Math 524 Homework 3

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1. Suppose $\sum a_n$ is an absolutely convergent series. Show that the trigonometric series $\sum a_n \sin nx$ is absolutely and uniformly convergent.

   Proof. We have $|\sin nx| \leqslant 1$ for all n and x. So, for all n and x,

   $$|a_n \sin(nx)| \leqslant |a_n|$$

   By the M-Weierstrass test, since $\sum a_n$ absolutely converges, so does $\sum a_n \sin nx$. QED

2. $$\sum_{n=0}^{\infty} a_n x^n = \frac{1}{2} + \frac{1}{3}x + \frac{1}{2^2}x^2 + \frac{1}{3^2}x^3 + dots$$

   (a) Show that the limit $\lim_{n\to\infty} |a_{n+1}/a_n|$ does not exist.

   $$a_{n+1}/a_n = \begin{cases} \frac{2}{3} & \text{for n odd} \\ \frac{3}{2} & \text{for n even} \end{cases}$$

   So the limit $\lim_{n\to\infty} |a_{n+1}/a_n|$ does not exist.

   (b) Use the Cauchy-Hadamard Thm to determine radius of convergence.

   $$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \frac{3}{2} \implies R = \frac{2}{3}$$

   So the radius of convergence is $\frac{2}{3}$.

3. If $0 < p \leqslant |a_n| \leqslant q$ for all $n \in \mathbb{N}$, find the radius of convergence of $\sum a_n x^n$.

   By Cauchy-Hadamard,

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}} \leqslant \frac{1}{q} \leqslant \frac{1}{p}$$

   So

$$R \leqslant \frac{1}{q} \leqslant \frac{1}{p}$$

   (I feel like there's more we can say about R)

4. Let $f(x) = \sum a_n x^n$ for $|x| < R$. If f is an even function on $(-R, R)$, show that $a_n = 0$ for all odd n.

   Proof. Since f is even, we have that $f(x) = f(-x)$. So

$$\begin{aligned} f(x) &= \sum a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots \\ f(-x) &= \sum a_n (-x)^n = a_0 - a_1 x + a_2 x^2 - a_3 x^3 + \dots \\ f \text{ even } &\implies a_1 = -a_1,\ a_3 = -a_3,\ \dots \\ &\implies a_{2k+1} = 0 \end{aligned}$$

   As required. QED

5. Use Lagrange's Remainder Thm to show that if f is defined for $|x| < r$ and if there exists a constant B such that $|f^{(n)}(x)| \leqslant B$ for all $|x| < r$ and all $n \in \mathbb{N}$, then the Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

   converges to $f(x)$ for all $|x| < r$.

Proof. Lagrange's Remainder:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

For some c between x and a.
Let $c = 0$, we have $\exists B$ such that $|f^{(n+1)}(0)| \leqslant B$ so

$$\frac{f^{(n+1)}(0)}{(n+1)!} \leqslant \frac{B}{(n+1)!}$$

Note $\lim \frac{B}{(n+1)!} = 0$, so

$$\lim_{n\to\infty} R_n(x) = 0$$

Since the remainder $R_n(x) \to 0$, we have $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n + 0$, so the Taylor series converges to $f(x)$. QED

6. Assume $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on $(-R, R)$.

   (a) Show that

   $$F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

   has radius of converge R, and satisfies $F'(x) = f(x)$.

   By Cauchy-Hadamard,

   $$\frac{1}{r} = \limsup_{n\to\infty} \frac{a_n}{n+1}^{\frac{1}{n}} = \frac{\limsup_{n\to\infty} a_n^{\frac{1}{n}}}{\lim_{n\to\infty}(n+1)^{\frac{1}{n}}} = \limsup_{n\to\infty} a_n^{\frac{1}{n}} = \frac{1}{R}$$

   So the radius of convergence of F is the same as f. So, we can differentiate F on $(-R, R)$ to get $F'(x) = (n+1)\sum_{n=0}^{\infty} \frac{a_n}{n+1}x^n = \sum_{n=0}^{\infty} a_n x^n = f(x)$.

   (b) If g is an arbitrary function satisfying $g'(x) = f(x)$ on $(-R, R)$, find a power series representation of g.

   A power series representation of g would just be F with an extra constant term added on. Because when you differentiate, the constant goes away, so any g that looks like F with a constant added will have derivative $= f$. For example:

$$g(x) = 57 + \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} \; g'(x) = F'(x) = f(x)$$

(c) Derive a power series representation for $g(x) = -ln(1-x)$
Hint: $g'(x)$

We have $g'(x) = \frac{1}{1-x}$ which looks like the sum of a geometric series, so we can write

$$g'(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Which has antiderivative

$$g(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

and radius of convergence $R = 1$. (Changed the indexing from n=0 to n=1 to avoid dividing by 0 :))