17 Show that the distribution

$$F(z_1, z_2) = \exp\left[-\left\{z_1^{-1} + z_2^{-1} + (z_1z_2)^{-1}\right\}\right], \quad z_1, z_2 > 0,$$

has unit Fréchet margins. Is it max-stable? If not, what is the limiting distribution of appropriately rescaled componentwise maxima from F?
Setting $z_1 = \infty$ shows that the margin for z_2 is unit Fréchet, and the result for z_1 follows by symmetry. As $t \to \infty$,

$$F(tz_1, tz_2)^t = \exp\left[-\left\{z_1^{-1} + z_2^{-1} + (tz_1z_2)^{-1}\right\}\right] \to \exp\left[-\left\{z_1^{-1} + z_2^{-1}\right\}\right],$$

so it is not max-stable, and the limiting distribution for rescaled maxima is independence.

18 Derive (5.28) by noting that if the intensity function $\dot{\mu}$ exists, then

$$V(0) = V(z) + \int_0^{z_1} \cdots \int_0^{z_D} \dot{\mu}(x_1, \ldots, x_D)\, \mathrm{d}x_1 \cdots \mathrm{d}x_D.$$

If this argument makes you queasy, find a better one.
As $V(0) = \mu(\mathcal{E})$ is constant, differentiation by $z_{C+1}, \ldots, z_D$ yields

$$0 = \frac{\partial^{D-C}}{\partial z_{C+1} \cdots \partial z_D} V(z_1, \ldots, z_D) + \int_0^{z_1} \cdots \int_0^{z_C} \dot{\mu}(x_1, \ldots, x_C, z_{C+1}, \ldots, z_D)\, \mathrm{d}x_1 \cdots \mathrm{d}x_C,$$

which gives the required result on setting $z_1 = u_1, \ldots z_C = u_C$. Queasiness arises because the left-hand side and the integral on the right-hand side both equal infinity. Alternatively one can integrate the equality

$$\dot{\mu}(z_1, \ldots, z_D) = -\frac{\partial^D V(z_1, \ldots, z_D)}{\partial z_1 \cdots \partial z_D}$$

over the set $\{(z_1, \ldots z_D) : 0 \le z_d \le u_d, d = 1, \ldots, C\}$.

19 (a) Check the argument below (5.29), noting that the constraints $\sum_d w_d^* = \sum_d w_d = 1$ imply that

$$w_d^* = \frac{m_d w_d}{m_D + \sum_{c=1}^{D-1}(m_c - m_D)w_c}, \quad d = 1, \ldots, D-1.$$

and the Jacobian is

$$\frac{\partial w^*}{\partial w} = \frac{\partial(w_1^*, \ldots, w_{D-1}^*)}{\partial(w_1, \ldots, w_{D-1})}.$$

You may require the matrix determinant lemma $|A + ab^{\mathrm{T}}| = |A|(1 + b^{\mathrm{T}}A^{-1}a)$, where A is an invertible $p \times p$ matrix and a and b are $p \times 1$ vectors.
(b) Show that $\max_d m_d \ge m^{\mathrm{T}}w \ge \min_d m_d$ for all $w \in \mathcal{S}_{D-1}$, and give an algorithm for simulation from $\dot{\nu}$ by noting that

$$\dot{\nu}(w) \propto \Pr(U \le \min_d m_d/m^{\mathrm{T}}w \mid W = w) \times f(w),$$

where $f(w)$ is the density of the variable W defined in (5.29) and $U \sim U(0,1)$ is independent of W. What is the efficiency of your algorithm?
(Coles and Tawn, 1991, Theorem 2)
(a) If we write $w_d^* = m_d w_d/m^{\mathrm{T}}w$, where $m^{\mathrm{T}}w = \sum_{c=1}^{D-1}(m_c - m_D)w_c$, then

$$\begin{aligned}
\partial w_d^*/\partial w_d &= m_d/(m^{\mathrm{T}}w) - w_d(m_d - m_D)/(m^{\mathrm{T}}w)^2,\\
\partial w_d^*/\partial w_c &= m_d w_d(m_c - m_D)/(m^{\mathrm{T}}w)^2, \quad c \ne d.
\end{aligned}$$

