Math 531 Homework 5

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1 Section 3.3

- Problem 10: Construct a group of order 12 that is not abelian.
  The dihedral group D_6, the symmetries of an regular hexagon.

- Problem 12:

  (a) Let $C_1 = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} | a = b\}$. Show that C_1 is a subgroup of $\mathbb{Z} \times \mathbb{Z}$.

  Proof.

  Theorem 1 (Subgroup Test)**.** *Let G be a group and let H be a nonempty subset of G. If for all $a, b \in H$, $ab^{-1} \in H$, then $H \leqslant G$.*

  Proof of 1 here. Using the above theorem, consider two elements of C_1, $(a, b), (c, d) \in C_1$, N2S: $(a, b) - (c, d) \in C_1$. By definition, $a = b, c = d$, thus $a - c = b - d$, therefore $(a, b) - (c, d) = (a - c, b - d) \in C_1$. As required. Thus, C_1 is a subgroup. QED

  (b) Show that $C_n = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} | a \equiv b \mod n\}$ is a subgroup of $\mathbb{Z} \times \mathbb{Z}$.

  Proof. Again using Theorem 1, consider 2 elements in $C_n : (a, b), (c, d)$, similarly, we N2S: $(a, b) - (c, d) \in C_n$. Since subtraction of congruence classes is well defined, this is true. ($a \equiv b \mod n, c \equiv d \mod n$, so $a - c \equiv b - d \mod n = (a, b) - (c, d) \in C_n$). QED

  (c) Show that every proper subgroup of $\mathbb{Z} \times \mathbb{Z}$ that contains C_1 has the form C_n, for some pos. int. n.

- Problem 14: Let G_1 and G_2 be groups, and let G be the direct product $G_1 \times G_2$. $H = \{(x_1, x_2) \in G_1 \times G_2 | x_2 = e\}$ and $K = \{(x_1, x_2) \in G_1 \times G_2 | x_1 = e\}$

  (a) Show that H and K are subgroups of G.

  Proof. By the subgroup test, if $ab^{-1} \in H, K$, then $H \leqslant G$ and $K \leqslant G$.
  ab^{-1} in H is the direct product of (a, e) and (b^{-1}, e), which is equal to $(ab^{-1}, e) \in H$. Therefore $H \leqslant G$. Similarly, ab^{-1} in K is the direct product of (e, a) and (e, b^{-1}), which is $(e, ab^{-1}) \in K$. Therefore $K \leqslant G$. QED

  (b) Show that $HK = KH = G$.

  Proof. The direct product HK is defined as the direct product $(x_1, e) \times (e, x_2) = (x_1e, ex_2) = (x_1, x_2) = G$. Similarly, KH is defined as the direct product $(e, x_2) \times (x_1, e) = (ex_1, x_2e) = (x_1, x_2) = G$. QED

  (c) Show that $H \cap K = (e, e)$.

  Proof. The intersection of two groups is the elements they share. In this case, the direct products (x_1, e) and (e, x_2) share one and only one element, namely when both $x_1 = x_2 = e$. This is the element (e, e), and it occurs in both H and K, and is the only element for which this is true. QED

2 Section 3.4

- Problem 4: Show that $\mathbb{Z}_{10}^{\times}$ is isomorphic to the additive group $\mathbb{Z}_4$.

Proof. We have an isomorphism $\phi : \mathbb{Z}_4 \to \mathbb{Z}_{10}^{\times}$ via the mapping:

- $0 \to 1$
- $1 \to 3$
- $2 \to 5$
- $3 \to 7$

Since 3 is a generator of $\mathbb{Z}_{10}^{\times}$, ϕ is a surjection. and since we laid out the mapping, it is easy to see that ϕ is an injection as well. (I just realized this is problem 1, whoops!) QED

- Problem 4: Show that $\mathbb{Z}_5^{\times}$ is not isomorphic to $\mathbb{Z}_8^{\times}$ by showing that the first group has an element of order 4 but the second does not.

  Proof. $\mathbb{Z}_5^{\times}$:

  - 1, $|1| = 1$
  - 2, $|2| = 4$
  - 3, $|3| = 4$
  - 4, $|4| = 2$

  $\mathbb{Z}_8^{\times}$:

  - 1, $|1| = 1$
  - 2, $|2| = \infty$
  - 3, $|3| = 2$
  - 4, $|4| = \infty$
  - 5, $|5| = 2$
  - 6, $|6| = \infty$
  - 7, $|7| = 2$

  QED

- Problem 5: Show that the group $(\mathbb{Q}, +)$ is not isomorphic to the group $(\mathbb{Q}^+, \cdot)$.

  Proof. The group $(\mathbb{Q}, +)$ has one, and only one, element of finite order, namely 0. $\mathbb{Q}^+, \cdot$ on the other hand, has two elements of finite order, $-1, 1$. Therefore, by proposition 3.4.3 in the book, there exists no isomorphism between these two. QED

- Problem 9: Prove that any group with three elements must be isomorphic to $\mathbb{Z}_3$.

Proof. Consider some arbitrary group $(G, *)$ with 3 distinct elements. Then it is safe to say that the elements are $G = \{e, a, b\}$. Since

$$ab = a \implies b = e \text{ and } ab = b \implies a = e$$

we can conclude that

$$ab = e$$

Also, since

$$a^2 = a \implies a = e \text{ and } a^2 = e \implies a = e$$

we can conclude that

$$a^2 = b$$

and thus

$$a^3 = a^2 a = ba = e$$

Therefore, a generates the whole set, so G is cyclic, and since $|G| = 3$, $(G, +)$ must be isomorphic to $\mathbb{Z}_3$. QED

- Problem 14: Let G be the following matrices over $\mathbb{R}$.
  $e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, a = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, b = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, c = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$
  Show that G is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. It is easy to see that $|e| = 1$, $|b| = 2$, $|c| = 2$, $|d| = 2$. The group $\mathbb{Z}_2 \times \mathbb{Z}_2$ has 4 elements, if their orders are $1, 2, 2, 2$, then there exists an isomorphism, by proposition 3.4.3 in the book. The elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ are as follows:

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0), (1,0), (0,1), (1,1)\}$$

We check:
$|e = (0,0)| = 1$.
$|(1,0)| = 2$, since $(1,0) + (1,0) = (2,0) = (0,0) \mod 2 = e$.
$|(0,1)| = 2$, since $(1,0) + (1,0) = (2,0) = (0,0) \mod 2 = e$.
$|(1,1)| = 2$, since $(1,1) + (1,1) = (2,2) = (0,0) \mod 2 = e$.
Thus, by proposition 3.4.3 in the book, G is isomorphic is $\mathbb{Z}_2 \times \mathbb{Z}_2$. QED