Math 523 Homework 4

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1. Recall that Cantor's Nested Intervals Theorem states the following: Consider a nested sequence of bounded closed intervals:

$$[a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_3, b_3] \supseteq \cdots \supseteq [a_k, b_k] \supseteq \cdots$$

Then the intersection $\bigcap_{k=1}^{\infty} [a_k, b_k]$ is not empty. Give an example of a nested sequence of bounded open intervals whose intersection is empty.

$$(0,1) \supseteq \left(0, \frac{1}{2}\right) \supseteq \left(0, \frac{1}{4}\right) \supseteq \cdots \supseteq \left(0, \frac{1}{2^k}\right)$$

There is no point they all have in common.

2. Show directly from the definition that if $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences, then so is the sequence $\{a_n + b_n\}$.

If $\{a_n\}$ is Cauchy, then for any $\varepsilon > 0$, $\exists n^* \in \mathbb{N}$ such that $|a_m - a_n| < \frac{\varepsilon}{2}$, $\forall m, n \geqslant n^*$. And similarly, if $\{b_n\}$ is Cauchy, then for any $\varepsilon > 0$, $\exists m^* \in \mathbb{N}$ such that $|b_m - b_n| < \frac{\varepsilon}{2}$, $\forall m, n \geqslant m^*$. Then "$\{a_n + b_n\}$ is Cauchy" $\iff$ for all $\varepsilon > 0$, $\exists \ell \in \mathbb{N}$ such that $|(a_m + b_m) - (a_n - b_n)| < \varepsilon$, $\forall m, n \geqslant \ell$. Choose $\ell = \max(n^*, m^*)$, then

$$|(a_m + b_m) - (a_n - b_n)| \leqslant |a_m - a_n| + |b_m - b_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

For $m, n \geqslant \ell$.

3. Consider a sequence such that $x_1 > 0$ and $x_{n+1} = 1/(2 + x_n)$ for all $n \in \mathbb{N}$. Show that $x_n > 0$ for all n. Then show that $\{x_n\}$ is a contractive sequence. Evaluate the limit: $\lim_{n\to\infty} x_n$.

Proof. The denominator is a positive real $\forall n$, and the numerator is always 1, therefore $x_n > 0\ \forall n$.

$$|x_{n+2} - x_{n+1}| = \left|\frac{1}{2 + x_{n+1}} - \frac{1}{2 + x_n}\right| = \frac{1}{(2 + x_{n+1})(2 + x_n)}|x_{n+1} - x_n|$$

Since that fraction is always between 0 and 1, $\{x_n\}$ is contractive.

$$\begin{aligned} L &= \frac{1}{2+L} \\ L^2 + 2L - 1 &= 0 \\ L &= -1 \pm \sqrt{2} \\ L &= -1 + \sqrt{2} && \text{(This limit must be nonnegative)} \end{aligned}$$

QED

4. Let $\{x_n\}$ be a bounded sequence, and let $s = \sup\{x_n \mid n \in \mathbb{N}\}$. Show that if $s \neq x_n$ for all n, then there is a subsequence of $\{x_n\}$ that converges to s.

Proof. By definition of supremum, and with $s \neq x_n$, $\forall n$, we have that for any $\varepsilon > 0$, one can find an n such that $s - \varepsilon < x_n < s$. There are infinitely many such n, so write the increasing sequence $\{n_k\}$, such that $s - \frac{1}{n_k} < x_{n_k} < s$. Then $\{n_k\}$ is a subsequence of $\{x_n\}$, and it is epsilon-close to s, so $\{n_k\}$ converges to s. QED