Math 551 Homework 2

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1 Section 2.3

- 2: Let x be a number and A a subset of $\mathbb{R}$.

  (a) Prove that if $d(x, A) > 0$, then $d(x, y) > 0$ for all $y \in A$.

  Proof. Let $x \in \mathbb{R}$ and $A \subset \mathbb{R}$ such that $A \neq \emptyset$. Assume $d(x, A) > 0$. Thus $d(x, A) > 0$ is the infimum of all distances $|x - y|$ for $y \in A$. Thus by definition of infimum, $d(x, A) \leq |x - y| = d(x, y)$ for all y in A. Therefore $d(x, y) > 0$. QED

  (b) Give an example for which $d(x, y) > 0$ for all $y \in A$, but $d(x, A) = 0$.
  Consider $A = (0, 1)$ and $x = 0$. Then $d(x, A) = 0$ but $d(x, y) > 0$ for all $y \in A$.

- 3: Prove that a subset of $\mathbb{R}$ is bounded if and only if it has both upper and lower bounds.

  Proof. ($\Longrightarrow$): Suppose A has a lower bound x and an upper bound y. Then $A \subseteq [x, y]$ and using the triangle inequality, $d(a, b) < y - x$. Therefore A is bounded.
  ($\Longleftarrow$): Suppose that A is bounded, then there is some n such that $d(a, b) < n$ for all $a, b \in A$. Then $A \subseteq [a - n, a + n]$ for some $a \in A$. and therefore it has lower and upper bounds. QED

- 4: If $\{C_i\}_{i=1}^n$ is a finite family of closed sets, then $\cup_{i=1}^n C_i$ is closed.

Proof. We must show that $\mathbb{R}\backslash\{C_i\}_{i=1}^n$ is open.

$$\cup_{i\in(1,n)}(\mathbb{R}\backslash C_i) = \mathbb{R}\backslash \cap_{i\in(1,n)} C_i$$

Is open by theorem 2.7. Therefore, by definition, $\cup_{i=1}^n C_i$ is closed. QED

- 8: Show that if x is the limit of the sequence $\{a_n\}_{n=1}^\infty$ of real numbers and all the terms of the sequence are distinct, then x is a limit point of the range of the sequence. Give an example to show that the limit of a sequence may not be a limit point of the range of the sequence if the terms of the sequence are not distinct.
  Consider some x where it is the limit of the sequence $\{a_n\}_{n=1}^\infty$ of real numbers. Then given $\epsilon > 0$ there is a positive integer N such that if $n \geq N$ then $|a_n - x| < \epsilon$. Therefore if we consider the range of the sequence, $\{a_0, a_n\}$, then we must show that x is a limit point. Recall that x is a limit point of this range iff every neighborhood of x contains a separate point of the range.

  $$a_n \in (x - \epsilon, x + \epsilon)$$

  Therefore every neighborhood of x contains a separate point of the range.
  If the terms are not distinct, then we can consider the sequence $\{b_n\}_{n=1}^\infty$ where $b_n = x$ for all $n \in \mathbb{N}$. Then this sequence certainly converges to x, however you can not find a point other than x that is contained in the range.

- 9: Let $x \in \mathbb{R}$ and $A \subset \mathbb{R}$.

  (a) Prove that x is a limit point of A if and only if there is a sequence of distinct points of A which converges to x.

  Proof. See the above problem for one direction.
  For the other direction, assume there is a sequence of distinct points of A which converge to x. Then, by the question above, x is a limit point of the range of A. QED

  (b) Prove that x is a limit point of A if and only if every open set containing x contains infinitely many points of A.

Proof. ($\Longrightarrow$): Assume x is a limit point of A, then for every open interval around x, there is an element of $A \neq x$ in that interval, therefore, by varying the size of the interval, you can get infinitely many points.
($\Longleftarrow$): Assume there are infinitely many points of A such that every open set containing x contains a distinct point of A which is not x. Then, by definition, x is a limit point. QED

2 Section 2.4

- 2: Give an example of a nested sequence $\{[a_n, b_n]\}_{n=1}^{\infty}$ whose intersection is empty.
Let $a_n = 0, \forall n \in \mathbb{N}$, and let $b_n = \frac{1}{n}, \forall n \in \mathbb{N}$. Then their intersection is empty, since any element would be greater than 0, yet less than $\frac{1}{n}$ for all $n \in \mathbb{N}$. No such element exists.

- 3: Consider $[0, 1]$ and the family of open intervals $O = \{(-0.001, 0.001), (0.999, 1.001)\} \cup \{\frac{1}{n}, 1\}_{n=1}^{\infty}$. Find a finite subcollection of O whose union contains $[0, 1]$.
*I don't understand this problem

- 4: Prove the Bolzano-Weierstrass Theorem. Every bounded, infinite subset of $\mathbb{R}$ has a limit point.

  Proof. Begin with a bounded sequence (x_n) (Call it $[a, b]$): Using the bisection argument, we can show that $[a, b]$ is a sequence of nested intervals, since they are nested, the intersection of all of these intervals is nonempty, thus there is a number x which is in each interval of $[a, b]$. This is a limit point of (x_n). QED

- 8: Show that every uncountable subset of $\mathbb{R}$ has a limit point.

  Proof. Suppose that $A \subset \mathbb{R}$ is uncountable. For $n \in \mathbb{Z}$, let $A_n = A \cap [n, n+1]$. Some A_n, call it A_N, is infinite. Since $A_N \subseteq [N, N+1]$, A_N has a limit point (call it x) in $[N, N+1]$. This point is also a limit point of A in $\mathbb{R}$, since if there exists a neighborhood of x in $\mathbb{R}$, then (Said neighborhood $\cap\, [N, N+1]$) is a neighborhood of x in $[N, N+1]$. Therefore it contains a point of A_N other than x. QED

3 Section 3.1

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