Math 523 Homework 3

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Section 2.3

5. If $\{a_n\}$ and $\{b_n\}$ diverge to $+\infty$, prove that $\{a_n + b_n\}$ and $\{a_nb_n\}$ also diverge to $+\infty$.

   Proof. By limit properties, we have that $\lim_{n\to\infty}\{a_n+b_n\} = \lim_{n\to\infty}\{a_n\}+\lim_{n\to\infty}\{b_n\}$, and $\lim_{n\to\infty}\{a_nb_n\} = \lim_{n\to\infty}\{a_n\} \cdot \lim_{n\to\infty}\{b_n\}$, so

$$\lim_{n\to\infty}\{a_n + b_n\} = +\infty + (+\infty) = +\infty$$

$$\lim_{n\to\infty}\{a_nb_n\} = (+\infty)(+\infty) = +\infty$$

QED

Section 2.4

15(b). If $a_1 = 1$ and $a_{n+1} = \sqrt{1 + a_n}$, show that $\{a_n\}$ converges to α.

Proof. We will show $\{a_n\}$ is strictly increasing and bounded above, and thus converges.

- $a_2 > a_1$ because $\sqrt{2} > 1$, assume $a_{k+1} > a_k$ for some k. We seek $a_{k+2} > a_{k+1}$, so $a_{k+2} = \sqrt{1 + a_{k+1}} > \sqrt{1 + a_k} = a_{k+1}$. So $\{a_n\}$ is strictly increasing.
- $\{a_n\}$ is bounded above by 3, $\sqrt{1+1} < 3$. Assume $a_k < 3$ for some k, we seek $a_{k+1} < 3$, $a_{k+1} = \sqrt{1 + a_k} < \sqrt{1+3} < 3$.

So $\{a_n\}$ converges, to find the value we will take the limit of both sides of the recursion formula.

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{1+a_n} \implies A = \sqrt{1+A}$$

This gives

$$A^2 - A - 1 = 0$$

And since A is nonnegative, $A = \frac{1+\sqrt{5}}{2}$ QED

Section 2.5

2(c). Find the set of accumulation points of $S = \{a_n \mid n \in \mathbb{N}\}$ Where

$$a_n = \begin{cases} 0 & n \text{ is odd} \\ \frac{n}{n+1} & n \text{ is even} \end{cases}$$

$\frac{n}{n+1} = 1 - \frac{1}{n+1}$, so as n increases, a_n tends to 1. Therefore for any nbd of 1, there are infinitely many n such that a_n is in the nbd. $\{1\}$ is the set of accumulation points.

Additional Problems

1. The mistake in the argument is treating the limit of the sequence, L as a number. When in fact it is ∞, which is not a number. If it were you could also claim $\infty = \infty + 1 \implies 0 = 1$, which is clearly untrue.

2. Using the axiom of completeness, consider a nonempty set of reals B which is bounded below. Let $A = -B$, since B is bounded below, we have some x for which $x \leqslant b \ \forall b \in B$, then $-x \geqslant -b \ \forall (-b) \in A$. So lower bounds for B are upper bounds for A. Since A is bounded above, it has a lub, call it y. Then I claim $-y$ is the greatest lower bound for B. It is indeed a lower bound, if not then $b < -y$ for some $b \in B$, but that would mean $-b > y$ for some $-b \in A$, which is false. A similar argument shows nothing greater than $-y$ is a lower bound.

3. Let $x, \bar{x}$ both be suprema of a set of reals A and $x \neq \bar{x}$. Then since $x \in A$ and $\bar{x}$ is a supremum, $x \leqslant \bar{x}$. Similarly since $\bar{x} \in A$ and x is a supremum, $\bar{x} \leqslant x$. Thus $x = \bar{x}$ contradicts $x \neq \bar{x}$. Thus there must only be one.

4. Let $q \neq 0 \in \mathbb{Q}^*$ and $x \in \mathbb{R}^* \setminus \mathbb{Q}^*$. By way of contradiction, assume $qx = \frac{a}{b}$ for nonzero integers a, b. Since $q \in \mathbb{Q}^*$, one can write $q = \frac{m}{n}$ for nonzero integers m, n. So $x = \frac{na}{mb}$, and since all four of a, b, m, n are nonzero integers, this is a fraction of nonzero integers, so $x \in \mathbb{Q}^*$, contradiction.

5. Prove for $x < y$, (x, y) contains infinitely many rationals.

   Proof. By density of $\mathbb{Q}$ in $\mathbb{R}$, there exists at least one $a \in \mathbb{Q}$ such that $x < a < y$. Let n be an integer such that $\frac{1}{n} < y - x$, then consider $x + \frac{1}{n}, x + \frac{1}{n+1}, \ldots$ Since $\frac{1}{n} < y - x$, and $0 < \frac{1}{n+k} < \frac{1}{n}$, this is an infinite sequence of rationals between x and y. QED