Math 723 Homework 4

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Assignment 4

Chapter 3

16. Fix a positive number α. Choose $x_1 > \sqrt{\alpha}$, and defined $x_2, x_3, x_4 \dots,$ by the recursion formula:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right)$$

(a). Prove that $\{x_n\}$ decreases monotonically and that $\lim x_n = \sqrt{\alpha}$.

Proof. By induction on $\{x_n\}$.

$$x_1 - x_2 = \frac{x_1^2 - \alpha}{2x_1}$$

We have $x_1 > \sqrt{\alpha}$ by assumption, so this is positive, so $x_1 > x_2$. This calculation holds for all x_n, because $x_n^2 > \alpha$ for all $n \in \mathbb{N}$, so the difference is always positive. It is also bounded below so therefore the limit exists. Letting $L = \lim x_n$, we have

$$L = \frac{1}{2}\left(L + \frac{\alpha}{L}\right) \implies L^2 = \alpha$$

And so $L = \lim x_n = \sqrt{\alpha}$. □

(b). Put $\varepsilon_n = x_n - \sqrt{\alpha}$, and show that

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}$$

So that, setting $\beta = 2\sqrt{\alpha}$,

$$\varepsilon_{n+1} < \beta \left(\frac{\varepsilon_1}{\beta} \right)^{2^n}$$

Proof.

$$\varepsilon_{n+1} = x_{n+1} - \sqrt{\alpha} = \frac{1}{2} \left(x_n + \frac{\alpha}{x_n} \right) - \sqrt{\alpha}$$

We have $\varepsilon_n^2 = x_n^2 - 2x_n\sqrt{\alpha} + \alpha$, and $\varepsilon_{n+1} = \frac{x_n}{2} + \frac{\alpha}{2x_n} - \sqrt{\alpha}$. Clearing denominators:

$$2x_n\varepsilon_{n+1} = x_n^2 - 2x_n\sqrt{\alpha} + \alpha = \varepsilon_n^2$$

So $\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n}$. By above, we always have $x_n > \sqrt{\alpha}$, so the denominator is bigger, and therefore

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}$$

Then, we must show for all n that,

$$\varepsilon_{n+1} < \beta \left(\frac{\varepsilon_1}{\beta} \right)^{2^n}$$

Let $n = 1$, we have

$$\varepsilon_2 < \frac{\varepsilon_1^2}{\beta} = \beta \left(\frac{\varepsilon_1}{\beta} \right)^2$$

Assume for some $k \in \mathbb{N}$,

$$\varepsilon_{k+1} < \beta \left(\frac{\varepsilon_1}{\beta} \right)^{2^k}$$

Then,

$$\varepsilon_{k+2} < \beta \left(\frac{\varepsilon_{k+1}}{\beta} \right)^2 < \beta \left[\frac{\beta \left(\frac{\varepsilon_1}{\beta} \right)^{2^k}}{\beta} \right]^2 = \beta \left(\frac{\varepsilon_1}{\beta} \right)^{2^{k+1}}$$

As required. $\square$

(c). If $\alpha = 3$ and $x_1 = 2$, show that $\frac{\varepsilon_1}{\beta} < \frac{1}{10}$ and therefore

$$\varepsilon_5 < 4 \cdot 10^{-16},\ \varepsilon_5 < 4 \cdot 10^{-32}$$

Proof. We have $\varepsilon_1 = 2 - \sqrt{3}$, and $\beta = 2\sqrt{3} < 4$. So

$$\frac{\varepsilon_1}{\beta} = \frac{2-\sqrt{3}}{2\sqrt{3}} = \frac{1}{2\sqrt{3}(2+\sqrt{3})} < \frac{1}{10}$$

Because $2\sqrt{3}(2+\sqrt{3}) = 6 + 4\sqrt{3} > 10$. Therefore

$$\varepsilon_5 < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^4} < 4\left(\frac{1}{10}\right)^{16} \qquad \text{(By above)}$$

Similarly

$$\varepsilon_6 < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^5} < 4\left(\frac{1}{10}\right)^{32}$$

□

Chapter 4

1. Suppose f is a real function defined on $\mathbb{R}^1$ which satisfies

$$\lim_{h\to 0}\left[f(x+h) - f(x-h)\right] = 0$$

For every $x \in \mathbb{R}$. Does this imply that f is continuous?

Proof. Yes, because this limit gives $f(x+h) = f(x-h)$, for small h, which is as we have defined $f(x+)$ and $f(x-)$. Since we have that they are equal always, there are no discontinuities and therefore f is continuous. □

18. Consider the function f on $\mathbb{R}$ defined by:

$$f(x) = \begin{cases} 0 & \text{if } x \text{ irrational} \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \end{cases}$$

Prove that f is continuous at irrational points, and that f has a simple discontinuity at every rational point.

Proof. Let $k \in \mathbb{R} \setminus \mathbb{Q}$, then $f(k) = 0$. We also have

$$\lim_{t \to k} f(t) = \text{Limit of sequences of rationals converging to k}$$

By density of the reals in $\mathbb{Q}$, we have that the denominators in each sequence $\{t_n\}$ grow, so $\lim_{t \to k} f(t) = 0 = f(k)$, so f is continuous at irrationals.
Let $k = \frac{m}{n} \in \mathbb{Q}$. We have $f(k) = \frac{1}{n}$, select a sequence of irrationals $\{t_n\}$ which converge to k as $n \to \infty$. Then the limit

$$\lim_{t \to k} f(t) = 0 \neq \frac{1}{n} = f(k)$$

Therefore there is a discontinuity of the first kind, because $f(k+)$ and $f(k-)$ both exist, they are just not equal to $\lim_{t \to k} f(t)$. □

Chapter 5

2. Suppose $f'(x) > 0$ in (a, b). Prove that f is strictly increasing in (a, b), and let g be its inverse function. Prove g is differentiable, and

$$g'(f(x)) = \frac{1}{f'(x)} \quad (a < x < b)$$

Proof. We have $f'(x) > 0$ in (a, b), so

$$\lim_{t \to x} \frac{f(t) - f(x)}{t - x} > 0$$

Approaching from the left, we have that $t < x$ so the denominator is negative, and since the limit is > 0, the numerator is also negative, then $f(t) < f(x)$ as required. Approaching from the right we get the denominator is positive, so the numerator must also be positive, giving $f(t) > f(x)$, as required.
Let g be the inverse of f, then

$$g'(f(x)) = \lim_{t \to x} \frac{g(f(t)) - g(f(x))}{f(t) - f(x)}$$

But we have $g(f(k)) = k$ for all k, so

$$g'(f(x)) = \lim_{t \to x} \frac{t - x}{f(t) - f(x)} = \frac{1}{f'(x)}$$

And so g is differentiable □