Math 341 Homework 8

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October 2020

1 Practice problems

1.1 Problem 8.3

Let $f : X \to Y$ and $g : Y \to Z$ be maps, prove that if the composition $g \circ f$ is injective then f is injective.

Proof. Consider the injective composition $g \circ f : X \to Z$, and suppose $x_1 \neq x_2$ this implies $g(f(x_1)) \neq g(f(x_2))$. By the definition of a function, this implies that $f(x_1) \neq f(x_2)$, thus f is an injection. QED

1.2 Problem 8.9

Construct a bijection from $\mathbb{N}$ to $2\mathbb{Z}^+$ (the set of positive, even integers).

Proof. Consider a function $f : \mathbb{N} \to 2\mathbb{Z}^+$, defined by $f(n) = 2n$. This is the mapping, $(1 \to 2), (2 \to 4), (3 \to 6), ...$
N2S (need to show): f is bijective, or f is both injective and surjective.

Remark. *Recall that a function $g : X \to Y$, is injective if $\forall x_1 \neq x_2 \in X$, $g(x_1) \neq g(x_2)$.*

Remark. *A function $g : X \to Y$, is surjective if $\forall y \in Y$, $\exists x \in X$ such that $g(x) = y$.*

1. Injectivity: $\forall n_1 \neq n_2 \in \mathbb{N}$, then the mapping is $f(n_1) = 2n_1$ and $f(n_2) = 2n_2$, if $f(n_1) = f(n_2)$ then $2n_1 = 2n_2 \implies 2(n_1 - n_2) = 0$. However since the integers have no nontrivial zero divisors, $(n_1 - n_2)$ must be equal to zero, therefore $n_1 = n_2$. And since this is true,

the contrapositive must be true (if $n_1 \neq n_2$, then $f(n_1) \neq f(n_2)$), as required. Therefore the function f is injective.

2. Surjectivity: $\forall z \in 2\mathbb{Z}^+$, $\exists n \in \mathbb{N}$ such that $f(n) = z$. If we choose some arbitrary $z \in 2\mathbb{Z}^+$, by definition, $z = 2n$, and since z is a positive, even integer, we can write this as $z = f(n)$. Therefore f is surjective.

Since $f(n) = z$ is both injective and surjective, then it is a bijection from $\mathbb{N} \to 2\mathbb{Z}^+$, As required. QED

1.3 Problem 8.10

Prove that there is a bijection $\mathbb{N} \to \mathbb{Z}$.

Proof. Consider $f : \mathbb{N} \to \mathbb{Z}$, defined by $f(n) = \begin{cases} k & n = 2k \\ -k & n = 2k+1 \end{cases}$

This maps $(1 \to 0), (2 \to 1), (3 \to -2), (4 \to 2), ...$

Remark. *A function $f : X \to Y$ is injective iff $f(x) = f(y) \implies x = y$.*

N2S: Bijectivity

1. Injectivity: For some $n_1, n_2 \in \mathbb{N}$, suppose $f(n_1) = f(n_2)$. Then there are 3 cases:

   (a) If they are both even, then $f(n_1) = k$, where $n_1 = 2k$. Also $f(n_2) = k$, where $n_2 = 2k$, it is clear that, $n_1 = n_2 = 2k$.

   (b) If they are both odd, then $f(n_1) = -k$, where $n_1 = 2k+1$. Also $f(n_2) = -k$, where $n_2 = 2k+1$, again it is clear that $n_1 = n_2 = 2k+1$.

   (c) If one is even and one is odd, then $f(n_1) = k$, where $n_1 = 2k$ and $f(n_2) = -k$, where $n_2 = 2k+1$. However in this case $n_1 \neq n_2$, because one is $2k$, and one is $2k+1$. $2k \neq 2k+1$, clearly. $\lightning$

   Therefore f is an injection.

2. Surjectivity: $\forall z \in \mathbb{Z}$, $\exists n \in \mathbb{N}$ s.t. $f(n) = z$. Take some $z \in \mathbb{Z}$, z has a sign (positive or negative), or it is 0.

   (a) If z is positive, then z is produced by $f(2n)$. Ex. $(z = 1, n = 2$, $(z = 2, n = 4)$,...

(b) If z is negative, then z is produced by $f(2n+1)$. Ex. $(z = -1, n = 3)$, $(z = -2, n = 5)$,...

(c) If $z = 0$, it is produced by $f(1)$.

And since each z has a unique producer in terms of n, the function f is surjective.

Since f is both injective and surjective, then it is a bijection from $\mathbb{N} \to \mathbb{Z}$, as required. QED