

Mini - Project OR

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Simplex Method

Introduction to the Simplex Method:

Simplex method also called simplex technique or simplex algorithm was developed by G.B. Dantzeg, An American mathematician. Simplex method is suitable for solving linear programming problems with a large number of variable. The method through an iterative process progressively approaches and ultimately reaches to the maximum or minimum values of the objective function.

Principle of Simplex Method:

It has not been possible to obtain the graphical solution to the LP problem of more than two variables. For these reasons mathematical iterative procedure known as 'Simplex Method' was developed. The simplex method is applicable to any problem that can be formulated in-terms of linear objective function subject to a set of linear constraints.

The simplex method provides an algorithm which is based on the fundamental theorem of linear programming. This states that **"the optimal solution to a linear programming problem if it exists, always occurs at one of the corner points of the feasible solution space."**

The simplex method provides a systematic algorithm which consist of moving from one basic feasible solution to another in a prescribed manner such that the value of the objective function is improved. The procedure of jumping from vertex to the vertex is repeated. The simplex algorithm is an iterative procedure for solving LP problems.

It consists of:

- (i) Having a trial basic feasible solution to constraints equation,
- (ii) Testing whether it is an optimal solution,
- (iii) Improving the first trial solution by repeating the process till an optimal solution is obtained.

Schematic diagram of simplex Table:

