Math 523 Homework 2

Theo Koss

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Section 2.1

2. Determine whether the given sequence $\{a_n\}$ converges or diverges with a_n as given. In each case, prove your conclusion.

   b. $a_n = \frac{n}{n^2-2}$. Converges to $\lim_{n\to\infty} \frac{n}{n^2-2} = 0$.

   Proof. Choose an $\epsilon > 0$, and choose N so that $|\frac{n}{n^2-2}| < \epsilon$ when $n \geqslant N$. Whenever $n \geqslant N$, we get $|\frac{n}{n^2-2}| = \frac{|n|}{|n^2-2|}$ $|n| = n$ if $n \geqslant 0$, and $|n^2 - 2| = n^2 - 2$ if $n > 1$. So

   $$|\frac{n}{n^2-2}| = \frac{n}{n^2-2}$$

   if $n > 1$.

   $$\frac{n}{n^2-2} < \frac{2n}{n^2} = \frac{2}{n}$$

   If $n > 2$. This is less than ϵ when $n > \frac{2}{\epsilon}$. So choose $N > \max\{2, \frac{2}{\epsilon}\}$. Then $|\frac{n}{n^2-2}| < \epsilon$ for $n \geqslant N$. QED

   c. $a_n = \frac{1}{n^p}$ where p positive constant. Converges to 0 for $p \geqslant 1$.

   Proof. By Theorem 2.1.13 in the text, if $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$. Then note that $\frac{1}{n^p} = (\frac{1}{n})^p$, and furthermore, $|(\frac{1}{n})^p| < 1$ for all $p \geqslant 1$ and $n > 1$. So choose $N > 1$, then if $n > N$,

   $$|\frac{1}{n^p}| < \epsilon$$

   . QED

11. Give an example of a sequence that is bounded but not convergent.

$$a_n = (-1)^n$$

Series is bounded, as it fluctuates between -1 and 1 for n odd and even, respectively, so it is bounded by 1, however it does not converge.

17. Use the binomial theorem to prove that $\lim_{n\to\infty} nr^n = 0$ if $|r| < 1$.

Proof. Let $|r| = \frac{1}{1+x}$, by binomial thm, $(1+x)^n \geqslant 1 + nx + ... > nx + n(n-1)\frac{x^2}{2}$ for $n > 2$.

Let $\epsilon > 0$ be given, then let $N > \ell$

$$|nr^n| = \frac{|n|}{(1+x)^n} < \frac{2n}{2nx + n(n-1)x^2} < \frac{2}{2x + (n-1)x^2} = \frac{2}{nx^2 - x^2 + 2x} < \epsilon$$

Solving for n:

$$n > \frac{-x^2\epsilon + 2x\epsilon - 2}{-x^2\epsilon}$$

Call $\ell = \frac{-x^2\epsilon+2x\epsilon-2}{-x^2\epsilon}$. Then for $n \geqslant \ell$, we have that nr^n converges to 0. QED

Section 2.2

3. Part d, prove that $\lim_{n\to\infty} a_n = A$ implies $\lim_{n\to\infty} a_n^p = A^p$ for $p \in \mathbb{N}$.

Proof. We will use induction, base case: $p = 2$, then by part (b) of this theorem, $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} a_n = A$ implies $\lim_{n\to\infty} a_n a_n = AA$.
Assume it holds for p, then

$$\lim_{n\to\infty} a_n^{p+1} = A^{p+1}$$

$$\iff \lim_{n\to\infty} a_n^{p-1} \cdot a_n = A^{p-1} \cdot A$$

$$\implies \text{(By assumption) } a_n = A$$

Which is true by assumption. QED

Negative case of part (a):

Proof. Let ϵ be given, since a_n and b_n converge, let $|a_n - A| < 3\epsilon$ and $|b_n - A| < 2\epsilon$.

$|a_n - b_n - A - B| = |(a_n - A) - (b_n - B)| \leqslant |a_n - A| - |b_n - B| < 3\epsilon - 2\epsilon = \epsilon$

QED

12. Consider the sequences $\{a_n\}$ and $\{b_n\}$, where sequence $\{a_n\}$ converges to zero. Is it true that the sequence $\{a_n b_n\}$ converges to zero? Explain. (See Theorem 2.2.7.)

It is true when b_n is a bounded sequence, by theorem, however when it is unbounded, the product may not converge to 0, if b_n "grows faster" than a_n.

Additional Questions

1. *Proof.* By way of contradiction, assume $|-1^n - A| < \epsilon = \frac{1}{2}$ for $n \geqslant N$. If n is even:

$$|a_n - A| < \frac{1}{2} \implies |-1 - A| < \frac{1}{2} \implies -\frac{1}{2} < -1 - A < \frac{1}{2}$$

Then consider $n + 1$, which is odd:

$$|a_n - A| < \frac{1}{2} \implies |1 - A| < \frac{1}{2} \implies -\frac{1}{2} < 1 - A < \frac{1}{2}$$

Therefore we have that A is between $-\frac{1}{2}$ and $-\frac{3}{2}$, and it is between $\frac{1}{2}$ and $\frac{3}{2}$. This is true of no real number, so a_n must not converge to a real. QED

2. Show that $\lim_{n\to\infty}(\frac{1}{n} - \frac{1}{n+1}) = 0$.

Proof. Remark 2.1.8(b) states: If $\{a_n\}$ and $\{b_n\}$ differ from each other in only a finite number of terms, then both sequences converge to the same value or they both diverge. Given that $a_n = \{\frac{1}{n}\}$ converges to

0 (problem 2(b) above), notice that $a_n = b_{n-1}$, where $b_n = \{\frac{1}{n+1}\}$ so they have all the same terms, offset by 1, except the starting point. Therefore b_n also converges to 0. By theorem 2.2.7(a), their difference diverges to the difference $0 - 0 = 0$. QED

3. Show that if $\{a_n\}$ converges to A, then $\{|a_n|\}$ converges to $|A|$.

Proof. By the triangle inequality, we have that

$$||a_n| - |A|| \leqslant |a_n - A|$$

Since a_n converges to A, we have an N s.t. for any $\epsilon > 0$, $n \geqslant N$ implies $|a_n - A| < \epsilon$. Then, by our inequality above,

$$||a_n| - |A|| \leqslant |a_n - A| < \epsilon$$

For $n \geqslant N$, therefore we have shown $\{|a_n|\}$ converges to $|A|$ for $n \geqslant N$. QED