

Math 524 Homework 3

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1. Let f be any function on $[a, b]$ with no upper bound. Prove that the upper sum $U(f, P)$ is infinite for every partition P of $[a, b]$. Conclude that f is not integrable.

Proof.

$$M_k(f) = \sup \{f(x) \mid x \in [x_{k-1}, x_k]\}$$

Upper sum for partition $P = \{x_0, x_1, x_2, \dots, x_n\}$:

$$U(P, f) = \sum_{k=1}^n M_k \Delta x_k$$

However since f has no upper bound on $[a, b]$, we have that for each partition P , M_k is infinite, therefore the sum $\sum_{k=1}^n M_k \Delta x_k$ is infinite. Therefore f is not integrable. QED

2. (a) Let f be a bounded function on a set A , and consider:

$$M = \sup \{f(x) \mid x \in A\} \quad m = \inf \{f(x) \mid x \in A\}$$

$$M' = \sup \{|f(x)| \mid x \in A\} \quad m' = \inf \{|f(x)| \mid x \in A\}$$

Show that $M - m \geq M' - m'$.

$$M - m = \sup(|f(x) - f(y)| \mid x, y \in A)$$

and

$$M' - m' = \sup(|f(x)| - |f(y)| \mid x, y \in A)$$

By the triangle inequality, we have:

$$|f(x) - f(y)| \geq ||f(x)| - |f(y)||$$

So,

$$M - m \geq M' - m'$$

- (b) Show that if f is integrable on $[a, b]$ then $|f|$ is also integrable on $[a, b]$. Let f be integrable on $[a, b]$, then we have that

$$\int_a^b f(x)dx = \overline{\int_a^b f(x)dx} = B \quad (\text{Where } B \text{ is a real number})$$

Then, we have:

$$U(P, f) = \sum_{k=1}^n M \Delta x_k \quad L(P, f) = \sum_{k=1}^n m \Delta x_k$$

Subtracting, we get:

$$U(P, f) - L(P, f) = \sum_{k=1}^n (M - m) \Delta x_k$$

By above we get:

$$U(P, f) - L(P, f) \geq U(P, |f|) - L(P, |f|)$$

And since f is Riemann integrable, we have $U(P, f) - L(P, f) = 0$, so

$$0 \geq U(P, |f|) - L(P, |f|)$$

And all $U(P, |f|)$ are greater than or equal to all $L(P, |f|)$, so we have $U(P, |f|) = L(P, |f|)$ therefore $|f|$ is Riemann integrable.

3. Let $f(x) = |x|$ and define $F(x) = \int_{-1}^x f$. Find a piecewise algebraic formula for $F(x)$ for all x . Where is F continuous? Where is F differentiable?

$$f(-2) = |-2| = 2 \quad F(-2) = \int_{-1}^{-2} f =$$

Piecewise formula:

$$F(x) = \begin{cases} \int_{-1}^x (-f) = \frac{1}{2}(1 - x^2) & \text{if } x < 0 \\ F(0) + \int_{-1}^x (f) = \frac{1}{2}(1 + x^2) & \text{if } x > 0 \end{cases}$$

F is continuous everywhere, and it is differentiable everywhere (everywhere except 0 is obvious, at $F(0)$ we get $F'(0) = \lim_{h \rightarrow 0} \frac{F(x) - F(0)}{h - 0}$, and $|F(x) - F(0)| \leq |x|^2$) Therefore F is differentiable at 0 as well.

4. Let $L(x) = \int_1^x \frac{1}{t} dt$ for all $x > 0$.

(a) Evaluate $L(1)$. Explain why L is differentiable and find $L'(x)$.

$$L(1) = \int_1^1 \frac{1}{t} dt = 0$$

L is differentiable by the fundamental theorem of calculus, we have $\frac{d}{dx} \int_1^x \frac{1}{t} dt = \frac{1}{x}$.

(b) If $E : \mathbb{R} \rightarrow (0, \infty)$ is given by $E(x) = e^x$, it is known that $E'(x) = e^x$. Let $\ln : (0, \infty) \rightarrow \mathbb{R}$ denote the inverse function of E . Use the inverse function theorem to prove that the derivative of $\ln(x)$ equals $\frac{1}{x}$.

