Math 835 Homework 3

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Chapter 13

Section 3

1. Prove that it is impossible to construct the regular 9-gon.

   Proof. If the 9-gon were constructable, so would $\cos(\frac{2\pi}{9})$, but that has minimal polynomial $8x^3 - 6x + 1$, which is degree 3. Therefore the field extension is degree 3 (not a power of 2), so the 9-gon is not constructable. (Equivalently, note that 140 degrees is not divisible by 3.) $\square$

2. Prove that Archimedes' construction (With Ruler, not straightedge) actually trisects θ.

   Proof. We wish to show that $3\alpha = \theta$. Labelling the points from left to right, $A, B, C,$ and D. We have that $180 - \theta$ is the measure of angle ACD, because it is a supplementary angle (they lie on a straight line). We also have that the measure of angle BCD is $180 - 2\beta$, because the angles of a triangle add to 180, and $\beta = \gamma$. And the measure of angle ACB is α because the triangle is isosceles. So now we have

   $$180 - \theta = (180 - 2\beta) + (\alpha)$$

   Furthermore, the problem gives us $\beta = \gamma = 2\alpha$, so

   $$180 - \theta = 180 - 3\alpha$$

   Therefore, $\theta = 3\alpha$, as required. $\square$

5. Use the fact that $\alpha = 2\cos(\frac{2\pi}{5})$ satisfies the equation $x^2 + x - 1$ to conlude that the regular 5-gon is constructable by straightedge and compass.

   Proof. Since α is a root of $x^2 + x - 1$, and that polynomial is irred. over $\mathbb{Q}$, we have that adjoining α to $\mathbb{Q}$ is a degree 2 (quadratic) extension. From class, all extensions which have degree power of 2 are constructable. Therefore α is constructable, which gives us the central angle of a regular pentagon. To complete the construction, simply draw 5 angles which connect to eachother (so that they all share 1 point), then draw a circle of radius 1 around all of them. Where the angles connect to the circle are the vertices of the regular pentagon, so connect them with straight lines and you're done :) (I guess actually we just have that the cosine of the angle is constructable, so construct the adjacent sides and the hypotenuses. But this is the same as constructing the 72° angle). $\square$