Math 305 Final Review

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Problem 1

Draw a flow diagram and write down a continuous or a discrete model that could describe the changing populations below.

(a) 100 fish per unit time are removed from a pond that has a population of fish that grows at a steady 2% per year.

$$P(t+1) = 1.02P(t) - 100$$

(b) Baleen whales have a maximum growth rate of 5% per year and a carrying capacity of 400000 in the Antarctic fishery.

$$\frac{dW}{dt} = 0.05W(1 - \frac{W}{400000})$$

(c) A rabbit population in the absence of predators would grow at a yearly rate of r%. Wolves consume rabbits in a way that can be reasonably modeled by a mass-action term. If there are no rabbits to eat, wolves die at a rate proportional to their population size. Rabbits:

$$\frac{dR}{dt} = rR - aRW$$

Wolves:

$$\frac{dW}{dt} = cRW - dW$$

Problem 2

Consider an SIR model for a diphtheria epidemic in a school with 400 students, where each infected individual has a constant number, c, contacts with others in the population per unit time, and the average infectious period, $\frac{1}{\gamma}$, is 3 days. Probability of transmission when an infectious and a susceptible person 'meets' is $a = 0.2$.

(a) Write down a continuous SIR model describing this situation.

$$\frac{dS}{dt} = -\frac{ac}{\gamma}\frac{SI}{N}$$
$$\frac{dI}{dt} = \frac{ac}{\gamma}\frac{SI}{N} - \frac{I}{\frac{1}{\gamma}}$$
$$\frac{dR}{dt} = \frac{I}{\frac{1}{\gamma}}$$

(b) Give a condition for c that guarantees that an epidemic does not start, that is, $I(t) \leq I(0)$ for all $t \geq 0$.

$$\frac{dI}{dt} \leq 0$$
$$\frac{ac}{\gamma}\frac{SI}{N} - \frac{I}{\frac{1}{\gamma}} \leq 0$$
$$acS - \gamma^2 N \leq 0$$
$$c \leq \frac{\gamma^2 N}{aS}$$

(c) Assume that the contact number B is 5 for diphtheria, where $B = \frac{ac}{\gamma}$. What is the smallest percentage of the population that need to be immunized before somebody comes down with the disease to guarantee that an epidemic does not start?

$$I \geq \frac{ac}{\gamma}$$
$$I \geq B$$
$$p \geq \frac{B}{N} \cdot 100$$

Problem 3

Jack drinks 5 cups of strong coffee in succession just before his race in an athletic competition at the Olympics. Caffeine is very rapidly absorbed in the body. *A* strong cup of coffee has about 150 mg of caffeine. The half-life of caffeine in the body is about 3 hours.

(a) Formulate and solve a model that describes the amount of caffeine in his body.

$$
\begin{aligned}
C(0) &= 750mg \\
\frac{dC}{dt} &= -k \cdot C(t) \\
\frac{1}{2} &= e^{-k\cdot 3} \\
\implies k &= \frac{ln2}{3} \\
C(t) &= 750 \cdot e^{-\frac{ln2}{3}\cdot t}
\end{aligned}
$$

(b) It is estimated that only about 8% of caffeine in the body actually ends up in the plasma. A person has about. 0.04 liters of plasma per kg of body weight, and Jack weighs 80 kg. The International Olympic Committee limits caffeine plasma concentration to about 16.8 mg per liter. Is Jack in danger of violating this limit if he is tested 30 minutes after the race?

$$
\begin{aligned}
P(t) &= .08 \cdot C(t) \\
P(.5) &= .08 \cdot 750 \cdot e^{-(\frac{ln2}{3}\cdot .5)} \\
&\approx 53.45mg \\
\text{Jack has } 3.2L \text{ of plasma so concentration } &= \frac{53.45}{3.2} mg/L \\
&\approx 16.7mg/L < 16.8mg/L
\end{aligned}
$$

Jack is not in danger of violating the limit.