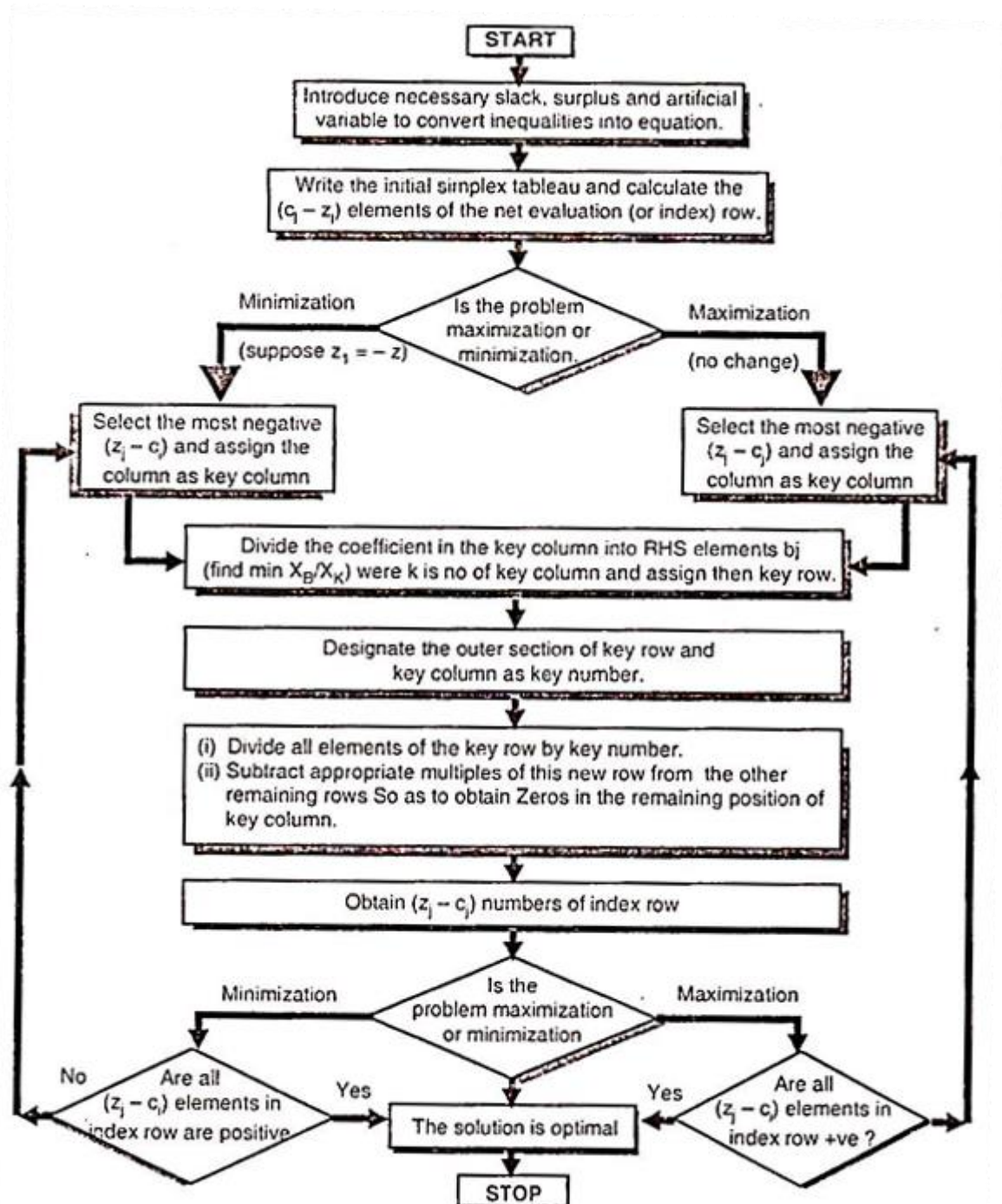
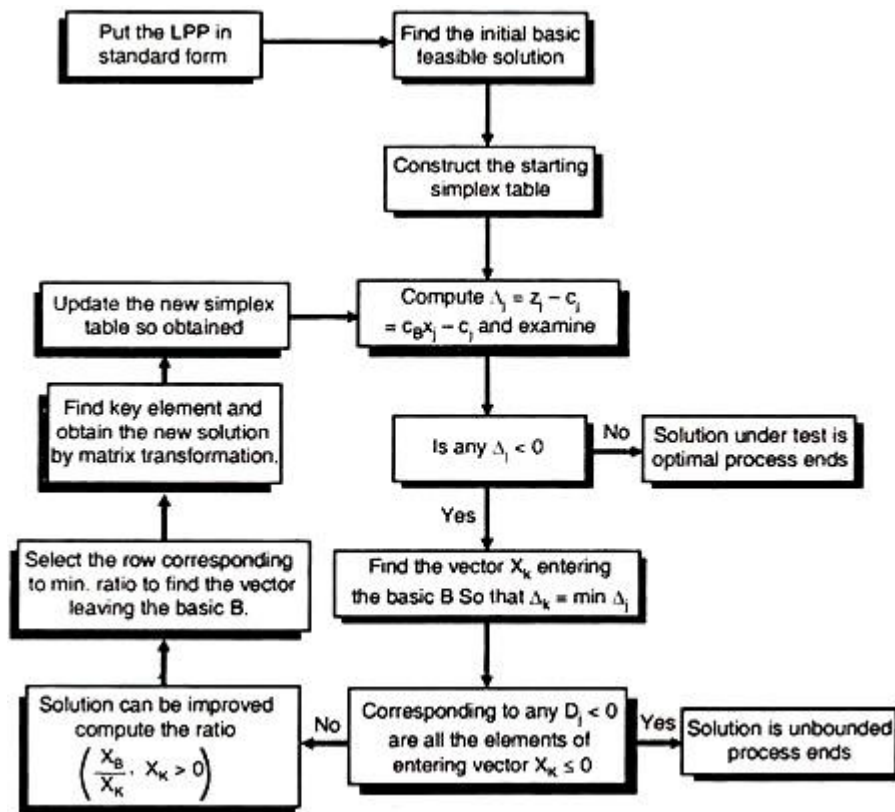


Fig. 4.6. Schematic Diagram of simplex algorithm.

Flow Chart of Simplex Method:



Computational Procedure of Simplex Method:

The computational aspect of the simplex procedure is best explained by a simple example.

Example :**Consider the linear programming problem:**

$$\text{Maximize } z = 3x_1 + 2x_2$$

$$\text{Subject to } x_1 + x_2, \leq 4$$

$$x_1 - x_2, \leq 2$$

$$x_1, x_2, \geq 0$$

$$x_1, x_2 > 0$$

Solution:**The steps in simplex algorithm are as follows:****Step 1:****Formulation of the mathematical model:**

(i) Formulate the mathematical model of given LPP.

