Math 553 Homework

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1 Section 1.4

- Problem 2: A plane P contained in $\mathbb{R}^3$ is given by the equation $ax + by + cz + d = 0$. Show that the vector $v = (a, b, c)$ is perpendicular to the plane and that $|d|/\sqrt{a^2 + b^2 + c^2}$ measures the distance from the plane to the origin $(0, 0, 0)$.

  Proof. Take two points on the plane: $(x_1, y_1, z_1), (x_2, y_2, z_2)$. They both satisfy:

$$ax_1 + by_1 + cz_1 = -d$$

$$ax_2 + by_2 + cz_2 = -d$$

Then this gives $(x_1 - x_2, y_1 - y_2, z_1 - z_2) \cdot (a, b, c) = 0$. In other words, any vector on the plane is perpendicular to (a, b, c). QED

- Problem 4: Given two planes $a_i x + b_i y + c_i z + d_i = 0$, $i = 1, 2$ prove that a necessary and sufficient condition for them to be parallel is

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Where the convention is made that if a denominator is zero, the corresponding number is also zero.

Proof. $(\Longrightarrow)$: Consider two parallel planes,

$$a_1 x + b_1 y + c_1 z + d_1$$

$$a_2 x + b_2 y + c_2 z + d_2$$

Since they are parallel, this implies that $a_1 = a_2, b_1 = b_2, c_1 = c_2$. From this it is immediate that $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

$(\Longleftarrow)$: Assume that $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$. Then, the ratio of a_1 to a_2 must be exactly the ratio of b_1 to b_2, and that of c_1 and c_2. WLOG, consider these 3 ratios to be k. Then $ka_2 = a_1$, $kb_2 = b_1$ and $kc_2 = c_1$. This shows that the normal vector of plane 2 is simply a multiple of the normal vector of plane 1. Therefore, the planes must be parallel. QED

- Problem 5: Show that an equation of a plane passing through three noncolinear points $p_1 = (x_1, y_1, z_1)$, $p_2 = (x_2, y_2, z_2)$, $p_3 = (x_3, y_3, z_3)$ is given by

$$(p_3 - p_1) \wedge (p_3 - p_2) \cdot (p - p_3) = 0$$

where $p = (x, y, z)$ is an arbitrary point of the plane and $p - p_1$, for instance, means the vector $(x - x_1, y - y_1, z - z_1)$.

Proof. By the triple product determinant,

$$[(p_3 - p_1) \times (p_3 - p_2)] \cdot (p - p_3) = det \begin{vmatrix} x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_3 - x_2 & y_3 - y_2 & z_3 - z_2 \\ x - x_3 & y - y_3 & z - z_3 \end{vmatrix} =$$

$= [(x_3 - x_1) \cdot ((y_3 - y_1)(z - z_3) - (z_3 - z_2)(y - y_3))]$
$-[(y_3 - y_1) \cdot ((x_3 - x_2)(z - z_3) - (z_3 - z_2)(x - x_3))]$
$+[(z_3 - z_1) \cdot ((x_3 - x_2)(y - y_3) - (y_3 - y_2)(x - x_3))]$
Which is equal to 0. QED

2 Section 1.5

- Problem 1: Given the parametrized curve (helix)

$$\alpha(s) = \left(a \cos\left(\frac{s}{c}\right), a \sin\left(\frac{s}{c}\right), b\frac{s}{c}\right)$$

Where $c^2 = a^2 + b^2$.

a. Show that s is the arc length.

Arc length of helix given by: Arc length$=\sqrt{(a^2+b^2)}\frac{s}{c}$

$$= \sqrt{c^2}\frac{s}{c}$$

$$= c \cdot \frac{s}{c} = s$$

Therefore the arc length is s.

b. Determine the curvature and torsion of α.

$$\kappa = \frac{a}{a^2+b^2} = \frac{a}{c^2}$$

$$\tau = \frac{b}{a^2+b^2} = \frac{b}{c^2}$$

c. Determine the osculating plane of α.

$$\begin{bmatrix} z_1 - acos\left(\frac{s}{c}\right) & z_2 - asin\left(\frac{s}{c}\right) & z_3 - \frac{bs}{c} \\ -asin\left(\frac{s}{c}\right) & acos\left(\frac{s}{c}\right) & b \\ -acos\left(\frac{s}{c}\right) & -asin\left(\frac{s}{c}\right) & 0 \end{bmatrix} = 0$$

$$(z_1 - acos(\frac{s}{c}))(basin(\frac{s}{c})) - (z_2 - asin(\frac{s}{c}))(bacos(\frac{s}{c})) + (z_3 - \frac{bs}{c})a$$

$$z_1 bsin(\frac{s}{c}) - absin(\frac{s}{c})cos(\frac{s}{c}) - z_2 bcos(\frac{s}{c}) + abcos(\frac{s}{c})sin(\frac{s}{c}) + (z_3 -$$

$$z_1 bsin(\frac{s}{c}) + z_2 bcos(\frac{s}{c}) + (z_3 - \frac{bs}{c})a = 0$$

d. Show that the lines containing $n(s)$ and passing through $\alpha(s)$ meet the z axis under a constant angle equal to $\frac{\pi}{2}$. *Not sure

e. Show that the tangent lines to α make a constant angle with the z axis.

