Math 531 Homework 7

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Sorry I didn't really have time to format these nicely, hopefully they are still legible!

531 3.7 – 3.8

3.7

1. Write down the formulas for all homomorphisms from $\mathbb{Z}_6$ into $\mathbb{Z}_9$

By ex. 3.7.7, any homomorphism from $\mathbb{Z}_n$ into $\mathbb{Z}_k$ must be of the form $\phi([x]_n) = [mx]_k$ for all $[x]_n \in \mathbb{Z}_n$, Where $k \mid mn$ ~~order of m divides n~~. In this case, that is $9 \mid 6m$, this is true for $m = 3, 6, 9$.

Thus, all surjective homomorphisms $\phi: \mathbb{Z}_6 \to \mathbb{Z}_9$ are of the form $\phi([x]_6) = [mx]_9$ where $m = 3, 6, 9$. (Just realized $9 \equiv 0 \pmod 9$)

b. for $\mathbb{Z}_{24}$ into $\mathbb{Z}_{18}$?

All m such that $18 \mid 24m$, which are $m = 0, 3, 6, 9, 12, 15, \pmod{18}$.

$\phi: \mathbb{Z}_{24} \to \mathbb{Z}_{18}$ defined by $\phi([x]_{24}) = [mx]_{18}$ for the m listed above.

9. Which of the following are homomorphisms.

(a). $\phi: \mathbb{R}^{\times} \to GL_2(\mathbb{R})$ defined by $\phi(a) = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$.
Let $a, b \in \mathbb{R}^{\times}$, $\phi(ab) = \begin{bmatrix} ab & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} b & 0 \\ 0 & 1 \end{bmatrix} = \phi(a)\phi(b)$.

(b). Let $a, b \in \mathbb{R}$, $\phi(ab) = \begin{bmatrix} 1 & 0 \\ a+b & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} = \phi(a)\phi(b)$.

(c). $\phi: M_2(\mathbb{R}) \to \mathbb{R}$ def. by $\phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a$. Let $\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) \in M_2($
then $\phi\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) = \phi\left(\begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}\right) = a_1 + a_2$
and since $a_1 + a_2 = \phi\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + \phi\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$, ϕ is a homomorphis

★ I don't really understand d–f. ★

(d.) $\phi : GL_2(\mathbb{R}) \to \mathbb{R}^{\times}$ def. by $\phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ab$.

Let $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in GL_2(\mathbb{R})$. Then

$\phi\left(\begin{bmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{bmatrix}\right) = (a_1a_2 + b_1c_2) \cdot (a_1b_2 + b_1d_2)$

$= \phi\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) \cdot \phi\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$

(e.)

10. Let $\phi : G_1 \to G_2$ be a group homomorphism of G_1 onto G_2.

(a.) Prove that if G_1 is abelian then so is G_2.

Pf. Let $x, y \in G_2$. Since ϕ is an onto homomorphism, $\exists\, a, b \in G_1$ s.t. $\phi(a) = x$, $\phi(b) = y$.

Then: $x \cdot y = \phi(a) \cdot \phi(b)$

$= \phi(a \cdot b)$, (since ϕ is homomorphism)

$= \phi(b \cdot a)$, (since G_1 is abelian)

$= \phi(b) \cdot \phi(a)$

$= y \cdot x$.

Thus G_2 is abelian. $\square$

b.) Prove that if G_1 is cyclic then G_2 is cyclic.

Pf.

Let G_1 be cyclic and $G_1 = \langle a \rangle$. Then every element of G_1 is equal to a^n, for some integer n.

Let $y \in G_2$. Since ϕ is an onto homomorphism $\exists b \in G_1$ s.t. $\phi(b) = y$. However, by above, since $b \in G_1$, $b = a^m$.

$$\text{Thus, } y = \phi(b) = \phi(a^m) = [\phi(a)]^m.$$

This shows all $y \in G_2$ are of the form $[\phi(a)]^m$, where m is an integer thus G_2 is a cyclic group. ∎