Let $x(t)$ be a real valued signal (i.e. a function of time). Let $h(t)$ denote its Hilbert transform, defined as:

Eq. 1 $$h(t)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{x(t)}{\tau-t}d\tau$$

where the principle value of the integral is used.

The complex analytic signal of $x(t)$ is defined as:

Eq. 2 $$z(t)=x(t)+ih(t)$$

where $\iota=\sqrt{-1}$.

There are two important aspects to discuss here:
1. A practical, numerical algorithm for computing the Hilbert transform and analytic signal for a given discrete time series.
2. The properties of the Hilbert transform and analytic signal that make them useful in time series analysis.

In some literature, $z(t)$ in Eq. 2 not only goes by the name of analytic signal, but also by the name of pre-envelope or complex envelope.

The envelope is defined by some as the modulus $|z(t)|$, and its squared value gives the instantaneous power of the signal. The phase of $z(t)$ gives the instantaneous phase of the signal. And the derivate with respect to time of the phase of $z(t)$ is the instantaneous frequency of the signal.

One of the properties is that the original time series $x(t)$ and the Hilbert transform $h(t)$ are orthogonal:

Eq. 3 $$\int_t x(t)h(t)dt=0$$

The numerical algorithm for computing the analytic signal follows:
1. Given the discrete time series x_t, for $t=0...N_T-1$.
2. Compute its discrete Fourier transform:

Eq. 4 $$y_\omega=\sum_{t=0}^{N_T-1}x_t\exp\left(-i2\pi\omega t/N_T\right)$$

3. Define:

Eq. 5 $$u_\omega=\begin{cases}y_\omega\ ,\ for\ \ \omega=0\\ 2y_\omega\ ,\ for\ \ \omega=1...\left(N_T/2-1\right)\\ y_\omega\ ,\ for\ \ \omega=N_T/2\\ 0\ ,\ for\ \ \omega=\left(N_T/2+1\right)...\left(N_T-1\right)\end{cases}$$

4. Compute its inverse discrete Fourier transform:

Eq. 6 $$z_t=\frac{1}{N_T}\sum_{\omega=0}^{N_T-1}u_\omega\exp\left(+i2\pi\omega t/N_T\right)$$

which yields the complex discrete-time “analytic” signal.

Note that the negative frequencies were set to zero in Eq. 5, which is part of the definition of the Hilbert transform. And also note that the complex valued frequencies were multiplied by two, thus conserving the total energy (Paseval's theorem).

Consider the computation of the DHT (discrete Hilbert transform) and the analytic signal within some predefined frequency band. Then the natural way to do this is to set to zero the frequencies outside the band of interest in Eq. 5, and then invert the FT (Eq. 6).

However, note that if one filters the signal to a single discrete frequency, then the instantaneous amplitude, phase, and frequency are deterministic over time:

Eq. 7
$$\left\{ \begin{array}{l} H(\sin) = -\cos \\ H(\cos) = \sin \end{array} \right\}$$

Using general notation, the instantaneous phase at time *t* is:

Eq. 8
$$\phi = \arctan\frac{\mathrm{Im}}{\mathrm{Re}} = \arctan\frac{x}{y}$$

where *x* is the imaginary part of the analytic signal z_t from Eq. 6, and *y* is the real part.

The instantaneous frequency is defined as:

Eq. 9
$$\omega = \frac{1}{2\pi}\frac{d\phi}{dt}$$

Note that:

Eq. 10
$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial x}\dot{x} + \frac{\partial\phi}{\partial y}\dot{y}$$

Recall that:

Eq. 11
$$\frac{d}{du}\arctan(u) = \frac{1}{1+u^2}$$

Which all taken together gives:

Eq. 12
$$\frac{d\phi}{dt} = \frac{y\dot{x} - x\dot{y}}{y^2 + x^2}$$

The next pages correspond to the main publication on the practical computation of the discrete Hilbert transform.

