Math 524 Homework 2

Theo Koss

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1. Give another proof of the Comparison Test using the Monotone Convergence Theorem. In other words, assume that $0 < a_n \leqslant b_n$ for all n and that $\sum b_n$ converges. Show that the sequence $(s_n\ )$ of partial sums of the series $\sum a_n$ is monotone and bounded above.

   Proof. Let $\sum a_n$ and $\sum b_n$ be series' such that $0 < a_n \leqslant b_n$ for all n and $\sum b_n$ converges. Let $\{s_k\}$ be the sequence of partial sums of a_n, and let $L = \sum_{n=1}^{\infty} b_n$. We must check 2 things:

   (a) $\{s_k\}$ is monotone increasing. Since we are given $a_n > 0$ (strictly), $s_{k+1} > s_k$ for all k. Thus $\{s_k\}$ is monotone increasing.

   (b) $\{s_k\}$ is bounded above. We are given $a_n \leqslant b_n$ for all n, so

   $$s_k = \sum_{n=1}^{k} a_n \leqslant \sum_{n=1}^{k} b_n \leqslant \sum_{n=1}^{\infty} b_n = L$$

   So $\{s_k\}$ is bounded above.

   Therefore $\{s_k\}$ converges and thus so does $\sum a_n$ QED

2. Give another proof of the Alternating Series Test using Cantor's Nested Intervals Theorem: If $[a_1, b_1] \supset [a_2, b_2] \supset \cdots$ is a nested sequence of closed, bounded intervals and $\lim_{n\to\infty} |b_n - a_n| = 0$ then the intersection $\bigcap [a_n, b_n]$ consists of a single point.

Proof. Let $\sum a_n$ where $a_n = (-1)^n b_n$ and $b_n \geqslant 0 \ \forall n$. Such that $\lim_{n\to\infty} b_n = 0$ and $\{b_n\}$ is a decreasing sequence. We must show that $\sum a_n$ is convergent.

To construct a set of nested closed, bounded intervals, we will look at the even and odd partial sums of a_n.

$$\{s_{2k} = s_2, s_4, s_6, \ldots, s_{2k}\}$$

$$\{s_{2k-1} = s_1, s_3, s_5, \ldots, s_{2k-1}\}$$

Each s looks like:

$$s_2 = a_1 - a_2, \ s_4 = (a_1 - a_2) + (a_3 - a_4), \ldots$$

$$s_1 = a_1, \ s_3 = a_1 - (a_2 - a_3), \ldots$$

Observe that s_{2k} (even partial sums) are monotone increasing. (Also, since these partial sums are never infinite, we never run into the grouping terms problem which was shown in class). And similarly, s_{2k+1} are monotone decreasing.
Now setting up our intervals, we have

$$[0, s_1] \supset [s_2, s_3] \supset [s_4, s_5] \supset \cdots \supset [s_{2k}, s_{2k+1}]$$

Which are nested, closed, and bounded, and we were given that $\lim_{n\to\infty} b_n = 0$. Therefore we have $\bigcap [s_{2k}, s_{2k+1}] \ni x$ for some x. Therefore $\sum a_n$ converges. QED

3. In this problem, you will examine a specific rearrangement of a conditionally convergent series for which you will see that the value of the sum has definitely changed! Let S be the sum of the alternating harmonic series: $S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$. Multiplying each term by $1/2$ gives a new series that converges to $S/2$. Add these two series together to get a third series. Show that this third series is a rearrangement of

the alternating harmonic series, but that its sum is now $(3/2)S$.

$$\begin{aligned}
S &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \\
\frac{S}{2} &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} + \dots \\
\frac{3}{2}S &= 1 + \frac{1}{3} - \frac{2}{4} + \frac{1}{5} + \frac{1}{7} - \frac{2}{8} + \dots \\
&= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots
\end{aligned}$$

The even terms will cancel out if mutliple of 2 (but not 4), and will add together if multiple of 4. (Giving the one which was cancelled out always)

4. Consider the infinite series $\sum_{n=0}^{\infty} \frac{1}{2n+1}$. Determine whether this series converges or diverges. Explain.
Using the limit comparison test with the harmonic series:

$$\lim_{n\to\infty} \frac{\frac{1}{2n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$$

Which is finite and nonzero, and since the harmonic series diverges, so does this series.

5. For each power series, find the set of all values of x for which the series converges or diverges. [Hint: The Ratio Test is often useful for finding an interval of convergence, but there may be points at which it gives an inconclusive result.]

   (a) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

   (b) $\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots$