Math 341 Problem Set 1

Theo Koss

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1 Practice Problems

Problem 1

Prove by induction: for all $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} (2i-1) = n^2.$$

Proof. Basis: $n = 1$ Check: $\sum_{i=1}^{1} (2i-1) = 1 = n^2$.
Induction step: Assume the formula holds for $n = k$

$$\sum_{i=1}^{k} (2i-1) = k^2.$$

We must show it holds for $n = k+1$:

$$\sum_{i=1}^{k+1} (2i-1) = (k+1)^2 = k^2 + 2k + 1.$$

The sum is the sum of the first k terms, plus the last one.

$$\sum_{i=1}^{k+1} (2i-1) = \sum_{i=1}^{k} (2i-1) + 2k + 1$$

Using assumption:

$$\sum_{i=1}^{k}(2i-1)+2k+1=k^2+2k+1=(k+1)^2$$

As required. Therefore, by induction, the formula is true for all $n\in\mathbb{N}$ QED

Problem 2

Prove by induction: for all $n\in\mathbb{N}$

$$\sum_{i=1}^{n}i^2=\frac{n\,(n+1)\,(2n+1)}{6}$$

Proof. Basis: $n=1$ Check: $\sum_{i=1}^{1}i^2=1=\frac{1(1+1)(2+1)}{6}$
Induction step: Assume the formula holds for $n=k$:

$$\sum_{i=1}^{k}i^2=\frac{k\,(k+1)\,(2k+1)}{6}=\frac{2k^3+3k^2+k}{6}$$

We must show it holds for $n=k+1$:

$$\sum_{i=1}^{k+1}i^2=\frac{(k+1)\,(k+2)\,(2k+3)}{6}=\frac{2k^3+9k^2+13k+6}{6}$$

The sum on the left is equal to the sum from $i=1$ to k, plus the term $i=k+1$:

$$\sum_{i=1}^{k+1}i^2=\sum_{i=1}^{k}i^2+k^2+2k+1$$

Using assumption:

$$\sum_{i=1}^{k}i^2+k^2+2k+1=\frac{k\,(k+1)\,(2k+1)}{6}+k^2+2k+1$$

Simplifying:

$$=\frac{2k^3+3k^2+k}{6}+\frac{6k^2+12k+6}{6}=\frac{2k^3+9k^2+13k+6}{6}=\sum_{i=1}^{k+1}i^2$$

$\therefore$ the formula is true for all $n\in\mathbb{N}$.

Problem 3

Prove by induction: for all $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2.$$

Proof. Basis: $n = 1$ Check: $\sum_{i=1}^{1} i^3 = 1 = \left(\sum_{i=1}^{1} i\right)^2$.
Induction step: Assume the formula holds for $n = k$:

$$\sum_{i=1}^{k} i^3 = (1 + 2 + 3 + ... + k)^2$$

Show that the formula holds for $n = k + 1$:

$$\sum_{i=1}^{k+1} i^3 = \left(1 + 8 + 27 + ... + k^3 + (k+1)^3\right)$$

The sum from 1 to $k + 1$ is equal to the sum from 1 to k, plus the $i = k + 1$ term, so:

$$\sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k+1)^3$$

Using our assumption, and the fact that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$:

$$\sum_{i=1}^{k+1} i^3 = (1 + 2 + 3 + ... + k)^2 + (k+1)^3 = \left(\frac{k\,(k+1)}{2}\right)^2 + (k+1)^3$$

Simplifying:

$$\left(\frac{k\,(k+1)}{2}\right)^2 + (k+1)^3 = \frac{k^2\,(k+1)^2}{4} + (k+1)^3 = \frac{k^2\,(k+1)^2 + 4\,(k+1)^3}{4}$$

More simplifying:

$$\frac{k^2\,(k+1)^2 + 4\,(k+1)^3}{4} = \frac{k^4 + 6k^3 + 13k^2 + 12k + 4}{4} = \frac{(k+1)^2\,(k+2)^2}{4}$$

Factoring out a square:

$$\frac{(k+1)^2\,(k+2)^2}{4} = \left(\frac{(k+1)\,(k+2)}{2}\right)^2$$

And since $\left(\frac{(k+1)(k+2)}{2}\right)^2$ Is equal to $\left(\sum_{i=1}^{k+1} i\right)^2$, we have proven the induction case, and therefore the formula is true for all $n \in \mathbb{N}$.