Computing the Discrete-Time "Analytic" Signal via FFT

S. Lawrence Marple, Jr.

Abstract— **Starting with a real-valued N-point discrete-time signal, frequency-domain algorithms are provided for computing**

1) **the complex-valued *standard* N-point discrete-time "analytic" signal of the same sample rate;**
2) **the complex-valued *decimated* $N/2$-point discrete-time "analytic" signal of half the original sample rate;**
3) **the complex-valued *interpolated* NM-point discrete-time "analytic" signal of M times the original sample rate.**

Special adjustment of transform end points are shown to be necessary in order to generate proper discrete-time "analytic" signals.

I. Introduction

In recognition of the recent 50th anniversary since the introduction of the continuous-time analytic signal by Gabor [4] and Ville [20], this correspondence presents transform-based algorithmic techniques to compute an analytic-like signal in the finite duration discrete-time case. There are pitfalls in applying the continuous-time analytic signal properties to the discrete-time situation. Applying appropriate scaling factors to account for the periodicity of the discrete-time signal spectrum will be shown to avoid the pitfalls.

A real-valued continuous-time signal has the property that its Fourier transform (spectrum) is complex symmetric. Thus, the negative frequency half of the signal spectrum contains redundant information with respect to the positive frequency half. The objective of Gabor and Ville in creating the analytic signal was to remove this spectral redundacy by deleting the negative frequency half of the signal transform. The resultant complex-valued signal with one-sided spectrum preserves all information as contained in the original real-valued signal. The analytic signal has proven to be more than just an academic curiousity. Use of the analytic signal in lieu of the original real-valued signal for many signal processing applications has been demonstrated to mitigate estimation biases and to eliminate undesirable cross-term artifacts between negative and positive frequency components that would occur had the original signal been used. These applications include spectral analysis [8], [9], [10], [19], linear and quadratic time-frequency signal analysis such as the Wigner transform [1], time-delay estimation [11], and estimation of the envelope amplitude of a signal in AGC loops of high-end digital receivers.

