Math 531 Homework 10

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1 Section 5.1

- Problem 1: Which of the following sets are subrings of the field $\mathbb{Q}$ of rational numbers? Assume that $m, n \in \mathbb{Z}$ with $n \neq 0$ and $(m, n) = 1$.

  (a) $\{\frac{m}{n} | n \text{ is odd}\}$

  Proof. Call this set H. Clearly H is a subset of $\mathbb{Q}$. So we need to show: $\forall a, b \in H$, $a - b \in H$ and $a \cdot b \in H$. Let $a = \frac{m_1}{n_1}$ and $b = \frac{m_2}{n_2}$. Therefore, $(m_1, n_1) = (m_2, n_2) = 1$. And n_1, n_2 are odd numbers.

  i $a - b = \frac{m_1}{n_1} - \frac{m_2}{n_2} = \frac{m_1 n_2 - m_2 n_1}{n_1 n_2} \in H$, since $n_1 n_2$, the product of 2 odd numbers, is odd.

  ii $a \cdot b = \frac{m_1 m_2}{n_1 n_2} = \frac{c}{d}$, clearly $(c, d) = 1$ and d is odd by the argument above.

  Thus, H is a subring of $\mathbb{Q}$. QED

  (b) $\{\frac{m}{n} | n \text{ is even}\}$.

  Proof. Let $a = \frac{m_1}{n_1}$, $b = \frac{m_2}{n_2}$

  $$a - b = \frac{m_1}{n_1} - \frac{m_2}{n_2} = \frac{m_1 n_2 - m_2 n_1}{n_1 n_2} \in H$$

  Since $m_1 n_2 - m_2 n_1 \in \mathbb{Q}$ and $n_1 n_2$ is even.

  $$a \cdot b = \frac{m_1 m_2}{n_1 n_2} \in H$$

  Since $m_1 m_2 \in \mathbb{Q}$, and $n_1 n_2$ is even. Thus H is a subring. QED

(c) $\{\frac{m}{n} | 4 \not| n\}$

(d) $\{\frac{m}{n} | (n, k) = 1\}$ where k is a fixed positive integer.

Proof. Suppose $a = \frac{m_1}{n_1}$ and $b = \frac{m_2}{n_2}$ are in the set. Then by definition, $(n_1, k) = 1$ and $(n_2, k) = 1$ and thus, $(n_1 n_2, k) = 1$. Now we need to show that $\forall a, b \in H$, $a - b \in H$ and $a \cdot b \in H$.

i $a - b = \frac{m_1}{n_1} - \frac{m_2}{n_2} = \frac{m_1 n_2 - m_2 n_1}{n_1 n_2} \in H$, because as we showed above, $(n_1 n_2, k) = 1$.

ii $a \cdot b = \frac{m_1 m_2}{n_1 n_2} \in H$, of course.

Thus this subset forms a subring. QED

- Problem 2: Which of the following sets are subrings of the field $\mathbb{R}$ of real numbers?

(a) $A = \{m + n\sqrt{2} | m, n \in \mathbb{Z} \text{ and } n \text{ is even}\}$

Proof. Let $a_1, a_2 \in A$, then $a = m_1 + n_1\sqrt{2}$ and $b = m_2 + n_2\sqrt{2}$. Now,

$$a_1 - a_2 = (m_1 - m_2) + (n_1 - n_2)\sqrt{2} \in A$$

subtraction of two integers is an integer, and subtraction of two even integers is another even integer.

$$\begin{aligned} a_1 a_2 &= (m_1 + n_1\sqrt{2})(m_2 + n_2\sqrt{2}) \\ &= m_1 m_2 + m_1 n_2\sqrt{2} + n_1 m_2\sqrt{2} + 2n_1 n_2 \\ &= (m_1 m_2 + 2n_1 n_2)(m_1 n_2 + n_1 m_2)\sqrt{2} \in A \end{aligned}$$

Clearly, $m_1 n_2 + n_1 m_2$ is even because n_1 and n_2 are even. Thus A is a subring of $\mathbb{R}$. QED

(b) $B = \{m + n\sqrt{2} | m, n \in \mathbb{Z} \text{ and } n \text{ is odd}\}$

Proof. Again, let $a_1, a_2 \in B$, then $a = m_1 + n_1\sqrt{2}$ and $b = m_2 + n_2\sqrt{2}$. Then

$$a_1 - a_2 = (m_1 - m_2) + (n_1 - n_2)\sqrt{2}$$

$m_1 - m_2$ may not be odd, therefore $a_1 - a_2 \notin B$, and thus B is not a subring of $\mathbb{R}$. QED

(c) $C = \{a + b\sqrt[3]{2} | a, b \in \mathbb{Q}\}$

Proof. Let $m_1 = a_1 + b_1\sqrt[3]{2}$, $m_2 = a_2 + b_2\sqrt[3]{2}$.

$$m_1 m_2 = (a_1 + b_1\sqrt[3]{2})(a_2 + b_2\sqrt[3]{2})$$

$$= a_1 a_2 + (2^{\frac{2}{3}} b_1 b_2) + (a_1 b_2 + a_2 b_1)\sqrt[3]{2}$$

$2^{\frac{2}{3}}$ is not a rational number, therefore $m_1 m_2 \notin C$ and thus C is not a subring of $\mathbb{R}$. QED

(d) $D = \{a + b\sqrt[3]{3} + c\sqrt[3]{9} | a, b, c \in \mathbb{Q}\}$

Proof. Let $m_1 = a_1 + b_1\sqrt[3]{3} + c_1\sqrt[3]{9}$ and $m_2 = a_2 + b_2\sqrt[3]{3} + c_2\sqrt[3]{9}$.

$$m_1 m_2 = (a_1 + b_1\sqrt[3]{3} + c_1\sqrt[3]{9})(a_2 + b_2\sqrt[3]{3} + c_2\sqrt[3]{9})$$

$$= a_2 a_2 + a_1 b_2\sqrt[3]{3} + a_1 c_2\sqrt[3]{9} + b_1 a_2\sqrt[3]{3} + b_1 b_2 3^{\frac{2}{3}} + ...$$

There's more stuff but I know $3^{\frac{2}{3}} \notin \mathbb{Q}$ so I'll stop there. QED

(e) $E = \{m + nu\}$

Proof. QED

(f) $F = \{\}$

Proof. QED

- Problem 6: Show that no proper nontrivial subset of $\mathbb{Z}$ can form a ring under the usual operations of addition and multiplication.

  Proof. Suppose, for the sake of contradiction, that $(M, +, \cdot)$ is a ring, where M is a nontrivial subset of the integers. Then by definition of ring, the multiplicative identity $1 \in M$, and also by definition, each element should have an additive inverse in M. Thus, $-1 \in M$.
  Now for any integer $n \in \mathbb{Z}^+$, we have:

  $$\underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} = n$$

  $$\implies n \in M \implies \mathbb{Z}^+ \subseteq M$$

And by definition of a ring, $\forall n \in M$, $\exists -n \in M$, where

$$n + (-n) = 0 \implies \underbrace{-1 + (-1) + \cdots + (-1)}_{n \text{ times}} = -n$$

$$\implies \mathbb{Z} \subseteq M \implies M = \mathbb{Z}$$

This is a contradiction, because we said M was a proper subset of $\mathbb{Z}$, implying $M \neq \mathbb{Z}$. QED