Math 724 Homework 2

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Chapter 2

14. Give an example of an open cover of $(0,1)$ which has no finite subcover.

    The set
    $$\bigcup_{n=1}^{\infty}(0, 1-\frac{1}{n})$$
    Is an open cover of $(0,1)$, because for any $x \in (0,1)$, there is an N such that $1-\frac{1}{N} < x$ (So $x \in (0, 1-\frac{1}{N})$), and clearly the sets are open. There is no finite subcover because assume there is, then there is a maximum N. But then $\frac{1}{2N} \in (0,1)$ but it is not in the subset.

25. Prove that every compact metric space K has a countable base, and that K is therefore separable.

    Proof. Let K be a compact metric space. For every $n \in \mathbb{N}$, consider the open ball $B(x, \frac{1}{n})$ with radius $\frac{1}{n}$ centered around some $x \in K$. Clearly, this is an open cover of K. Since we have that K is compact, there is a finite subcover $V_n =$ The union of finitely many of the neighborhoods. Say $V_n = \bigcup_{i=1}^{N} B(x_i, \frac{1}{n})$.Now, taking the union of all of these finite subcovers for each $n \in \mathbb{N}$ is countable, and furthermore, this union is a base of K.

    To show K is separable, we must find a coutable, dense subset. Pick one point from each open ball of our base (countably many). Then clearly this subset is countable, and it is dense because V_n covers K. □

Chapter 3

6. Investigate the behavior (convergence or divergence) of $\sum a_n$ if:

   (a). $a_n = \sqrt{n+1} - \sqrt{n}$ The k-th partial sum:

   $$S_k = \sqrt{k+1} - 1 \qquad \text{(Everything less than k cancels out!)}$$

   Diverges as $k \to \infty$, so the series diverges.

   (b). $a_n = \frac{\sqrt{n+1}-\sqrt{n}}{n}$ Similar to above, the k-th partial sum:

   $$S_k = \frac{\sqrt{k+1} - 1}{k} \to 0 \text{ As } k \to \infty$$

   So the series converges.

   (c). $a_n = (\sqrt[n]{n} - 1)^n$ $\limsup_{n\to\infty}(\sqrt[n]{n} - 1) = 0$ (By theorem 3.20 (c), $\lim_{n\to\infty} \sqrt[n]{n} = 1$). So it converges by the root test.

   (d). $a_n = \frac{1}{1+z^n}$ For complex z.

   If $|z| \leqslant 1$, the series diverges because the terms will always be at least $\frac{1}{2}$, so the limit of partial sums does not converge to 0.
   For $|z| > 1$, we can compare to the geometric series $\sum \frac{1}{|z|^n}$ which converges by theorem 3.26. Therefore our series converges.

7. Prove that the convergence of $\sum a_n$ implies the convergence of $\sum \frac{\sqrt{a_n}}{n}$.

   Proof. By theorem 1.35, the Schwarz inequality, we have,

   $$\left(\sum a_i b_i\right)^2 \leqslant \left(\sum a_i^2\right)\left(\sum b_i^2\right)$$

   In our case, $a_i \coloneqq \sqrt{a_n}$ and $b_i \coloneqq \frac{1}{n}$. So,

   $$\left(\sum \frac{\sqrt{a_n}}{n}\right)^2 \leqslant (\sum a_n)(\sum \frac{1}{n^2})$$

   $\sum a_n$ converges by assumption, and $\sum \frac{1}{n^2}$ converges by theorem 3.28. Therefore $\left(\sum \frac{\sqrt{a_n}}{n}\right)^2$ converges and so $\sum \frac{\sqrt{a_n}}{n}$ also converges. $\square$