Math 524 Homework 1

Theo Koss

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1. Let $0.a_1a_2a_3\cdots$ be an infinite, but not periodic, decimal expansion. Consider the sets

$$A = \{x \in \mathbb{Q} \mid x \leqslant 0.a_1a_2\cdots a_k \text{ for some } k \geqslant 1\}$$
$$B = \{x \in \mathbb{Q} \mid x \geqslant 0.a_1a_2\cdots a_k \text{ for all } k \geqslant 1\}.$$

Show that (A, B) is a gap in $(\mathbb{Q}, \leqslant)$.

Proof. We must show the following:

(a) A and B are nonempty, disjoint, and $A \cup B = \mathbb{Q}$.

(b) If $a \in A$ and $b \in B$, then $a < b$.

(c) A has no greatest element, and B has no least element.

(a) A and B are both nonempty because for example $0 \in A$ and $1 \in B$. Suppose $a \in A$, then we have that

$$a \leqslant 0.a_1a_2\ldots a_k \implies a \not> 0.a_1a_2\ldots a_k$$

Note: $a \neq 0.a_1a_2\ldots a_k$ for <u>ALL</u> k, therefore $a \notin B$. So $A \cap B = \emptyset$.Finally, any $q \in \mathbb{Q}$ is either greater than x, less than x, or equal to x for some k. Therefore $A \cup B = \mathbb{Q}$

(b) Let $a \in A$ and $b \in B$. Then from above, we have shown $a \notin B$, negating the condition for set B we get $\exists k \geqslant 1$ such that $a < 0.a_1a_2\ldots a_k$, thus $a < b$.

(c) Suppose $a = 0.a_1a_2\dots a_k \in A$ is a greatest element for some $k \geqslant 1$, then consider $a' = 0.a_1a_2\dots a_ka_{k+1} > a$, therefore A has no greatest element. (Also note every finite decimal is $\in \mathbb{Q}$.)
Similar argument holds for B, assume b is a least element, then write it as $b = \frac{x}{y}$, but then we have $b' = \frac{x}{y+1} < b$.

QED

2. Let F be the set of all rational numbers that have a decimal expansion with only a finite number of nonzero digits. Show that F is dense in $\mathbb{Q}$.

Proof. Fix $a, b \in \mathbb{Q}$ with $a < b$. By definition we have $a = \frac{x}{y}$ for $x, y \in \mathbb{Z}$ and $y \neq 0$ and $b = \frac{w}{z}$.Then consider $b - a = \frac{p}{q}$ for integers p, q (by closure), then we have $b-a = \frac{p}{q} \geqslant \frac{1}{q} > \frac{1}{10^n}$ for some $n \in \mathbb{N}$. So there is some n such that $\frac{1}{10^n} < b - a$. Then, let $X = \{\frac{k}{10^n} | k \in \mathbb{Z}, n \in \mathbb{N}\}$. Elements of X are finite decimal expansions, and there is a largest $c \in X$ such that $c \leqslant a$. Then, simply add $c + \frac{1}{10^n}$ (Choose the n you need). Then $a < c + \frac{1}{10^n} < b$. QED

3. Let D (the dyadic rationals) be the set of all numbers $m/2^n$ where m is an integer and n is a natural number. Show that D is dense in $\mathbb{Q}$. [Hint: Consider base 2 expansions.]

Proof. Let x and y be two rationals with $x < y$. Then expand them in base 2, such that x =Some string of 1's and 0's, and y also is some string of 1's and 0's. Then the dyadic rationals $D = \{\frac{m}{2^n} \mid m \in \mathbb{Z}, n \in \mathbb{N}\}$ look like numbers in base 2 with terminating decimals. Specifically, numbers in D with $m = 1$ look like $.00\dots1$.

$$\frac{1}{2}_2 = .1 \ \frac{1}{4}_2 = .01 \ \frac{1}{8}_2 = .001$$

Then, write x and y down and find the first digit at which they differ. Say k, then $\frac{1}{2^k}$ can fit inside, so then find the largest dyadic rational z smaller than x. (This just means cut off the decimal at some point). Then

$$x < z + \frac{1}{2^k} < y$$

QED

4. In the construction of the real numbers in terms of the rational numbers, we defined the sum of two real numbers by the rule $a + b = \inf\{r + s \mid r, s \in \mathbb{Q} \text{ and } x \leqslant r, y \leqslant s\}$. Prove that addition of real numbers is commutative and associative and satisfies the law $a + 0 = a$ for all real numbers a.