Proof. By inverse function theorem, we have that for a point $b = f(a)$:

$$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}$$

So,

$$\frac{d}{dx} \ln(b) = \frac{1}{e'(a)} = \frac{1}{b}$$

as required. QED

(c) Explain why $L(x) = \ln(x)$ for all $x > 0$.

The two functions both have the same derivative (by (a) and (b)), and $L(1) = \ln(1) = 0$, so they are “aligned”. It breaks for $x \leq 0$ because that is not in the domain of $\ln : (0, \infty) \rightarrow \mathbb{R}$.

(d) Let

$$\gamma_n = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) - L(n)$$

Prove (γ_n) converges.

Proof. By the integral test we have (γ_n) converges iff $\int_1^\infty f(x)dx$ converges, where $f(x) = \gamma_x$.
 Suppose $\int_1^\infty f(x)dx$ converges, then:

$$S_{k+1} - \gamma_1 \leq \int_1^{k+1} f(x)dx \leq \int_1^\infty f(x)dx$$

So S_{k+1} is bounded, and we have by induction that (γ_k) is increasing, therefore (γ_k) converges. QED

- (e) Show how consideration of the sequence $(\gamma_{2n} - \gamma_n)$ leads to a new proof of the identity:

$$L(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

RHS is equal to

$$\sum_{n=1}^{2m} (-1)^{n+1} \frac{1}{n}$$

Splitting into even and odd terms gives:

$$= \sum_{n=1}^m \frac{1}{2n-1} - \sum_{n=1}^m \frac{1}{2n}$$

Which is

$$\gamma_{2m} - 2\gamma_m + L(2)$$

Which limits to $L(2)$.

5. If \vec{u} and \vec{v} are vectors in \mathbb{R}^3 , use the triangle inequality to prove that:

- (a) $\|\vec{u}\| - \|\vec{v}\| \leq \|\vec{u} - \vec{v}\|$.

By theorem 9.2.2 in the text, we have that $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$.

$$\|\vec{u}\| = \|u - v + v\| \leq \|u - v\| + \|v\| \quad (\text{By triangle inequality})$$

Subtract $\|\vec{v}\|$ from both sides:

$$\|\vec{u}\| - \|\vec{v}\| \leq \|\vec{u} - \vec{v}\|$$

$$(b) \quad ||\vec{u}|| - ||\vec{v}|| \leq ||\vec{u} - \vec{v}||$$

$$||\vec{v}|| = ||v - u + u|| \leq ||v - u|| + ||u|| \quad (\text{By triangle inequality})$$

Subtracting the two above equations gives:

$$||\vec{u}|| - ||\vec{v}|| \leq ||u - v|| + ||v|| - (||v - u|| + ||u||)$$

$$\implies ||\vec{u}|| - ||\vec{v}|| \leq ||\vec{u} - \vec{v}||$$

(Taking abs value forces $||u - v|| = ||v - u||$)

6. Suppose $f : [a, b] \rightarrow \mathbb{R}$ is differentiable and f' is continuous. The *graph* of f is the curve $y = f(x)$ in \mathbb{R}^2 . Show that the arclength L of the graph of f for $a \leq x \leq b$ is given by:

$$L = \int_a^b \sqrt{1 + (dy/dx)^2} dx$$

Proof. The *graph* of f is the same as a smooth parameterization of f , which looks like $y = (t, f(t))$. Then, we have that the length of y is defined by:

$$L(y) = \sup \{L_Q \mid Q \text{ is a partition of } [a, b]\}$$

Using this smooth parameterization, we can invoke Theorem 9.7.2, it states that the length of a smooth parameterization $\gamma(t) = (f(t), g(t))$ for $t \in [a, b]$ is finite and equal to:

$$L(\gamma) = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} dt$$

In our case, we have $y = (t, f(t))$, so the arc length is given by $t' = 1$, and $f'(t) = dy/dx$:

$$L = \int_a^b \sqrt{1 + (dy/dx)^2} dx$$

QED