Math 553 Homework 5

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1 Section 2.5

- Problem 1: Compute the first fundamental forms of the following parametrized surfaces where they are regular:

  a. $X(u, v) = (a \sin u \cos v, b \sin u \sin v, c \cos u)$; ellipsoid

$$X_u = (a \cos u \cos v, b \cos u \sin v, -c \sin u)$$

$$X_v = (-a \sin u \sin v, b \sin u \cos v, 0)$$

$$E = X_u \cdot X_u = (a^2 \cos^2 v + b^2 \sin^2 v) \cos^2 u + c^2 \sin^2 u$$

$$F = X_u \cdot X_v = \frac{1}{4}(b^2 - a^2) \sin(2u) \sin(2v)$$

$$G = X_v \cdot X_v = (a^2 \sin^2 v + b^2 \cos^2 v) \sin^2 u$$

First fundamental form is always:

$$Edu^2 + 2Fdudv + Gdv^2$$

(I'll omit writing it out, just going to write down the coefficients)

b. $X(u, v) = (au \cos v, bu \sin v, u^2)$; elliptic paraboloid.

$$X_u = (a \cos v, b \sin v, 2u), X_v = (-au \sin v, bu \cos v, 0)$$

$$E = a^2 \cos^2 v + b^2 \sin^2 v + 4u^2$$
$$F = \frac{u}{2}(b^2 - a^2) \sin(2v)$$
$$G = (a^2 \sin^2 v + b^2 \cos^2 v)u^2$$

c. $X(u, v) = (au \cosh v, bu \sinh v, u^2)$; hyperbolic paraboloid.

$$X_u = (a \cosh v, b \sinh v, 2u), X_v = (au \sinh v, bu \cosh v, 0)$$

$$E = a^2 \cosh^2 v + b^2 \sinh^2 v + 4u^2$$
$$F = \frac{u}{2}(b^2 + a^2) \sinh 2v$$
$$G = (a^2 \sinh^v + b^2 \cosh^2 v)u^2$$

d. $X(u, v) = (a \sinh u \cos v, b \sinh u \sin v, c \cosh u)$; hyperboloid of two sheets.

$$X_u = (a \cosh u \cos v, b \cosh u \sin v, c \sinh u)$$

$$X_v = (-a \sinh u \sin v, b \sinh u \cos v, 0)$$
$$E = (a^2 \cos^2 v + b^2 \sin^2 v) \cosh^2 u + c^2 \sinh^2 u$$

$$F = \frac{1}{4}(b^2 - a^2)\sinh(2u)\sin(2v)$$
$$G = (a^2\sin^2 v + b^2\cos^2 v)\sinh^2 u$$

- Problem 3: Obtain the first fundamental form of the sphere in the parametrization given by the stereographic projection.

$$X(u,v) = \left(\frac{4u}{u^2+v^2+4}, \frac{4v}{u^2+v^2+4}, \frac{2(u^2+v^2)}{u^2+v^2+4}\right)$$

Therefore,

$$X_u = \frac{4}{(u^2+v^2+4)^2}(-u^2+v^2+4, -2uv, 2u)$$

and

$$X_v = \frac{4}{(u^2+v^2+4)^2}(-2uv, u^2-v^2+4, 2v)$$

Then:

$$E = \frac{16}{(u^2+v^2+4)^4}[(u^2+v^2+4)^2 - 12u^2]$$
$$F = -\frac{192uv}{(u^2+v^2+4)^4}$$
$$G = \frac{16}{(u^2+v^2+4)^4}[(u^2+v^2+4)^2 - 12v^2]$$

Let $\zeta = u^2+v^2+4$ then,

$$E = \frac{16(\zeta^2 - 12u^2)}{\zeta^4}, F = -\frac{192}{\zeta^4}, G = \frac{16(\zeta^2 - 12v^2)}{\zeta^4}$$

- Problem 5: Show that the area A of a bounded region R of the surface $z = f(x, y)$ is

$$A = \iint_Q \sqrt{1 + f_x^2 + f_y^2} dxdy$$

where Q is the normal projection of R onto the xy plane.

