Math 551 Homework 1

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1 Section 2.1

- Problem 2: Explain why a set of real numbers cannot have more than one least upper bound or more than one greatest lower bound.
  Consider some set $A \subseteq \mathbb{R}$, where $s \in A$ is the least upper bound. For there to be more than one l.u.b, there would have to be some $a \in A$ such that a is an upper bound, and $a \leq s$. If $a < s$, then a is the least upper bound, and if $e = s$, then there is only 1 l.u.b, and it's $s = a$.
  A similar argument follows for g.l.b. Consider some $l \in A$ where l is a greatest lower bound. Then again, for there to be another g.l.b, there would have to be some $b \in A$ such that b is both a lower bound, and greater than all other lower bounds. If $b > l$, then b is the g.l.b, and if $b = l$, then there is 1 g.l.b, $l = b$.

- Problem 3: Prove that a set cannot contain more than one of its upper bounds or more than one of its lower bounds.

  Proof. Assume the contrapositive is true, that is, consider a nonempty set A, and 4 elements of A, $a, b, c, d \in A$ such that $a \neq b$ are upper bounds, and $c \neq d$ are lower bounds. If a, b are both upper bounds, one of them must be the least upper bound, since they are both in A. The only way for them both to be upper bounds and in A is if $a = b$ (See section 2.1, problem 2). $\lightning$. Thus, our assumption, the fact that $a \neq b \in A$ were both upper bounds is false.
  Similarly, if c, d are both lower bounds, one of them must be the greatest lower bound. However, again, the only way for both to be lower bounds

and in A is if $c = d$, which is a contradiction. Thus, we have proved that if there are 2 upper or lower bounds in an arbitrary set A, then those bounds *must* be equal to each other. QED

- Problem 4: Prove the Greatest Lower Bound Property assuming the Least Upper Bound Property as an axiom.

  Proof. N2S: The LUB of the set of lower bounds of a set is the GLB of that set.
  Consider a nonempty set A, and suppose that $\exists l \in A$ such that l is a lower bound. Consider the set of all lower bounds of A, call it B. Then B is nonempty ($l \in B$). Notice that each element of A is an upper bound of B. Then, by the LUB property, B has a L.U.B. Call it γ.

  Remark. Claim: γ is a lower bound of A. If not, then there exists some $b \in A$ such that $b < \gamma$. But as noted earlier, each member of A is an upper bound of B, and therefore b is an upper bound of B that is smaller than γ, the LUB of B.$\lightning$

  And since $\gamma = SupB$, $\forall l$ where l is a lower bound of A, $l \leq \gamma$. Therefore γ must be the GLB of A. QED

2 Section 2.2

- Problem 4: Prove that every subset of a countable set is countable.

  Proof.

  Definition 2.1 (Countable Sets)**.** A set S is countable when there is an injection $f : S \to \mathbb{Z}^+$.

  Consider some countable set A. Then, by definition, $\exists f : A \to \mathbb{Z}^+$. Where f is an injection. Then for any $B \subset A$, there exists some function $f|_B : B \to \mathbb{Z}^+$, and since $f : S \to \mathbb{Z}^+$ is injective, so is $f|_B : B \to \mathbb{Z}^+$. Therefore, any subset of an arbitrary countable set is countable. QED

- Problem 5: Prove that any two non-degenerate closed and bounded intervals have the same cardinal number.

Proof.

Definition 2.2 (Cardinality)**.** Two sets A, B have the same *cardinality* or *cardinal number* iff there is a bijection $f : A \to B$.

N2S: $f : [0, 1] \to [a, b]$ is bijective.
For some $a, b \in \mathbb{R}$, where $b \neq 0$ (or if $b = 0$, then $a < 0$ and we would define f to send x to ax).
Consider $f : [0, 1] \to [a, b]$ such that $\forall x \in [0, 1]$, $f(x) = bx$. We check:

1. **Injectivity**. $f(x) = f(y) \implies x = y$.
   Consider some $x, y \in [0, 1]$ where $f(x) = f(y)$. By definition of f, this means $bx = by$. Remember that b is some nonzero real number, then of course,
   $$bx = by \implies x = y$$
   Thus f is injective.
2. **Surjectivity**. $\forall y \in [a, b] \exists x \in [0, 1]$ such that $f(x) = y$.
   Take some $y \in [a, b]$, we want to show that $y = f(x)$, that is, $y = bx$. Now simply divide by b and now $x = \frac{y}{b}$. Recall again that b is some nonzero real number. (If it were, we would define f to send x to ax.) Then
   $$f(x) = b \cdot \frac{y}{b} = y$$
   As required.

