Math 551 Homework 4

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1 Section 3.4

- Problem 6: Let X be a metric space with metric d and A a nonempty subset of X. Define $f : X \to \mathbb{R}$ by

  $$f(x) = d(x, A), \quad x \in X$$

  Show that f is continuous.

  Proof. We wish to show that $\forall x, y \in X, \quad |f(x) - f(y)| \leq d(x, y)$. Consider the distance from x, y to a third point $z \in A$. There are 2 cases:

  i. $d(x, z) \geq d(y, z)$.

  ii. $d(x, z) < d(y, z)$.

  i.

  $$\forall z \in A, \quad f(x) \leq d(x, z) \qquad \text{(By definition of infimum)}$$
  $$\exists \epsilon > 0 \text{ s.t. } f(x) + \epsilon > d(x, z)$$

  Theorem 1 (Inverse Triangle Inequality)**.**

  $$\forall x, y \in \mathbb{R}, \quad |x - y| \geq ||x| - |y||$$

  This follows directly from the Triangle Inequality.

Using Thm. 1,

$$\begin{aligned}
&d(x,z) - d(y,z) \le d(x,y) \\
\implies& f(x) - d(y,z) \le d(x,y) \\
\implies& f(x) - f(y) - \epsilon \le d(x,y) \\
\implies& f(x) - f(y) \le d(x,y) + \epsilon
\end{aligned}$$

Thus $f(x) - f(y) \le d(x,y)$, As required.
ii.
This case is similar, and the proof is nearly the same. QED

2 Section 4.1

- Problem 3: Show that the finite complementary topology is a topology for any set X.

  Proof.

  Definition 1 (Topology for X)**.** Let $\mathcal{T}$ be a family of subsets of X, $\mathcal{T}$ is called a topology for X if the following conditions are met:

  1. The set X and the empty set $\emptyset$ belong to $\mathcal{T}$.
  2. The union of any family of members of $\mathcal{T}$ is a member of $\mathcal{T}$.
  3. The intersection of any finite family of members of $\mathcal{T}$ is a member of $\mathcal{T}$.

  Recall that the finite complementary topology is the set containing the following:

  – The empty set, $\emptyset$
  – All subsets $\mathcal{O}$ of X for which $X \backslash \mathcal{O}$ is a finite set.

  Call this set $\mathcal{T}'$. We must show the three conditions from definition 1 hold for any set X.
  There are 2 cases:

  i. X is finite.
  ii. X is infinite.

Case i. is trivial, because if X is finite, then the finite complementary topology coincides with the discrete topology, which is clearly a topology.
Case ii. requires a bit more work.

1. $\emptyset$ is contained in the definition. The compliment $X \backslash X = \emptyset$ is finite, and therefore X is contained in $\mathcal{T}$, so condition 1 is met.
2. Let $A = \bigcup_{i \in I} U_i$ be an arbitrary union of sets whose complements are finite. Then, by De Morgan's law,
$$A = X - (\bigcap_{i \in I} V_i) \quad \text{for finite sets } V_i$$
By definition of intersection, $\bigcap_{i \in I} V_i \subseteq V_i$ for all $i \in I$. V_i is finite, therefore so is any of its subsets. Therefore the complement $X \backslash A$ is finite, so A is an open set, thus condition 2 is met.
3. Consider the intersection of any two finite sets $U_k \cap U_j$. Again using De Morgan's law, this is equivalent to $X \backslash (V_k \cup V_j)$. Since the intersection of two finite sets is finite, $U_k \cap U_j = X \backslash (V_k \cup V_j)$ is contained within $\mathcal{T}$. Since an intersection will *never* increase when you add a set, confirming with 2 sets is sufficient to prove it for any finite number of sets. Therefore condition 3 is met.

QED

3 Section 4.2

- 1, part (b): Let A, B be subsets of a space X. Show that $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$.

  Proof. We know that $A \cap B \subset A$, by definition, and $A \cap B \subset B$, also by definition. Also, $A \subset \bar{A}$ and $B \subset \bar{B}$. (Since $\bar{H}$ is the smallest closed set containing H). Thus we can conclude that $A \cap B$ is a subset of each of $\bar{A}$ and $\bar{B}$. Therefore, $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$. QED

4 Section 4.5

- Problem 4: Let $f : X \to Y$ be a continuous function on the indicated spaces and A a subspace of X. Prove that the restriction $f|_A : A \to Y$ of f to A is continuous.

Proof. Let $(X, \mathcal{T})$ be one space, and $(Y, \mathcal{J})$ the other. Consider some $H \in \mathcal{J}$, then $f|_A(H) = f^{-1}(H) \cap A$ where $f^{-1}(H)$ is open. Therefore $f^{-1}(H) \cap A$ is open in the subspace. Thus $f|_A : X \to Y$ is continuous. QED

- Problem 9: Prove that a finite subset A of a Hausdorff space X has no limit points. Conclude that A must be closed.

  Proof. Let $A = \{x_1, x_2, \dots, x_n\}$ and let p be a limit point of A. This means for every open set U containing p, it must also contain at least one other element of A. In mathematical terms:

  $$U \cap A \backslash \{p\} \neq \emptyset$$

  Since X is Hausdorff, there are open sets U_i containing p and open sets V_i containing x_i such that $U_i \cap V_i = \emptyset$ by definition. Let $U = \bigcap_{i=1}^{n} U_i$ and let $V = \bigcup_{i=1}^{n} V_i$. Then the following are true:

  1. $p \in U$
  2. $A \subset V$
  3. $U \cap V = \emptyset$

  Therefore we have found a neighborhood of p which doesn't intersect A. Thus, p cannot be a limit point, and there are no limit points of A. The second part follows, because $A' = \emptyset$ since A has no limit points, and $\bar{A} = A \cup A' = A$, so A is closed. QED