Math 553 Homework

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1 Section 2.5

- Problem 2: Let $x(\phi, \theta) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ be a parametrization of the unit sphere S^2. Let P the plane $x = z\cot\alpha, 0 < \alpha < \pi$, and β be the acute angle which the curve $P \cap S^2$ makes with the semimeridian $\phi = \phi_0$. Compute $\cos\beta$.

$$X_\phi = (-\sin\theta\sin\phi, \sin\theta\cos\phi, 0)$$
$$X_\theta = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$$

From p. 115, the loxodrome calculation:

$$\cos\beta = \frac{\theta'}{\sqrt{(\theta')^2 + (\phi')^2\sin^2\theta}}$$

We know that $\frac{x}{z} = \tan\theta\cos\phi = \cot\alpha$. Differentiating w.r.t. θ:

$$\sec^2\theta\cos\phi = 0$$

w.r.t. ϕ:

$$-\tan\theta\sin\phi = 0$$

These hold whence $\phi = \frac{\pi}{2}$ and $\theta = 0$. (I'm stuck here, I know I'm supposed to use the eqn I put above, but I don't understand what θ' and ϕ' are, since there is no parametrization. I tried differentiating the other eqn that the problem gave me but I don't understand the result.)

- Problem 12: Show that the area of a regular tube of radius r around a curve α is $2\pi r$ times the length of α.

Proof. Let $X(s,v) = \alpha(s) + r(n(s)\cos v + b(s)\sin v), r \neq 0, s \in I$ be the parametrization for such a tube. From the text, area is defined as

$$A(R) = \iint_Q |X_s \times X_v| ds dv, \quad Q = X^{-1}(R)$$

For parametrization $X(s,v)$.

$$X_s = t(s) + r((-\kappa t(s) + \tau b(s))\cos v - \tau n(s)\sin v)$$

$$X_v = r(-n(s)\sin v + b(s)\cos v)$$

Rewriting in terms of the Frenet Frame:

$$
\begin{aligned}
X_s &= (1 - r\kappa \cos v)\mathbf{t} - r\tau \sin v\mathbf{n} + r\tau \cos v\mathbf{b} \\
X_v &= -r \sin v\mathbf{n} + r \cos v\mathbf{b} \\
X_s \times X_v &= -r(1 - r\kappa \cos v)(\mathbf{n} \cos v + \mathbf{b} \sin v) \\
|X_s \times X_v| &= \sqrt{r^2(n \cos v + b \sin v - br\kappa \sin v \cos v - nr\kappa \cos^2 v)^2} \\
&= \sqrt{r^2((\cos v - r\kappa \cos^2 v)n)^2 + r^2((\sin v - r\kappa \sin v \cos v)
\end{aligned}
$$

Again I got lost, I am not understanding this section

QED

- Problem 14: The gradient of a differentiable function $f : S \to \mathbb{R}$ is a differentiable map grad $f : S \to \mathbb{R}^3$ which assigns to each point $p \in S$ a vector grad $f(p) \in T_pS \subset \mathbb{R}^3$ such that

  $$\langle \text{grad} f(p), v\rangle_p = df_p(v) \quad \text{For all } v \in T_pS.$$

  Show that:

  a. If E, F, G are coefficients of the first fundamental form in a parametrization $x : U \subset \mathbb{R}^2 \to S$, then grad f on $x(U)$ is given by

  $$\text{grad} f = \frac{f_u G - f_v F}{EG - F^2} x_u + \frac{f_v E - f_u F}{EG - F^2} x_v$$

  In particular, if $S = \mathbb{R}^2$ with coordinates x, y:

  $$\text{grad} f = f_x e_1 + f_y e_2$$

Let $p = X(u, v)$ be a point, if $f : S \to \mathbb{R}$ is a differentiable function then grad $f(p) \in T_pS$. Thus

$$\text{grad} f(p) = \alpha X_u + \beta X_v$$

for functions α, β defined on U. Using this, we achieve the following:

$$f_u = \alpha E + \beta F \quad f_v = \alpha F + \beta G$$

Now, solving for α from this system:

$$f_u - f_v = \alpha(E - F) + \beta(F - G)$$

$$\alpha = \frac{f_u E - f_u F - f_v E + f_v F}{\beta F - \beta G}$$

Then doing a similar thing for β and plugging in to the eqn above:

$$\text{grad} f(p) = \frac{f_u G - f_v F}{EG - F^2} x_u + \frac{f_v E - f_u F}{EG - F^2} x_v$$

b. If you let $p \in S$ be fixed and v vary in the unit circle $|v| = 1$ in T_pS, then $df_p(v)$ is maximum iff $v = \text{grad} f / |\text{grad} f|$.

$$\langle \text{grad} f, v \rangle = |\text{grad} f(p)| \cos\theta \leq |\text{grad} f(p)|$$

if $|v| = 1$. Then the upper bound must be given by $v = \text{grad} f / |\text{grad} f|$.

c.