Math 723 Homework 5

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Assignment 5

Chapter 5

15. Suppose $a \in \mathbb{R}$, f is a twice differentiable real function on (a, ∞), and M_0, M_1, M_2 are least upper bounds of $|f(x)|, |f'(x)|, |f''(x)|$, respectively on (a, ∞).

    (a). Prove that
    $$M_1^2 \leqslant 4M_0 M_2$$

    Proof. By Taylor's theorem, for $h > 0$ we have the approximation:
    $$f'(x) = \frac{1}{2h}\left[f(x + 2h) - f(x)\right] - hf''(\zeta)$$

    For some $\zeta \in (x, x + 2h)$. Therefore,
    $$M_1 = |f'(x)| \leqslant hM_2 + \frac{M_0}{h}$$

    Taking the derivative of the right (w.r.t. h) gives $M_2 - \frac{M_0}{h^2}$. Since we want to minimize RHS, we let the derivative equal 0. Then
    $$M_2 = \frac{M_0}{h^2} \implies h^2 = \frac{M_0}{M_2} \implies h = \sqrt{\frac{M_0}{M_2}}$$

    Plugging this value for h above gives
    $$M_1 \leqslant \sqrt{\frac{M_0}{M_2}} \cdot M_2 + \frac{M_0}{\sqrt{\frac{M_0}{M_2}}} = 2\sqrt{M_0 M_2}$$

Squaring both sides:
$$M_1^2 \leqslant 4M_0M_2$$

$\square$

(b). Show that equality can actually happen. Define

$$f(x) = \begin{cases} 2x^2 - 1 & \text{if } (-1 < x < 0) \\ \frac{x^2-1}{x^2+1} & \text{if } (0 \leqslant x < \infty) \end{cases}$$

We have $M_0 = 1$ because $|\lim_{x \to -1} f(x)| = 1$. Therefore least upper bound is 1. $M_1 = 4$ because $|\lim_{x \to -1} 4x| = 4$, and similarly, $M_2 = 4$ because $f''(x) = 4$ for $-1 < x < 0$. (and second derivative of the other piece is never greater than 4). Therefore

$$M_1^2 = 16 = 4M_0M_2$$

(c). Does this inequality hold for vector valued functions? Not necessarily, because the mean value theorem need not hold for vector valued functions, and Taylor's theorem depends on the MVT. (Remark 5.16)

22. Suppose f is a real function on $(-\infty, \infty)$. Call x a <u>fixed point</u> if $f(x) = x$.

(a). If f is differentiable and $f'(t) \neq 1$ for all real t, prove that f has at most one fixed point.

Proof. Let f be differentiable and $f'(t) \neq 1$ for all real t. Then define $g(x) = f(x) - x$, then we have roots of g are fixed points of f. To find roots of g, we can look at the derivative $g'(x) = f'(x) - 1$ (it is differentiable because f is, and clearly x is.) However we have that $f'(x) \neq 1$ for all real x, so the derivative is never equal to 0. This means the function g is monotonic, so it never crosses the x axis more than once, giving at most one fixed point of f. $\square$

(b). Show that
$$f(t) = t + (1 + e^t)^{-1}$$
Has no fixed point, although $0 < f'(t) < 1$ for all real t.

Proof. Fixed point of f implies $f(t) = t$ which gives $(1+e^t)^{-1} = 0$. However, $\frac{1}{1+e^t}$ is never 0. So there are no fixed points of f, but the derivative is

$$f'(t) = 1 - \frac{1 + e^t}{(1 + e^t)^2}$$

Which is between 0 and 1. $\qquad\square$

(c). If there is a constant $A < 1$ such that $|f'(t)| \leqslant A$ for all real t, prove that a fixed point of f exists, and that $x = \lim x_n$, where x_1 is an arbitrary real, and

$$x_{n+1} = f(x_n)$$

Proof. Again define $g(x) = f(x) - x$. We again never have $g'(x) = 0$ because $|f'(x)| < 1$. However, $\exists a,\ g(a) > 0$ and $\exists b,\ g(b) < 0$. Therefore, by the intermediate value theorem, there is a point c with $g(c) = 0$, which gives a fixed point x^* of f.
Now, considering the sequence $\{x_n\}$, with

$$x_{n+1} = f(x_n)$$

and $x_1 \in \mathbb{R}$. We have $x_{n+1} - x^* = f(x_n) - f(x^*)$. Then, by the mean value theorem, there exists some ζ such that $x_{n+1} - x^* = f'(\zeta)(x_n - x^*)$. This gives

$$|x_{n+1} - x^*| \leqslant A|x_n - x^*| \leqslant A^2|x_{n-1} - x^*| \leqslant \dots$$

And, since $|A| < 1$, this sequence converges to 0, so

$$\{x_n\} \to x^*$$

$\qquad\square$

(d). I'm not sure what is meant by the zig-zag path.