Proof. If the surface is given by:

$$z = f(x, y)$$

Then R can be parametrized by $x : Q \to S$ given by:

$$X(u, v) = (u, v, f(u, v))$$

Then, taking partial derivatives:

$$X_u = (1, 0, f_u(u, v))$$

$$X_v = (0, 1, f_v(u, v))$$

Then, taking the cross product:

$$X_u \times X_v = (-f_u(u, v), -f_v(u, v), 1)$$

By definition, the area is then

$$A = \iint_Q ||X_u \times X_v|| dudv = \iint_Q \sqrt{1 + f_x^2 + f_y^2} dxdy$$

As required. QED

- Problem 7: The coordinate curves of a parametrization $X(u, v)$ are a Tchebyshef net if the lengths of the opposite sides of any quadrilateral formed by them are equal. Show that a necessary and sufficient condition for this is

$$\frac{\partial E}{\partial v} = \frac{\partial G}{\partial u} = 0$$

Proof. ($\Longrightarrow$): Suppose that the coordinate curves of X form a Tschebyshef net. Let $X(u_1, v_1)$ and $X(u_2, v_2)$ be the opposite vertices of a quadrilateral. Then, clearly, the other two vertices are $X(u_1, v_2)$ and $X(u_2, v_1)$. Since the system is Tschebyshef, opposite sides have the same length, therefore:

$$\int_{u_1}^{u_2} ||X_u(u, v_1)||du = \int_{u_1}^{u_2} ||X_u(u, v_2)||du$$

and

$$\int_{v_1}^{v_2} ||X_v(u_1, v)||dv = \int_{v_1}^{v_2} ||X_v(u_2, v)||dv$$

Then, if $u_1 = u_0$ and $v_1 = v_0$ are constants and $u_2 = u$, $v_2 = v$ we have:

$$\int_{u_0}^{u} ||X_u(t, v_0)||dt = \int_{u_0}^{u} ||X_u(t, v)||dt$$

Since $E = X_u \cdot X_u$,

$$\int_{u_0}^{u} \sqrt{E(t, v_0)}dt = \int_{u_0}^{u} \sqrt{E(t, v)}dt$$

Now differentiating both sides with respect to v, we achieve:

$$\int_{u_0}^{u} \underbrace{\partial_v \sqrt{E(t, v_0)}}_{=0} \, dt = \int_{u_0}^{u} \partial_v \sqrt{E(t, v)} dt$$

$$\implies 0 = \int_{u_0}^{u} \frac{1}{2\sqrt{E(t, v)}} \cdot \frac{\partial E}{\partial v}(t, v) dt$$

Since this holds for all u, this implies that

$$\frac{\partial E}{\partial v}(t, v) = 0$$

(Since $\frac{1}{2\sqrt{E(t,v)}}$ can't).

Similarly, using equation 2, we achieve:

$$\frac{\partial G}{\partial u} = 0$$

$(\Longleftarrow)$: Suppose that

$$\frac{\partial E}{\partial v} = 0$$

This necessarily implies

$$E = E(u)$$

Then

$$\int_{u_1}^{u_2} ||X_u(t, v_1)|| dt = \int_{u_1}^{u_2} \sqrt{E(t)} dt = \int_{u_1}^{u_2} ||X_u(t, v_2)|| dt$$

Again, similarly,

$$\frac{\partial G}{\partial u} = 0$$

implies

$$\int_{v_1}^{v_2} ||X_v(u_1, t)||dt = \int_{u_1}^{u_2} \sqrt{G(t)}dt = \int_{u_1}^{u_2} ||X_v(u_2, t)||dt$$

QED

- Problem 10: Let $P = \{(x, y, z) \in \mathbb{R}^3; z = 0\}$ be the xy plane and let $X : U \to P$ be a parametrization of P given by

$$X(\rho, \theta) = (\rho\cos\theta, \rho\sin\theta)$$

where

$$U = \{(\rho, \theta) \in \mathbb{R}^2; p > 0, 0 < \theta < 2\pi\}$$

Compute the coefficients of the first fundamental form of P in this parametrization.

$$P = (\rho\cos\theta, \rho\sin\theta, 0)$$

$$P_\rho = (\cos\theta, \sin\theta, 0)$$
$$P_\theta = (-\rho\sin\theta, \rho\cos\theta, 0)$$
$$E = 1 \quad F = 0 \quad G = \rho^2$$
$$\text{FFF} = dudu + \rho^2 dvdv$$

2 Section 2.6

1. Let S be a regular surface covered by coordinate neighborhoods V_1 and V_2. Assume that $V_1 \cap V_2$ has two connected components, W_1, W_2, and that the Jacobian of the change of coordinates is positive in W_1 and negative in W_2. Prove that S is nonorientable.

   Not sure