Thus, since f is both injective and surjective, by definition 2.2, $[0, 1]$ has the same cardinality as $[a, b]$ where $a, b \in \mathbb{R}$, and $b \neq 0$. QED

- Problem 7: Prove that every (non-degenerate) open inverval is equivalent to $\mathbb{R}$.

Proof. Once again, we N2S:$f : \mathbb{R} \to (a, b)$ is bijective.
Consider $f : \mathbb{R} \to (a, b)$ where $f(x) = \frac{x}{1+|x|}$. First we check that this "works", that is, the denominator never equals 0. Of course, the only way for this to be true is if $|x| = -1$, this is impossible. Now we chek bijectivity:

1. **Injectivity**. $f(x) = f(y) \implies x = y$.
   Consider some $x, y \in \mathbb{R}$ where $f(x) = f(y)$. Then

$$\frac{x}{1+|x|} = \frac{y}{1+|y|}$$

   Rearranging,

$$y(1+|x|) = x(1+|y|) \Leftrightarrow x + x|y| = y + y|x|$$

   This is true iff $x = y$. Thus f is injective.

2. **Surjectivity**. $\forall y \in (a,b) \exists x \in \mathbb{R}$ such that $f(x) = y$.
   Take some $y \in (a,b)$. We want to show now that $y = f(x)$, that is,

$$y = \frac{x}{1+|x|} \Leftrightarrow x = \frac{y}{y+1}$$

   This is unique, thus we have found exactly one $x \in \mathbb{R}$ such that $f(x) = y$. As required.

QED

- Problem 8:

  (a) Prove that any two non degenerate intervals have the same cardinal number.
  We have proven that any 2 open intervals have the same cardinality, and that any two closed intervals have the same cardinality. Thus we only need to prove $f : (0,1) \to [a,b)$ exists and is bijective, and $g : (0,1) \to (a,b]$ exists and is bijective.

  Proof. First, we define $f : [a,b) \to [0,1)$ by $x \mapsto \frac{x-a}{b-a}$. This is a bijection. Next, we must find a bijection $\phi : [0,1) \to (0,1)$, this is easy. Let $\phi(x) = a_0$ if $x = 0$. Or $\phi(x) = a_{n+1}$ if $x = a_n$. Else $\phi(x) = x$. This function uses the concept of Hilbert's hotel to "shift" each element over by 1, and add in 0 at the a_0 spot. This is obviously a bijection. Since there is a bijection from $[a,b) \to [0,1)$, and from $[0,1) \to (0,1)$, and from $(0,1) \to (a,b)$ this means they all have the same cardinal number. A similar argument follows for $(a,b]$. We define $f : (a,b] \to [0,1)$ once again by $x \mapsto \frac{x-a}{b-a}$. We

have already defined the bijection from $[0, 1)$ to $(0, 1)$, thus $(a, b]$ has the same cardinality as $(0, 1)$.
Thus, we have showed there exists a bijection for each of the following:

(i) $[a, b] \to (0, 1)$
(ii) $(a, b) \to (0, 1)$
(iii) $[a, b) \to (0, 1)$
(iv) $(a, b] \to (0, 1)$

Thus, any two non degenerate intervals have the same cardinality.
QED

(b) Prove that every non-degenerate interval is uncountable.

Proof. Since we have showed a bijection $(0, 1) \to \mathbb{R}$, $(0, 1)$ is uncountable, and by our answer to the question above, we have showed a bijection between every non-degenerate interval and $(0, 1)$. QED

(c) Prove that every non-degenerate interval contains both rational and irrational numbers.

Proof. Consider $\frac{1}{\sqrt{2}}$. This is irrational.
Recall that the sum *and* product of 1 rational and 1 irrational is always irrational.
Now WLOG, to show how this method works, let (a, b) be a non degenerate interval, then if $|b - a| > \frac{1}{\sqrt{2}}$, define $z = a + \frac{1}{\sqrt{2}}$. This is an irrational in the interval. Otherwise, if $|b - a| < \frac{1}{\sqrt{2}}$, define $z = \frac{1}{\sqrt{2}}(b - a)$. This again is an irrational in the interval. This method can be applied for any $a, b \in \mathbb{R}$. QED