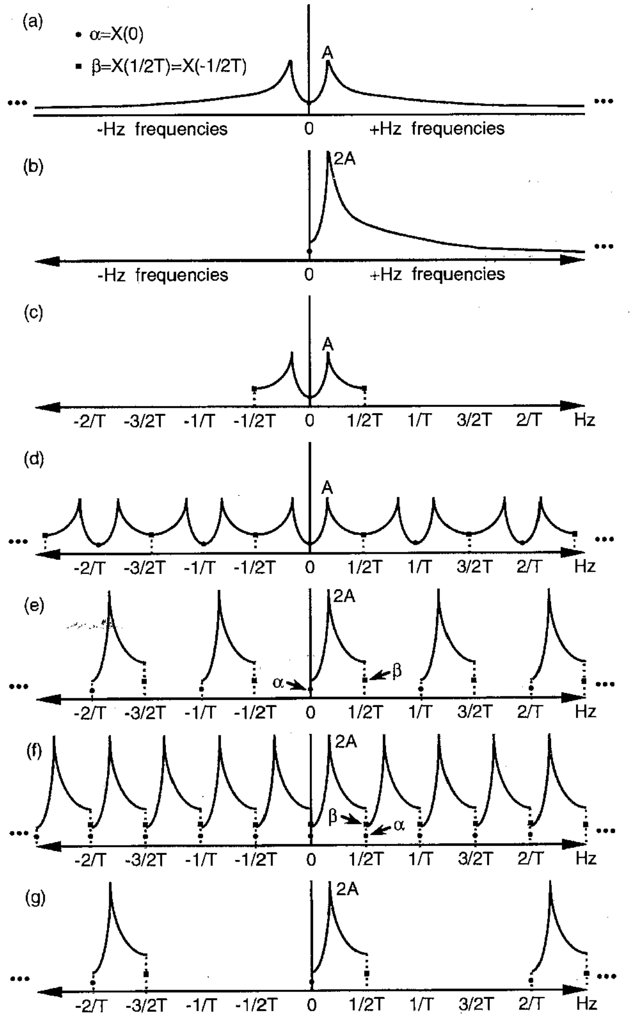

Fig. 1. Continuous-time and discrete-time magnitude spectra for real and analytic signals. (a) Symmetric spectrum of typical nonbandlimited real continuous-time signal. (b) One-sided spectrum of continuous-time analytic signal for (a). (c) Symmetric spectrum of bandlimited real continuous-time signal. (d) Periodic spectrum of real discrete-time signal after sampling (period T) bandlimited signal of (c). (e) Periodic spectrumof *standard* discrete-time "analytic" signal for (d). (f) Periodic spectrum of *decimated* discrete-time "analytic" signal for (d) of half sample rate. (g) Periodic spectrum of *interpolated* discrete-time "analytic" signal for (d) of twice sample rate.

II. Continuous-Time Analytic Signal

Let $x(t)$ be a real-valued finite energy signal defined over the temporal interval $-\infty < t < \infty$ with *continuous-time Fourier transform* (CTFT) [10, ch. 2]

$$X(f) = \int_{-\infty}^{\infty} x(t)\exp(-j2\pi ft)\,dt \tag{1}$$

defined over the frequency interval $-\infty < f < \infty$. Because $x(t)$ is real, the CTFT is complex conjugate symmetric $X(-f) = X^*(f)$. The CTFT magnitude spectrum $|X(f)|$ symmetry and infinite frequency extent for a typical nonbandlimited real-valued signal is depicted in Fig. 1(a).

The continuous-time analytic signal $z(t)$ corresponding to $x(t)$ is most simply defined in the frequency domain as

$$Z(f) = \begin{cases} 2X(f), & \text{for } f > 0 \\ X(0), & \text{for } f = 0 \\ 0, & \text{for } f < 0 \end{cases} \tag{2}$$

which is inverse transformed to obtain $z(t)$. Note that the value of $Z(f)$ at $f = 0$ is defined here to reflect the mathematical behavior of the Fourier transform at a discontinuity, which yields the average of the values on either side of the discontinuity; some literature, however, defines $Z(0) = 2X(0)$. Fig. 1(b) depicts the magnitude spectrum of the analytic signal corresponding to Fig. 1(a).

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The mathematical term *analytic* is applied to $z(t)$ because it is an analytic function of a continuous complex variable.

Due to its one-sided spectral definition, the analytic signal will necessarily be complex-valued and can therefore be represented in terms of its "real" and "imaginary" components $z(t) = z_r(t) + jz_i(t)$, for which $z_r(t) = \mathrm{Re}\{z(t)\}$ and $z_i(t) = \mathrm{Im}\{z(t)\}$ are both real-valued functions. It can be shown [4], [7], [16] that

$$z_r(t) = x(t) \quad \text{and} \quad z_i(t) = \mathrm{HT}\{x(t)\} \tag{3}$$

in which HT designates the Hilbert transform operation, which is a two-sided ($-\infty < t < \infty$) time-domain convolution of $x(t)$ with the function $\frac{1}{\pi t}$ [16, Sec. 7.4]. The Fourier transform of the real component is $Z_r(f) = X(f)$, which is a conjugate symmetric (even) function. The Fourier transform of the imaginary component is

$$Z_i(f) = \begin{cases} X(f), & \text{for } f > 0 \\ 0, & \text{for } f = 0 \\ -X^*(-f), & \text{for } f < 0 \end{cases}$$

which is a conjugate antisymmetric (odd) function. Combining $Z_r(f)$ and $Z_i(f)$ then yields the definition of (2).

Two equivalent approaches for the creation of a continuous-time analytic signal from a real-valued signal are therefore possible, specifically, a frequency-domain approach by forming the one-sided spectrum of (2) and a time-domain filtering approach via the Hilbert transform of (3). The analytic signal created by either of these approaches has one important property, specifically, the orthogonality between the real and imaginary components of the analytic signal

