Math 553 Final Exam

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1. Let v and w be vector fields along a parameterized curve $\alpha : I \to S$ on an oriented regular surface S, with unit normal vector $N : S \to \mathbb{R}^3$. Show that $\frac{d}{dt}\langle v, w\rangle = \langle \frac{Dv}{dt}, w\rangle + \langle v, \frac{Dw}{dt}\rangle$.

   Proof.
   $$\frac{d}{dt}\langle v, w\rangle = \langle v', w\rangle + \langle v, w'\rangle$$
   Then, since v' and w' are both tangents to S, we can break them into components:
   $$v' = \frac{Dv}{dt} + v_n$$
   and likewise for w'. Where $\frac{Dv}{dt}$ represents the tangential component of v' and v_n represents the normal component. Then, by definition, v_n is orthogonal to w and

therefore $\langle v_n, w\rangle = 0$. Similarly $\langle v, w_n\rangle = 0$. Now differentiating:

$$\begin{aligned}
\frac{d}{dt}\langle v, w\rangle &= \langle v', w\rangle + \langle v, w'\rangle \\
&= \langle \frac{Dv}{dt} + v_n, w\rangle + \langle v, \frac{Dw}{dt} + w_n\rangle \\
&= \langle \frac{Dv}{dt}, w\rangle + \langle v_n, w\rangle + \langle v, \frac{Dw}{dt}\rangle + \langle v, w_n\rangle \\
&= \langle \frac{Dv}{dt}, w\rangle + \langle v, \frac{Dw}{dt}\rangle
\end{aligned}$$

As required. QED

2. Suppose that S has a coordinate patch (U, ϕ). Recall that the curve $(u(t), v(t))$ in U determines a geodesic on S provided the geodesic equations are satisfied. i.e.

$$u'' + (u')^2\Gamma^1_{11} + 2u'v'\Gamma^1_{12} + (v')^2\Gamma^1_{22} = 0$$

$$v'' + (u')^2\Gamma^2_{11} + 2u'v'\Gamma^2_{12} + (v')^2\Gamma^2_{22} = 0$$

Now suppose as well that (U, ϕ) is isothermal, in that the coefficients of the first fundamental form satisfy $E = G = \lambda$ and $F = 0$. Show that the geodesic equations become:

$$2\lambda u'' + (u')^2\lambda_u + 2u'v'\lambda_v - (v')^2\lambda_u = 0$$

$$2\lambda v'' + (u')^2\lambda_v + 2u'v'\lambda_u - (v')^2\lambda_v = 0$$

Proof. According to Do Carmo pp. 239, working out the Christoffel symbols given $E = G = \lambda$ and $F = 0$:

$$\Gamma_{11}^1 = \frac{1}{2}\frac{\lambda_u}{\lambda},\ \Gamma_{11}^2 = -\frac{1}{2}\frac{\lambda_v}{\lambda}$$

$$\Gamma_{12}^1 = \frac{1}{2}\frac{\lambda_v}{\lambda}\ \Gamma_{12}^2 = \frac{1}{2}\frac{\lambda_u}{\lambda}$$

$$\Gamma_{22}^1 = -\frac{1}{2}\frac{\lambda_u}{\lambda},\ \gamma_{22}^2 = \frac{1}{2}\frac{\lambda_v}{\lambda}$$

Therefore the first geodesic equation becomes

$$u'' + (u')^2\frac{1}{2}\frac{\lambda_u}{\lambda} + 2u'v'\frac{1}{2}\frac{\lambda_v}{\lambda} + (v')^2(-\frac{1}{2}\frac{\lambda_u}{\lambda}) = 0$$

Then, multiplying everything by 2λ to get rid of the denominator:

$$\implies 2\lambda u'' + (u')^2\lambda_u + 2u'v'\lambda_v - (v')^2\lambda_u = 0$$

As required. The second equation follows similarly. QED

3. Consider the surface $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 : y > 0\}$, endowed with first fundamental form $I_{(x,y)} = \frac{dx^2+dy^2}{y^2}$.

   (i) Compute curvature of $\mathbb{H}$ using $K = \frac{1}{2\lambda}\Delta(\log\lambda)$.

