Math 531 Homework 1

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1 Section 1.1

- Problem 1, Let $m, n, r, s \in \mathbb{Z}$. If $m^2 + n^2 = r^2 + s^2 = mr + ns$, prove that $m = r$ and $n = s$.

  Proof. Assume, to the contrary, that $m \neq r$ and $n \neq s$. Then it follows that $m = r + a$ and $n = s + b$, for some $a, b \neq 0 \in \mathbb{Z}$. Then,

  $$m^2 + n^2 = (r+a)^2 + (s+b)^2 = r^2 + 2ra + a^2 + s^2 + 2sb + b^2 \tag{1}$$

  Also,

  $$mr + ns = r^2 + ra + s^2 + sb \tag{2}$$

  And since $r^2 + s^2 = mr + ns$, we deduce that

  $$r^2 + ra + s^2 + sb = r^2 + s^2 \implies ra + sb = 0 \tag{3}$$

  Since 1 and 2 are equal, we can set them equal and cancel terms, therefore:

  $$r^2+2ra+a^2+s^2+2sb+b^2 = r^2+ra+s^2+sb \implies a^2+b^2+ra+sb = 0 \tag{4}$$

  By 3, $ra + sb = 0$, so

  $$a^2 + b^2 = 0$$

  $\lightning$ QED

- Problem 2, List all the numbers between 6 and the next perfect number, including divisors and sums of those divisors. (Sorry, formatting is weird)

Numbers	7	8	9	10	11	12	13
Divisors	1,7	1,2,4,8	1,3,9	1,2,5,10	1,11	1,2,3,4,6,12	1,13
Sum of divisors	1	7	4	8	1	16	1
14	15	16	17	18	19	20	21
1,2,7,14	1,3,5,15	1,2,4,8,16	1,17	1,2,3,6,9,18	1,19	1,2,4,5,10,20	1,3,
10	9	15	1	21	1	22	11
22	23	24	25	26	27	28	
1,2,11,22	1,23	1,2,3,4,6,8,12,24	1,5,25	1,2,13,26	1,3,9,27	1,2,4,7,14,28	
14	1	36	6	16	13	28	

- Problem 9, Let $a, b, c \in \mathbb{Z}$ such that $a + b + c = 0$, show that if $n \in \mathbb{Z}$ divides two of the integers, it divides all three.

  Proof. If $n|a$ and $n|b$, then $a = np$, and $b = nq$. Then $a + b + c = np + nq + c = 0$. So $c = -n(p + q)$, therefore c is some multiple of n, thus $n|c$. QED

- Problem 15, For what $n \in \mathbb{Z}^+$ is $\gcd(n, n + 2) = 2$. Conjecture: $n = 2k, k \in \mathbb{N}$.

