Math 341 Exam 1

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September 2020

1 Problem 1

Prove by induction: $7^{n+1} + 8^{2n-1} \vdots 57$ for all $n \in \mathbb{N}$.

Proof. Basis: Check for $n = 1$: $7^2 + 8 \vdots 57 = 57 \vdots 57$.
Inductive step: Assume that the formula holds for $n = k$. We must show it holds for $n = k + 1$. So, by assumption: $7^{k+1} + 8^{2k-1} \vdots 57$. We must show:

$$7^{k+2} + 8^{2k+1} \vdots 57$$

Simplifying:

$$= 7 \cdot 7^{k+1} + 64 \cdot 8^{2k-1}$$

Rearranging:

$$= 7 \cdot 7^{k+1} + 7 \cdot 8^{2k-1} + 57 \cdot 8^{2k-1} = 7(7^{k+1} + 8^{2k-1}) + 57 \cdot 8^{2k-1}$$

By our assumption:

$$7^{k+1} + 8^{2k-1} \vdots 57 \Longrightarrow 7(7^{k+1} + 8^{2k-1}) \vdots 57$$

and by definition:

$$57 \cdot 8^{2k-1} \vdots 57$$

Finally:

$$7(7^{k+1} + 8^{2k-1}) \vdots 57 \text{ and } 57 \cdot 8^{2k-1} \vdots 57$$
$$\Longrightarrow 7^{k+2} + 8^{2k+1} \vdots 57$$

As required. QED

2 Problem 2

Prove by induction: if $a \geq -1$, then, $(1+a)^n \geq 1 + na$ for all $n \geq 0$.

Proof. *Basis*: Check for $n = 1$: $1 + a \geq 1 + a$.
Inductive step:
Assume the formula holds for $n = k$, so $(1+a)^k \geq (1+ka), a \geq -1$ We must show it holds for $n = k + 1$, that is:

$$(1+a)^{k+1} \geq (1 + (k+1)a)$$

Simplifying:

$$(1+a) \cdot (1+a)^k \geq (1 + ka + a)$$

QED

3 Problem 3

Prove if $a \vdots c$ and $b \vdots c$, then for any x and y, $ax + by \vdots c$.

Proof. We must show that $ax + by = cl$, for some $l \in \mathbb{Z}$. $a \vdots c$, by definition means $a = cn, n \in \mathbb{Z}$, similarly, $b \vdots c$ by definition means $b = cm, m \in \mathbb{Z}$. Then for any x, y, $x \cdot cn + y \cdot cm \vdots c$, since we can factor out a c: $ax + by = c(xn + ym)$ let $l = xn + ym$, then we have: $ax + by = cl, l \in \mathbb{Z}$, which, by definition, means $ax + by \vdots c$. QED

4 Problem 4

Prove that $12 \not\vdots 5$.

Proof. Assume 12 *is* divisible by 5. Then 12=5c, for some **integer** c. This means that $c = \frac{12}{5}$, which is not an integer, thus we have a contradiction, so $12 \not\vdots 5$. QED

5 Problem 5

Let $a, b, c \in \mathbb{N}$ and suppose $a > c$ and $b > c$. True or false: if $ab \vdots c$, then $a \vdots c$ or $b \vdots c$. Prove.

Proof. This is true, since $ab \vdots c$, by definition, means $ab = cn$, for some $n \in \mathbb{N}$. QED