Math 723 Homework 3

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Assignment 3

1. Show that

$$|\frac{f_{n+1}}{f_n} - \varphi| = \frac{1}{f_n} \cdot \frac{1}{\varphi^{n+1}} \text{ and } \lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \varphi$$

Where φ is the unique positive root of $\varphi^2 - \varphi - 1 = 0$.

Proof. Induction base case:

$$|\frac{f_1}{f_0} - \varphi| = \frac{1}{f_0} \cdot \frac{1}{\varphi^1}$$

$$\implies |1 - \varphi| = \frac{1}{\varphi} \implies \varphi^2 - \varphi - 1 = 0$$

Which is exactly as we have defined φ. Assume

$$|\frac{f_{k+1}}{f_k} - \varphi| = \frac{1}{f_k} \cdot \frac{1}{\varphi^{k+1}}$$

Then, we have

$$
\begin{aligned}
|\frac{f_{k+2}}{f_{k+1}} - \varphi| &= |\frac{f_{k+1} + f_k}{f_{k+1}} - \varphi| \\
&= |1 + \frac{f_k}{f_{k+1}} - \varphi| \\
&= |1 + (\varphi + \frac{1}{f_k} \cdot \frac{1}{\varphi^{k+1}}) - \varphi| \\
&= |1 + \frac{1}{f_k} \cdot \frac{1}{\varphi^{k+1}}| \\
&= \frac{1}{f_{k+1}} \cdot \frac{1}{\varphi^{k+2}} && (1 = \varphi - \tfrac{1}{\varphi})
\end{aligned}
$$

Then, since this equality holds and the RHS $\to 0$ as $n \to \infty$, we have that $\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \varphi$ □

2. With the Fibonacci numbers, define $f(z) = \sum_{n=0}^{\infty} f_n z^n$. Show that this power series has radius of convergence $R = \frac{1}{\varphi}$, and for $|z| < R$,

$$f(z) = \frac{1}{1 - z - z^2}$$

Proof. By the Cauchy-Hadamard Theorem, we have

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}$$

We have $a_n = f_n$, and by above, $\limsup_{n\to\infty} |f_n|^{1/n} = \varphi$. Thus $R = \frac{1}{\varphi}$. Let $|z| < R$, then

$$f(z) = \sum_{n=0}^{\infty} f_n z^n = 1 + f_1 z + \sum_{n=2}^{\infty} (f_{n-1} + f_{n-2}) z^n$$

Split the sum:

$$= 1 + f_1 z + \sum_{n=2}^{\infty} a_{n-1} z^n + \sum_{n=2}^{\infty} a_{n-2} z^n$$

Pull out z from the first sum, and z^2 from the second:

$$= 1 + f_1 z + z\sum_{n=2}^{\infty} a_{n-1}z^{n-1} + z^2\sum_{n=2}^{\infty} a_{n-2}z^{n-2}$$

Now each sum is exactly $f(z)$, so

$$= 1 + zf(z) + z^2 f(z) \implies f(z) = \frac{1}{1-z-z^2}$$

As required □

3. The generalized binomial coefficient for $z \in \mathbb{C}$ and $k = 0, 1, 2, \ldots$ is 1 if $k = 0$ and
$$\binom{z}{k} = \frac{z(z-1)\cdots(z-k+1)}{k!}$$
Find the radius of convergence of $B_s(z) = \sum_{n=0}^{\infty}\binom{s}{n}z^n$.

Proof. Apply the ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{\binom{s}{n+1}z^{n+1}}{\binom{s}{n}z^n} = \frac{s-n}{n+1}z$$

Then
$$\lim_{n\to\infty}|\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty}\left|\frac{s-n}{n+1}z\right| = |z|$$
Since $\frac{s-n}{n+1}$ converges to 1 for fixed s.
Therefore to converge we require $|z| < 1$. So $R = 1$. □

4. For every sequence $\mathbf{a} = (a_n) \in \{0, 2\}^{\mathbb{N}}$ define $x(\mathbf{a}) = \sum_{n=1}^{\infty}\frac{a_n}{3^n}$. Show that the set of all such values is the cantor set.

Proof. Recall the Cantor set is defined by first taking $[0, 1]$, then removing the middle third, $(\frac{1}{3}, \frac{2}{3})$. Then we get two parts: $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. At $n = 1$, the sum can take two values, 0 or $\frac{2}{3}$. The value 0 is in the first part, and 2/3 is in the second part. Then at level 2, we have 4 options:

- $(0,0) \mapsto 0$
- $(0,2) \mapsto \frac{2}{9}$
- $(2,0) \mapsto \frac{2}{3}$
- $(2,2) \mapsto \frac{2}{3} + \frac{2}{9} = \frac{8}{9}$

All of which lie in the four parts in the second layer (in fact, one in each), which are

$$\left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$$

Each segment after n steps has length $\frac{1}{3^n}$, and there are 2^n of them. A value of 0 keeps you in the segment you are in, while a value of 2 moves you over to the next segment (of the Cantor set), skipping over the middle third. Therefore there is no way to get outside of the set with a sequence of 0's and 2's. And, each sequence **a** corresponds to a specific value of the cantor set, by choosingleft or right (0 or 2) however many times. $\square$