  Proof. Suppose $n = 2k, k \in \mathbb{N}$, (n is even). Then

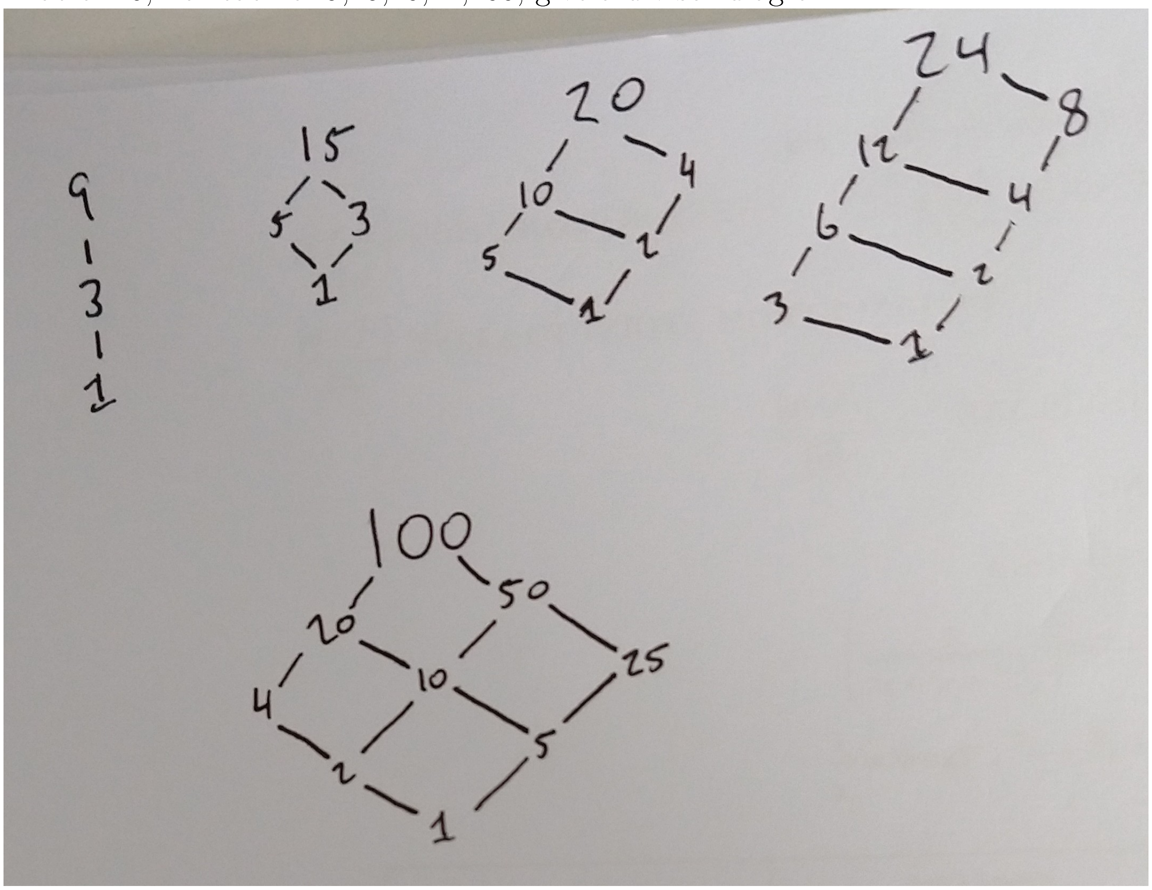
  $$\gcd(n, n + 2) = \gcd(2k, 2k + 2) = \gcd(2k, 2)$$

  Since $2k$ is always even, $2|2k$, and 2 is the largest divisor because the largest number than can divide 2 is 2. Therefore $\gcd(n, n + 2) = 2$ when $n = 2k$. QED

2 Section 1.2

- Problem 4, $\{1, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 49, 53, 59\}$. (For my work, sieved out all multiples of 2,3 and 5, this is what remained.)

- Problem 6, For each of 9,15,20,24,100, give a divisor diagram.



- Problem 10, Prove that $n^4 + 4$ is composite if $n > 1$.

  Proof.
  $$n^4 + 4 = (n^2 - 2n + 2)(n^2 + 2n + 2)$$
  And since
  $$(n^2 - 2n + 2) \in \mathbb{Z}$$
  and
  $$(n^2 + 2n + 2) \in \mathbb{Z}$$
  and
  $$(n^2 - 2n + 2) \neq (n^2 + 2n + 2)$$
  for $n > 1 \in \mathbb{Z}$, this is a product of two integers, therefore it is composite.
  QED

- Problem 24, Prove that every positive integer can be uniquely expressed as the product of a square and a square-free integer. (Kinda lost on this one but I'll give it a try)

  Proof. By the FTA, every positive integer can be uniquely expressed as the product of primes, like so:

$$a = (p_1^{n_1} p_2^{n_2} p_3^{n_3} ... p_k^{n_k}) = \prod_{i=1}^{k} p_i^{k_i}$$

  (Ok I'm confused here, for example how is 6 created with a square? Does 1 count as a square? It must because there's no other way to get all the primes.) I was thinking about splitting a into parts, one square part and one square-free part, like:

$$a = \underbrace{(p_1^{n_1} p_2^{n_2} p_3^{n_3} ... p_k^{n_k})^2}_{\text{Squares}} \cdot \underbrace{(q_1^{m_1} q_2^{m_2} q_3^{m_3} ... q_l^{m_l})}_{\text{Square-frees}}$$

  But I'm not sure if that's mathematically rigorous. QED