Math 341 Homework 4

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1 Practice problems

1.1 Problem 1

Let $a \in \mathbb{Z}$, $b \in \mathbb{N}$. Suppose a is divided by b with remainder r and quotient q. Divide $a+1$ by b with remainder.

Proof. By definition of divisibility with remainder,

$$a = bq + r \text{ and } 0 \leq r < b$$

Also by theorem 4.1, $q, r \in \mathbb{Z}$ are unique $\forall a$. So

$$a + 1 = bq' + r'$$

There are 2 cases:

1. $r \in \{0, 1, ...b-2\}$: In this case $r' = r + 1$, and since $r \leq b - 2$ we know $r' \leq b - 1$ and thus $r' < b$. Also because we haven't "overflowed*" so to speak, $q' = q$.
   (* By this I mean since $r' < b$, q remains the same, whereas if $r' = b$, then $q' = q + 1$.)

2. $r = b - 1$: In this case, $r' = b$, since $r' = r + 1 = (b-1) + 1 = b$. And since $r' = b$, this is division with remainder 0, (or normal division). Also since $r' = b$, $q' = q + 1$.

So $a + 1 = bq + r + 1$ when $r \in \{0, 1, ...b-2\}$.
And $a + 1 = b(q+1) + 0$ when $r = b - 1$. QED

1.2 Problem 3

Let $b \in \mathbb{N}$ and let $a \in \mathbb{Z}^{<}$. Prove the existence of $q, r \in \mathbb{Z}$, such that $a = bq+r$ and $0 \leq r < b$.

Proof. By definition, $a = bq+r$, and $0 \leq r < b$, then $-a = bq'+r'$, $0 \leq r < b$. $q' = -q$, however r' can be one of two things, either:

1. $r' = b - r$: In this case, $-a = bq' + r' = -bq + b - r = -b(q-1) - r$. Therefore $q, r \in \mathbb{Z}$ exist for $-a$.

2. $r' = b + r$: In this case, $-a = bq' + r' = -bq + b + r = -b(q-1) + r$. Therefore q, r exist, and are unique, for $-a$.

QED

1.3 Problem 4

Let $b \in \mathbb{N}$ and suppose $-b < r < b$. Prove that if $r \vdots b$, then $r = 0$.

Proof. Recall that by definition of divisibility, $r \vdots b \implies r = bn,$ for some $n \in \mathbb{Z}$. Also since $-b < r < b$, then

$$-b < bn < b$$

again where $n \in \mathbb{Z}$. We can divide everything by b to get:

$$-1 < n < 1, n \in \mathbb{Z}$$

There are no integers n between -1 and 1, except 0. Thus $n = 0$ and since $r = bn$, $r = 0$. QED