Math 724 Homework 1

Theo Koss

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1 Chapter 1

1. If $r \neq 0$ is rational, and x is irrational, prove that $r+x$ and rx are irrational.

   Proof. Let $r \neq 0$ be rational and x be irrational.

   (a) Assume, by way of contradiction, that $r+x$ is rational. Then $r+x = \frac{p}{q}$ for $p, q \in \mathbb{Z}$ and $q \neq 0$. We have that r is rational, so $r = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ and $b \neq 0$. Then we have

   $$x = \frac{p}{q} - \frac{a}{b} = \frac{bp - aq}{bq}$$

   since the ring $(\mathbb{Z}, +, \cdot)$ is closed under addition and multiplication so numerator and denominator are integers, and since $\mathbb{Z}$ is a domain, $b \neq 0$ and $q \neq 0 \implies bq \neq 0$. So then we have $x \in \mathbb{Q}$ which is a contradiction.

   (b) Assume, by way of contradiction, that $rx = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$ and $q \neq 0$. We have $r = \frac{a}{b}$ for $a, b \in \mathbb{Z}$ and $b \neq 0$. Then

   $$x = \frac{rx}{r} = \frac{\frac{p}{q}}{\frac{a}{b}} = \frac{p}{q} \cdot \frac{b}{a} = \frac{pb}{qa}$$

   Again since $\mathbb{Z}$ is closed under multiplication and $r \neq 0$ by assumption, we have $a \neq 0$ and therefore $qa \neq 0$, so we are not dividing by 0 and $x \in \mathbb{Q}$ contradiction.

   QED