Math 524 Homework 5

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1 Section 10.2

2(a).

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} = \lim_{r\to 0}\left(\frac{r\cos\theta(r\sin\theta)^2}{(r\cos\theta)^2+(r\sin\theta)^2}\right) = \lim_{r\to 0}(r\sin^2\theta\cos\theta) = 0$$

This agrees with definition 10.2.1 because $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} = 0$ iff for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x,y) - 0| < \varepsilon$ whenever $(x,y) \in D$ and $0 < \sqrt{x^2+y^2} < \delta$.

Proof. Let ε be given. Then we want $\exists\delta$ such that $f(x,y) = \frac{xy^2}{x^2+y^2} < \varepsilon$ when $0 < \sqrt{x^2+y^2} < \delta$. So we need $0 < |x| < \delta$ and $0 < |y| < \delta$.

$$\begin{aligned} |f(x,y)| &= \left|\frac{xy^2}{x^2+y^2}\right| \\ &= \left|\frac{r\cos\theta(r\sin\theta)^2}{(r\cos\theta)^2+(r\sin\theta)^2}\right| \\ &= |r\sin^2\theta\cos\theta| \\ &< |r| \\ \text{Let } \delta &= \varepsilon \end{aligned}$$

Since $0 < r < \delta$, $\delta = \varepsilon$ forces $|f(x,y)| < \varepsilon$. As required. QED

3(g). Determine if the given limit is finite.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^4}{x^2 + y^4}$$

Approach from $x = 0$,

$$\lim_{y\to 0} \frac{-y^4}{y^4} = -1$$

Approach from $x = y$,

$$\lim_{y\to 0} \frac{y^2 - y^4}{y^2 + y^4} = \frac{1 - y^2}{1 + y^2} = 1$$

Therefore the limit is infinite.

2 Section 10.4

3(b). Show that $f(x, y) = \sqrt{x^2 + y^2}$ is not differentiable at the origin by showing $f_x(0, 0)$ does not exist. $f(x, 0) = \sqrt{x^2} = |x|$ which is not differentiable.

4. $f(x, y) = \sqrt[3]{xy}$

   (a) Show that $f_x(0, 0) = 0 = f_y(0, 0)$.

   $$f(x, 0) = \sqrt[3]{0} = 0 \ f(0, y) = \sqrt[3]{0} = 0$$

   (b) Find $\nabla F = (0, 0)$.

   (c) Show that f is not differentiable at $(0, 0)$.

   Proof. From definition 10.4.1, we must show $f(P + h) = f(P) + m \cdot h + \varepsilon\|h\|$ has
   $\varepsilon \not\to 0$ as $h \to 0$. QED

   (d) Is f continuous at $(0, 0)$? Yes:

   Proof. Let $\varepsilon > 0$ be given. Choose $\delta = \sqrt[3]{\varepsilon^2}$ and suppose that $0 < |(x, y)| < \delta$.

   $$0 < \sqrt{x^2 + y^2} < \delta$$
   $$0 < \sqrt{x^2 + y^2} < \sqrt{\varepsilon^3}$$

And, $x < \sqrt{x^2+y^2}$ $y < \sqrt{x^2+y^2}$

$$\begin{aligned}
|f(x,y)| &= |\sqrt[3]{xy}| \\
&< |\sqrt[3]{(\sqrt{x^2+y^2})(\sqrt{x^2+y^2})}| \\
&< |\sqrt[3]{(\sqrt{\varepsilon^3})(\sqrt{\varepsilon^3})}| \\
&= |\sqrt[3]{\varepsilon^3}| \\
&= \varepsilon
\end{aligned}$$

QED

3 Section 10.5

2. (a) Show $D_i f = -D_{-i} f$, provided f_x exists.
Suppose f_x exists, so $D_i f = f_x = \lim_{h\to 0} \frac{f(P+hi)-f(P)}{h}$ exists.

$$-D_{-i} f = -\left[\lim_{h\to 0} \frac{f(P-hi)-f(P)}{h}\right] =$$

$$= -\left[-\left[\lim_{h\to 0} \frac{f(P+hi)-f(P)}{h}\right]\right]$$

$$= -(-f_x) = f_x$$

(b) Show that if f is differentiable at (a,b) then for any unit vector u, $D_{-u} f(a,b) = -D_u f(a,b)$.

Proof. By theorem 10.5.2, since f is differentiable at (a,b), we have $D_u f(a,b)$ exists in any direction u, and that

$$D_u f(a,b) = \nabla f(a,b) \cdot u$$

So,

$$D_{-u} f(a,b) = \nabla f(a,b) \cdot -u = -(\nabla f(a,b) \cdot u) = -D_u f(a,b)$$

QED

(c) If $D_u f$ exists for a unit vector u, show that $D_{-u} f = -D_u f$.

