Math 551 Homework 5

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1 Section 4.5

- Problem 10: Let X be a Hausdorff space, A a subset of X, and x a limit point of A. Prove that every open set containing x contains infinitely many members of A.

  Proof. By way of contradiction, suppose M is an open set which contains x and is finite. Then we can index the points $a_1, a_2, \dots, a_n \in M$. By definition, since X is a Hausdorff space, for each $a_k \in M$, where $(1 \leq k \leq n)$,

  $$\exists U_k, V_k \subset M \quad \text{Such that } a_k \in U_k, x \in V_k \quad \text{and} \quad U_k \cap V_k = \emptyset$$

  Let

  $$V = \bigcap_{k=1}^{n} V_k$$

  Clearly V is an open set since it is the intersection of finitely many open sets. And for each k between 1 and n, $U_k \cap V = U_k \cap V_k$. Then $U_k \cap V = \emptyset$, equivalently, $a_k \notin V$.

  Thus we have found an open subset of M which contains x but no other points of A. Which means x is not a limit point of A. Contradiction. QED

2 Section 5.1

- Problem 4: Let $(\mathbb{R}, \mathcal{T}')$ be the space of real numbers with the finite complement topology. Is $(\mathbb{R}, \mathcal{T}')$ connected or disconnected? Prove your answer.

  Proof. The space is connected.

  By way of contradiction, assume it is disconnected. Then there exists two sets U and V which are non-empty, open and disjoint sets such that $\mathbb{R} = U \cup V$. However:

  $$U \cap V = \emptyset \implies U \subset \mathbb{R} \backslash V, \quad \text{and} \quad V \subset \mathbb{R} \backslash U.$$

  Since U and V are both nonempty and we are using the finite complement topology, $\mathbb{R} \backslash V$ and $\mathbb{R} \backslash U$ are both finite. By the equation above, U and V must be finite themselves.

  Thus $\mathbb{R} = U \cup V$ is finite, which is a contradiction because of course $\mathbb{R}$ is infinite. QED

3 Section 5.2

- Problem 4: Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of connected subsets of a space X such that for each integer $n \geq 1$, A_n has at least one point in common with one of the preceding sets $A_1, \ldots, A_{n-1}$. Then $\bigcup_{n=1}^{\infty} A_n$ is connected.

  Proof. By way of contradiction, assume $\bigcup_{n=1}^{\infty} A_n$ is not connected. Then there must exist a separation of it, U, V where U, V are nonempty, open disjoint sets.

  Consider some arbitrary $A_i \in \{A_n\}$. Then A_i lies entirely within U or V. There are 2 cases:

  1. All of the A_i are entirely in U or V.
  2. Some of the A_i are in U and some are in V.

In case 1, if they are all in U, then U, V are not a separation of $\bigcup_{n=1}^{\infty} A_n$. Same argument if they are all in V. Contradiction.

In case 2, if they are split then the $A_i \in U$ are disjoint from the $A_i \in V$. This contradicts the hypothesis that for each integer $n \geq 1$, A_n has at least one point in common with one of the preceding sets $A_1, \ldots, A_{n-1}$. Thus this is a contradiction.

$$\therefore \bigcup_{n=1}^{\infty} A_n \quad \text{is connected.}$$

QED

- Problem 5: Determine whether each of the following subspaces of $\mathbb{R}^2$ is connected or disconnected. Give a reason for each answer.
  (a) Asymptotic curves: Disconnected, since even though they will get infinitely close to each other, they will never touch so you can find 2 nonempty disjoint open sets.
  (b) Asymptotic curves and the asymptote: connected, since the Topologist's sine curve is connected and it's kind of the same deal.
  (c) Intersecting lines: Connected, since it is clearly path-connected, which is stronger than simply being connected.
  (d) Figure 5.2: Connected, as you can not find 2 nonempty disjoint open sets such that the union is the set.
- Problem 6: Prove that every countable subset of $\mathbb{R}$ is totally disconnected.

Proof. We must show for every countable subset of $\mathbb{R}$ each component is a single point. Also recall that any interval in $\mathbb{R}$ is uncountable. (Since there is a bijection from arbitrary $(a, b) \to (0, 1)$ and $(0, 1)$ uncountable.)

