Math 531 Homework 6

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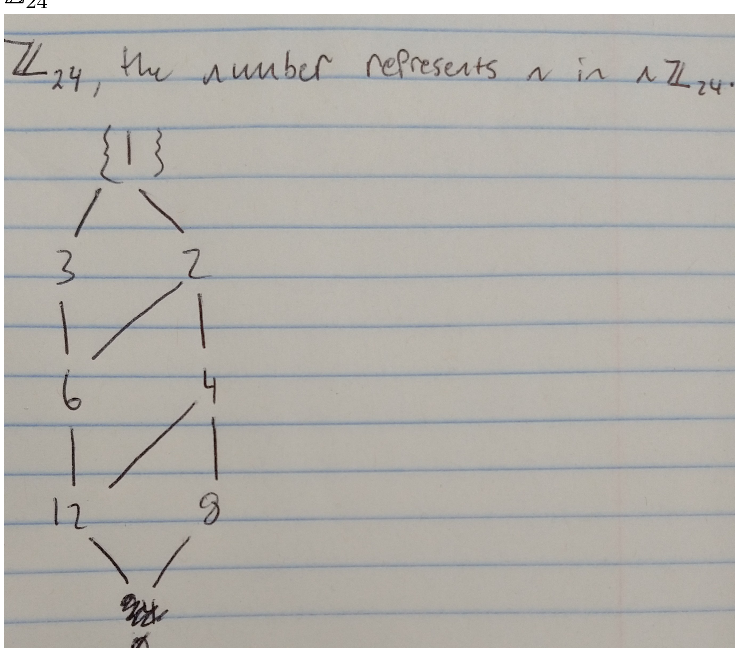
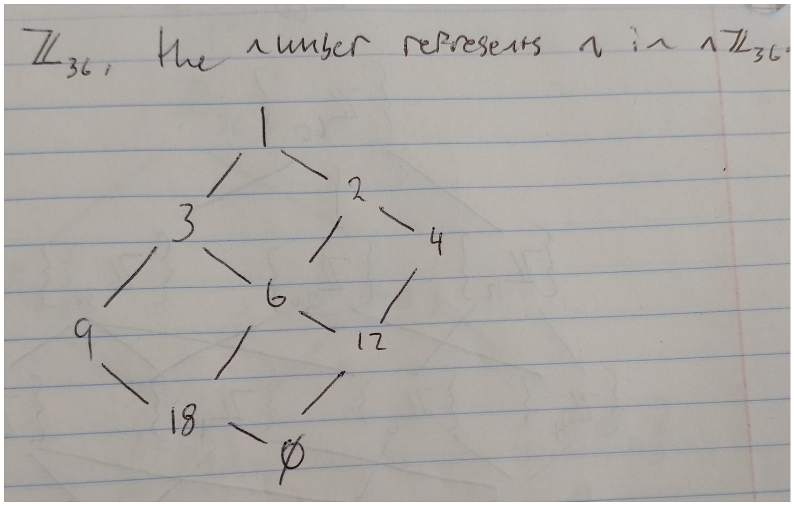
1 Section 3.5

- Problem 3: Give the subgroup diagrams of the following groups.

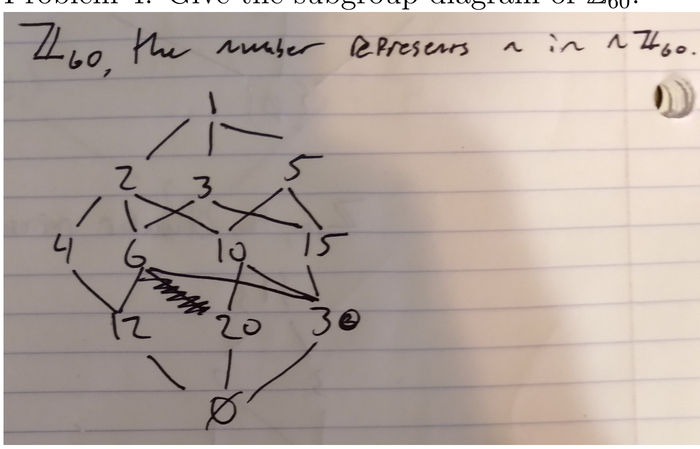
  (a) $\mathbb{Z}_{24}$

  (b) $\mathbb{Z}_{36}$



- Problem 4: Give the subgroup diagram of $\mathbb{Z}_{60}$.



- Problem 11: Which of the multiplicative groups, $\mathbb{Z}_7^\times$, $\mathbb{Z}_{10}^\times$, $\mathbb{Z}_{12}^\times$, $\mathbb{Z}_{14}^\times$ are isomorphic?

Remark. *Two finite cyclic groups of the same order are isomorphic.* *Proof.*

1. $\mathbb{Z}_7^\times$ is cyclic with generator 3, $|(\mathbb{Z}_7^\times)| = 6$.
2. $\mathbb{Z}_{10}^\times$ is cyclic with generator 3. $|\mathbb{Z}_{10}^\times| = 9$.
3. $\mathbb{Z}_{12}^\times$ is isomorphic to the klein 4-group, which is noncyclic. $|\mathbb{Z}_{12}^\times| = 4$.