Problem 4

Find:

$$\sum_{k=1}^{n} k\,(k!)$$

Let $f(n) = \sum_{k=1}^{n} k\,(k!)$

n	$f(n)$
1	1
2	5
3	23
4	119
5	719

Conjecture: $\sum_{k=1}^{n} k\,(k!) = (n+1)! - 1, n \in \mathbb{N}$.

Proof. Basis: Check $n = 1$: $\sum_{k=1}^{1} k\,(k!) = 1 = (1+1)! - 1$
Inductive step: Assume the conjecture holds for $k = n$:

$$\sum_{k=1}^{n} k\,(k!) = (n+1)! - 1$$

Show the conjecture holds for $k = n+1$:

$$\sum_{k=1}^{n+1} k\,(k!) = (n+2)! - 1$$

The sum from 1 to $n+1$ includes the entire sum from 1 to n, plus the $k = n+1$ term:

$$\sum_{k=1}^{n+1} k\,(k!) = \sum_{k=1}^{n} k\,(k!) + (n+1)\,((n+1)!)$$

Using our assumption:

$$\sum_{k=1}^{n+1} k\,(k!) = (n+1)! - 1 + (n+1)\,((n+1)!)$$

Note that:

$$(n+1)! = (n+1) \times n \times (n-1) \times ... \times 1$$

For simplicity's sake, let $x = (n+1)$:

$$\sum_{k=1}^{n+1} k\,(k!) = x! - 1 + x \times x! = (x+1) \times x! - 1$$

By the definition of a factorial:

$$(x+1) \times x! = (x+1)!$$

So:

$$\sum_{k=1}^{n+1} k\,(k!) = (x+1) \times x! - 1 = (x+1)! - 1$$

Substitute $(n+1)$ for x:

$$\sum_{k=1}^{n+1} k\,(k!) = (n+2)! - 1.$$

As required, $\therefore$ our conjecture is true, $\sum_{k=1}^{n} k\,(k!) = (n+1)! - 1, n \in \mathbb{N}$.

Problem 5

Prove *Bernoulli's inequality*: if $a > -1$ then $(1+a)^n \geq 1 + na$ for all $n \in \mathbb{N}$
Proof. Basis: $n = 1$ Check: $(1+a)^1 = 1 + a$.
Inductive step: Assume the inequality holds for $n = k$:

$$(1+a)^k \geq 1 + ka$$

Prove it holds for $n = k + 1$:

$$(1+a)^{k+1} \geq 1 + (k+1)\,a$$

Splitting up the exponent:

$$(1+a)^{k+1} = (1+a)^k \times (1+a)$$

Using our assumption $(1+a)^k \geq 1 + ka$:

$$(1+a)^k \times (1+a) \geq (1+ka)\,(1+a)$$

Simplifying:

$$(1+ka)\,(1+a) = 1 + (k+1)\,a + ka^2$$

And since

$$(1+a)^{k+1} \geq 1 + (k+1)\,a + ka^2$$

It must also be greater than $1 + (k+1)\,a$, as required. Therefore by induction, the inequality is true for $a > -1$

Problem 6

Prove by completing the square:

$$2n + 1 < n^2 : n \geq 3$$

Proof. Moving everything to one side:

$$n^2 - 2n - 1 > 0 : n \geq 3$$

Completing the square:

$$(n-1)^2 - 2 > 0 : n \geq 3$$

Getting n alone:

$$(n-1)^2 > 2 : n \geq 3$$

The absolute smallest that $(n-1)^2$ can be is 4, when $n = 3$, which is greater than 2. Therefore, $(n-1)^2 > 2 : n \geq 3$, and similarly: $2n + 1 < n^2 : n \geq 3$.

Problem 7

Prove by induction: $2^n \geq n^2 : n \geq 4$

Proof. Basis: $n = 4$ Check:

$$2^4 \geq 4^2$$

Indeed it is, *Induction step*: Assume the conjecture holds for $n = k$:

$$2^k \geq k^2 : k \geq 4$$

Show it holds for $n = k + 1$:

$$2^{k+1} \geq (k+1)^2 : k \geq 3$$

Splitting up the exponent:

$$2^k \times 2 \geq k^2 + 2k + 1$$

Using our assumption $2^k \geq k^2$:

$$2^k \times 2 \geq 2k^2 \geq k^2 + 2k + 1 = (k+1)^2 : (k \geq 3)$$

And since

$$2^k \times 2 = 2^{k+1}$$

The following must be true:

$$2^{k+1} \geq (k+1)^2 : (k \geq 3)$$

As required. Therefore, by induction the inequality is true.