Math 553 Exam 1

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1. Consider the ellipse $\gamma(t) = (a\cos t, b\sin t)$, where $a, b > 0$. Compute the curvature of γ at t, and find the points on the trace of γ where the curvature achieves its maximum and minimum values.

$$\gamma'(t) = (-a\sin t, b\cos t), \gamma''(t) = (-a\cos t, -b\sin t)$$
$$\kappa(t) = \frac{||\gamma'(t) \times \gamma''(t)||}{||\gamma'(t)||^3} = \frac{||\gamma'(t) \times \gamma''(t)||}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}$$

Cross product:

$$\begin{aligned}\gamma'(t) \times \gamma''(t) &= (-a\sin t, b\cos t) \times (-a\cos t, -b\sin t)\\ &= (0, 0, ab\sin^2 t + ab\cos^2 t) = (0, 0, ab)\end{aligned}$$
$$||(0, 0, ab)|| = ab$$

Therefore,

$$\kappa(t) = \frac{ab}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}$$

Extreme points:
If $b > a$,

$$\text{Max: } t = n\pi + \frac{\pi}{2}$$
$$\text{Min: } t = n\pi$$
$$n \in \mathbb{Z}$$

If $a > b$,

$$\text{Max: } t = n\pi$$
$$\text{Min: } t = n\pi + \frac{\pi}{2}$$
$$n \in \mathbb{Z}$$

If $a = b$, curvature is constant, therefore there is no max or min. (makes sense as the curve is an ellipse, with axes a and b. If $a > b$ then the ellipse is horizontal, and if $b > a$ then the ellipse is vertical.)

2. Find a value x so that the arc length of the tractrix $\alpha(t) = (\sin t, \cos t + \log\tan\frac{t}{2})$ on the interval $[\frac{\pi}{2}, x]$ is equal to 1.

$$\alpha'(t) = (\cos t, -\sin t + \csc t)$$

$$\begin{aligned}
|\alpha'(t)| &= \sqrt{\cos^2 t + (\csc t - \sin t)^2} \\
&= \sqrt{\csc^2 t + \sin^2 t - 2 + \cos^2 t} \\
&= \sqrt{\csc^2 t - 1} = \sqrt{\cot^2 t} \\
&= |\cot t|
\end{aligned}$$

$$\mathcal{L} = \int_{\frac{\pi}{2}}^{x} |\alpha'(t)|dt = \int_{\frac{\pi}{2}}^{x} |\cot t| \; dt = \left[\frac{\cot t \ln|\sin t|}{|\cot t|}\right]_{\pi/2}^{x}$$

$$= \frac{\cot x \ln|\sin x|}{|\cot x|} - \underbrace{\frac{\cot\frac{\pi}{2}\ln|\sin\frac{\pi}{2}|}{\cot\frac{\pi}{2}}}_{=0,\text{ by L'Hospital's}} = 1$$

$$\implies \ln|\sin x| = \frac{|\cot x|}{\cot x} = \pm 1$$

$$\ln|\sin x| = -1$$

At $x = \arcsin\frac{1}{e} \approx 2.76486514.$

3. Find the curvature and torsion of the parameterized curve $\alpha(t) = (3t, 3t^2, 2t^3)$.

$$\begin{aligned}
\kappa(t) &= \frac{|\alpha'(t) \times \alpha''(t)|}{|\alpha'(t)|^3} \\
&= \frac{|(36t^2, -36t, 18)|}{|(3, 6t, 6t^2)|^3} \\
&= \frac{36t^2 + 18}{216t^6 + 324t^4 + 162t^2 + 27} \\
&= \frac{12t^2 + 6}{72t^6 + 108t^4 + 54t^2 + 9} \\
&= \left(\frac{6}{9}\right) \cdot \left(\frac{2t^2 + 1}{8t^6 + 12t^4 + 6t^2 + 1}\right)
\end{aligned}$$

$$
\begin{aligned}
\tau(t) &= -\frac{(\alpha'(t) \times \alpha''(t)) \cdot \alpha'''(t)}{|\alpha'(t) \times \alpha''(t)|^2} \\
&= \frac{12}{(2t^2+1)^2} \\
&= \frac{12}{4t^4 + 4t^2 + 1}
\end{aligned}
$$

4. Let $I \subset \mathbb{R}$ be an open interval, and let $\alpha : I \to \mathbb{R}^2$ be a regular curve, not necessarily PBAL. Suppose there is a value $t_0 \in I$ where the map $t \mapsto |\alpha(t)|$ achieves a local maximum. Show that the curvature $\kappa(t_0)$ at t_0 satisfies $|\kappa(t_0)| \geqslant \frac{1}{|\alpha(t_0)|}$.

