Math 553 Homework

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1 Section 1.3

- Problem 1: Find a parameterized curve $\alpha(t)$ whose trace is the circle $x^2 + y^2 = 1$ such that $\alpha(t)$ runs clockwise around the circle and $\alpha(0) = (0, 1)$.

$$\alpha(t) = (\sin(t), \cos(t))$$

- Problem 2: Let $\alpha(t)$ be a parameterized curve which does not pass through the origin. If $\alpha(t_0)$ is the point of the trace of α closest to the origin and $\alpha'(t_0) \neq 0$. Show that the position vector $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$.

Proof. N2S: $\alpha(t_0) \cdot \alpha'(t_0) = 0$.

We know from the problem that

$$|\alpha(t_0)| < \alpha(t)$$

for all t in the domain except t_0. This implies

$$|\alpha(t_0)|^2 < |\alpha(t)|^2$$

Again for all t except t_0. This inequality implies that $\alpha(t) \cdot \alpha(t)$ is minimized at t_0. Therefore the derivative equals 0:

$$\frac{d}{dt}[\alpha(t_0) \cdot \alpha(t_0)] = 2(\alpha'(t_0) \cdot \alpha(t_0)) = 0 \implies \alpha'(t_0) \cdot \alpha(t_0) = 0$$

As required. QED

- Problem 3: A parameterized curve $\alpha(t)$ has the property that its second derivative $\alpha''(t)$ is identically 0. What can be said about the curve?

$\alpha(t)$ must be a straight line, since the first derivative will be constant, then the second derivative must be 0.

- Problem 5: Let $\alpha : I \to \mathbb{R}^3$ be a parameterized curve, with $\alpha'(t) \neq 0$ for all $t \in I$. Show that $|\alpha(t)|$ is a nonzero constant iff $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$.

Proof. ($\Longrightarrow$) : Assume $|\alpha(t)|$ is a nonzero constant, then

$$|\alpha(t)| = C > 0$$

then,

$$\alpha(t) \cdot \alpha(t) = |\alpha(t)|^2 = C^2$$

Is also constant, and

$$\alpha'(t) \cdot \alpha(t) = \frac{1}{2}\frac{d}{dt}[\alpha(t) \cdot \alpha(t)] = \frac{1}{2}\frac{d}{dt}C^2 = 0$$

Therefore $\alpha'(t) \cdot \alpha(t) = 0$ as required.

($\Longleftarrow$) : Assume $\alpha'(t) \cdot \alpha(t) = 0$, then:

$$\alpha'(t) \cdot \alpha(t) = \frac{1}{2}\frac{d}{dt}[\alpha(t) \cdot \alpha(t)] = 0$$

implies

$$\alpha(t) \cdot \alpha(t) = C^2$$

For some real C. Then $C > 0$ because $C = 0 \implies \alpha'(t) = 0$ contradiction to what the problem says. Then we have showed that

$$|\alpha(t)|^2 = C^2 > 0 \implies \alpha(t) = C > 0$$

As required. QED

2 Section 1.3

- Problem 1: Show that the tangent lines with the regular parameterized curve $\alpha(t) = (3t, 2t^2, 2t^3)$ make a constant angle with the line $y = 0, z = x$ (equivalent to $(x, 0, x)$).

  Proof. The direction of the given line is $u = (1, 0, 1)$. Now we must show that $\theta = \arccos\left(\frac{(u \cdot v)}{|u||v|}\right)$ is constant, where v is the tangent line, $v = (3, 4t, 6t^2)$.

  $$\theta = \arccos\left(\frac{(u \cdot v)}{|u||v|}\right) = \arccos\left(\frac{3 + 6t^2}{\sqrt{18 + 32t^2 + 72t^4}}\right)$$

  Is not constant?? Although I notice that it *is* constant if instead $\alpha(t) = (3t, 3t^2, 2t^3)$, then $\theta = \arccos\left(\frac{3+6t^2}{\sqrt{18+72t^2+72t^4}}\right) = \frac{\pi}{4}$ QED

- Problem 2: A circular disk of radius 1 in the xy plane rolls without slipping along the x axis.

  a. Obtain a parameterized curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ the trace of which is the cycloid, and determine its singular points.

     $\alpha(t) = (t - sint, 1 - cost) \implies \alpha'(t) = (1 - cost, sint)$

     To find singular points, set $\alpha'(t) = 0$.

     $$1 - cost = 0 \quad sint = 0 \implies t = 2\pi k, k \in \mathbb{Z}$$

b. Compute the arc length of the cycloid corresponding to a complete rotation of the disk.

$$L|_0^{2\pi} = \int_0^{2\pi} |\alpha'(t)|dt$$

$$= \int_0^{2\pi} \sqrt{(1-cost)^2 + (sint)^2}dt$$

$$= \int_0^{2\pi} \sqrt{1 - 2cost + cos^2t + sin^2tdt}$$

$$= \int_0^{2\pi} \sqrt{2 - 2cost}dt$$

$$= 2\int_0^{2\pi} \sqrt{\frac{1-cost}{2}}dt$$

$$= 2\int_0^{2\pi} sin\left(\frac{t}{2}\right) dt$$

$$= -4cos\left(\frac{t}{2}\right)|_0^{2\pi}$$

$$= -4(-1-1) = 8$$

- Problem 10: Let $\alpha : I \to \mathbb{R}^3$ be a parameterized curve. Let $[a, b] \subset I$ and set $\alpha(a) = p, \alpha(b) = q$.

  a. Show that for any constant vector v, $|v| = 1$,

$$(q-p) \cdot v = \int_a^b \alpha'(t) \cdot v \ dt \leq \int_a^b |\alpha'(t)|dt$$

Proof. By the Cauchy-Schwarz inequality:

$$\alpha'(t) \cdot v \leq |\alpha'(t)||v| = |\alpha'(t)|$$

Therefore

$$\int_a^b \alpha'(t) \cdot v \, dt \leq \int_a^b |\alpha'(t)| dt$$

As required QED

b. Set $v = \frac{q-p}{|q-p|}$ and show that $|\alpha(b) - \alpha(a)| \leq \int_a^b |\alpha'(t)|dt$. That is, the curve of shortest length between $\alpha(a)$ and $\alpha(b)$ is the straight line joining these two points.

Proof. Inserting $v = \frac{q-p}{|q-p|}$, $p = \alpha(a)$ and $q = \alpha(b)$ into the inequality from a. gives us:

$$\int_a^b |\alpha'(t)| dt \geq (q-p) \cdot v$$

$$= (\alpha(b) - \alpha(a)) \cdot \frac{q-p}{|q-p|}$$

$$= \frac{|\alpha(b) - \alpha(a)|^2}{|\alpha(b) - \alpha(a)|}$$

$$= |\alpha(b) - \alpha(a)|$$

As required. QED