Problem 4

Consider a customer service model with three states: satisfied, neutral, and dissatisfied. The transition probabilities between these states are as follows:

- If a customer is satisfied, there is a 0.8 probability that they will remain satisfied in the next interaction, a 0.15 probability that they will become neutral, and a 0.05 probability that they will become dissatisfied.
- If a customer is neutral, there is a 0.3 probability that they will become satisfied, a 0.6 probability that they will remain neutral, and a 0.1 probability that they will become dissatisfied.
- If a customer is dissatisfied, there is a 0.2 probability that they will become satisfied, a 0.3 probability that they will become neutral, and a 0.5 probability that they will remain dissatisfied.

(a) What is the probability that if a customer starts in the neutral state they will be: satisfied after 3 interactions?

- $N \to S \to S \to S$
- $N \to S \to N \to S$
- $N \to S \to D \to S$
- $N \to N \to S \to S$
- $N \to N \to N \to S$
- $N \to N \to D \to S$
- $N \to D \to S \to S$
- $N \to D \to N \to S$
- $N \to D \to D \to S$

Are all the possible sequences which end in S. Now to calculate the probabilities:

- $P(N \to S \to S \to S) = 0.192$
- $P(N \to S \to N \to S) = 0.0135$
- $P(N \to S \to D \to S) = 0.003$
- $P(N \to N \to S \to S) = 0.144$
- $P(N \to N \to N \to S) = 0.108$
- $P(N \to N \to D \to S) = 0.012$
- $P(N \to D \to S \to S) = 0.016$
- $P(N \to D \to N \to S) = 0.009$
- $P(N \to D \to D \to S) = 0.01$

$$\begin{aligned}
P(N \to x \to x \to S) = {} & P(N \to S \to S \to S) \\
& + P(N \to S \to N \to S) \\
& + P(N \to S \to D \to S) \\
& + P(N \to N \to S \to S) \\
& + P(N \to N \to N \to S) \\
& + P(N \to N \to D \to S) \\
& + P(N \to D \to S \to S) \\
& + P(N \to D \to N \to S) \\
& + P(N \to D \to D \to S) \\
= {} & 0.192 + 0.0135 + 0.003 + 0.144 \\
+ {} & 0.108 + 0.012 + 0.016 + 0.009 + 0.01 \\
= {} & 0.5085
\end{aligned}$$

(b) Do you expect a steady-state for this problem, that is, will the probabilities of being in the different states settle after a long time? Why or why not?

Problem 5

In a board gaming club, there are five players (Alice, Bob, Charlie, David, and Emily) who regularly compete against each other in various board games. They use the Elo rating system to rank their skills. The current Elo ratings for each player are as follows:

- Alice: 1800
- Bob: 2000
- Charlie: 2100
- David: 1900
- Emily: 2200

They decide to have a board game tournament consisting of three matches. The results of the matches are as follows:

- Match 1: Alice defeats Bob.
- Match 2: Charlie defeats David.
- Match 3: Emily defeats Alice.

Using the Elo rating system, calculate the new Elo ratings for each player after the tournament. Assume the k-factor (the parameter that determines the amount of rating change) is 32. Determine the new rankings of the players based on their updated Elo ratings. New ratings:

- Alice: 1780.52
- Bob: 1988.03
- Charlie: 2112.97
- David: 1875.03
- Emily: 2211.48

Problem 6

Consider a one-dimensional cellular automaton with a binary state, where each cell can be either 0 or 1 . The evolution of the cellular automaton follows the following rule:

- If the cell is currently 0 and both of its neighboring cells are also 0 , it will remain 0 in the next generation.
- If the cell is currently 0 and exactly one of its neighboring cells is 1 , it will become 1 in the next generation.

- If the cell is currently 0 and both of its neighboring cells are 1 , it will remain 0 in the next generation.
- If the cell is currently 1 , it will become 0 in the next generation.

Consider an initial configuration of the cellular automaton in which all cells are 0 except for a segment of 10 cells. In this 10 cell segment the first cell is 1 , the second cell is 0 , and the remaining cells are randomly assigned either 0 or 1 .

(a) Write down the initial configuration of your cellular automaton.

(b) Calculate the configuration of the cellular automaton after one generation.

(c) Determine the configuration of the cellular automaton after two generations.

(d) Find the steady state configuration of the cellular automaton, if any.