$$\theta = arccos\left(\frac{t(s)\cdot(0,0,1)}{|t(s)||(0,0,1)|}\right) = \frac{(-asin\frac{s}{c}, acos\frac{s}{c}, b)\cdot(0,0,1)}{|(-asin\frac{s}{c}, acos\frac{s}{c}, b)||(0,0,1)|}$$

$$= \frac{b}{\sqrt{a^2+b^2}} = \frac{b}{c}$$

- Problem 2: Show that the torsion τ of α is given by

$$-\frac{\alpha'(s)\wedge\alpha''(s)\cdot\alpha'''(s)}{|k(s)|^2}$$

Proof. By defintion $\alpha'(s) = t(s)$. Differentiating:

$$\alpha''(s) = t'(s) = \kappa(s)n(s)$$

Again:

$$\alpha'''(s) = \kappa'(s)n(s) + \kappa(s)n'(s)$$

$$n' = -\kappa t - \tau b$$

Thus:

$$\alpha'''(s) = \kappa'(s)n(s) - \kappa(s)^2 t(s) - \kappa(s)\tau(s)b(s)$$

Computing the cross product of $\alpha'(s)$ and $\alpha''(s)$:

$$\alpha'(s)\times\alpha''(s) = \kappa(s)b(s)$$

So

$$(\alpha'(s) \times \alpha''(s)) \cdot \alpha'''(s) = -\kappa(s)^2 \tau(s)$$

Therefore

$$\tau = -\frac{\alpha'(s) \wedge \alpha''(s) \cdot \alpha'''(s)}{|k(s)|^2}$$

QED

- Problem 4: Assume that all normals of a parametrized curve pass through a fixed point. Prove that the trace of the curve is contained in a circle.

Proof. Call the fixed point p, the curve $a(s)$, and a unit normal vector of the curve $n(s)$. Then $\alpha'(s) \cdot n(s) = 0$. Since $n(s)$ passes through p, we have $\alpha(s) - p = kn(s)$ for some scalar k. So then:

$$\frac{d}{ds}(|\alpha(s)-p|^2) = \frac{d}{ds}(\alpha(s)-p)\cdot(\alpha(s)-p)+(\alpha(s)-p)\cdot\frac{d}{ds}(\alpha(s)-p)$$

$$= 2\frac{d}{ds}(\alpha(s) - p) \cdot (\alpha(s) - p)$$

$$= 2\alpha'(s) \cdot (\alpha(s) - p)$$

$$= 2\alpha'(s) \cdot kn(s)$$

$$= 2k\alpha'(s) \cdot n(s)$$

$$= 0$$

Therefore, $|\alpha(s) - p|^2$ is constant, thus $|\alpha(s) - p|$ is also constant. In other words, the distance between the curve $\alpha(s)$ and the fixed point p is constant for all $s \in I$. This implies that the trace of α is a circle. QED

- Problem 6:

  a. Demonstrate that the norm of a vector and the angle θ between two vectors, $0 \leq \theta \leq \pi$, are invariant under orthogonal transformations with positive determinant.

  Norm: Orthogonal transformation implies: $\exists \rho : \mathbb{R}^3 \to \mathbb{R}^3$ Such that $\rho v \cdot \rho u = v \cdot u$. Need to show that $||v|| = ||\rho v||$.

  $$||\rho v||^2 = \rho v \cdot \rho v = v \cdot v = ||v||^2$$

  Therefore norm is preserved.

  Angle: Recall $cos\theta = \frac{u \cdot v}{||u||||v||}$.

  $$\cos\theta_\rho = \frac{\rho u \cdot \rho v}{||\rho u||||\rho v||} = \frac{u \cdot v}{||u||||v||} = cos\theta$$

  Therefore the angle is preserved under ρ.

  b. Show that the vector product of two vectors is invariant under orthogonal transformations with positive determinant. Is the assertion still true if we drop the condition on the determinant?

$$\rho u \times \rho v = (det\rho)(u \times v)$$

Therefore, the vector product of two vectors is invariant if $det\rho = 1$. If, on the other hand, $det\rho = -1$, then $\rho u \times \rho v \neq u \times v$.

c. Show that the arc length, the curvature, and the torsion of a parametrized curve are invariant under rigid motions.

Proof. Let $\alpha : I \to \mathbb{R}^3$ be a parametrized curve and $T : \mathbb{R}^3 \to \mathbb{R}^3$ a rigid motion. Let $\mathcal{L}$ be its length, κ its curvature and τ its torsion. Since translations and orthogonal transformations preserve norm, T preserves norm.

