Math 724 Homework 1

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1 Chapter 1

1. If $r \neq 0$ is rational, and x is irrational, prove that $r + x$ and rx are irrational.

   Proof. Let $r \neq 0$ be rational and x be irrational.

   (a) Assume, by way of contradiction, that $r + x$ is rational. Then $r + x = \frac{p}{q}$ for $p, q \in \mathbb{Z}$ and $q \neq 0$. We have that r is rational, so $r = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ and $b \neq 0$. Then we have

   $$x = \frac{p}{q} - \frac{a}{b} = \frac{bp - aq}{bq}$$

   since the ring $(\mathbb{Z}, +, \cdot)$ is closed under addition and multiplication so numerator and denominator are integers, and since $\mathbb{Z}$ is a domain, $b \neq 0$ and $q \neq 0 \implies bq \neq 0$. So then we have $x \in \mathbb{Q}$ which is a contradiction.

   (b) Assume, by way of contradiction, that $rx = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$ and $q \neq 0$. We have $r = \frac{a}{b}$ for $a, b \in \mathbb{Z}$ and $b \neq 0$. Then

   $$x = \frac{rx}{r} = \frac{\frac{p}{q}}{\frac{a}{b}} = \frac{p}{q} \cdot \frac{b}{a} = \frac{pb}{qa}$$

   Again since $\mathbb{Z}$ is closed under multiplication and $r \neq 0$ by assumption, we have $a \neq 0$ and therefore $qa \neq 0$, so we are not dividing by 0 and $x \in \mathbb{Q}$ contradiction.

$\square$

17. Prove that

$$|x + y|^2 + |x - y|^2 = 2|x|^2 + 2|y|^2$$

if $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^k$.

Proof.

$$|x + y|^2 = |x|^2 + 2|xy| + |y|^2, \ |x - y|^2 = |x|^2 - 2|xy| + |y|^2 \ \text{(By FOIL)}$$

So

$$|x + y|^2 + |x - y|^2 = 2|x|^2 + 2|y|^2$$

As required. Geometrically, the LHS is the sum of squares of the diagonals ($x + y$ is the long diag, $x - y$ is the short). RHS is 2 times sum of squares of the side lengths. The parallelogram has corners $0, x, y,$ and $x + y$. $\square$

2 Chapter 2

2. Prove that the set of all algebraic numbers is countable.

Proof. For every positive $N \in \mathbb{Z}$, we have that the set of

$$n + |a_0| + |a_1| + \cdots + |a_n| = N$$

is finite. (compositions of N?) By theorem 2.12 in the text,

$$S = \bigcup_{n=1}^{\infty} E_n$$

is countable if E_n is a sequence of countable sets. For every positive integer N, we have a countable (in fact finite) set of terms which satisfy the above sum. Therefore, since the index set $\mathbb{Z}$ is countable, the union of these sets is countable. Then we biject each set of terms to an algebraic number, so the set of algebraic numbers is at most countable. $\square$

12. Let $K \subset R^1$ consist of 0 and the numbers $1/n$, for $n = 1, 2, 3, \ldots$. Prove that K is compact directly from the definition. (Without the Heine-Borel theorem).

Proof. We must show that every open cover of K contains a finite subcover. Consider an open cover $\{G_\alpha\}$, it must contain 0, so in particular one of the sets of the cover contains 0. Then, we consider a neighboorhood around 0, say $G_0 = (-\varepsilon, \varepsilon)$ for some $\varepsilon \in \mathbb{R}$. This neighboorhood contains infinitely many points, and outside of the neighboorhood there are only finitely many (the maximum is $1/1 = 1$). So we may enumerate them, say $x_1, x_2, \ldots, x_m$ for some $m \in \mathbb{Z}$. Then, consider the open set G_1 containing x_1, then another open set G_2 containing x_2, do this for all x_m. Since there are finitely many, there are finitely many open sets, and clearly their union is K, so K is compact. $\square$