We know $E = G = \lambda = \frac{1}{y^2}$. So $\frac{1}{2\lambda} = \frac{y^2}{2}$.
The Laplacian is then:

$$\begin{aligned}
\Delta(\log\lambda) &= \left(\frac{\partial^2 \log\lambda}{\partial x^2}\right) + \left(\frac{\partial^2 \log\lambda}{\partial y^2}\right) \\
&= \left(\frac{\partial^2 \log\lambda}{\partial y^2}\right) \\
&= \frac{\partial}{\partial y}\left(-\frac{2}{y}\right) \\
&= \frac{2}{y^2}
\end{aligned}$$

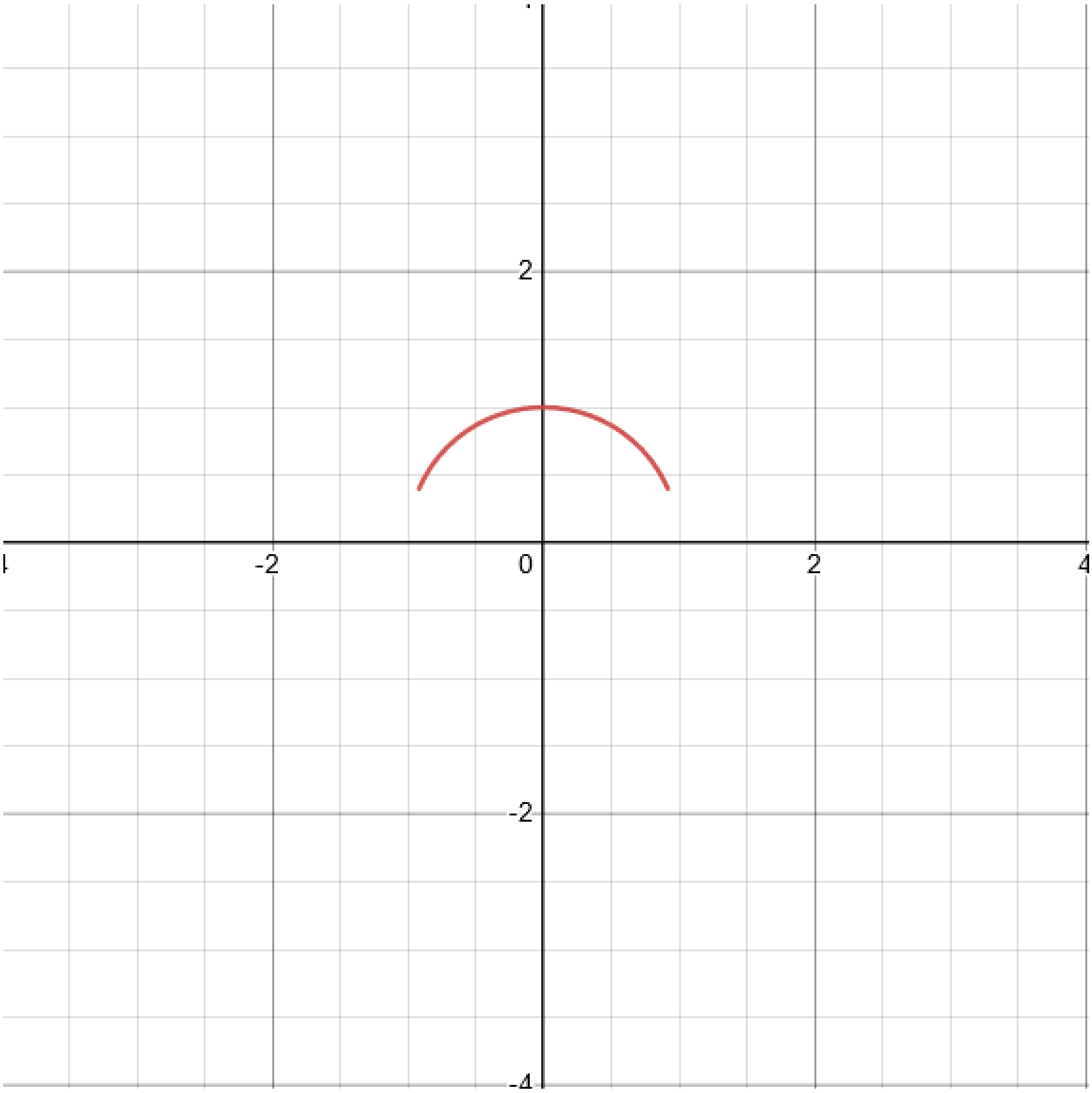
So $K = \frac{1}{2\lambda}\Delta(\log\lambda) = \frac{y^2}{2} \cdot \frac{2}{y^2} = 1$.

