Math 531 Homework 12

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April 2021

1 Section 5.4

- Problem 2: Show that the associative and commutative laws hold for addition in $\mathbb{Q}(D)$.

  Proof. (a) Associative: Let

  $$x = a_1 + b_1 D$$
  $$y = a_2 + b_2 D$$
  $$z = a_3 + b_3 D$$

  N2S: $(x+y)+z = x+(y+z)$.

  $$\begin{aligned}(x+y)+z &= (a_1 + b_1 D + a_2 + b_2 D) + a_3 + b_3 D \\ &= ((a_1+a_2) + (b_1+b_2)D) + a_3 + b_3 D \\ &= (a_1+a_2) + a_3 + (b_1+b_2)D + b_3 D \\ &= a_1 + (a_2+a_3) + ((b_1+b_2)+b_3)D \\ &= a_1 + (a_2+a_3) + b_1 D + (b_2+b_3)D \\ &= a_1 + b_1 D + [(a_2+a_3) + (b_2+b_3)D] \\ &= a_1 + b_1 D + [a_2 + b_2 D + a_3 + b_3 D] \\ &= x + (y+z)\end{aligned}$$

  As required.

(b) Commutative: Let

$$x = a_1 + b_1 D$$

$$y = a_2 + b_2 D$$

$$x + y = (a_1 + b_1 D) + (a_2 + b_2 D)$$

$$= (a_1 + a_2) + (b_1 + b_2) D$$

$$= (a_2 + a_1) + (b_2 + b_1) D$$

$$= a_2 + a_1 + b_2 D + b_1 D$$

$$a_2 + b_2 D + a_1 + b_1 D$$

$$= y + x$$

As required.

QED

- Problem 3: Show that the associative and commutative laws hold for multiplication in $\mathbb{Q}(D)$.

Proof. (a) Associative: Let

$$x = a_1 + b_1 D$$

$$y = a_2 + b_2 D$$

$$z = a_3 + b_3 D$$

N2S: $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

$$x \cdot (y \cdot z) = (a_1 + b_1 D) \cdot ((a_2 + b_2 D) \cdot (a_3 + b_3 D))$$

$$= (a_1 + b_1 D) \cdot ((a_2 a_3 - b_2 b_3) + (a_2 b_3 + b_2 a_3) D)$$

$$= (a_1(a_2 a_3 - b_2 b_3) - b_1(a_2 b_3 + b_2 a_3)) + (a_1(a_2 b_3 + b_2 a_3) + b_1(a_2 a_3 - b_2 b_3)) D$$

$$= (a_1 a_2 a_3 - a_1 b_2 b_3 - b_1 a_2 b_3 - b_1 b_2 a_3) + (a_1 a_2 b_3 + a_1 b_2 a_3 + b_1 a_2 a_3 - b_1 b_2 b_3) D$$

$$(x \cdot y) \cdot z = ((a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_1 a_2) D) \cdot (a_3 + b_3 D)$$

$$= (a_1 a_2 a_3 - a_1 b_2 b_3 - b_1 a_2 b_3 - b_1 b_2 a_3) + (a_1 a_2 b_3 + a_1 b_2 a_3 + b_1 a_2 a_3 - b_1 b_2 b_3) D$$

$$= x \cdot (y \cdot z)$$

As required, therefore multiplication is associative in $\mathbb{Q}(D)$.

(b) Commutative: Let

$$x = a_1 + b_1 D$$

$$y = a_2 + b_2 D$$

$$x \cdot y = (a_1 + b_1 D) \cdot (a_2 + b_2 D) = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_1 a_2) D$$

$$y \cdot x = (a_2 + b_2 D) \cdot (a_1 + b_1 D) = (a_2 a_1 - b_2 b_1) + (b_2 a_1 + a_2 b_1) D$$

$$= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_1 a_2) D$$

$$\therefore x \cdot y = y \cdot x$$

Thus multiplication is commutative in $\mathbb{Q}(D)$.

QED