# Math 835 Homework 1

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## 1 Chapter 13

### 1.1 Chapter 1

1. Show that $p(x) = x^3 + 9x + 6$ is irreducible in $\mathbb{Q}[x]$. Let $\theta$ be a root of $p(x)$, find the inverse of $1 + \theta$ in $\mathbb{Q}(\theta)$.

   *Proof.* By the Eisenstein criterion, let $p = 3$, $p \mid 9$ and $p \mid 6$ but $p^2 = 9 \nmid 6$ and $p \nmid 1$. $\square$

   Following example 4 in 13.1, (pp. 515), apply Euclidean Algorithm in $\mathbb{Q}[x]$ to get $a(x)$ and $b(x)$ such that

$$a(x)(1 + x) + b(x)(x^3 + 9x + 6) = 1$$

$x^2 - x + 10$ $R = -4$

$x + 1 \,\big|\, x^3 + 0x^2 + 9x + 6$

$-(x^3 + x^2)$

$-x^2 + 9x$

$-(-x^2 - 1)$

$10x + 6$

$-(10x + 10)$

$-4$

$$
\begin{aligned}
a(x) &= \frac{(x^2 - x + 10)}{4},\ b(x) = -\frac{1}{4} \quad \text{(Div. both sides by 4)} \\
\frac{(1+x)(x^2 - x + 10) - (x^3 + 9x + 6)}{4} &= 1 \\
\frac{(1+\theta)(\theta^2 - \theta + 10)}{4} &= 1 \qquad\qquad \text{(Since } p(\theta) = 0\text{)} \\
\implies (1+\theta)^{-1} &= \frac{\theta^2 - \theta + 10}{4}
\end{aligned}
$$

3. Show that $x^3+x+1$ is irreducible over $\mathbb{F}_2$ and let $\theta$ be a root. Compute the powers of $\theta$ in $\mathbb{F}_2$.

*Proof.* $p(x) = x^3 + x + 1$ is irreducible by the rational root theorem, and checking $p(0) = 0^3 + 0 + 1 \neq 0$ and $p(1) = 1^3 + 1 + 1 \neq 0$. $\square$

Powers of $\theta$:

$$
\begin{aligned}
\theta^1 &= \theta \\
\theta^2 &= \theta^2 \\
\theta^3 &= \theta + 1 \\
\theta^4 &= \theta^2 + \theta \\
\theta^5 &= \theta^3 + \theta^2 = \theta^2 + \theta + 1 \\
\theta^6 &= \theta^2 + 1 \\
\theta^7 &= 1 = \theta^0 \\
&\vdots
\end{aligned}
$$