   Proof. First, WLOG, assume $\alpha(t)$ is PBAL. If not, reparametrize such that it is, and since curvature is invariant in reparametrization, the result does not change. Now define a new function $f(t) := |\alpha(t)|^2$. Differentiating twice:

$$
\begin{aligned}
f'(t) &= 2\alpha'\alpha \\
f''(t) &= 2\alpha''\alpha + 2|\alpha'|^2 \\
|\alpha'|^2 &= 2\alpha''\alpha - f''(t)
\end{aligned}
$$

   From the problem, we know there exists a value t_0 where $|\alpha|$ is a local maximum, and therefore the second derivative of $f(t)$ at $t = t_0$ will be negative. Therefore:

$$
|\alpha'(t_0)|^2 < \alpha''(t_0)\alpha(t_0)
$$

Now since α is PBAL, $|\alpha'| = 1$, so:

$$1 = |\alpha'(t_0)|^2 < \alpha''(t_0) \cdot \alpha(t_0) \leqslant |\alpha''(t_0)| \cdot |\alpha(t_0)|$$

Then, since $|\kappa| = |\alpha''|$, it holds that $|\kappa(t_0)| \geqslant \frac{1}{|\alpha(t_0)|}$.
QED

5. Let $I \subset \mathbb{R}$ be an open interval, and let $\kappa : I \to \mathbb{R}$ be a differentiable function, and let $\alpha : I \to \mathbb{R}^2$ be a regular curve, PBAL, so that the curvature of α at $s \in I$ is given by $\kappa(s)$.

   a. Show that there is a function $\theta : I \to \mathbb{R}$ so that $\alpha'(s) = (\cos\theta(s), \sin\theta(s))$.

      Let $\alpha'(s) = (x, y)$ for some $x, y \in \mathbb{R}$.

$$\begin{aligned}
(\cos\theta(s), \sin\theta(s)) &= \alpha'(s) \\
(\cos\theta(s), \sin\theta(s)) &= (x, y) \\
(\theta(s), \theta(s)) &= (\arccos(x), \arcsin(y)) \\
\theta(s) &= \arccos(x) = \arcsin(y)
\end{aligned}$$

b. Show that $\theta'(s) = \kappa(s)$.

$$\kappa(s) = |\alpha''(s)| = \sqrt{(-\sin\theta(s))^2 + (\cos\theta(s))^2} = 1$$

$$\begin{aligned}
\theta'(s) &= -\frac{1}{\sqrt{1-x^2}} \\
&= -\frac{1}{\sqrt{1-\cos^2\theta(s)}} \\
&= -\frac{1}{\sin\theta(s)} \\
&= -\frac{1}{y} \\
\theta'(s) &= \frac{1}{\sqrt{1-y^2}} \\
&= \frac{1}{x} \\
&\implies x = -y \\
&\iff x = \sqrt{1-(-x)^2} \\
&\implies x = \frac{1}{\sqrt{2}}
\end{aligned}$$

$$\theta'(s) = \arccos\frac{\sqrt{2}}{2} = \frac{\pi}{4} \neq 1$$

Hmm. Not sure where I messed up, I don't know what I'm doing :(

c. Let $s_0 \in I$. Show that there is a constant $\theta_0 \in \mathbb{R}$ so that:

$$\theta(s) = \theta_0 + \int_{s_0}^{s} \kappa(t)dt.$$

$$\theta_0 = \int_{s_0}^{s} \kappa(t)dt - \theta(s)$$

(IDK)

d. Show that there are constants $a, b \in \mathbb{R}$ so that:

$$\alpha(s) = (a + \int_{s_0}^{s} \cos\theta(t)dt, b + \int_{s_0}^{s} \sin\theta(t)dt).$$

(IDK)

e. Now suppose that $\alpha, \beta : I \to \mathbb{R}$ are both parameterized by arc length, so that α and β both have curvature $\kappa(s)$ at $s \in I$, and so that $\alpha'(s_0) = \beta'(s_0)$. Show that there is a translation $T : \mathbb{R}^2 \to \mathbb{R}^2$ so that $\alpha = \beta \circ T$. What happens if one doesn't assume that $\alpha'(s_0) = \beta'(s_0)$? (IDK)