Math 723 Homework 4

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Assignment 5

Chapter 5

15. Suppose $a \in \mathbb{R}$, f is a twice differentiable real function on (a, ∞), and M_0, M_1, M_2 are least upper bounds of $|f(x)|, |f'(x)|, |f''(x)|$, respectively on (a, ∞).

    (a). Prove that
    $$M_1^2 \leqslant 4M_0M_2$$

    Proof. By Taylor's theorem, for $h > 0$ we have the approximation:
    $$f'(x) = \frac{1}{2h}\left[f(x+2h) - f(x)\right] - hf''(\zeta)$$
    For some $\zeta \in (x, x+2h)$. Therefore,
    $$M_1 = |f'(x)| \leqslant hM_2 + \frac{M_0}{h}$$
    Taking the derivative of the right (w.r.t. h) gives $M_2 - \frac{M_0}{h^2}$. Since we want to minimize RHS, we let the derivative equal 0. Then
    $$M_2 = \frac{M_0}{h^2} \implies h^2 = \frac{M_0}{M_2} \implies h = \sqrt{\frac{M_0}{M_2}}$$
    Plugging this value for h above gives
    $$M_1 = \sqrt{\frac{M_0}{M_2}} \cdot M_2 + \frac{M_0}{\sqrt{\frac{M_0}{M_2}}} = 2\sqrt{M_0M_2}$$

Squaring both sides:

$$M_1^2 = 4M_0M_2$$

□

(b). Show that equality can actually happen. Define

$$f(x) = \begin{cases} 2x^2 - 1 & \text{if } (-1 < x < 0) \\ \frac{x^2-1}{x^2+1} & \text{if } (0 \leqslant x < \infty) \end{cases}$$

We have $M_0 = 1$ because $|\lim_{x\to-1} f(x)| = 1$. Therefore least upper bound is 1. $M_1 = 4$ because $|\lim_{x\to-1} 4x| = 4$, and similarly, $M_2 = 4$ because $f''(x) = 4$ for $-1 < x < 0$. (and second derivative of the other piece is never greater than 4). Therefore

$$M_1^2 = 16 = 4M_0M_2$$

(c). Does this inequality hold for vector valued functions? Not necessarily, because the mean value theorem need not hold for vector valued functions. (Remark 5.16)

22. Suppose f is a real function on $(-\infty, \infty)$. Call x a <u>fixed point</u> if $f(x) = x$.

(a). If f is differentiable and $f'(t) \neq 1$ for all real t, prove that f has at most one fixed point.

Proof. Let f be differentiable and $f'(t) \neq 1$ for all real t. Then define $g(x) = f(x) - x$, then we have roots of g are fixed points of f. To find roots of g, we can look at the derivative $g'(x) = f'(x) - 1$ (it is differentiable because f is, and clearly x is.) However we have that $f'(x) \neq 1$ for all real x, so the derivative is never equal to 0. This means the function g is monotonic, so it never crosses the x axis more than once, giving at most one fixed point of f. □

(b). Show that

$$f(t) = t + (1 + e^t)^{-1}$$

Has no fixed point, although $0 < f'(t) < 1$ for all real t.

Proof. Fixed point of f implies $f(t) = t$ which gives $(1+e^t)^{-1} = 0$. However, $\frac{1}{1+e^t}$ is never 0. So there are no fixed points of f, but the derivative is

$$f'(t) = 1 - \frac{1+e^t}{(1+e^t)^2}$$

Which is between 0 and 1. □

(c). If there is a constant $A < 1$ such that $|f'(t)| \leqslant A$ for all real t, prove that a fixed point of f exists, and that $x = \lim x_n$, where x_1 is an arbitrary real, and

$$x_{n+1} = f(x_n)$$

Proof. Again define $g(x) = f(x) - x$. We again never have $g'(x) = 0$ because $|f'(x)| < 1$. However, $\exists a,\ g(a) > 0$ and $\exists b,\ g(b) < 0$. Therefore, by the intermediate value theorem, there is a point c with $g(c) = 0$, which gives a fixed point x^* of f.
Now, considering the sequence $\{x_n\}$, with

$$x_{n+1} = f(x_n)$$

and $x_1 \in \mathbb{R}$. We have $x_{n+1} - x^* = f(x_n) - f(x^*)$. Then, by the mean value theorem, there exists some ζ such that $x_{n+1} - x^* = f'(\zeta)(x_n - x^*)$. This gives

$$|x_{n+1} - x^*| \leqslant A|x_n - x^*| \leqslant A^2|x_{n-1} - x^*| \leqslant \dots$$

And, since $|A| < 1$, this sequence converges to 0, so

$$\{x_n\} \to x^*$$

□

(d). I'm not sure what is meant by the zig-zag path.

