Math 341 Homework 12

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1 Practice problems

1.1 Problem 12.1

Prove that for every $n \geq 0$, $11^{12n+6} + 1 \vdots 13$.

Proof. N2S: $11^{12n+6} \equiv -1 \mod 13$. By definition,

$$11^{12n+6} = (11^{12})^n \cdot (11)^6$$

And by FlT, $a^{12} \equiv 1 \mod 13$, so

$$(11^{12})^n \cdot (11)^6 \equiv (1)^n \cdot (11)^6 \mod 13$$

For $n \geq 0$, $1^n = 1$. Also, $11 \equiv -2 \mod 13$ So

$$(1)^n \cdot (11)^6 = (-2)^6 = 64 \equiv -1 \mod 13$$

As required. QED

1.2 Problem 12.3

Let p be an odd prime. Prove that

$$\sum_{n=1}^{p-1} n \vdots p$$

Proof. Since p is odd, it must be of the form $p = 2k + 1$, for some $k \in \mathbb{Z}$. By example 1.1,

$$\sum_{n=1}^{p-1} n = \frac{(p-1)(p)}{2}$$

So

$$2\sum_{n=1}^{p-1} n = p(p-1)$$

Using $p = 2k + 1$,

$$p(p-1) = p(2k) = 2kp$$

Since twice the sum is equal to $2kp$, the sum must be equal to kp. By definition, $kp \vdots p$, As required. QED

1.3 Problem 12.6

Solve $x^{21} \equiv 6 \mod 7$.

Proof. Using FlT, $x^6 \equiv 1 \mod 7$. Since $x^{18} = (x^6)^3$, $x^{18} \equiv 1 \mod 7$. So $x^{21} \equiv x^3 \mod 7$.

We N2S that $x^3 - 6 \equiv 0 \mod 7$, in other words, $x^3 - 6 = 7n$, for some $n \in \mathbb{N}$.

$$x^3 - 6 = 7n$$

$$x^3 = 7n + 6$$

$$x = \sqrt[3]{7n + 6}$$

If $n = 3, 17, 30$, clearly $x \equiv 3, 5, 6 \mod 7$ are solutions, respectively. They are also the only solution because it is clear that $\nexists n \in \mathbb{N}$ such that:

$$0 = 7n + 6$$

$$1 = 7n + 6$$

$$8 = 7n + 6$$

$$64 = 7n + 6$$

And since these are all of the $x \in \mathbb{Z}_7$, we have found all the solutions and proven that there are no others. QED