Math 835 Homework 6

Theo Koss

October 2024

Chapter 13

Section 6

1. Suppose m and n are coprime positive integers. Let ζ_m be a primitive m-th root of unity and let ζ_n be a primitive n-th root of unity. Prove that $\zeta_m\zeta_n$ is a primitive mn-th root of unity.

   Proof. ζ_m and ζ_n are primitive resp. m and n-th roots of unity, so we have $\zeta_m^m = 1$ and $\zeta_n^n = 1$ and $\zeta_m^k \neq 1$, and $\zeta_n^l \neq 1$ for any $1 \leqslant k \leqslant m-1$ and $1 \leqslant l \leqslant n-1$. Then

   $$(\zeta_m\zeta_n)^{mn} = 1$$

   And because m and n are coprime $\mathrm{lcm}(m,n) = mn$, so we do not have a smaller i for which $(\zeta_m\zeta_n)^i = 1$. Therefore $\zeta_m\zeta_n$ is a primitive mn-th root of unity. □

2. Let ζ_n be a primitive $n-$th root of unity and let d be a divisor of n. Prove that ζ_n^d is a primitive n/d-th root of unity.

   Proof. Let $q = n/d$, we have

   $$(\zeta_n^d)^q = \zeta_n^{dq} = \zeta_n^n = 1$$

   So ζ_n^d is clearly a $q-$th root of unity. Now it remains to show that ζ_n^d is primitive in μ_q. We have $\zeta_n^d = \zeta_q$ and since 1 is coprime to q, ζ_q^1 is primitive in μ_q. □

13.