Chapter 6

3. Define three functions $\beta_1, \beta_2, \beta_3$ as follows: $\beta_j(x) = 0$ if $x < 0$, $\beta_j = 1$ if $x > 0$ for $j = 1, 2, 3$; and $\beta_1(0) = 0,\ \beta_2(0) = 1,\ \beta_3(0) = \frac{1}{2}$. Let f be a bounded function on $[-1, 1]$.

(a). Prove that $f \in R(\beta_1)$ iff $f(x+) = f(0)$ and that then

$$\int f d\beta_1 = f(0)$$

Proof. ($\Longrightarrow$) Let $f \in R(\beta_1)$ and pick a $\varepsilon > 0$. Then, by theorem 6.6, we have for every $\varepsilon > 0$, there exists a partition P such that

$$U(P, f, \beta_1) - L(P, f, \beta_1) < \varepsilon$$

So, choose a parition $P = \{x_1, x_2, \ldots, x_n\}$ containing 0, say $x_k = 0$. Because $\Delta\beta_i = 0$ everywhere *except* when β_1 jumps to 1, we have:

$$U(P, f, \beta_1) = M_k = \sup f(x) \text{ For } (0 = x_k < x < x_{k+1})$$

And

$$L(P, f, \beta_1) = m_k = \inf f(x) \text{ For } (0 = x_k < x < x_{k+1})$$

Are epsilon-close when x is between x_k and x_{k+1}. We also have that M_k and m_k are respectively the supremum and infimum of $f(x)$, so

$$m_k \leqslant f(x) \leqslant M_k \implies |f(x) - f(0)| < \varepsilon$$

and so $f(x+) = f(0)$.
($\Longleftarrow$) Let $f(x+) = f(0)$. Then we have for any $\varepsilon > 0$, there exists a $\delta > 0$ with $|f(x) - f(0)| < \varepsilon$ when $0 < x < \delta$. Now consider a partition Q of $[-1, 1]$ which contains a point $x_k = 0$. By the equality above, we have for $x \in (0, \delta)$, that

$$\begin{aligned} |f(x) - f(0)| &\leqslant M_k - m_k \\ &= U(Q, f, \beta_1) - L(Q, f, \beta_1) \\ &< \varepsilon \end{aligned}$$

And so the Riemann-Stieltjes sums are epsilon-close, which gives

$$\int f d\beta_1$$

Exists, and equals $f(0)$. $\square$

(b). State and prove a similar result for β_2.
Similarly, we have $f \in R(\beta_2) \iff f(x-) = f(0)$.

Proof. Same as above! $\square$

(c). Prove that $f \in R(\beta_3)$ iff f is continuous at 0.

Proof. ($\implies$) Let $f \in R(\beta_3)$. Then there is a partition P for which the Riemann-Stieltjes sums are epsilon-close,

$$U(P, f, \beta_3) - L(P, f, \beta_3) < \varepsilon$$

In this case, because $\beta_3 = \frac{1}{2}$ for $x = 0$. We have that at the point in the partition where $x_k = 0$, $\Delta\beta_3 = \frac{1}{2}$ from above and below, so

$$\Delta\beta_3(x_k) = \frac{1}{2},\ \Delta\beta_3(x_{k-1}) = \frac{1}{2}$$

And $\Delta\beta_3 = 0$ everywhere else. Thus,

$$U(P, f, \beta_3) = \frac{M_{k-1} - M_k}{2}$$

And

$$L(P, f, \beta_3) = \frac{m_{k-1} - m_k}{2}$$

So then

$$U(P, f, \beta_3) - L(P, f, \beta_3) = \frac{(M_{k-1} - M_k) - (m_{k-1} - m_k)}{2} < \varepsilon$$

But $|f(x) - f(0)|$ is at most the max of $M_k - m_k$ or $M_{k-1} - m_{k-1}$. This gives

$$|f(x) - f(0)| \leqslant (M_k - m_k) + (M_{k-1} - m_{k-1}) < \varepsilon$$

As required.
($\impliedby$) Let $\varepsilon > 0$ be given and $|f(x) - f(0)| < \varepsilon$ for $x \in (0, \delta)$. Then,

$$|f(x) - f(0)| \leqslant (M_k - m_k) + (M_{k-1} - m_{k-1}) < \varepsilon$$

Gives that the Riemann-Stieltjes sums are epsilon-close, so $f \in R(\beta_3)$. $\square$

(d). If f is continuous at 0, prove that

$$\int f d\beta_1 = \int f d\beta_2 = \int f d\beta_3 = f(0)$$

Proof. Let f be continuous at 0, then clearly

$$f(x+) = f(x-) = \lim_{x \to 0} f(x) = f(0)$$

Gives

$$\int f d\beta_1 = \int f d\beta_2 = \int f d\beta_3 = f(0)$$

By (a), (b), and (c). □