$$\int_{-\infty}^{\infty} z_r(t) z_i(t)\, dt = 0. \tag{4}$$

III. Discrete-Time "Analytic" Signal Properties

We next consider the properties appropriate for an analytic-like discrete-time signal $\mathrm{z}[n]$ corresponding to a real-valued discrete-time signal $\mathrm{x}[n]$ of finite duration. We assume that $\mathrm{x}[n]$ for $0 \le n \le N-1$ is obtained by sampling a bandlimited real-valued continuous-time signal $x(nT) = \mathrm{x}[n]$ at periodic time intervals of T seconds, in which T is selected to prevent aliasing. The spectrum of a bandlimited continuous-time signal is illustrated in Fig. 1(c). Note that the spectral magnitude at the foldover frequencies ($\pm 1/2T$ Hz) has been exaggerated in Fig. 1(c) in order to illustrate the properties being presented here. Actual continuous-time signals are normally passed through antialiasing filters that attenuate the spectral response such that $X(\pm 1/2T) \approx 0$.

The spectrum of the discrete-time signal $\mathrm{x}[n]$ is obtained from the *discrete-time Fourier transform* (DTFT) [10, p. 40]

$$\mathrm{X}(f) = T \sum_{n=0}^{N-1} \mathrm{x}[n] \exp(-j2\pi f n T) \tag{5}$$

which has a periodic structure, as shown in Fig. 1(d). Note that the DTFT is identical to the CTFT $X(f)$ over the baseband interval covering $|f| \le 1/2T$ Hz. The DTFT is evaluated with a fast Fourier transform (FFT), which computes the DTFT at a discrete set of frequencies. $\mathrm{X}(f_m) = \mathrm{X}[m]$, for which $f_m = m/NT$ Hz, and $0 \le m \le N-1$.

There are two properties we wish to satisfy in order for $\mathrm{z}[n] = \mathrm{z}_r[n] + j\mathrm{z}_i[n]$ to be an analytic-like discrete-time signal. First, the real part must exactly yield the original discrete-time sequence

$$\mathrm{z}_r[n] = \mathrm{x}[n] \quad \text{for } 0 \le n \le N-1. \tag{6}$$

Second, the real and imaginary components must be orthogonal over the finite interval

$$T \sum_{n=0}^{N-1} \mathrm{z}_r[n] \mathrm{z}_i[n] = 0. \tag{7}$$

Now, consider three cases of the analytic-like discrete-time signal that differ in their sample rates.

IV. Computing Standard Discrete-Time "Analytic" Signal of SameSample Rate

A formal analytic signal is a complex-valued continuous-time function with a Fourier transform that vanishes for negative frequencies. A complex sequence, such as the discrete-time signal $\mathrm{z}[n]$, cannot be considered in a formal mathematical sense to be an analytic function since it is a function of an integer variable rather than a continuous variable. In addition, the periodic structure of the DTFT spectrum [see Fig. 1(d)] will prevent the spectrum from vanishing for all negative frequencies. However, due to the legacy of the term *analytic* in the continuous-time signal processing literature, we shall refer to the discrete-time "analytic" signal in quotes to alert the reader that $\mathrm{z}[n]$ simply shares some similar properties as the continuous-time analytic signal $z(t)$ in the signal processing sense, even though it is not an analytic function in the mathematical sense.

A time-domain complex filtering approach to generate an approximation to a discrete-time "analytic" signal from a real-valued discrete-time signal has been suggested by Oppenheim and Schafer [15, Sec. 10.4.1 and Fig. 10.10] and a design procedure described by Reilly *et al.* [17]. The approach essentially uses dual quadrature FIR filters to jointly produce $\mathrm{z}_r[n]$ and $\mathrm{z}_i[n]$. The quadratic filter approach will satisfy the orthogonality property (7) but will not preserve the original data values (6). Another frequently used approach [17] applies only a single FIR filter that approximates the Hilbert transform of (3), thereby generating $\mathrm{z}_i[n]$; $\mathrm{z}_r[n]$ is obtained simply by equating to $x[n]$. This alternative approach thus preserves the original data values (6), but the orthogonality property (7) will not be satisfied.

An alternative frequency-domain approach based on (2) is proposed here to create a one-sided periodic spectrum in which the negative frequency half of each spectral period is set to zero, yielding the periodic one-sided spectrum illustrated in Fig. 1(e). Assuming N is even, the specific procedure for creating a complex-valued N-point discrete-time "analytic" signal from a real-valued N-point discrete-time signal, which preserves the original sample rate and satisfies properties (6) and (7), is as follows.

- Compute the N-point DTFT $\mathrm{X}[m]$ (5) using an FFT of the N real data samples
- Form the N-point one-sided discrete-time "analytic" signal transform

$$\mathrm{Z}[m] = \begin{cases} \mathrm{X}[0], & \text{for } m = 0 \\ 2\mathrm{X}[m], & \text{for } 1 \le m \le \frac{N}{2} - 1 \\ \mathrm{X}[\frac{N}{2}], & \text{for } m = \frac{N}{2} \\ 0, & \text{for } \frac{N}{2} + 1 \le m \le N - 1. \end{cases} \tag{8}$$

- Compute, using an N-point inverse DTFT

$$\mathrm{z}[n] = \frac{1}{NT} \sum_{m=0}^{N-1} \mathrm{Z}[m] \exp(+j2\pi mn/N)$$

to yield the complex discrete-time "analytic" signal of same sample rate as the original signal $\mathrm{x}[n]$.

Assuming a real data vector x of N data values, the Matlab code shown at the bottom of the next page computes the standard discrete-time "analytic" signal. Based on (2), it would seem that $\mathrm{Z}[\frac{N}{2}]$, which corresponds to the foldover frequency $1/2T$ Hz, should be defined as $2\mathrm{X}[\frac{N}{2}]$ rather than as $\mathrm{X}[\frac{N}{2}]$. However, omitting the factor of 2 is required in order to satisfy properties (6) and (7). To illustrate, consider the eight-point real data vector $\mathrm{x} = [4, 2, -2, -1, 3, 1, -3, 1]$. Using $\mathrm{Z}[\frac{N}{2}] = \mathrm{X}[\frac{N}{2}]$ in (8) yields $\mathrm{z} = [4 - j0.396, 2 + j3, -2 + j1.811, -1 - j2.293, 3 - j1.104, 1 + j3, -3 - j0.311, 1 - j3.707]$ as the "analytic" data vector (note that the real part is equal to the original data), and the orthogonality sum (7) evaluates to 0. Using $\mathrm{z}[\frac{N}{2}] = 2\mathrm{x}[\frac{N}{2}]$ yields $\mathrm{z} = [3.875 - j0.396, 2.125 + j3, -2.125 + j1.811, -0.875 - j2.293, 2.875 - j1.104, 1.125 + j3, -3.125 - j0.311, 1.125 - j3.707]$ as the "analytic" data vector (note that the real part is *not* equal to the original data). One intuitive justification for choosing (8) to define the discrete-time "analytic" signal transform is noting that 0 Hz and Nyquist frequency transform terms are shared boundaries between negative and positive frequency halves of the periodic spectrum, and division into respective one-sided positive and negative spectra requires that these terms be split between the two spectral halves.

A formal justification for (8) is by construction based on the even and odd function properties analogous to those noted in Section II for the real and imaginary component transforms of $Z(f)$. For the N-point DTFT, the discrete-time versions have the form $\mathrm{Z}_r[m] = \mathrm{X}[m]$ for $0 \le m \le N-1$ and

$$\mathrm{Z}_i[m] = \begin{cases} 0, & \text{for } m = 0 \\ \mathrm{X}[m], & \text{for } 1 \le m \le \frac{N}{2} - 1 \\ 0, & \text{for } m = \frac{N}{2} \\ -\mathrm{X}[m], & \text{for } \frac{N}{2} + 1 \le m \le N - 1. \end{cases}$$

The term $\mathrm{Z}_i[\frac{N}{2}]$ is zero in order to satisfy both the odd function property and the periodicity property, in which the value at $m = \frac{N}{2}$ is shared between the positive and negative portions of the odd function period. Combining $\mathrm{Z}_r[m]$ and $\mathrm{Z}_i[m]$ then yields (8).

