\vec{a} \times \vec{b}),$$

where $\vec{\hat{\sigma}} = [\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_x]$ is the Pauli vector and $\vec{a}$, $\vec{b}$ are 3-dimentional vectors.

3 Mixed states (10 points)

Consider a system which has a probability of $\frac{1}{3}$ of being in the state $|\psi\rangle$ and a probability of $\frac{2}{3}$ of being in the state $|\phi\rangle$ where,

$$|\psi\rangle = \frac{3}{\sqrt{10}}|+\rangle + \frac{1}{\sqrt{10}}|-\rangle,$$
$$|\phi\rangle = \frac{1}{\sqrt{10}}|+i\rangle - \frac{3}{\sqrt{10}}|-i\rangle.$$

Note that $|+\rangle$ and $|-\rangle$ are the state vectors of the Hadamard gate on $|0\rangle$ and $|1\rangle$, respectively. $|\pm i\rangle$ are the eigenstates of $\hat{\sigma}_y$.

(1) Write down the density matrix $\hat{\rho}_M$ representing this state.

(2) Compare $\hat{\rho}_M^2$ with $\hat{\rho}_M$.

4 Density Matrix - I

Consider the density matrix of a state, $\hat{\rho}$, that is written in terms of a so-called Bloch vector $\vec{a}$ as:

$$\hat{\rho} = \frac{1}{2}(\hat{I} + \vec{\sigma}\cdot\vec{a}),$$

where $|\vec{a}| \leq 1$.

(1) Show that the eigenvalues of $\hat{\rho}$ are $\frac{1}{2}(1 \pm |\vec{a}|)$.

(2) The purity of a state is sometimes defined as $P = \mathrm{tr}(\hat{\rho}^2)$. Show that $P = \frac{1}{2}(1 + |\vec{a}|^2)$. Why is it called the purity?

5 Density Matrix - II (10 points)

Consider the 2-qubit state,

$$|\psi\rangle_{AB} = \frac{1}{2}(\sqrt{2}|00\rangle + |01\rangle + |11\rangle).$$

(1) Find $\hat{\rho}_B = \mathbf{Tr}_A|\psi\rangle\langle\psi|$ and $\hat{\rho}_A = \mathbf{Tr}_B|\psi\rangle\langle\psi|$.

(2) Show that these matrices have the same eigenvalues. What are the eigenvalues?

6 Rotations to gates (10 points)

Find one single-qubit rotation that implements each gate below up to a global phase.

(1) Pauli-Y gate,

$$\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

(2) Hadamard gate,

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

7 Kraus operators

Consider a qubit whose ground state and excited state are denoted as $|0\rangle^{(s)}$ and $|1\rangle^{(s)}$, respectively. The environment is modeled as a qubit whose initial state is in its ground state, $|0\rangle^{(e)}$. If the energy decay of the qubit to the environment is described by the following evolution:

$$\begin{aligned} &|00\rangle^{(s\otimes e)} \to |00\rangle^{(s\otimes e)}, \\ &|10\rangle^{(s\otimes e)} \to \sqrt{1 - e^{-t/T_1}}|01\rangle^{(s\otimes e)} + \sqrt{e^{-t/T_1}}|10\rangle^{(s\otimes e)}., \end{aligned}$$

where T_1 is a constant, whose significance will be discussed in future lectures on Nuclear Magnetic Resonance. Note that we assume $t > 0$.

Find the Kraus operators.

8 Measurement (10 points)

Consider the state

$$|\psi\rangle = \frac{1}{3}|00\rangle + \frac{1}{\sqrt{3}}|01\rangle + \frac{\sqrt{2}}{3}|10\rangle + \frac{1}{\sqrt{3}}|11\rangle.$$

The first qubit is measured to be in the state $|1\rangle$.

(1) What is the state of the system after this measurement?

(2) What is the probability then a subsequent measurement of qubit 2 will be in the state $|0\rangle$?