Math 553 Homework

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1 Section 2.3

- Problem 1: Let $S^2 = \left\{(x, y, z) \in \mathbb{R}^3;\ x^2 + y^2 + z^2 = 1\right\}$ be the unit sphere and let $A : S^2 \to S^2$ be the antipodal map $A(x, y, z) = (-x, -y - z)$. Prove that A is a diffeomorphism.

  Proof. For A to be a diffeomorphism it must be:

  1. Differentiable (or smooth).

     The map $(x, y, z) \mapsto (-x, -y, -z)$ from $\mathbb{R}^3$ to itself is smooth. A is a restriction of this map, therefore A is smooth.

2. Has inverse A^{-1} which is also smooth.

   All A does is flip the signs of each coordinate, so A^{-1} must also simply flip the signs. Therefore $A^{-1} = A$. Since we proved above that A is smooth, then A^{-1} must also be smooth.

Thus A is a diffeomorphism. QED

- Problem 5: Let $S \subset \mathbb{R}^3$ be a regular surface, and let $d : S \to \mathbb{R}$ be given by $d(p) = |p - p_0|$, where $p \in S$, $p_0 \neq \mathbb{R}^3$, $p_0 \notin S$; that is, d is the distance from p to a fixed point p_0 not in S. Prove that d is smooth.

  Proof.

  Remark. Ex. 1 in the text states, the square of the distance from a fixed point $p_0 \in \mathbb{R}^3$, $f(p) = |p - p_0|^2$ is differentiable. Furthermore it states that the need for the square comes from the fact that the distance $|p-p_0|$ is not differentiable only at $p = p_0$.

  Using this, we now simply need to show that p is never equal to p_0. This is simple, as $p \in S$ and $p_0 \notin S$, therefore they must never be equal. Therefore d is smooth QED

- Problem 6: Prove that the definition of a differentiable map between surfaces does not depend on the parametrizations chosen.

Proof. A differentiable map is a smooth, continuous map $\phi : S_1 \to S_2$. ϕ is differentiable at $p \in S_1$ if given parametrizations: $f : U \in \mathbb{R}^2 \to S_1$ and $g : W \in \mathbb{R}^2 \to S_2$. With $p \in f(U)$ and $\phi(f(U)) \subset g(W)$, the map $g^{-1} \circ \phi \circ f : U \to W$ is differentiable at $q = f^{-1}(p)$.

$$\begin{aligned} g_1^{-1} \circ \phi \circ f_1 &= g_1^{-1} \circ g_2 \circ (g_2^{-1} \circ \phi \circ f_2) \circ f_2^{-1} \circ f_1 \\ &= G \circ (g_2^{-1} \circ \phi \circ f_2) \circ F \end{aligned}$$

and

$$\begin{aligned} g_2^{-1} \circ \phi \circ f_2 &= g_2^{-1} \circ g_1 \circ (g_1^{-1} \circ \phi \circ f_1) \circ f_1^{-1} \circ f_2 \\ &= G^{-1} \circ (g_1^{-1} \circ \phi \circ f_1) \circ F^{-1} \end{aligned}$$

The maps G and F are diffemorphisms, therefore they are smooth, so this composition must be smooth. Therefore we have shown if $g_1^{-1} \circ \phi \circ f_1$ is smooth, then $g_2^{-1} \circ \phi \circ f_2$ is smooth. Therefore the choice of f_1 and g_1 or f_2 and g_2 does not matter. QED

- Problem 7: Prove that the relation "S_1 is diffeomorphic to S_2" is an equivalence relation.

Proof. If $S_1 \simeq S_2$ then there is a smooth, bijective map $f : S_1 \to S_2$ and its inverse $f^{-1} : S_2 \to S_1$ is also smooth.

1. Trivially, $S \simeq S$ for all manifolds S.

2. Since the function and its inverse necessarily exist, $S_1 \simeq S_2 \iff S_2 \simeq S_1$.

3. If $S_1 \simeq S_2$, and $S_2 \simeq S_3$, then there exist functions $f : S_1 \to S_2$ and $g : S_2 \to S_3$, therefore the composition $g \circ f : S_1 \to S_3$ necessarily exists, is smooth (composition of two smooth functions is smooth), and is bijective (composition of two bijections is a bijection).

Therefore, "S_1 is diffeomorphic to S_2" is an equivalence relation. QED

2 Section 2.4

- Problem 1: Show that the equation of the tangent plane at (x_0, y_0, z_0) of a regular surface given by $f(x, y, z) = 0$, where 0 is a regular value of f is,

$$f_x(x_0, y_0, z_0)(x-x_0)+f_y(x_0, y_0, z_0)(y-y_0)+f_z(x_0, y_0, z_0)(z-z_0) = 0$$

- Problem 7: Let $f : S \to R$ be given by $f(p) = |p - p_0|^2$, where $p \in S$ and p_0 is a fixed point of $\mathbb{R}^3$. Show that $df_p(w) = 2w \cdot (p - p_0), w \in T_p(S)$.

  Choose a differentiable curve $\alpha : (-\epsilon, \epsilon) \to S$ with $\alpha(0) = p$, $\alpha'(0) = w$. Then $f(\alpha(t)) = |\alpha(t) - p_0|^2$ and

$$df_p(w) = \frac{d}{dt} f(\alpha(t))|_{t=0} = 2\alpha'(t) \cdot (\alpha(t) - p_0)|_{t=0}$$
$$= 2w \cdot (p - p_0)$$

  As desired.

- Problem 10: Let $\alpha : I \to \mathbb{R}^3$ be a regular parametrized curve with nonzero curvature everywhere and PBAL. Let

$$x(s, v) = \alpha(s) + r(n(s) \cos v + b(s) \sin v, \quad r = \text{const.} \neq 0, s \in I$$

  be a parametrized surface (the tube of radius r around α), where n is the normal vector and b is the binormal vector of α. Show that, when x is regular, its unit normal vector is

$$N(s, v) = -(n(s) \cos v + b(s) \sin v)$$

$$N = \frac{x_s \wedge x_v}{|x_s \wedge x_v|}$$

$$
\begin{aligned}
x_s &= t + rn'(s)\cos v + rb'(s)\sin v \\
&= t + r(-kt - \tau b)\cos v + r\tau n \sin v \\
&= (1 - rk\cos v)t - (r\tau\cos v)b + (r\tau\sin v)n
\end{aligned}
$$

$$
\begin{aligned}
x_v &= 0 - rn\sin v + rb\cos v \\
&= (-r\sin v)n + (r\cos v)b
\end{aligned}
$$

$$
\begin{aligned}
N &= \frac{(0, (r\sin(v)(-rk\cos(v)+1))b, (r\cos(v)(-rk\cos(v)+1))n)}{\sqrt{r^2}\sqrt{(1-rk\cos v)^2}\sqrt{\sin^2 v + \cos^2 v}} \\
&= -(b(s)\sin v + n(s)\cos v)
\end{aligned}
$$

- Problem 21: Sorry ran out of time :(