Math 553 Homework 7

Theo Koss

May 2022

1 Section 3.2

- Problem 3: Let $C \subset S$ be a regular curve on a surface S with Gaussian curvature $K > 0$. Show that the curvature k of C at p satisfies

  $$|k| \geq \min(|k_1|, |k_2|)$$

  Where k_1 and k_2 are the principal curvatures of S at p.

  Proof. By definition the normal curvature k_n of C at p is,

  $$k_n = k \cos \theta$$

  where $\theta =$ the angle between $n(p)$ and $N(p)$, the normal vectors of C and S at p, respectively. Suppose $|k_1| \leq |k_2|$, since k_n lies between these 2, and $K > 0$, we have that k_1 and k_2 are nonzero and have the same sign. And,

$$\begin{aligned} |k_1| \leq |k_n| \leq |k_2| \\ \implies |k_1| \leq k|\cos\theta| \leq |k_2| \end{aligned}$$

Since the maximum of $|\cos\theta| = 1$, we have

$$k \geq k|\cos\theta| \geq |k_1| \implies k \geq \min(|k_1|, |k_2|)$$

QED

- Problem 4: Assume that a surface S has the property that $|k_1| \leq 1$, $|k_2| \leq 1$ everywhere. Is it true that the curvature k of a curve on S also satisfies $|k| \leq 1$?

  Proof. Let C be a curve on S with curvature k. By definition

  $$k_n = k\cos\theta \implies |k_n| = |k\cos\theta|$$

  Now, applying Euler's formula,

  $$\begin{aligned} k_n &= k\cos\theta \\ k_n &= k_1\cos^2\theta + k_2\sin^2\theta \\ |k\cos\theta| &= |k_1\cos^2\theta + k_2\sin^2\theta| \\ |k\cos\theta| &\leq |k_1\cos^2\theta| + |k_2\sin^2\theta| \\ |k\cos\theta| &\leq \cos^2\theta + \sin^2\theta = 1 \end{aligned}$$

  This implies $|k| \leq \frac{1}{|\cos\theta|}$, since $|\cos\theta| \leq 1$, $|k| \leq 1$ only when $\theta = n\pi$. QED

- Problem 6: Show that the sum of the normal curvatures for any pair of orthogonal directions, at a point $p \in S$, is constant.

  Proof. According to Euler's formula,

  $$k_n(\theta) = k_1 \cos^2 \theta + k_2 \sin^2 \theta$$

  Where θ is some direction on S at p. Then for an orthogonal direction, we consider $\theta + \frac{\pi}{2}$. Call the vector in this direction v. Its normal curvature is then given by:

  $$\begin{aligned} k_v(\theta) &= k_n\left(\theta + \frac{\pi}{2}\right) \\ &= k_1 \cos^2\left(\theta + \frac{\pi}{2}\right) + k_2 \sin^2\left(\theta + \frac{\pi}{2}\right) \\ &= k_1\left(\cos\left(\theta + \frac{\pi}{2}\right)\right)^2 + k_2\left(\sin\left(\theta + \frac{\pi}{2}\right)\right)^2 \\ &= k_1((\cos\theta)(0) - (\sin\theta)(1))^2 + k_2((\sin\theta)(0) + (\cos\theta)(1))^2 \\ &= k_1 \sin^2\theta + k_2 \cos^2\theta \end{aligned}$$

  Therefore the sum of the normal curvatures for these two directions is:

  $$\begin{aligned} k_n(\theta) + k_v(\theta) &= (k_1 \cos^2\theta + k_2 \sin^2\theta) + (k_1 \sin^2\theta + k_2 \cos^2\theta) \\ &= k_1(\cos^2\theta + \sin^2\theta) + k_2(\sin^2\theta + \cos^2\theta) \\ &= k_1 + k_2 \end{aligned}$$

Which is constant. QED

- Problem 17: Show that if $H \equiv 0$ on S and S has no planar points, then the Gauss map $N : S \to S^2$ has the following property:

$$\langle dN_p(w_1), dN_p(w_2)\rangle = -K(p)\langle w_1, w_2\rangle$$

for all $p \in S$ and all $w_1, w_2 \in T_p(S)$. Show that the above condition implies that the angle of two intersecting curves on S and the angle of their spherical images are equal up to a sign.

Proof. Let e_1, e_2 be the principal directions of $p \in S$. Then we have:

$$w_1 = a_1 e_1 + a_2 e_2$$

and

$$w_2 = b_1 e_1 + b_2 e_2$$

Then

$$dN_p(w_1) = dN_p(a_1 e_1 + a_2 e_2) = -a_1 k_1 e_1 - a_2 k_2 e_2$$

And

$$dN_p(w_2) = dN_p(b_1 e_1 + b_2 e_2) = -b_1 k_1 e_1 - b_2 k_2 e_2$$

Then the inner product

$$\langle dN_p(w_1), dN_p(w_2)\rangle = a_1 b_1 k_1^2 + a_2 b_2 k_2^2$$

Then, since $H \equiv 0$ on S, $k_1 = -k_2 \implies K(p) = -k_1^2 = -k_2^2$. Since S has no planar points, $K(p)$ is not 0. Thus,

$$\langle dN_p(w_1), dN_p(w_2) \rangle = -K(p)\langle w_1, w_2 \rangle$$

QED