Arc length:

$$\mathcal{L}_T(\alpha) = \int_{t_0}^{t} ||T\alpha(t)||dt = \int_{t_0}^{t} ||\alpha(t)|| = \mathcal{L}(\alpha)$$

Curvature, $\kappa(s) = ||\alpha''(s)||$. Since T is a linear transformation, it holds that $(T\alpha(s))'' = T\alpha''(s)$. Therefore:

$$\kappa_T(s) = ||(T\alpha(s))''|| = ||T\alpha''(s)|| = ||\alpha''(s)|| = \kappa(s)$$

For torsion, recall $\tau(s)n(s) = b'(s)$.

$$\tau(s)n(s) = b'(s) = (t(s) \wedge n(s))'$$

$$= t'(s) \wedge n(s) + t(s) \wedge n'(s)$$

$$= \kappa(s)n(s) \wedge n(s) + \alpha'(s) \wedge n'(s)$$

$$= \alpha'(s) \wedge n'(s)$$

Then

$$\tau_T(s)Tn(s)$$

$$= T\alpha'(s) \wedge Tn'(s)$$

$$= T(\alpha'(s) \wedge n'(s))$$

$$= T(\tau(s)n(s))$$

$$= \tau(s)Tn(s)$$

So $\tau_T(s) = \tau(s)$. (Regretting using T as the rigid motion, τ_T looks weird.) QED

- Problem 12: Let $\alpha : I \to \mathbb{R}^3$ be a regular parametrized curve and let $\beta : J \to \mathbb{R}^3$ be a reparametrization of $\alpha(I)$ by the arc length $s = s(t)$, measured from $t_0 \in I$. Let $t = t(s)$ be the inverse function of s and set $\frac{d\alpha}{dt} = \alpha'$, $\frac{d^2\alpha}{dt^2} = \alpha''$, etc. Prove that:

  a. $\frac{dt}{ds} = \frac{1}{|\alpha'|}$, $\frac{d^2t}{ds^2} = -\left(\frac{\alpha' \cdot \alpha''}{|\alpha'|^4}\right)$.

$s = s(t) = \int_{t_0}^{t} |\alpha'(t)| dt$ so we have $\frac{ds}{dt} = |\alpha'|$. Then $\frac{dt}{ds} = \frac{1}{|\alpha'|}$ since they are invertible functions. Now

$$\frac{d^2t}{ds^2} = \frac{d}{ds}\frac{dt}{ds} = \frac{d/dt(dt/ds)}{ds/dt} = \frac{d/dt(|\alpha'(t)|^{-1})}{|\alpha'(t)|}$$

$$= \frac{-d/dt(|\alpha'(t)|)}{|\alpha'(t)|^3} = \frac{-1}{|\alpha'(t)|^3}\frac{d}{dt}[(\alpha' \cdot \alpha')^{1/2}] = \frac{2\alpha' \cdot \alpha''}{|\alpha'|^3 \cdot 2|\alpha'|}$$

$$= -\left(\frac{\alpha' \cdot \alpha''}{|\alpha'|^4}\right)$$

b. The curvature of α at $t \in I$ is:

$$\kappa(t) = \frac{|\alpha' \wedge \alpha''|}{|\alpha'|^3}$$

We have $\alpha' = |\alpha'|\bar{T}$ where $\bar{T}$ is the unit tangent vector. So

$$\alpha'' = \frac{d}{dt}\alpha' = \frac{d/ds\alpha'}{dt/ds} = |\alpha'|\frac{d}{ds}(|\alpha'|\bar{T})$$

$$\frac{d}{ds}(|\alpha'|\bar{T}) = \frac{d/dt|\alpha'|}{ds/dt} + |\alpha'|\kappa\bar{N} = \frac{\alpha' \cdot \alpha''}{|\alpha'|^2}\bar{T} + |\alpha'|\kappa\bar{N}$$

Now

$$\alpha'' = \frac{\alpha' \cdot \alpha''}{|\alpha'|}\bar{T} + |\alpha'|^2\kappa\bar{N}$$

and

$$\alpha' \wedge \alpha'' = |\alpha'|\bar{T} \wedge \left(\frac{\alpha' \cdot \alpha''}{|\alpha'|}\bar{T} + |\alpha'|^2\kappa\bar{N}\right) = |\alpha'|^3\kappa\bar{T} \wedge \bar{N} = |\alpha'|^3\kappa$$

Computing norms:

$$|\alpha' \wedge \alpha''| = \kappa |\alpha'|^3$$

As required.

c. The torsion of α at $t \in I$ is:

$$\tau(t) = -\frac{(\alpha' \wedge \alpha'') \cdot \alpha'''}{|\alpha' \wedge \alpha''|^2}$$

*Not sure

d. If $\alpha : I \to \mathbb{R}^3$ is a plane curve $\alpha(t) = (x(t), y(t))$, the signed curvature of α at t is:

$$\kappa(t) = \frac{x'y'' - x''y'}{((x')^2 + (y')^2)^{\frac{3}{2}}}$$

*Not sure