Math 723 Final Exam

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1. Let $s_1 = \sqrt{2}$ and $s_{n+1} = \sqrt{2 + \sqrt{s_n}}$. Prove that (s_n) converges.

Proof. We must show that the sequence is:

(a) Bounded above

(b) Increasing.

The sequence is bounded above by 2, by induction, $\sqrt{2} \leqslant 2$. Assume $s_n \leqslant 2$ for some n. Then

$$\begin{aligned} s_{n+1} &= \sqrt{2 + \sqrt{s_n}} \\ &\leqslant \sqrt{2 + \sqrt{2}} \\ &\leqslant \sqrt{4} = 2 && (2 + \sqrt{2} \leqslant 4) \end{aligned}$$

The sequence is increasing again by induction. Clearly $\sqrt{2} < \sqrt{2 + \sqrt{2}}$, because $2 \leqslant 2 + \sqrt{2}$. Assume $s_n \geqslant s_{n-1}$ for some n. Then

$$\begin{aligned} s_{n+1} &= \sqrt{2 + \sqrt{s_n}} \\ &\geqslant \sqrt{2 + \sqrt{s_{n-1}}} && \text{(Assumption)} \\ &= s_n \end{aligned}$$

Therefore (s_n) is bounded above and increasing, therefore converges. □

2. For a complex sequence (s_n) define the sequence of *arithmetic means* by $\sigma_n = \frac{1}{n+1}\sum_{k=0}^{n} s_k$. Show that if $\lim_{n\to\infty} s_n = s$, then $\lim_{n\to\infty} \sigma_n = s$. Give an example that the converse is not true.

   Proof. Let $\lim_{n\to\infty} s_n = s$, then there exists some N such that when $n \geqslant N$, $|s_n - s| < \varepsilon$. Then, we can split the sum based on N,

$$\sigma_n = \frac{1}{n+1}\left(\sum_{k=0}^{N-1} s_k + \sum_{k=N}^{n} s_k\right)$$

   Then since $\sum_{k=N}^{n} s_k = (n-N+1)\cdot s + \sum_{k=N}^{n}(s_k - s)$, we have

$$\sigma_n = \frac{1}{n+1}\left(\sum_{k=0}^{N-1} s_k + (n-N+1)\cdot s + \sum_{k=N}^{n}(s_k - s)\right)$$

   Then, as $n \to \infty$, the first sum is finite so it is negligible. The second sum tends to 0 because past N, s_k is epsilon-close to s. Therefore, in passing to the limit, we get:

$$\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} \frac{(n-N+1)\cdot s}{n+1} = s$$

   □

   An example when the converse is not true is the sequence (s_n) where

$$s_n = \begin{cases} 0 & \text{if n even} \\ 1 & \text{if n odd} \end{cases}$$

   Has arithmetic mean $\frac{1}{2}$, but the limit of s_n does not exist.

3. A map $f : X \to Y$ between metric spaces is open if $f(V)$ is open in Y for every open $V \subset X$. Show that every continuous open map from $\mathbb{R}$ to $\mathbb{R}$ is monotonic.

   Proof. Assume, BWOC, that f is a continuous open map from $\mathbb{R}$ to $\mathbb{R}$ which is not monotonic. That is, there exist 3 points $x_1 < x_2 < x_3$ such that $f(x_1) < f(x_2)$ but $f(x_2) > f(x_3)$. But since f is continuous and open, we have that the image of every interval is an open interval. So we should have

$$f((x_1, x_3)) \text{ open interval in } \mathbb{R}$$

But $f(x_2) > f(x_3)$ so this is not an open interval. Contradiction, so f must be monotonic. $\square$

4. Suppose that $f : [0, \infty) \to \mathbb{R}$ is continuous and differentiable for $x > 0$. Assume further that $f(0) = 0$ and that f' is monotone increasing. Let $g(x) = \frac{f(x)}{x}$, $x > 0$, and show that g is monotone increasing.

   Proof. We have for $x_1 < x_2$, $f'(x_1) \leqslant f'(x_2)$. And g is differentiable because it is the quotient of two differentiable functions, so

   $$g'(x) = \frac{xf'(x) - f(x)}{x^2}$$

   We want to show that $xf'(x) \geqslant f(x)$ for all x, which would give a nonnegative derivative and therefore an increasing function. At $x = 0$, we have $0 \geqslant 0$ indeed. Assume $x_n f'(x_n) \geqslant f(x_n)$ for some n.

   $$\begin{aligned} x_{n+1}f'(x_{n+1}) &\geqslant x_{n+1}f'(x_n) \\ &\geqslant x_n f'(x_n) && (x_n \leqslant x_{n+1} \text{ and } f' \text{ monotonic}) \\ &\geqslant f(x_n) && \text{(Assumption)} \\ &\geqslant f(x_{n+1}) && (f \text{ nondecreasing}) \end{aligned}$$

   So $xf'(x) \geqslant f(x)$ for all x therefore g' is nonnegative, so g is monotone increasing. $\square$

5. Define two curves in $\mathbb{C}$ by

   $$\gamma_1(t) = e^{2it}, \quad \gamma_2(t) = e^{2\pi it\sin\left(\frac{1}{t}\right)}, \quad t \in [0, 2\pi).$$

   Determine whether these curves are rectifiable and if so, find their length.

Proof. The curves are rectifiable if

$$L = \int_a^b |\gamma'(t)|dt$$

is finite. For $\gamma_1(t)$, we have $\gamma_1'(t) = 2ie^{2it}$, so $|\gamma_1'| = 2$. Therefore

$$L = \int_a^b |\gamma'(t)|dt = \int_0^{2\pi} 2dt = 4\pi$$

So γ_1 is rectifiable, with length 4π.
For γ_2, we have

$$\gamma_2'(t) = 2\pi i \left(\sin\left(\frac{1}{t}\right) + t\cos\left(\frac{1}{t}\right) \cdot \left(-\frac{1}{t^2}\right)\right) e^{2\pi i t \sin(\frac{1}{t})}$$

By the chain rule. This has magnitude:

$$|\gamma_2'(t)| = 2\pi \left| \sin\left(\frac{1}{t}\right) - \frac{\cos\left(\frac{1}{t}\right)}{t} \right|$$

Since sin and cos are bounded by 1, we have

$$|\gamma_2'(t)| \leqslant 2\pi \left| 1 - \frac{1}{t} \right|$$

But as $t \to 0$, this gets infinitely large, so the curve is not rectifiable as $|\gamma_2'(t)|$ is unbounded. □

6. Let $f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2x}$. On what intervals does the series converge uniformly?

Proof.
- For $x = 0$, we have $f(0) = \sum_{n=1}^{\infty} 1 = \infty$ diverges.
- Fix $x > 0$, we have $1 + n^2x \geqslant n^2x$ so

$$\sum_{n=1}^{\infty} \frac{1}{n^2x} \geqslant f(x)$$

And since x is fixed, we can pull out a constant $\frac{1}{x}$

$$\sum_{n=1}^{\infty} \frac{1}{n^2x} = \frac{1}{x} \cdot \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{x} \cdot \frac{\pi^2}{6}$$

Converges by comparison to $\zeta(2)$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2 x}$ converges and is greater than $f(x)$, $f(x)$ converges for $x > 0$.

Since we have examined $x = 0$ and $x > 0$, we don't even need to examine $x < 0$, because any interval containing a negative number and a positive one contains 0, and the series does not converge for $x = 0$. Therefore the interval of convergence is $I = (0, \infty)$. $\square$