Math 523 Homework 1

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Section 1.3

2. Use induction to prove the following:

f. If $x \in (0, 1)$ is a fixed real number, then $0 < x^n < 1$ for all $n \in \mathbb{N}$.

Proof. Let $x \in (0, 1)$ and $n \in \mathbb{N}$
Base case: $n = 1$, $x^1 = x \in (0, 1) \implies 0 < x < 1$ by definition.
Inductive step: Suppose $0 < x^k < 1$ for some $k \in \mathbb{N}$. Consider $n = k + 1$, then

$$x^{k+1} = x^k \cdot x$$

By induction hypothesis,

$$0 < x^k < 1$$

multiply by x:

$$0 < x^{k+1} < x$$

Since $x < 1$ (by assumption)

$$0 < x^{k+1} < 1$$

QED

h. $2^n < n!$ for all natural numbers $n \geqslant 4$.

Proof. Let $n \geqslant 4 \in \mathbb{N}$.
Base case: $n = 4$,

$$2^4 = 16,\ 4! = 24,\ 16 < 24\checkmark$$

Inductive step: Suppose $2^k < k!$ for some $k \geqslant 4$. Consider $k+1$. Then

$$2^{k+1} = 2^k + 2^K$$

$$\begin{aligned}(k+1)! &= k \cdot k! + k! \\ &< k \cdot 2^k + 2^k && \text{(By hypothesis)} \\ &< 2^k + 2^k \\ &= 2^{k+1}\end{aligned}$$

So

$$(k+1)! > 2^{k+1}$$

QED

j. $\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}$ is an integer for every $n \in \mathbb{N}$.

Proof. Let $n \in \mathbb{N}$.
Base case: $n = 1$

$$\frac{1}{5} + \frac{1}{3} + \frac{7}{15} = \frac{3}{15} + \frac{5}{15} + \frac{7}{15} = 1$$

Inductive step: Suppose the expression is an integer for some $k \in \mathbb{N}$ consider $k+1$. Then we seek

$$\frac{(k+1)^5}{5} + \frac{(k+1)^3}{3} + \frac{7(k+1)}{15}$$

to be an integer. i.e. we want

$$3(k+1)^5 + 5(k+1)^3 + 7(n+1) = 15j$$

for some $j \in \mathbb{N}$. (Divisible by 15)
Expanding:

$$(3k^5 + 5k^3 + 7k) + 15(k^4 + 2k^3 + 3k^2 + 2k + 1)$$

By induction assumption the first part is divisible by 15, and we factored out a 15 from second part so that part is clearly divisible by 15. (Sum of 2 things that are divisible by n is divisible by n trivially, but just in case: Suppose a, b are both divisible by n

$$a = x \cdot n, \; b = y \cdot n$$

so

$$a + b = (x + y) \cdot n$$

) QED

5. **a.** Prove

$$1 + nx \leqslant (1 + x)^n$$

for all $n \in \mathbb{N}$ with $x \geqslant -1$ a fixed real.

Proof. Let $n \in \mathbb{N}$ and $x \geqslant -1 \in \mathbb{R}$.
Base case: $1 + x = 1 + x\checkmark$
Inductive step: Assume $1 + kx \leqslant (1 + x)^k$

$$\begin{aligned}(1 + x)^{k+1} &= (1 + x)(1 + x)^k \\ &\geqslant (1 + x)(1 + kx) \\ &= 1 + (k + 1)x + kx^2 \\ &\geqslant 1 + (k + 1)x\end{aligned}$$

QED

b. Use binomial theorem to prove

$$1 + nx \leqslant (1 + x)^n$$

for all $n \in \mathbb{N}$ with $x \geqslant 0$ a fixed real.

Proof. By binomial theorem, RHS becomes:

$$(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}$$

Simplifying:

$$= x^n + \binom{n}{n-1} x^{n-1} + \cdots + \binom{n}{2} x^2 + nx + 1$$

LHS becomes:

$$1 + nx$$

Since $x \geqslant 0$, every term in RHS is non-negative. Therefore:

$$\begin{aligned}(1+x)^n &= x^n + \binom{n}{n-1} x^{n-1} + \cdots + \binom{n}{2} x^2 + nx + 1 \\ &= 1 + nx + \ell \\ &\geqslant 1 + nx\end{aligned}$$

So $1 + nx \leqslant (1+x)^n$ QED

Section 1.4

2. Prove that if q^2 is divisible by 3, then so is q.

Proof. We will prove the contrapositive, assume $3 \not| q$. That is, $q \equiv 1$ or $q \equiv 2 \mod 3$.