(ii) If objective function is of minimisation type then convert it into one of maximisation by following relationship

$$\text{Minimise } Z = - \text{Maximise } Z^*$$

$$\text{When } Z^* = -Z$$

(iii) Ensure all b_i values [all the right side constants of constraints] are positive. If not, it can be changed into positive value on multiplying both side of the constraints by -1.

In this example, all the b_i (right side constants) are already positive.

(iv) Next convert the inequality constraints to equation by introducing the non-negative slack or surplus variable. The

coefficients of slack or surplus variables are zero in the objective function.

In this example, the inequality constraints being ' \leq ' only slack variables s_1 and s_2 are needed.

Therefore given problem now becomes:

Therefore given problem now becomes

$$\text{Max } z = 3x_1 + 2x_2 + 0s_1 + 0s_2$$

$$\text{Subject to } x_1 + x_2 + s_1 + 0s_2 = 4$$

$$x_1 - x_2 + 0s_1 + s_2 = 2$$

$$x_1, x_2, s_1, s_2 \geq 0$$

(v) Now present the constraints equations in matrix form :

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Step 2:

Set up the initial solution.

Write down the coefficients of all the variables in given LPP in the tabular form, as shown in table below to get an initial basic Feasible solution.

$$x_B = B^{-1} b$$

Table 1: Initial Simplex Table

$C_j \rightarrow$			c_1	c_2	\dots	C_n	$0..0-0$
Coefficient of Basic	Variables in Basic	Value of basic Variables	Variables – Variables				
Variables (C_B)	B	$b_{(=x_B)}$	x_1	x_2	\dots	x_n	$s_1 s_2 \dots s_n$
C_{B1}	s_1	$x_{B1} = b_1$	a_{11}	a_{12}	a_{1n}	1	0 0
C_{B2}	s_2	$x_{B2} = b_2$	a_{21}	a_{22}	a_{2n}	1	0 0
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
C_{Bm}	s_m	$x_{Bm} = b_m$	a_{m1}	a_{m2}	a_{mn}	0	0 1
$Z = \sum C_{Bi} \cdot X_{Bi}$ $= (\text{B.V. Coefficients}) = \sum (\text{B.V. Coefficients})$ $\times (\text{Values of B.V.}) \times (\text{J}^{\text{th}} \text{ column of data matrix})$ $c_j - z_j$			0	0	0	0	0 0
			$c_1 - z_1$	$c_2 - z_2$	\dots	$c_n - z_n$	0 0 \dots 0

The first row in table indicates the coefficient c_j of variables in objective function, which remain same in successive tables. These values represent cost or profit per unit of objective function of each of the variables.

The second row gives major column headings for the simple table. Column C_B gives the coefficients of the current basic variables in the objective function. Column x_B gives the current values of the corresponding variables in the basic.

Number a_{ij} represent the rate at which resource (i- 1, 2- m) is consumed by each unit of an activity j (j = 1,2 ... n).

The values z_j represents the amount by which the value of objective function Z would be decreased or increased if one unit of given variable is added to the new solution.

It should be remembered that values of non-basic variables are always zero at each iteration.

So $x_1 = x_2 = 0$ here, column x_B gives the values of basic variables in the first column.

So $s_1 = 4, s_2 = 2$, here; The complete starting feasible solution can be immediately read from table 2 as $s_1 = 4, s_2, x_1 = 0, x_2 = 0$ and the value of the objective function is zero.