Proof. Let $D_u f$ exist for some unit vector u. That is, the limit

$$\lim_{h \to 0} \frac{f(P + hu) - f(P)}{h}$$

exists.

$$\begin{aligned}
-D_{-u} f &= -\left[\lim_{h \to 0} \frac{f(P - hu) - f(P)}{h}\right] \\
&= -\left[-\left[\lim_{h \to 0} \frac{f(P + hu) - f(P)}{h}\right]\right] \\
&= \lim_{h \to 0} \frac{f(P + hu) - f(P)}{h} \\
&= D_u f
\end{aligned}$$

QED

4 Section 11.1

1(b). Prove part (b) of lemma 11.1.1: If Q is a partition of R and $P \subseteq Q$, then $L(P, f) \leqslant L(Q, f)$ and $U(Q, f) \leqslant U(P, f)$.

Proof. Let Q be a partition of R and $P \subseteq Q$. If $P = Q$, we are done because $L(P, f) = L(Q, f)$ and $U(Q, f) = U(P, f)$. So suppose $P = (x_0, x_1, \ldots, x_n)$, and Q contains all of P and one extra point of $[a, b]$, say $c \in [x_{i-1}, x_i]$ for some i between 1 and n. Then let,

$$\begin{aligned}
m_i &= \inf \{f(x) \mid x \in [x_{i-1}, x_i]\}, \\
r_1 &= \inf \{f(x) \mid x \in [x_{i-1}, c]\}, \\
r_2 &= \inf \{f(x) \mid x \in [c, x_i]\}
\end{aligned}$$

We have $m_i = \min(r_1, r_2)$, so now:

$$
\begin{aligned}
L(P,f) &= \sum_{k=1}^{n} m_k \Delta x_k \\
&= \sum_{k=1}^{i-1} m_k(x_k - x_{k-1}) + m_i(x_i - x_{i-1}) + \sum_{k=i+1}^{n} m_k \Delta x_k \\
&\leqslant \sum_{k=1}^{i-1} m_k(x_k - x_{k-1}) + r_1(c - x_{i-1}) + r_2(x_i - c) + \sum_{k=i+1}^{n} m_k \Delta x \\
&= L(Q,f)
\end{aligned}
$$

Then, for $U(Q,f) \leqslant U(P,f)$, we do the same argument except $M_i = \sup\{\dots\}$ above, and R_1 and R_2 are also supremum. Then $M_i = \max(R_1, R_2)$ and essentially the same argument follows. QED

1(c). Prove Theorem 11.1.3:

Theorem 1. *A bounded function $f(x,y)$ on a rectangle $R = [a,b] \times [c,d]$ is Riemann integrable iff for any $\varepsilon > 0$, there exists a partition P of R such that $U(P,f) - L(P,f) < \varepsilon$.*

Proof. $(\implies)$: Let $f(x,y)$ on R be Riemann integrable. Then by definition 11.1.2,

$$\underline{\iint_R} f = I = \overline{\iint_R} f$$

Therefore,

$$\sup\{L(P,f) \mid P \text{ is a partition}\} = \inf\{U(P,f) \mid P \text{ is a partition}\}$$

So there exists a partition P with $U(P,f) - L(P,f) = 0$ which is less than any ε.

$(\impliedby)$: Suppose for any $\varepsilon > 0$, there exists a partition P of R such that $U(P,f) - L(P,f) < \varepsilon$. As $\varepsilon \to 0$, $\sup L(P,f)$ gets closer and closer to $\inf U(P,f)$. $\sup L(P,f)$ is bounded above by

$\inf U(P,f)$ and $\inf U(P,f)$ is bounded below by $\sup L(P,f)$. So as $\varepsilon \to 0$, $\sup L(P,f) \to \inf U(P,f)$. Therefore, we have

$$\underline{\iint_R} f = \overline{\iint_R} f$$

So $f(x,y)$ is Riemann integrable. QED

5 Section 11.2

7(a). Suppose that $f : [a,b] \to \mathfrak{R}$ and $g : [c,d] \to \mathfrak{R}$ are Riemann integrable, and there is a rectangle $R = [a,b] \times [c,d]$. Prove that

$$\iint_R f(x)g(y) = \left[\int_a^b f\right]\left[\int_c^d g\right]$$

Proof. Let f and g be Riemann integrable. That is, their integrals $\int_a^b f$ and $\int_c^d g$ exist. Since f only depends on x and g only depends on y, we have that $\int_a^b f(x)dx$ "looks like" a constant w.r.t. g, and vice versa. Now since an integral times a constant is equal to the integral of the function times that constant:

$$\int cf = c \int f$$

We can now write

$$\iint_R f(x)g(y) = \int_c^d \left(\int_a^b f(x)dx\right) g(y)dy = \int_a^b f(x)dx \cdot \int_c^d g(y)dy$$

QED