(ii) Let $\gamma : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{H}$ be given by

$$\gamma(t) = (x_0 + r\tanh t, r\text{sech } t)$$

where $x_0 \in \mathbb{R}$. Draw a picture of $\gamma(t)$, and show that γ is PBAL, in that $I_{\gamma(t)}(\gamma'(t)) = 1$.

Picture (assuming $x_0 = 0$ and $r = 1$. It is moved left or right depending on x_0 and scaled by r.)



$$E = r^2 \text{sech}^4 t$$

$$F = -r^2 \tanh t \text{sech}^3 t$$

$$G = r^2 \tanh^2 t \text{sech}^2 t$$

$$L = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{r^2\text{sech}^4 t - r^2 \tanh t\text{sech}^3 t + r^2 \tanh^2 t\text{sech}^2 t dt}$$

(This integral gave me an answer $\neq 1$. Usually the curve is like $X(u, v) = ...$ but it's different here so I'm a bit confused. I tried a couple ways but none of them worked out.)

(iii) Use problem 2 to show the curve is geodesic.

(iv) Does $\mathbb{H}$ have any other geodesics? Explain.

4. Compute the Euler characteristic of a torus of revolution T, and explain why you can conclude that the integral of the Gaussian curvature over T is equal to 0.

Proof. By Do Carmo pp. 276, the torus is homeomorphic to a sphere with one handle, and from a result on that page,

$$g = \frac{2 - \chi(S)}{2}$$

Where g denotes the number of handles on a surface. Plugging in 1 for g it is easy to see that $\chi(S) = 0$. Since Euler characteristic is a topological invariant, showing that $\chi(S) = 0$ for $S \simeq T$ directly implies $\chi(T) = 0$. Then, by the Gauss-Bonnet theorem,

$$\iint_T K dA = 2\pi\chi(T)$$

And since $\chi(T) = 0$, we may conclude that $K = 0$. QED

(Side note, this proof kind of feels like cheating because the result was in the text. I was trying to find a triangulation of the torus of revolution to show this more concretely but I couldn't find one.)

5. Let $\alpha : I \to \mathbb{R}^2$ be a simple, closed, regular curve, PBAL, in the plane, and let $\kappa : I \to \mathbb{R}$ be the (signed) curvature of α. Use Gauss-Bonnet to show that

$$\int_{\alpha} \kappa(s) ds = 2\pi$$

Proof. Using the isometry $\mathbb{R}^2 \to \mathbb{R}^3$ sending $(x, y) \to (x, y, 0)$, consider the surface in 3 space given by $M = (x, y, 0)$. This is isometric to $\alpha = (x, y)$. Now consider the Gauss-Bonnet theorem, using notation from eqn. 3 of https://mathworld.wolfram.com/Gauss-BonnetFormula.html:

$$\iint_M K dA = 2\pi\chi(M) - \sum \varphi_i - \int_{\partial M} \kappa_g ds$$

Working through this notation:

- Our surface M is defined by $M := (x, y, 0)$ which is homeomorphic to a disk, and therefore has Euler characteristic $\chi(M) = 1$ and $K = 0$.
- There is no "jump angle" φ_i of α since α never intersects itself (it is simple), therefore the sum works out to be 0.

- ∂M is the boundary of M, which is clearly α. Then the geodesic curvature κ_g is simply the curvature of α.

Thus, the Gauss-Bonnet theorem simplifies to:

$$\int_{\partial M} \kappa_g ds = 2\pi$$
$$\implies \int_{\alpha} \kappa(s) ds = 2\pi$$

As required. QED