V. Computing Decimated Discrete-Time "Analytic" Signal of Half Sample Rate

It is usually desirable in digital hardware implementations of digital signal processing operations to use the lowest sample rate consistent with preservation of the signal information without aliasing. As there is no signal information contained within each zero region of the standard discrete-time "analytic" signal of Fig. 1(e), it is apparent that the sample rate can be reduced by a factor of one-half and still preserve the signal information. By decimating by a factor of two, the periodic spectrum illustrated in Fig. 1(f) will be the result.

Assuming N is even, the specific procedure for creating a complex-valued decimated $\frac{N}{2}$-point discrete-time "analytic" signal from a real-valued N-point discrete-time signal, which represents a decimation of the original sample rate from $1/T$ Hz to $1/2T$ Hz and satisfies properties (6) and (7), is as follows:

- Compute the N-point DTFT $\mathrm{X}[m]$ (5) using an FFT of the N real data samples.
- Form the $\frac{N}{2}$-point one-sided discrete-time "analytic" signal transform

$$\mathrm{Z}[m] = \begin{cases} \mathrm{X}[0] + \mathrm{X}[\frac{N}{2}], & \text{for } m = 0 \\ 2\mathrm{X}[m], & \text{for } 1 \le m \le \frac{N}{2} - 1. \end{cases} \tag{9}$$

- Compute the $\frac{N}{2}$-point inverse DTFT using a FFT and scale by factor $\frac{1}{2}$ to yield the decimated discrete-time "analytic" signal of half the original sample rate.

Assuming a real data vector x of N data values, the following Matlab code computes the decimated discrete-time "analytic" signal:

```
N=length(x); X=fft(x,N);
z_dec=(1/2)*ifft([X(1)+X(N/2+1);2*X(2:N/2)],N/2);
```

When decimating by a factor of two, the Nyquist frequency term $\mathrm{X}[\frac{N}{2}]$ is aliased to 0 Hz, thus causing the addition shown in (9) for $\mathrm{Z}[0]$. Definition (9) reminds us that $X[0]$ and $X[\frac{N}{2}]$ are aliased to the same frequency component $Z[0]$ in the decimated case; therefore, the original $x[n]$ would not be recoverable by an inverse transform procedure applied to $Z[m]$. However, in practice, $\mathrm{X}[\frac{N}{2}] \approx 0$ due to antialiasing filtering prior to sampling and, therefore, $Z[0] \approx X[0]$; therefore, the decimated "analytic" signal will permit recovery of $x[n]$. This procedure positions the discrete-time 'analytic' signal baseband center to $1/4T$ Hz. An alternative approach based on complex demodulation of a bandpass signal centers the complex signal baseband at 0 Hz rather than at $1/4T$ Hz, producing a spectra like Fig. 1(f) shifted left by $1/4T$ Hz.

To interpolate $z[n]$ back to the original sample rate in order to recover $x[n]$, use the interpolation procedure presented in the next section for $M = 2$.

VI. Computing Interpolated Sample Rate Discrete-Time "Analytic" Signal

There are situations in which additional interpolated values of the discrete-time "analytic" signal are required, such as the case of time-delay estimation [11] in which a peak position must be determined with finer temporal resolution than that of the original sample interval. Using interpolating by $M = 2$, for example, the periodic one-sided spectrum illustrated in Fig. 1(g) may be obtained.

Trigonometric interpolation of any finite N-point real or complex discrete-time signal via FFT was first proposed by Gold and Rader [5, pp. 199–200] as a fast algorithmic alternative to trapezoidal numerical integration of the CTFT, such as described by Hamming [6]. The interpolation procedure of Gold and Rader [5], which is frequently cited in other texts [2, p. 199], [18, p. 216], consists of three steps:

1) computing an N-point FFT;
2) dividing the FFT results in half at the Nyquist frequency term and inserting $N(M-1)$ zeros between the two halves to create a stretched transform sequence,
3) computing an NM-point inverse FFT to yield the resultant interpolated signal sequence.

