Math 553 Homework 8

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1 Section 4.3

- Problem 2: Show that if x is an isothermal parametrization, that is, $E = G = \lambda(u,v)$ and $F = 0$, then
$$K = -\frac{1}{2\lambda}\Delta(\log\lambda)$$
where $\Delta\varphi$ denotes the Laplacian $(\delta^2\varphi/\delta u^2)+(\delta^2\varphi/\delta v^2)$ of the function φ. Conclude that when $E = G = (u^2+v^2+c^2)^{-2}$ and $F = 0$, then $K = \text{const.} = 4c$.

  Proof. By Exercise 1 in this section, if x is an orthogonal parametrization, then
$$K = -\frac{1}{2\sqrt{EG}}\left\{\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right\}.$$
Trivially, if x is an isothermal parametrization, it must also be an orthogonal parametrization, so it remains to show that

1. $2\sqrt{EG} = 2\lambda$
2. $\Delta(\log\lambda) = \left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u$

Since $E = G = \lambda > 0$, $\sqrt{EG} = \sqrt{\lambda^2} = \lambda$. Therefore $-\frac{1}{2\sqrt{EG}} = -\frac{1}{2\lambda}$.

Then,

$$\begin{aligned}
\Delta(\log\lambda) &= \delta_u^2(\log\lambda) + \delta_v^2(\log\lambda) \\
&= (\delta_v(\log\lambda))_v + (\delta_u(\log\lambda))_u \\
&= \left(\frac{\lambda_v}{\lambda}\right)_v + \left(\frac{\lambda_u}{\lambda}\right)_u \\
&= \left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u
\end{aligned}$$

As required. QED

Now, plugging in $E = G = \lambda = (u^2 + v^2 + c^2)^{-2}$. First calculating the Laplacian:

$$\begin{aligned}
\Delta(\log\lambda) &= \Delta(\log(u^2+v^2+c^2)^{-2}) \\
&= \left(\frac{-\frac{4v}{\left(u^2+v^2+c^2\right)^3}}{(u^2+v^2+c^2)^{-2}}\right)_v + \left(\frac{-\frac{4u}{\left(u^2+v^2+c^2\right)^3}}{(u^2+v^2+c^2)^{-2}}\right)_u \\
&= \left(-\frac{4v(u^2+v^2+c^2)^2}{\left(u^2+v^2+c^2\right)^3}\right)_v + \left(-\frac{4u(u^2+v^2+c^2)^2}{\left(u^2+v^2+c^2\right)^3}\right)_u \\
&= \left(-\frac{4v}{(u^2+v^2+c^2)}\right)_v + \left(-\frac{4u}{(u^2+v^2+c^2)}\right)_u \\
&= -\frac{4\left(-v^2+c^2+u^2\right)}{\left(v^2+u^2+c^2\right)^2} - \frac{4\left(v^2+c^2-u^2\right)}{\left(v^2+u^2+c^2\right)^2} \\
&= -8c\lambda
\end{aligned}$$

The Gaussian curvature is then,

$$\begin{aligned}
K &= -\frac{1}{2\lambda}\Delta(\log\lambda) \\
&= -\frac{1}{2\lambda}\cdot -8c\lambda \\
&= 4c
\end{aligned}$$

As required.

- **Problem.** Consider the unit disk $\mathbb{D} = \{(x,y) \in \mathbb{R}^2 : x^2+y^2<1\}$ endowed with FFF

$$\frac{dx^2+dy^2}{4(1-x^2-y^2)^2}$$

Use the previous problem to show that the Gaussian curvature of this surface satisfies $K(x,y) = -1$ for all $(x,y) \in \mathbb{D}$.

Since the first fundamental form is what it is, we know that $\lambda = E = G = \frac{1}{4(1-x^2-y^2)^2}$ and $F = 0$. Thus it follows that

$$K = -\frac{1}{2\lambda}\Delta(\log\lambda)$$

And we must show that

$$-\frac{1}{2\lambda}\Delta(\log\lambda) = -1$$

Therefore it will suffice to show that $2\lambda = \Delta(\log\lambda)$. Computing the Laplacian:

$$\begin{aligned}
\Delta(\log\lambda) &= \Delta\left(\log\left(\frac{1}{4(1-x^2-y^2)^2}\right)\right) \\
&= \left(\frac{\frac{x}{2\left(1-x^2-y^2\right)^2}}{\frac{1}{4(1-x^2-y^2)^2}}\right)_x + \left(\frac{\frac{y}{2\left(1-x^2-y^2\right)^2}}{\frac{1}{4(1-x^2-y^2)^2}}\right)_y \\
&= \left(\frac{2\lambda x}{\lambda}\right)_x + \left(\frac{2\lambda y}{\lambda}\right)_y \\
&= \delta_x(\lambda x) + \delta_y(\lambda y) \\
&= 2\lambda
\end{aligned}$$