- If $q \equiv 1 \mod 3$, then $q^2 \equiv 1 \mod 3 \implies 3 \not| q^2$.
- If $q \equiv 2 \mod 3$, then $q^2 \equiv 1 \mod 3 \implies 3 \not| q^2$.

QED

4. **a.** $\sqrt{3}$

Proof. By way of contradiction, assume $\sqrt{3} = \frac{a}{b}$ for coprime $a, b \in \mathbb{Z}$. Then $3 = \frac{a^2}{b^2} \implies 3b^2 = a^2$, thus we have shown a^2 is divisible by 3, therefore so is a. We also have $3b^2 = (3k)^2 = 9k^2$ (subbing in $3k$ for a) so b is also divisible by 3. $\lightning$ QED

b. $\sqrt{6}$

Proof. By way of contradiction, assume $\sqrt{6} = \frac{a}{b}$ for coprime $a, b \in \mathbb{Z}$. Then $6 = \frac{a^2}{b^2} \implies 6b^2 = a^2$, thus we have shown a^2 is divisible by 6, therefore so is a. We also have $6b^2 = (6k)^2 = 36k^2$ so b is also divisible by 6. $\lightning$ QED

c. $\sqrt[3]{2}$

Proof. By way of contradiction, assume $\sqrt[3]{2} = \frac{a}{b}$ for coprime $a, b \in \mathbb{Z}$. Then $2b^3 = a^3$, so a^3 is even, therefore a is even ($0^3 \mod 2 = 0$). So there is some $k \in \mathbb{Z}$ such that $2b^3 = (2k)^3 \implies b^3 = 4k^3$ so b is even. $\lightning$ QED

d. $\sqrt{2} + \sqrt{3}$

Proof. By way of contradiction, assume $\sqrt{2} + \sqrt{3}$ is rational, then so is $(\sqrt{2} + \sqrt{3})^2 = 5 + 2\sqrt{6}$. That would imply $\sqrt{6} \in \mathbb{Q}$, which we have disproven above. QED

5. Consider the statement P: the sum of two irrational numbers is irrational.

   a. Give example where P is true.
   As seen above:

   $$\sqrt{2} + \sqrt{3} \notin \mathbb{Q}$$

   b. Prove or disprove P by giving counterexample.

   $$\sqrt{2} + (-\sqrt{2}) = 0 \in \mathbb{Q}$$

Additional Problems

1. Let $|X| = n$ for some $n \geqslant 0$, and choose an integer k with $0 \leqslant k \leqslant n$. Let A be the collection of all subsets of X with k elements. Let B be the collection of all subsets of X with $n - k$ elements. Find a one to one correspondence $f : A \to B$. Conclude that A and B have the same number of elements.

   Proof. Define

   $$f : A \to B$$

as

$$Y_i \mapsto X \setminus Y_i$$

Where $A = \{Y_1, Y_2, \ldots, Y_r\}$, the Y's are subsets of X with k elements. As required, the subsets $X \setminus Y_i$ have order $n-k$. Since f is a bijection, $|A| = |B| \implies \binom{n}{k} = \binom{n}{n-k}$, i.e. the number of ways to choose k things from n is equal to the number of ways to choose $n-k$ things from n.

Intuitively, when you choose k things from n, you're also *not* choosing $n-k$ things from n. QED

2. Prove following with Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

a.

$$2^n = \sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$$

for any integer $n \geqslant 0$.

Proof. Let $n \geqslant 0 \in \mathbb{Z}$, using the binomial theorem, let $x = 1$ and $y = 1$. Then

$$(1+1)^n = 2^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k = \sum_{k=0}^{n} \binom{n}{k}$$

QED

b.

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

for any integer $n \geqslant 0$.

Proof. Let $n \geqslant 0 \in \mathbb{Z}$, using the binomial theorem, let $x = 1$ and $y = -1$. Then

$$(1-1)^n = 0 = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} (-1)^k = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$$

QED