Math 341 Exam 2

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1 Problem 1

Find $\gcd(2322, 654)$ using Euclid's Algorithm.

$$2322 = 654 \cdot 3 + 360$$

$$654 = 360 \cdot 1 + 294$$

$$360 = 294 \cdot 1 + 66$$

$$294 = 66 \cdot 4 + 30$$

$$66 = 30 \cdot +6$$

$$30 = 6 \cdot 5 + 0$$

Therefore $\gcd(2322, 654) = 6$.

2 Problem 2

Prove that for any $n \geq 2$, the numbers $n! + 2, n! + 3, ..., n! + n$ are composite.

Proof.

Remark. *Any number $n! \in \mathbb{N}$ can be written as $n! = 2 * 3 * 4 * ... * n$.*

Remark. *Any number $n!$ where $n \geq 2$ is even, since it will always be something times 2.*

For any number $n \geq 2$, we may write $n! = 2 * 3 * 4 * ... * n$. n is either even or odd.

1. $\forall n$, $n! + x$, where $x \in \{2, 4, ..., n\}$, is always even, and therefore composite, since $n!$ is even, by the remark above, so you are always able to factor out a 2 from $n! + x$, thus $n! + x$ is composite.

2. $\forall n$, $n!+y$, where $y \in \{3, 5, ..., n-1\}$, is always composite, because if you rearrange $n!$, and factor out y, you get $n! = y(z)$. Where $z = \frac{n!}{y} \in \mathbb{N}$. and we can factor out y from y, of course. Thus, we are able to rewrite $n! + y = y(z + 1)$. For example if $n = 4$, then $n! + 3 = 3(8 + 1)$ is composite. Therefore y is a factor, so $n! + y$ is composite.

Since n plus any even or odd number up to n is composite, then the statement is true. QED

3 Problem 3

Prove by strong induction that for any $n \in \mathbb{N}$, $n > 1$, there exists a prime factorization of n.

Proof. *Basis*: $n = 2$, the prime factorization is $2 = 2^1$.
Inductive step: Assume it is true for all $n > k$.
N2S: It is true for $n = k$. There are 2 cases:

1. If $n = k$ is prime, then the prime factorization is trivial, $k = k \cdot 1$.

2. If $n = k$ is composite, then $\exists m, n \in \mathbb{N}$, s.t. $k = m \cdot n$, and $1 < m, n \leq (k - 1)$. By our inductive assumption, m, n both have prime factorizations, so we may write them as $m = 2^{m_2} * 3^{m_3} * 5^{m_5} * ... * p^{m_p}$, and $n = 2^{n_2} * 3^{n_3} * 5^{n_5} * ... * p^{n_p}$. Then k must have a prime factorization, namely $k = 2^{m_2+n_2} * 3^{m_3+n_3} * 5^{m_5+n_5} * ... * p^{m_p+n_p}$.

Therefore, by strong induction, for any $n \in \mathbb{N}$, $n > 1$, there exists a prime factorization of n. QED

4 Problem 4

Find a solution to the equation $5x + 8y = 1$.
Since this is a Linear Diophantine equation, it can be solved via the reverse Euclidean Algorithm.

$$8 = 5 \cdot 1 + 3$$

$$5 = 3 \cdot 1 + 2$$
$$3 = 2 \cdot 1 + 1$$
$$2 = 1 \cdot 2 + 0$$

In reverse:

$$\begin{aligned} 1 &= (1 \cdot 3) + (-1 \cdot 2) \\ &= (-1 \cdot 5) + (2 \cdot 3) \\ &= (2 \cdot 8) + (-3 \cdot 5) \end{aligned}$$

Therefore $x = -3, y = 2$ is a solution to this equation.

5 Problem 5

Let $f(x) = x^2$. For each pair of sets X and Y below, determine if f defines a function from $X \to Y$, and if yes, whether this function is injective and/or surjective.

1. $X = \mathbb{N}$, $Y = \mathbb{N}$. Yes, f defines an injective function.

   Proof. f is injective because $\forall x_1 \neq x_2 \in \mathbb{N}$, their mapping is $f(x_1) = x_1^2$, and $f(x_2) = x_2^2$. If $f(x_1) = f(x_2)$, then $x_1^2 = x_2^2 \implies (x_1 - x_2)^2 = 0$, and since the natural numbers have no nontrivial zero divisors, x_1 must equal x_2. Since this is true, the contrapositive, (if $f(x_1) \neq f(x_2)$, then $x_1 \neq x_2$) must be true. Therefore f is injective.
   f is not surjective because any number $y \in \mathbb{N}$ which is not a perfect square, is not hit. QED

2. $X = \mathbb{Z}$, $Y = \mathbb{Z}$. Yes, f defines a function, however it is neither injective nor surjective.

   Proof. It is not injective because $x_1 = 2, x_2 = -2$ both map to 4. It is not surjective because any negative number, as well as any number that is not a perfect square, is not hit. QED

3. $X = \mathbb{Z}$, $Y = \mathbb{N}$. Yes, f defines a function, however it is neither injective nor surjective.

Proof. It is not injective because again, both $x_1 = 2, x_2 = -2$ map to 4. It is not surjective because any number that is not a perfect square is not hit. QED

4. $X = \mathbb{N}$, $Y = \mathbb{Z}$. Yes, f defines an injective function.

   Proof. Similarly to the first part, f is injective because $\forall x_1 \neq x_2 \in \mathbb{N}$, their mapping is $f(x_1) = x_1^2$, and $f(x_2) = x_2^2$. If $f(x_1) = f(x_2)$, then, by the first part of this problem, $x_1 = x_2$. Therefore the contrapositive is true, so f is an injection.
   f is not surjective because any negative integer, or integer that is not a perfect square, is not hit. QED

5. $X = \{0, 1\}$, $Y = \{0, 1\}$. Yes, f defines a bijective function.

   Proof. f is injective because we can check all $x \in X$, to see if we get different values of $y \in Y$. $f(0) = 0^2 = 0 \in Y$, and $f(1) = 1^2 = 1 \in Y$. We also just checked that every $y \in Y$ has a unique $x \in X$, so the function is both injective and surjective. QED