Chapter 6

3. Define three functions $\beta_1, \beta_2, \beta_3$ as follows: $\beta_j(x) = 0$ if $x < 0$, $\beta_j = 1$ if $x > 0$ for $j = 1, 2, 3$; and $\beta_1(0) = 0$, $\beta_2(0) = 1$, $\beta_3(0) = \frac{1}{2}$. Let f be a bounded function on $[-1, 1]$.

(a). Prove that $f \in R(\beta_1)$ iff $f(x+) = f(0)$ and that then

$$\int f d\beta_1 = f(0)$$

Proof. ($\Longrightarrow$) Let $f \in R(\beta_1)$ and pick a $\varepsilon > 0$. Then, by theorem 6.6, we have for every $\varepsilon > 0$, there exists a partition P such that

$$U(P, f, \beta_1) - L(P, f, \beta_1) < \varepsilon$$

So, choose a parition $P = \{x_1, x_2, \ldots, x_n\}$ containing 0, say $x_k = 0$. Because $\Delta\beta_i = 0$ everywhere *except* when β_1 jumps to 1, we have:

$$U(P, f, \beta_1) = M_k = \sup f(x) \text{ For } (0 = x_k < x < x_{k+1})$$

And

$$L(P, f, \beta_1) = m_k = \inf f(x) \text{ For } (0 = x_k < x < x_{k+1})$$

Are epsilon-close when x is between x_k and x_{k+1}. We also have that M_k and m_k are respectively the supremum and infimum of $f(x)$, so

$$m_k \leqslant f(x) \leqslant M_k \implies |f(x) - f(0)| < \varepsilon$$

and so $f(x+) = f(0)$.
($\Longleftarrow$) Let $f(x+) = f(0)$. Then we have for any $\varepsilon > 0$, there exists a $\delta > 0$ with $|f(x) - f(0)| < \varepsilon$ when $0 < x < \delta$. Now consider a partition Q of $[-1, 1]$ which contains a point $x_k = 0$. By the equality above, we have for $x \in (0, \delta)$, that

$$\begin{aligned} |f(x) - f(0)| &\leqslant M_k - m_k \\ &= U(Q, f, \beta_1) - L(Q, f, \beta_1) \\ &< \varepsilon \end{aligned}$$

And so the Riemann-Stieltjes sums are epsilon-close, which gives

$$\int f d\beta_1$$

Exists, and equals $f(0)$. □

(b). State and prove a similar result for β_2.
Similarly, we have $f \in R(\beta_2) \iff f(x-) = f(0)$.

Proof. Same as above! □

(c). Prove that $f \in R(\beta_3)$ iff f is continuous at 0.

Proof. ($\implies$) Let $f \in R(\beta_3)$. Then there is a partition P for which the Riemann-Stieltjes sums are epsilon-close,

$$U(P, f, \beta_3) - L(P, f, \beta_3) < \varepsilon$$

In this case, because $\beta_3 = \frac{1}{2}$ for $x = 0$. We have that at the point in the partition where $x_k = 0$, $\Delta\beta_3 = \frac{1}{2}$ from above and below, so

$$\Delta\beta_3(x_k) = \frac{1}{2}, \ \Delta\beta_3(x_{k-1}) = \frac{1}{2}$$

And $\Delta\beta_3 = 0$ everywhere else. Thus,

$$U(P, f, \beta_3) = \frac{M_{k-1} - M_k}{2}$$

And

$$L(P, f, \beta_3) = \frac{m_{k-1} - m_k}{2}$$

So then

$$U(P, f, \beta_3) - L(P, f, \beta_3) = \frac{(M_{k-1} - M_k) - (m_{k-1} - m_k)}{2} < \varepsilon$$

But $|f(x) - f(0)|$ is at most the max of $M_k - m_k$ or $M_{k-1} - m_{k-1}$. This gives

$$|f(x) - f(0)| \leqslant (M_k - m_k) + (M_{k-1} - m_{k-1}) < \varepsilon$$

As required.
($\impliedby$) Let $\varepsilon > 0$ be given and $|f(x) - f(0)| < \varepsilon$ for $x \in (0, \delta)$. Then,

$$|f(x) - f(0)| \leqslant (M_k - m_k) + (M_{k-1} - m_{k-1}) < \varepsilon$$

Gives that the Riemann-Stieltjes sums are epsilon-close, so $f \in R(\beta_3)$. □

(d). If f is continuous at 0, prove that

$$\int f d\beta_1 = \int f d\beta_2 = \int f d\beta_3 = f(0)$$

Proof. Let f be continuous at 0, then clearly

$$f(x+) = f(x-) = \lim_{x \to 0} f(x) = f(0)$$

Gives

$$\int f d\beta_1 = \int f d\beta_2 = \int f d\beta_3 = f(0)$$

By (a), (b), and (c). $\qquad\qquad\qquad\qquad\qquad\qquad\square$