Math 523 Homework 6

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Problems

1. Use definition of derivative to determine whether f is differentiable at 0, and if so find the value $f'(0)$, do the same for g.

   $$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h} = \lim_{h\to 0} \frac{h\sin\left(\frac{1}{h}\right)}{h} = \lim_{h\to 0} \sin\left(\frac{1}{h}\right)$$

   This does not exist. So f is not differentiable at 0. $g(x) = xf(x)$ so same calculation as above but:

   $$g'(0) = \cdots = \lim_{h\to 0} h\sin\left(\frac{1}{h}\right) = 0$$

   Does exist and equals 0 via the squeeze theorem.

2. Show that if f is even then f' is odd, and if f odd, f' even.

   Proof. Suppose f is even, so $f(-x) = f(x)$. Then by definition of derivative,

   $$f'(-x) = \lim_{h\to 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{h\to 0} \frac{f(-(x-h)) - f(x)}{h} =$$

   $$= \lim_{h\to 0} \frac{f(x-h) - f(x)}{h} = -\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = -f'(x)$$

   Suppose f is odd, so that $f(-x) = -f(x)$. Then

   $$f'(-x) = \lim_{h\to 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{h\to 0} \frac{f(-(x-h)) + f(x)}{h} =$$

$$= \lim_{h \to 0} \frac{-f(x-h) + f(x)}{h} = -\lim_{h \to 0} \frac{f(x-h) - f(x)}{h} =$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

So f' is even. QED

3. Show that the function $f(x) = x^3 + 2x + 1$ is strictly increasing on $\mathbb{R}$ and thus, has an inverse function f^{-1} on $\mathbb{R}$. Find the value of $(f^{-1})'(y)$ at the points corresponding to $x = 0, 1, -1$.

Proof. $f'(x) = 3x^2+2$ which is never negative for $x \in \mathbb{R}$, so f is strictly increasing on $\mathbb{R}$. By the inverse function theorem,

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(x)}$$

- At $x = 0$, $\left(f^{-1}\right)'(y) = \frac{1}{2}$
- At $x = 1$, $\left(f^{-1}\right)'(y) = \frac{1}{5}$
- At $x = -1$, $\left(f^{-1}\right)'(y) = \frac{1}{5}$

QED

4. (a) Use definition of derivative to prove: If f is an increasing function that is differentiable on an interval I, then $f'(x) \geqslant 0$ for all $x \in I$.
   (b) Is the following true or false? If f is strictly increasing and differentiable on I, then $f'(x) > 0$ on I. Explain

Proof. (a) : Let f be an increasing function which is differentiable on I. Then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

But $f(x+h) \geqslant f(x)$ since f is increasing, so numerator is nonnegative $\implies f'(x) \geqslant 0$ for x in the interval.
(b) : True, by f being strictly increasing, the statement goes from: $[f(x+h) \geqslant f(x)] \to [f(x+h) > f(x)]$ so instead of the numerator being nonnegative, it becomes strictly positive. QED

(WLOG assuming $h \to 0$ from the right, if from the left, denominator is negative as is numerator so still $f'(x) \geqslant 0$.)

5. Show that if $f'(c) = 0$ for some $c \in I$ and $f''(x) > 0$ for all $x \in I$, then f has a minimum value at c. Similarly, if $f'(c) = 0$ for some $c \in I$ and $f''(x) < 0$ then f has a maximum value at c.

   Proof. Let f be twice differentiable, $f''(x) > 0$ $\forall x \in I$ and $f'(c) = 0$ for some $c \in I$. Then

   $$f''(c) = \lim_{h\to 0} \frac{f'(c+h) - f'(c)}{h} = \lim_{h\to 0} \frac{f'(c+h)}{h} > 0$$

   So for small h: $\frac{f'(c+h)}{h} > 0$, which means if $h \to 0$ from the left, $f'(c+h) < 0$, so f is decreasing, and if $h \to 0$ from the right, $f'(c+h) > 0$, so that f is increasing. By the first derivative test, f has a local minimum at c.
   Similarly, let f be twice diff., $f''(x) < 0$ $\forall x \in I$ and $f'(c) = 0$ for some $c \in I$. Then follow the same argument but: $\frac{f'(c+h)}{h} < 0$ for small h. So if $h \to 0$ from left, $f'(c+h) > 0$ so f is increasing, and if $h \to 0$ from right, $f'(c+h) < 0$ so f is decreasing. Again apply first derivative test, $\implies$ f has a local maximum at c. QED

6. Evaluate the following limits:

   (a)

   $$\lim_{x\to 0^+} \frac{\sin x}{\sqrt{x}} \quad (0, \infty)$$

   $$\lim_{x\to 0^+} \frac{\sin x}{\sqrt{x}} = \lim_{x\to 0^+} \frac{2\sqrt{x}\cos x}{1} = 0$$

   (b)

   $$\lim_{x\to 0^+} \frac{\tan x - x}{x^3} \quad \left(0, \frac{\pi}{2}\right)$$

   $$\lim_{x\to 0^+} \frac{\tan x - x}{x^3} = \lim_{x\to 0^+} \frac{\sec^2 x - 1}{3x^2} = \lim_{x\to 0^+} \frac{2\sec^2 x \tan x}{6x} =$$

   $$= \lim_{x\to 0^+} \frac{-4\sec^2 x + 6\sec^4 x}{6} = \frac{1}{3}$$

(c)

$$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin x}\right)\ \left(0,\frac{\pi}{2}\right)$$

$$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin x}\right)=\lim_{x\to 0^+}\left(1-\frac{1}{\cos x}\right)=1-1=0$$

7. If $\lim_{x\to\infty} f(x)=\lim_{x\to\infty} g(x)=\infty$ and $\lim_{x\to\infty}\frac{f'(x)}{g'(x)}=L\in\mathbb{R}$, then $\lim_{x\to\infty}\frac{f(x)}{g(x)}=L$.

   Use result above to conclude the following: If $\lim_{x\to 0} f(x)=\lim_{x\to 0} g(x)=\infty$ and $\lim_{x\to 0}\frac{f'(x)}{g'(x)}=L\in\mathbb{R}$ then $\lim_{x\to 0}\frac{f(x)}{g(x)}=L$.

(a) *Proof.* Consider $\lim_{x\to 0^+}\frac{f(x)}{g(x)}$ where $\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+} g(x)=\infty$, and $\lim_{x\to 0^+}\frac{f'(x)}{g'(x)}=L\in\mathbb{R}$. QED