Consider an arbitrary countable subset of $\mathbb{R}$, call it X. Then consider some component A of X which contains 2 points. Call these points

a_1, a_2. Then $(a_1, a_2) \subset A$, which is uncountable, thus A itself is uncountable, and since $A \subset X$, X is also uncountable. Contradiction.

This reasoning can be extended if A contains more than 2 points. Thus each component must contain only 1 point. Therefore X is totally disconnected QED

4 Section 5.3

- Problem 3: Let $f : [a, b] \to [c, d]$ be a homeomorphism on the indicated intervals. Prove that f maps endpoints to endpoints.

  Proof. Note that a homeomorphism between two topological spaces X, Y is a function $f : X \to Y$ which has the following properties:

  i. f is a bijection.
  ii. f is continuous.
  iii. f^{-1} is continuous.

  Suppose, by way of contradiction, that $f(a)$ maps to neither c nor d. Then $f(a) = x \in (c, d)$, then the image of the interval $(a, b]$ must be $[c, x) \cup (x, d]$. However, we just showed that f must map an interval to $[c, x) \cup (x, d]$, which is *not* an interval, and since it is not an interval, it is not connected. This contradicts ii. above, that f is continuous.

  The same argument shows that b must also go to one of the endpoints. Thus homeomorphism $f : [a, b] \to [c, d]$ must map endpoints to endpoints. QED

  (Similarly I believe you could consider the non-cut points of the preimage and the image and notice that $(a, b]$ has 1 non-cut point and $[c, x) \cup (x, d]$ has 2, thus they are topologically different spaces so f not a homeomorphism.)

5 Section 5.4

- Problem 1: Prove that every polynomial having real coefficients and odd degree has a real root.

  Proof. Recall every polynomial on the real line is continuous. Since a polynomial is a sum of powers of x, powers of x are continuous thus every polynomial is continuous.

  We want to show that if $P(x) = a_n x^n + \cdots + a_1 x^1 + a_0$ is a polynomial such that n odd and $a_n \neq 0$, then there exists a real c such that $P(c) = 0$. If $a_n > 0$:

  $$\lim_{x\to\infty} P(x) = \infty \quad \text{and} \quad \lim_{x\to-\infty} P(x) = -\infty$$

  Thus there are real numbers $x_0 < x_1$ such that $P(x_0) < 0$ and $P(x_1) > 0$. (If $a_n < 0$, the argument is similar but $P(x_0) > 0$ and $P(x_1) < 0$.) Then, by the IVT, there is a real number $c \in [x_0, x_1]$ such that $P(c) = 0$. QED

6 Section 5.5

- Problem 7: Prove that a space X is path connected iff there is a point a in X such that each point of X can be joined to a by a path in X.

  Proof. $(\Longrightarrow)$: Assume X is a space where there is a point $a \in X$ such that each point of X can be joined to a by a path in X. We must show that $\forall x, y \in X$, the path starting at x and ending at y is in X. By the assumption, each point has a path to a. Also, for all $x \in X$, the path from $x \to a$ exists, as does the path from $a \to x$. Now notice that that the *path product* $p_1 * p_2 = \begin{cases} p_1(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ p_2(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$, where $p_1(1) = p_2(0)$, is also in X.

  Thus, $\forall x, y \in X$, there are two paths, $p_1 \in X$ from $x \to a$ and $p_2 \in X$ from $y \to a$. Then the reverse path $\bar{p_2}$ is in X. This is the path from

$a \to y$. Since $p_1(1) = a$ and $\bar{p_2}(0) = a$, the path product $p_1 * \bar{p_2} \in X$ by definition.
$(\Longleftarrow)$: Assume X is a path connected space, we must show that there is a point $a \in X$ such that each point of X can be joined to a by a path in X.

Since X is path connected, $\forall x, y \in X$, $\exists p : x \to y \in X$ where p is a path. Fix some $a \in X$. Then for each $x_{i \in I} \in X$, the paths $p_1, p_2, \ldots, p_i$ where $p_1 : x_1 \to a$, $p_2 : x_2 \to a$ and so on must be in X by definition of path connectedness. Thus we have showed that each point of X can be joined to a by a path in X. QED