4. $\mathbb{Z}_{14}^{\times}$ is cyclic with generator 3. $|\mathbb{Z}_{14}^{\times}| = 6$.

Therefore, by our remark, $\mathbb{Z}_{14}^{\times} \cong \mathbb{Z}_{7}^{\times}$, since they are both cyclic groups of the same order.

- Problem 21: Prove that if p and q are different odd primes, then $\mathbb{Z}_{pq}^{\times}$ is not a cyclic group.

  Proof. By the Chinese Remainder Theorem, since $\gcd(p, q) = 1$, it follows that $\mathbb{Z}_{pq}^{\times}$ is isomorphic to $\mathbb{Z}_{p}^{\times} \times \mathbb{Z}_{q}^{\times}$. Of course, $\mathbb{Z}_{p}^{\times}$ $\mathbb{Z}_{q}^{\times}$ are finite abelian groups, of order $p-1$ and $q-1$, respectively. And since p, q are odd primes, $p-1$ and $q-1$ are both divisible by 2, therefore they are not coprime. By the Fundamental Theorem of Finitely Generated Abelian Groups, the product of two finite abelian groups of non-coprime orders is never cyclic. QED

2 Section 3.6

- Problem 1: Find the orders of each of these permutations.

  (a) $(1, 2)(2, 3)(3, 4) = (1, 2, 3, 4)$, order 4.

  (b) $(1, 2, 5)(2, 3, 4)(5, 6) = (1, 2, 3, 4, 5, 6)$ order 6.

  (c) $(1, 3)(2, 6)(1, 4, 5) = (1, 4, 5, 3)(2, 6)$ order $lcm(4, 2) = 4$.

  (d) $(1, 2, 3)(2, 4, 3, 5)(1, 3, 2) = (1, 5, 3, 4)(2)$ order $lcm(4, 1) = 4$.

- Problem 3: Write out the addition table for $\mathbb{Z}_4 \times \mathbb{Z}_2$.

$\mathbb{Z}_4 = \{0,1,2,3\}$ $\quad$ $\mathbb{Z}_2 = \{0,1\}$

$$\mathbb{Z}_4 \times \mathbb{Z}_2 = \{\underset{a}{(0,0)}, \underset{b}{(0,1)}, \underset{c}{(1,0)}, \underset{d}{(1,1)}, \underset{e}{(2,0)}, \underset{f}{(2,1)}, \underset{g}{(3,0)}, \underset{h}{(3,1)}\}$$

+	a	b	c	d	e	f	g	h
a	a	b	c	d	e	f	g	h
b	b	a	d	c	f	e	h	g
c	c	d	e	f	g	h	a	b
d	d	c	f	e	h	g	b	a
e	e	f	g	h	a	b	c	d
f	f	e	h	g	b	a	d	c
g	g	h	a	b	c	d	e	f
h	h	g	b	a	d	c	f	e

(Now you can see why I usually use LaTex instead :))

- Problem 5: Show that no proper subgroup of S_4 contains both $(1,2,3,4)$ and $(1,2)$.

  Proof.

  Remark. *The group S_n is generated by $(1,2,...,n)$ and $(1,2)$.* *Proof.*

  Consider some subgroup $H \subseteq S_4$, such that $H \ni (1,2,3,4),(1,2)$. By our remark, any subgroup containing these two cycles must be exactly equal to S_n. Since both $(1,2,3,4),(1,2) \in H$, this proves that $H = S_4$ and thus $H \not\subset S_4$. QED

- Problem 10: Show that the following matrices form a subgroup of

$GL_2(\mathbb{C})$ isomorphic to D_4.
$\pm\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad \pm\begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix}, \quad \pm\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \quad \pm\begin{bmatrix}0 & i\\ -i & 0\end{bmatrix}.$

Proof. We first check, for $G =$ The matrices, $|G| = |D_4| = 8$. This is true. Therefore there *can* exist an isomorphism between the two. To show there must, $\forall a \in D_4$ we must find some $x \in G$ such that $|a| = |x|$. This will define the isomorphism $\phi : D_4 \to G$.

- $\rho_0 = e \in D_4$ has order 1. $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \in G$ also has order 1.
- $\rho_{90} \in D_4$ has order 4. $\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} \in G$ also has order 4.
- $\rho_{180} \in D_4$ has order 2. $\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} \in G$ also has order 2.
- $\rho_{270} \in D_4$ has order 4. $\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} \in G$ also has order 4.
- $\mu_1 \in D_4$ has order 2. $\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} \in G$ also has order 2.
- $\mu_1 \in D_4$ has order 2. $\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix} \in G$ also has order 2.
- $\mu_2 \in D_4$ has order 2. $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \in G$ also has order 2.
- $\mu_3 \in D_4$ has order 2. $\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \in G$ also has order 2.
- $\mu_4 \in D_4$ has order 2. $\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix} \in G$ also has order 2.

Where ρ_θ defines a rotation by θ degrees, and μ_1,μ_2,μ_3,μ_4, define a flip vertically, horizontally, diagonally about $\overline{13}$, and diagonally about $\overline{24}$, respectively. Therefore, for every element of D_4, there exists exactly one element of G with the same order, thus, $D_4 \cong G$. QED