		$C_j \rightarrow$		3	2	0	0	
Basic Variables	C_B	X_B	X_1	X_2	$S_1 (= x_1)$ B_1	$S_2 (=4)$ B_2		MIN Ratio (x_B/x_K)
S_1	0	4	1	1	1	0		to be completed in the next step
S_2	0	2	1	-1	0	1		
Z	$Z = C_B X_B$		$\Delta_1 = -3$ \uparrow	$\Delta_2 = -2$	$\Delta_3 = 0$	$\Delta_4 = 0$ \downarrow		$\Delta_j = z_j - c_j$

Step 3:

Test for optimality:

Now, proceed to test basic feasible for optimality by the rules given below. This is done by computing the "net evaluation" D_j for variable x_j by the formula

$$\begin{aligned} \Delta_j &= z_j - c_j \\ &= C_B X_j - C_j \end{aligned} \quad \dots(1)$$

Thus

$\Delta_1 = C_B X_1 - C_1$ $= (0,0)(1,1) - 3$ $= (0 \times 1 + 0 \times 1) - 3$ $= -3$	$\Delta_2 = C_B X_2 - C_2$ $= (0,0)(1,-1) - 2$ $= (0 \times 1 + 0 \times -1) - 2$ $= -2$	$\Delta_3 = C_B X_3 - C_3$ $= (0,0)(1,0) - 0$ $= (0 \times 1 + 0 \times 0) - 0$ $= 0$
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Optimality Test:

- (i) If all $\Delta_j \geq 0$, the solution under test will be optimal.
- (ii) If at least one Δ_j is negative, the solution under test is not optimal, then proceed to improve the solution in step 4.
- (iii) If corresponding to most negative Δ_j , all elements of the column X_j are negative or zero (≤ 0), then the solution under test will be unbounded

Applying this rule for testing the optimality of starting basic feasible solution, it is observed that Δ_1 , and Δ_2 both are negative. Hence proceed to improve this solution in step 4.

Step 4:

In order to improve this basic feasible solution, the vector or entering the basis matrix and the vector to be removed from the basis matrix are determined by the following rules, such vectors are usually named as "incoming vector" and "outgoing vector" respectively.

"Incoming Vector":

The incoming vector X_k is always selected corresponding to the most negative value of Δ_j . (say Δ_k). Here $\Delta_k = \text{Min} (\Delta_1, \Delta_2) = \text{Min} [-3, -2] = -3 = \Delta$.

Therefore $k = 1$ and column vector x_1 , must enter the basis matrix. The column x_1 is marked by an upward arrow (\uparrow).

"Outgoing Vector":

The outgoing vector β_r is selected corresponding to the minimum ratio of elements of X_B by the corresponding positive elements of predetermined incoming vector X_k . This rule is called the Minimum Ratio Rule. In mathematical form, this rule is written as,

$$\frac{x_{\beta_r}}{x_{r_k}} = \text{Min} \left[\frac{x_{\beta_r}}{x_{i_k}}, x_{i_k} > 0 \right]$$

or

$$\frac{x_{\beta_r}}{x_{r1}} = \text{Min} \left[\frac{x_{\beta1}}{x_{11}}, \frac{x_{\beta2}}{x_{21}} \right]$$

$$= \text{Min} \left[\frac{4}{1}, \frac{2}{1} \right]$$

\therefore

$$\frac{x_{\beta_r}}{x_{r1}} = \frac{2}{1} = \frac{x_{\beta2}}{x_{21}}$$

Comparing both sides of this equation, $r = 2$ so the vector B_2 marked with downward arrow (\downarrow) should be removed from the basis matrix.

Now starting table (2) is modified to table (3)

Table 3

		$C_j \rightarrow$					
		3	2	0	0		
Basic Variables	c_B	X_B	x_1	x_2	$s_1 (=x_3)$ (β_1)	$s_2 (=x_4)$ (β_2)	MinRatio (x_B/x_1)
s_1	0	4	1	1	1	0	$4/1 = 4$
s_2	0	2	$\leftarrow \boxed{1}$ key element	-1	0	1	$2/1 = 2$ main rate
z	$z = C_B X_B = 0$		-3	-2	0	0	$\leftarrow \Delta_j = z_j - c_j$
			\uparrow			\downarrow	

Step 5:

In order to bring $B_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in place of incoming vector $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ unity must occupy the marked ' \square ' position and zero at all other places of X_1 . If the number in the marked ' \square ' position is other than unity, divide all elements of that row by the 'key element' (the element at the intersection of minimum ratio arrow (\leftarrow) and incoming vector arrow (\uparrow) is called the key element). Then subtract appropriate multiples of this new row from the other remaining rows, so as to obtain zero in the remaining position of the column X_1 .

