Math 524 Homework 3

Theo Koss

March 2024

1. Let f be any function on $[a, b]$ with no upper bound. Prove that the upper sum $U(f, P)$ is infinite for every partition P of $[a, b]$. Conclude that f is not integrable.

   Proof.
   $$M_k(f) = sup\,\{f(x) \mid x \in [x_{k-1}, x_k]\}$$
   Upper sum for partition $P = \{x_0, x_1, x_2, \ldots, x_n\}$:
   $$U(P, f) = \sum_{k=1}^{n} M_k \Delta x_k$$
   However since f has no upper bound on $[a, b]$, we have that for each partition P, M_k is infinite, therefore the sum $\sum_{k=1}^{n} M_k \Delta x_k$ is infinite. Therefore f is not integrable. QED

2. (a) Let f be a bounded function on a set A, and consider:
   $$M = sup\,\{f(x) \mid x \in A\} \quad m = inf\,\{f(x) \mid x \in A\}$$
   $$M' = sup\,\{|f(x)| \mid x \in A\} \quad m' = inf\,\{|f(x)| \mid x \in A\}$$
   Show that $M - m \geqslant M' - m'$.
   $$M - m = sup(|f(x) - f(y)| \mid x, y \in A)$$
   and
   $$M' - m' = sup(||f(x)| - |f(y)||) \mid x, y \in A$$

By the triangle inequality, we have:

$$|f(x) - f(y)| \geqslant ||f(x) - |f(y)||$$

So,

$$M - m \geqslant M' - m'$$

(b) Show that if f is integrable on $[a, b]$ then $|f|$ is also integrable on $[a, b]$. Let f be integrable on $[a, b]$, then we have that

$$\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx = B \qquad \text{(Where B is a real number)}$$

Then, we have:

$$U(P, f) = \sum_{k=1}^{n} M\Delta x_k \quad L(P, f) = \sum_{k=1}^{n} m\Delta x_k$$

Subtracting, we get:

$$U(P, f) - L(P, f) = \sum_{k=1}^{n} (M - m)\Delta x_k$$

By above we get:

$$U(P, f) - L(P, f) \geqslant U(P, |f|) - L(P, |f|)$$

And since f is Riemann integrable, we have $U(P, f) - L(P, f) = 0$, so

$$0 \geqslant U(P, |f|) - L(P, |f|)$$

And all $U(P, |f|)$ are greater than or equal to all $L(P, |f|)$, so we have $U(P, |f|) = L(P, |f|)$ therefore $|f|$ is Riemann integrble.

3. Let $f(x) = |x|$ and define $F(x) = \int_{-1}^{x} f$. Find a piecewise algebraic formula for $F(x)$ for all x. Where is F continuous? Where is F differentiable?

$$f(-2) = |-2| = 2 \quad F(-2) = \int_{-1}^{-2} f =$$

Piecewise formula:

$$F(x) = \begin{cases} \int_{-1}^{x}(-f) = \frac{1}{2}(1-x^2) & \text{if } x < 0 \\ F(0) + \int_{-1}^{x}(f) = \frac{1}{2}(1+x^2) & \text{if } x > 0 \end{cases}$$

F is continuous everywhere, and it is differentiable everywhere (everywhere except 0 is obvious, at $F(0)$ we get $F'(0) = \lim_{h\to 0} \frac{F(x)-F(0)}{h-0}$, and $|F(x) - F(0)| \leqslant |x|^2$) Therefore F is differentiable at 0 as well.

4. Let $L(x) = \int_1^x \frac{1}{t}dt$ for all $x > 0$.

   (a) Evalute $L(1)$. Explain why L is differentiable and find $L'(x)$.

   $$L(1) = \int_1^1 \frac{1}{t}dt = 0$$

   L is differentiable by the fundamental theorem of calculus, we have $\frac{d}{dx}\int_1^x \frac{1}{t}dt = \frac{1}{x}$.

   (b) If $E : \mathbb{R} \to (0,\infty)$ is given by $E(x) = e^x$, it is known that $E'(x) = e^x$. Let $ln : (0,\infty) \to \mathbb{R}$ denote the inverse function of E. Use the inverse function theorem to prove that the derivative of $ln(x)$ equals $\frac{1}{x}$.

