Math 835 Homework 4

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Chapter 13

Section 4

1. Determine the splitting field and its degree over $\mathbb{Q}$ for $x^4 - 2$.

$$4 \text{ roots: } x = \{\pm\sqrt[4]{2}, \pm i\sqrt[4]{2}\}$$

Field: $\mathbb{Q}(\sqrt[4]{2}, i)$ has degree

$$[\mathbb{Q}(\sqrt[4]{2}, i) : \mathbb{Q}] = [\mathbb{Q}(\sqrt[4]{2}, i) : \mathbb{Q}(\sqrt[4]{2})] \cdot [\mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q}] = 4 \cdot 2 = 8$$

Minimal polynomial of $\sqrt[4]{2}$ is $x^4 - 2$, minimal polynomial of i is $x^2 + 1$.

2. Determine the splitting field and its degree over $\mathbb{Q}$ for $x^4 + 2$.

$$4 \text{ roots: } x = \{\pm\sqrt[4]{-2}, \pm i\sqrt[4]{-2}\}$$

Field: $\mathbb{Q}(\sqrt[4]{-2}, i)$ has degree

$$[\mathbb{Q}(\sqrt[4]{-2}, i) : \mathbb{Q}] = [\mathbb{Q}(\sqrt[4]{-2}, i) : \mathbb{Q}(\sqrt[4]{-2})] \cdot [\mathbb{Q}(\sqrt[4]{-2}) : \mathbb{Q}] = 4 \cdot 2 = 8$$

Minimal polynomial of $\sqrt[4]{-2}$ is $x^4 + 2$, minimal polynomial of i is $x^2 + 1$.

3. Determine the splitting field and its degree over $\mathbb{Q}$ for $x^4 + x^2 + 1$.

$$\text{Let } t = x^2,\ t^2 + t + 1 = 0 \text{ Quadratic formula: } t = \frac{-1 \pm \sqrt{-3}}{2}$$

Notice $y = -(x^2)$ also has $x^4 + x^2 + 1 = 0 \iff y^2 - y + 1 = 0$. So quadratic formula again,

$$y^2 - y + 1 = 0 \implies y = \frac{1 \pm \sqrt{-3}}{2}$$

So the 4 roots are

$$x = \{t, y\} = \left\{ \frac{-1 \pm \sqrt{-3}}{2}, \frac{1 \pm \sqrt{-3}}{2} \right\}$$

So we only need to adjoin $\sqrt{-3}$, which has minimal polynomial $x^2 + 3$, so $[\mathbb{Q}(\sqrt{-3}) : \mathbb{Q}] = 2$, need only a quadratic extension.