# Math 723 Homework 6

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## Assignment 6

### Chapter 6

8. Suppose $f$ is Riemann integrable on $[a, b]$ for every $b > a$ with $a$ fixed. Define
$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$$
if this limit exists and is finite. Then the integral is said to converge. Assume that $f(x) \geqslant 0$ and that $f$ decreases monotonically on $\overline{[1,\infty]}$. Prove that
$$\int_1^\infty f(x)dx$$
Converges iff
$$\sum_{n=1}^{\infty} f(n)$$
Converges.

*Proof.* ( $\Longrightarrow$ ) Assume $\int_1^\infty f(x)dx$ converges to some $L < \infty$. Consider a partition $\{x_0, \ldots, x_k\}$ of $[1, N]$ with length 1. The upper Riemann sum is then
$$\sum_{k=1}^{N} f(k)$$
And the lower sum is
$$\sum_{k=2}^{N} f(k)$$

And we have the inequality

$$\sum_{k=1}^{N} f(k) \leqslant \int_{1}^{N} f(x)dx \leqslant \sum_{k=2}^{N} f(k)$$

Taking the limit as $N \to \infty$ gives

$$\sum_{n=1}^{\infty} f(k) \leqslant \int_{1}^{\infty} f(x)dx = L$$

And so $\sum f(k)$ is bounded above and increasing, so it converges.

( $\Longleftarrow$ ) Let $\sum_{n=1}^{\infty} f(n)$ converge to some $L < \infty$. We then have the same inequality

$$\sum_{k=1}^{N} f(k) \leqslant \int_{1}^{N} f(x)dx \leqslant \sum_{k=2}^{N} f(k)$$

Which, when passed to the limit, gives

$$L \leqslant \int_{1}^{\infty} f(x)dx \leqslant L + f(1)$$

And so $\int_{1}^{\infty} f(x)dx$ converges. $\square$

16. For $1 < s < \infty$, defined

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Prove:

(a). $\zeta(s) = s\int_1^\infty \frac{\lfloor x \rfloor}{x^{s+1}}dx$

(b). $\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{x-\lfloor x \rfloor}{x^{s+1}}dx$

*Proof.* Let $N \in \mathbb{Z}^+$, then we have

$$\begin{aligned}
s\int_1^N \frac{\lfloor x \rfloor}{x^{s+1}}dx &= s\sum_{k=1}^{N} k\int_k^{k+1} \frac{1}{x^{s+1}}dx \\
&= \sum_{k=1}^{N} k\left[\frac{1}{k^s} - \frac{1}{(k+1)^s}\right] \\
&= \left[\frac{1}{1} - \frac{1}{2^s}\right] + 2\left[\frac{1}{2^s} - \frac{1}{3^s}\right] + \cdots + N\left[\frac{1}{N^s} - \frac{1}{(N+1)^s}\right] \\
&= \sum_{k=1}^{N} \frac{1}{k^s} \\
&= S_N && \text{(N-th partial sum)}
\end{aligned}$$

And since $N$ was arbitrary, we have that $\zeta(s) = s\int_1^\infty \frac{\lfloor x \rfloor}{x^{s+1}}dx$. Then, for part b, we have that $\frac{s}{s-1} = \int_1^\infty \frac{x}{x^{s+1}}$, so

$$\begin{aligned}
\zeta(s) &= s\int_1^\infty \frac{\lfloor x \rfloor}{x^{s+1}}dx \\
&= \int_1^\infty \frac{x}{x^{s+1}} - s\int_1^\infty \frac{x - \lfloor x \rfloor}{x^{s+1}}dx \\
&= \frac{s}{s-1} - s\int_1^\infty \frac{x - \lfloor x \rfloor}{x^{s+1}}dx
\end{aligned}$$

□

## Chapter 7

10. Letting $(x)$ denote the fractional part of a real number, consider

$$f(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2}$$

Find all discontinuities of $f$, and prove that they form a countable dense set. Show that $f$ is nevertheless Riemann integrable on every bounded interval.

*Proof.* Let $k$ be a fixed integer. A near-integer value of $kx$ will give a positive real number, however when

$$kx \in \mathbb{Z} \implies (kx) = 0$$

So we have a discontinuity for

$$kx \in \mathbb{Z} \implies x = \frac{a}{k} \in \mathbb{Q}$$

Specifically, the $k$-th term in the series

$$f_k(x) = \frac{(kx)}{k^2}$$

has a discontinuity at $x = \frac{a}{k}$ for each $a \in \mathbb{Z}$. We also see that if we consider $z \in \mathbb{R} \setminus \mathbb{Q}$, there exists *no* term in the series $f_k(x)$ for which $(kz) = 0$, that is, there is no number you can multiply $z$ by to get an integer (from assignment 1, rational times irrational is irrational). Therefore, the discontinuities of $f$ are $\mathbb{Q}$, a countable dense subset of $\mathbb{R}$.

Nevertheless, each term of the series is still Riemann integrable, since there are only countably many discontinuities. Then because each term is integrable, by Theorem 7.16, $f$ is also Riemann integrable. □

20. If $f$ is continuous on $[0,1]$ and if

$$\int_0^1 f(x)x^n dx = 0 \qquad \text{(n=1,2,3,...)}$$

Prove that $f(x) = 0$ on $[0,1]$.

*Proof.* Because $f$ is a continuous function, by the Stone-Weierstrass Theorem (7.26), there exist a sequence of polynomials $P_n$ such that

$$\lim_{n\to\infty} P_n = f(x)$$

Uniformly. Then consider the integral

$$\lim_{n\to\infty} \int_0^1 P_n f(x)dx = \int_0^1 f^2(x)dx$$

But the integral of each product $P_n f(x)$ is 0, so $\int_0^1 f^2(x)dx = 0$. This implies that $f(x) = 0$ on $[0,1]$ as required. □