Hence the Jacobian is the determinant of a matrix of the form $A + ab^{\mathrm{T}}$, where

$$\begin{aligned} A &= \operatorname{diag}(m_1, \ldots, m_{D-1})/(m^{\mathrm{T}} w), \\ a &= (m_1 w_1, \ldots, m_{D-1} w_{D-1})^{\mathrm{T}}, \\ b &= -(m_1 - m_D, \ldots, m_{D-1} - m_D)^{\mathrm{T}}/(m^{\mathrm{T}} w)^2. \end{aligned}$$

Now $|A| = (m^{\mathrm{T}} w)^{-(D-1)} \prod_{d=1}^{D-1} m_d$ and a little algebra gives $1 + b^{\mathrm{T}} A^{-1} a = m_D/(m^{\mathrm{T}} w)$, so the matrix determinant lemma gives $|\partial w^*/\partial w| = (m^{\mathrm{T}} w)^{-D} \prod_d m_d$. The probability density function for W is obtained on rewriting w^* in terms of w, and the rest of the argument is straightforward.
(b) As $m^{\mathrm{T}} w$ is a linear function of $w \in \mathcal{S}_{D-1}$, the inequalities follow at once. Let $m_{\min} = \min_d m_d$. A realisation of W is obtained by generating W^* from ν^* and then applying the transformation (5.29), but this has density f rather than $\dot\nu$. Now $m_{\min}/m^{\mathrm{T}} W \le 1$, so conditional on $W = w$, the event $U \le m_{\min}/m^{\mathrm{T}} w$ has probability $m_{\min}/m^{\mathrm{T}} w$, and thus the marginal density of those W for which this event occurs is proportional to $f(w)/(m^{\mathrm{T}} w)$ and must therefore be $\dot\nu$. The efficiency of the algorithm is the probability that a generated W is accepted, and this equals

$$\int \frac{m_{\min}}{m^{\mathrm{T}} w} f(w)\, \mathrm{d}w = D m_{\min} \int \dot\nu(w)\, \mathrm{d}w = D m_{\min},$$

which is greatest when all the m_d are equal to $1/D$, as might be expected; then $m^{\mathrm{T}} w = 1/D$ and hence $\dot\nu = \dot\nu^*$. The more the m_d vary, the less efficient the algorithm becomes.

20 Suppose that ε_1, ε_2, ε_3 are independent unit Fréchet variables, and let

$$Z_1 = \max\{\alpha\varepsilon_1, (1-\alpha)\varepsilon_2\}, \quad Z_2 = \max\{\beta\varepsilon_1, (1-\beta)\varepsilon_3\}, \quad 0 \le \alpha, \beta \le 1.$$

(a) Show that $\Pr(Z_1 \le z_1, Z_2 \le z_2)$ equals

$$\exp\{-(1-\alpha)/z_1 - (1-\beta)/z_2 - \max(\alpha/z_1, \beta/z_2)\}. \quad z_1, z_2 > 0,$$

and verify that the marginal distributions of Z_1 and Z_2 are unit Fréchet.
(b) Deduce that this joint distribution is bivariate extreme-value with discrete angular measure

$$\nu(\{0\}) = (1-\beta)/2, \quad \nu(\{1\}) = (1-\alpha)/2, \quad \nu(\{\alpha/(\alpha+\beta)\}) = (\alpha+\beta)/2,$$

and check that this has mean $1/2$ for all choices of α and β.
(c) Compute the extremal coefficient for this distribution, and discuss how it depends on the parameter values.
(d) Sketch the angular measure ν and investigate special cases corresponding to independence and complete dependence. Sketch typical datasets of values (z_1, z_2) corresponding to different possible choices of the parameters.

21 The multivariate asymmetric logistic extreme-value model has exponent function

$$V(z_1, \ldots, z_D) = \sum_{d=1}^{D} \frac{1-\psi_d}{z_d} + \sum_{d=1}^{D} \left\{ \left(\frac{\psi_d}{z_d} \right)^{1/\alpha} \right\}^{\alpha}, \quad z_1, \ldots, z_D > 0,$$

where $\psi_1, \ldots, \psi_D, \alpha \in (0, 1]$. Investigate how the placement of point masses and densities on corners and edges of $\mathcal{S}_{D-1}$ depends on the parameters, in the case $D = 3$.