Thus, the process can be fortified by simple matrix transformation as follow:

The intermediate coefficient matrix is:

	X_B	X_1	X_2	X_3	X_4	
R_1	4	1	1	1	0	
R_2	2	1	-1	0	1	
R_3	$Z=0$	3	2	0	0	$\leftarrow \Delta_j$

Apply $R_1 \rightarrow R_1 - R_2$
 $R_3 \rightarrow R_3 + 3R_2$

	X_B	X_1	X_2	X_3	X_4	
	2	0	2	1	0	
	2	1	-1	0	1	
	$z=6$	0	$1-5$	0	3	$\leftarrow \Delta_j$

Table 4:

Now construct the improved simple table as follows:

		$C_j \rightarrow$					
			3	2	0	0	
Basic Variables	C_B	X_B ($=B_j$)	X_1	X_2	s_1 ($=X_3$) B_2	s_2 ($=X_4$)	Mln Ratio ($X_B/X_2, X_2>0$)
s_1	0	2	0	$\leftarrow \boxed{2}$	1	1	$\frac{2}{2} = 1 \leftarrow \text{key row}$
x_1	3	2	1	-1	0	1	$2/-1 = -2$ negative Ratio is not considered
z	$z = C_B X_B$ $= 6$		0	$\boxed{-5}$ $\uparrow \text{key column}$	0	3	

From this table, the improved basic feasible solution is read as:

$$x_1 = 2, x_2 = 0, s_1 = 2, s_2 = 0$$

The improved value of $Z = 6$

Thus the optimal solution is obtained as

$$x_B = 3, x_2 = 1, \max z = 11$$

Step 6:

Now repeat step 3 through 5 as and when needed until an optimum solution is obtained in table 5.

$$\Delta_k = \text{Most negative } \Delta_j = -5 = \Delta_2$$

$\therefore k = 2$ and Hence X_2 should be entering vector (Key column)
By minimum ratio ratio

$$\text{Minimum Ratio} \left(\frac{X_B}{X_2}, x_2 > 0 \right) = \text{Min} \left(\frac{2}{2}, - \right)$$

Since negative Ratio is not counted,] so the second ratio is not considered.

Since first ratio is minimum, remove the first vector p , form the basis matrix. Hence the key element is 2.

Developing the first Row by the key elements 2, the intermediate coefficient matrix is obtained as

	X_B	X_1	X_2	X_3	X_4	
R_1	1	0	1	1/2	1/2	
R_2	2	1	-1	0	1	
R_3	$z=6$	0	-5	0	3	$\leftarrow \Delta_j$

Applying $R_2 \rightarrow R_2 + R_1$

$R_3 \rightarrow R_3 + R_1$

	X_B	X_1	X_2	X_3	X_4	
R_1	1	0	1	1/2	-1/2	
R_2	3	1	0	1/2	1/2	
R_3	$z=11$	0	0	5/2	1/2	$\leftarrow \Delta_j$

Now construct the next improved simplex table as follow :

Final Simplex Table 5 :

	C_j	\rightarrow	3	2	0	0	
Basis Variables	C_B	X_B	$X_1 (= \beta_1)$	$X_2 (= \beta_2)$	S_1	S_2	
$\rightarrow x_2$	2	1	0	1	1/2	-1/2	
x_1	3	3	1	0	1/2	1/2	
z	$z = C_B X_B = 11$		0	0	5/2	1/2	$\leftarrow \Delta_j$

This solution as read from this table is $x_1 = 3$, $x_2 = 2$, $S_1 = 0$, $S_2 = 0$ and $Z = 11$.

Also using the formula $\Delta_j = C_B X_j - C_j$ verify that all Δ_j 's are non-negative. Hence the optimum.

$$x_1 = 3$$

$$x_2 = 2$$

$$\text{Max } Z = 11$$

Simple Way for Simplex Computations:

Complete solution with its different computational steps can be more conveniently represented by a single table (6).