Proof. (a) Associative:

$$\begin{aligned}
(a+b)+c &= \inf\{r+s \mid r, s \in \mathbb{Q},\ (a+b) \leqslant r,\ c \leqslant s\} \\
&= \inf\{r+s \mid r, s \in \mathbb{Q},\ c \leqslant s, \inf\{l+k \mid l, k \in \mathbb{Q} \ldots\} \leqslant r\} \\
&= \inf\{(l+k)+s \mid l, k, s \in \mathbb{Q},\ c \leqslant s,\ x \leqslant l,\ y \leqslant k\} \\
&\qquad\qquad (x \text{ and } y \text{ came from second addition}) \\
&= \inf\{l+(k+s) \mid l, k, s \in \mathbb{Q} \mid \ldots\} \\
&\qquad\qquad (\text{By definition of rationals}) \\
&= \inf\{l+j \mid l, j \in \mathbb{Q},\ a \leqslant l, (b+c) \leqslant j\} \\
&= a+(b+c)
\end{aligned}$$

(b) Commutative:

$$\begin{aligned}
a+b &= \inf\{r+s \mid r, s \in \mathbb{Q}\, a \leqslant r,\ b \leqslant s\} \\
&= \inf\{s+r \mid r, s \in \mathbb{Q}\, b \leqslant s,\ a \leqslant r\} \\
&= b+a
\end{aligned}$$

(c) $0 + a = a$:

$$\begin{aligned}
0+a &= \inf\{r+s \mid r, s \in \mathbb{Q},\ a \leqslant r,\ b \leqslant s\} \\
&= []
\end{aligned}$$

QED

5. Consider the periodic base 3 expansion $(0.010101\cdots)_3$. Use geometric series to express this number as a ratio of two integers.

$$0.010101\ldots_3 = 0 \cdot 3 + 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{9} + 0 \cdot \frac{1}{27} + 1 \cdot \frac{1}{81} + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{1}{9}^n = \frac{a}{1-r} = \frac{1}{1-\frac{1}{9}} = \frac{9}{8}$$

6. In this problem, you will show that the p-series $\sum_{n=1}^{\infty} 1/n^p$ is convergent whenever $p > 1$. [Note that we have not yet studied integration, so the Integral Test may not be used at this point in the course.]

   (a) Show, by induction, that if $(x_n)_{n=1}^{\infty}$ is a sequence of positive numbers, then the partial sums $s_n = x_1 + \cdots + x_n$ are a monotone increasing sequence. Conclude that if the partial sums are bounded above, then the sum $\sum_{n=1}^{\infty} x_n$ converges.

   (b) Assume $p > 1$. Observe that

$$\frac{1}{1} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \left(\frac{1}{8^p} + \cdots + \frac{1}{15^p}\right) + \cdots$$
$$\leqslant 1\left(\frac{1}{1^p}\right) + 2\left(\frac{1}{2^p}\right) + 4\left(\frac{1}{4^p}\right) + 8\left(\frac{1}{8^p}\right) + \cdots$$

   Show that the right-hand side of this inequality converges, and hence the partial sums of the left hand-side are bounded above. Then conclude that $\sum_{n=1}^{\infty} 1/n^p$ is convergent.

(a) Base case: $n = 2$:

$$s_2 > s_1 \text{ Since } x_2 > 0$$

is indeed a monotone increasing sequence.
Inductive step: Assume $s_k > s_{k-1}$. Then we must show $s_{k+1} > s_k$.$s_{k+1} = s_k + x_{k+1}$. We have $x_{k+1} > 0$ by assumption, so $s_{k+1} > s_k$.
Then, by the monotone increasing+ bounded above theorem, if all the s_n's are bounded above, $\sum_{n=1}^{\infty}$ converges.

(b) RHS =

$$1\left(\frac{1}{1^p}\right) + 2\left(\frac{1}{2^p}\right) + 4\left(\frac{1}{4^p}\right) + 8\left(\frac{1}{8^p}\right) + \cdots$$

RHS converges by telescoping. Therefore the partial sums are bounded above, so

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ Converges for } p > 1$$