   Proof. By inverse function theorem, we have that for a point $b = f(a)$:

   $$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}$$

   So,

   $$\frac{d}{dx}ln(b) = \frac{1}{e(a)} = \frac{1}{b}$$

   as required. QED

   (c) Explain why $L(x) = ln(x)$ for all $x > 0$.
   The two functions both have the same derivative (by (a) and (b)), and $L(1) = ln(1) = 0$, so they are "aligned". It breaks for $x \leqslant 0$ because that is not in the domain of $ln : (0,\infty) \to \mathbb{R}$.

   (d) Let

   $$\gamma_n = (1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}) - L(n)$$

   Prove (γ_n) converges.

Proof. By the integral test we have (γ_n) converges iff $\int_1^\infty f(x)dx$ converges, where $f(x) = \gamma_x$.
Suppose $\int_1^\infty f(x)dx$ converges, then:

$$S_{k+1} - \gamma_1 \leqslant \int_1^{k+1} f(x)dx \leqslant \int_1^\infty f(x)dx$$

So S_{k+1} is bounded, and we have by induction that (γ_k) is increasing, therefore (γ_k) converges. QED

(e) Show how consideration of the sequence $(\gamma_{2n} - \gamma_n)$ leads to a new proof of the identity:

$$L(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

RHS is equal to

$$\sum_{n=1}^{2m} (-1)^{n+1} \frac{1}{n}$$

Splitting into even and odd terms gives:

$$= \sum_{n=1}^{m} \frac{1}{2n-1} - \sum_{n=1}^{m} \frac{1}{2n}$$

Which is

$$\gamma_{2m} - 2\gamma_m + L(2)$$

Which limits to $L(2)$.

5. If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^3$, use the triangle inequality to prove that:

(a) $\|\vec{u}\| - \|\vec{v}\| \leqslant \|\vec{u} - \vec{v}\|$.
By theorem 9.2.2 in the text, we have that $\|\vec{u} + \vec{v}\| \leqslant \|\vec{u}\| + \|\vec{v}\|$.

$$\|\vec{u}\| = \|u - v + v\| \leqslant \|u - v\| + \|v\| \quad \text{(By triangle inequality)}$$

Subtract $\|\vec{v}\|$ from both sides:

$$\|\vec{u}\| - \|\vec{v}\| \leqslant \|\vec{u} - \vec{v}\|$$

(b) $|\|\vec{u}\| - \|\vec{v}\|| \leqslant \|\vec{u} - \vec{v}\|$

$$\|\vec{v}\| = \|v - u + u\| \leqslant \|v - u\| + \|u\| \quad \text{(By triangle inequality)}$$

Subtracting the two above equations gives:

$$\|\vec{u}\| - \|\vec{v}\| \leqslant \|u - v\| + \|v\| - (\|v - u\| + \|u\|)$$

$$\implies |\|\vec{u}\| - \|\vec{v}\|| \leqslant \|\vec{u} - \vec{v}\|$$

(Taking abs value forces $\|u - v\| = \|v - u\|$)

6. Suppose $f : [a, b] \to \mathbb{R}$ is differentiable and f' is continuous. The *graph* of f is the curve $y = f(x)$ in $\mathbb{R}^2$. Show that the arclength L of the graph of f for $a \leqslant x \leqslant b$ is given by:

$$L = \int_a^b \sqrt{1 + (dy/dx)^2} dx$$

Proof. The *graph* of f is the same as a smooth parameterization of f, which looks like $y = (t, f(t))$. Then, we have that the length of y is defined by:

$$L(y) = sup\,\{L_Q \mid Q \text{ is a partition of } [a, b]\}$$

Using this smooth parameterization, we can invoke Theorem 9.7.2, it states that the length of a smooth parameterization $\gamma(t) = (f(t), g(t))$ for $t \in [a, b]$ is finite and equal to:

$$L(\gamma) = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} dt$$

In our case, we have $y = (t, f(t))$, so the arc length is given by $t' = 1$, and $f'(t) = dy/dx$:

$$L = \int_a^b \sqrt{1 + (dy/dx)^2} dx$$

QED