			$C_j \rightarrow$				3	3	0	0	
Basic Variable	C_B	X_B	X_1	X_2	S_1	S_2	$Min\ Ratio = X_B/X_k$				
S_1	0	4	1	1	1	0	4/1				
$\leftarrow S_2$	0	2	$\leftarrow \boxed{1}$...	-1	...	0	...	1	...	2/1 $\leftarrow Min$
$x_1 = x_2 = 0 \quad z = C_B X_B = 0$			-3	-2	0	0	0 \downarrow	$\Delta_j = Z_j - C_j$			
$\leftarrow S_1$	0	$\frac{R_1 \rightarrow R_1 - R_2}{2}$	0	$\leftarrow \boxed{2}$...	1	...	-1	...	2/2 = 1 Min	
$\rightarrow x_1$	3	2	1	-1	0	1	X				
$x_2 = S_2 = 0 \quad Z = C_B X_B = 0$			0	-5 \uparrow^*	0	3	$\leftarrow \Delta_j$				
x_2	2	1	0	1	1/2	-1/2					
x_3	3	3	1	0	1/2	+1/2					
$R_2 \rightarrow R_2 + \frac{R_1}{2}$											
$S_1 = S_2 = 0 \quad Z = C_B X_B = 11$			0	0	5/2	1/2	$\leftarrow All\ Z_j \geq 0$				

Thus the optimal solution is obtained as:

$$X_1 = 3, x_2 = 1, \max z = 11$$

Similarly for ,

Minimize $z = x_2 - 3x_3 + 2x_5$

Subject to $3x_2 - 2x_3 + 2x_5 \leq 7$

$$-2x_2 + 4x_3 \leq 12$$

$$-4x_2 + 3x_3 + 8x_5 \leq 10$$

$$x_2, x_3, x_5, 0$$

Solution : Equivalently $\max z^1 = -x_2 + 3x_3 - 2x_5$

where $z^1 = -z$, Introducing x_1, x_4 , and x_6 , as slack variable, the constraint equation becomes:

$$x_1 + 3x_2 - x_3 + 0x_4 + 2x_5 + 0x_6 = 7$$

$$0x_1 - 2x_2 + 4x_3 + 0x_4 + 0x_5 + 0x_6 = 12$$

$$0x_1 - 4x_2 + 3x_3 + 0x_4 + 8x_5 + x_6 = 10$$

Now proceeding as in above example the simplex computation are performed as follows :

		$C_j \rightarrow 0 \quad -1 \quad 3 \quad 0 \quad -2 \quad 0$							
Basic Variables	C_B	X_B	X_1	X_2	X_3	X_4	X_5	X_6	Min Ratio
x_1	0	7	1	3	-1	0	2	0	
x_4	0	12	0	-2	-1 key element	... 1 ...	0 ...	0	12/4=3 min
x_6	0	10	0	-4	3	0	8	1	10/3
$x_2 = x_3 = x_5 = 0$	$z^1 = C_B X_B = 0$		0	1	-3 \uparrow	0 \downarrow	2	0	$\leftarrow \Delta_1$
x_1	0	10	1	5/2 key element	... 0 ...	1/4 ...	2 ...	0 ...	4 \leftarrow min
x_3	3	3	0	-1/2	1	1/4	0	0	-
x_6	0	1	0	-5/2	0	-3/4	8	1	-
$x_2 = x_4 = x_5 = 0$	$z^1 = C_B X_B = 9$		0	-1/2* \uparrow	0	3/4	2	0	$\leftarrow \Delta_1$
x_2	-1	4	2/5	1	0	1/10	4/5	0	
x_3	3	5	1/5	0	1	3/10	2/5	0	
x_6	0	11	1	0	0	-1/2	10	1	
$x_1 = x_4 = x_5 = 0$	$z^1 C_B$ $X_B = 0 \quad z = -11$		1/5	0	0	4/5	12/5	0	$\leftarrow \Delta_1$

Thus optimal solution is $x_2 = 4, x_3 = 5, x_6 = 11$

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