However, this frequency-domain interpolation procedure produces incorrect results, as first reported by Nuttall [12], [13] and later rediscovered by Fraser [3]. First, the original data values are not preserved by the interpolation. Second, the interpolated signal samples of a real-valued sample sequence are complex-valued when strictly real-valued interpolated samples are expected.

The correct trigonometric interpolation procedure [3], [12], [13] for interpolating from N samples to NM samples divides the Nyquist frequency transform term using the following three-step approach:

- Compute the N-point DTFT $\mathrm{X}[m]$ using an FFT of N real or complex data values
- Form the NM-point split and zero-padded stretched transform

$$\mathrm{Y}[m] = \begin{cases} \mathrm{X}[m], & \text{for } 0 \le m \le \frac{N}{2} - 1 \\ \frac{1}{2}\mathrm{X}[\frac{N}{2}], & \text{for } m = \frac{N}{2} \\ 0, & \text{for } \frac{N}{2} + 1 \le m \le NM - \frac{N}{2} - 1 \\ \frac{1}{2}\mathrm{X}[\frac{N}{2}], & \text{for } m = NM - \frac{N}{2} \\ \mathrm{X}[k], & \text{for } \frac{N}{2} + 1 \le k \le N - 1 \\ & \text{and } m = k + N(M-1) \end{cases} \tag{10}$$

- Compute the NM-point inverse DTFT using a FFT and scale by M.

```
N=length(x); X=fft(x,N);
z_std=ifft([X(1);2*X(2:N/2);X(N/2+1);zeros(N/2-1,1)],N);
```

```
N=length(x); X=fft(x,N);
y=M*ifft([X(1:N/2);X(N/2+1)/2;zeros(N*(M-1)-1,1);X(N/2+1)/2;X(N/2+2:N)],M*N=);
```

```
N=length(x); X=fft(x,N);
z_int=M*ifft([X(1);2*X(2:N/2);X(N/2+1);zeros(M*N-N/2-1,1)],M*N);
```

The first Matlab code shown at the top of the page computes a trigonometrically interpolated discrete-time signal $y[n]$ from $x[n]$.

Assuming N is even and considering the issues that led to (8) and (10), the procedure for creating a complex-valued interpolated NM-point discrete-time "analytic" signal from an N-point real signal, which represents an interpolation of the original sample rate of $1/T$ Hz by a factor of M to M/T Hz and also satisfies properties (6) and (7), is as follows:

- Compute the N-point DTFT $\mathrm{X}[m]$ (5) using an FFT of the N real data samples.
- Form the NM-point one-sided discrete-time "analytic" signal transform

$$\mathrm{Z}[m] = \begin{cases} \mathrm{X}[0], & \text{for } m = 0 \\ 2\mathrm{X}[m], & \text{for } 1 \le m \le \frac{N}{2} - 1 \\ \mathrm{X}[\frac{N}{2}], & \text{for } m = \frac{N}{2} \\ 0, & \text{for } \frac{N}{2} + 1 \le m \le NM - 1 \end{cases} \tag{11}$$

- Compute the NM-point inverse DTFT using a FFT and scale by factor M to yield the interpolated discrete-time "analytic" signal of M times the original sample rate.

Assuming a real data vector x of N data values, the second Matlab code shown at the top of the page will compute the interpolated discrete-time "analytic" signal. Note that scaling by factor M compensates for the $\frac{1}{NMT}$ factor involved in the computation of the inverse DTFT.

VII. Conclusion

This correspondence has shown that we cannot simply use the continuous-time infinite-duration analytic signal formulation when creating a discrete-time analytic-like signal from a discrete-time finite-duration signal. When implemented using a frequency-domain approach, the DC and Nyquist frequency terms of the DFT require scaling by a factor of $1/2$, which is not required in or anticipated by the continuous-time signal case. This is a subtle difference that has eluded distinction in the DSP community.

Acknowledgment

The author would like to acknowledge the work of A. Nuttall on trigonmetric interpolation using the DFT [12], [13], which applied a similar subtle factor of $1/2$ to certain DFT terms as the inspiration for the insight that led to the results reported here.

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