Örnek 1

$$\begin{aligned} 15x_1 + 5x_2 - 5x_3 &= 29 \\ 5x_1 + 20x_2 + 10x_3 &= -3 \\ -5x_1 + 5x_2 + 15x_3 &= -7 \end{aligned}$$

denklem sistemini Gauss-Seidel iterasyon metodunu kullanarak, $\mathbf{x}^{(0)} = (2, -1, 1)$ *başlangıç noktası ve* $\delta = 10^{-2}$ *hata sınırı ile yaklaşık olarak çözünüz.*

Çözüm

$$\begin{aligned} |a_{11}| &= 15 > 10 = |a_{12}| + |a_{13}| \\ |a_{22}| &= 20 > 15 = |a_{21}| + |a_{23}| \\ |a_{33}| &= 15 > 10 = |a_{31}| + |a_{32}| \end{aligned}$$

olduğundan Gauss-Seidel yöntemi yakınsaktır.

$$x_1 = \frac{29 - 5x_2 + 5x_3}{15}$$
$$x_2 = \frac{-3 - 5x_1 - 10x_3}{20}$$
$$x_3 = \frac{-7 + 5x_1 - 5x_2}{15}$$

$\mathbf{x}^{(0)} = \left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right) = (2, -1, 1)$ *noktası ile başlayalım.*

1. iterasyon.

$$x_1^{(1)} = \frac{29 - 5x_2^{(0)} + 5x_3^{(0)}}{15} = \frac{29 - 5 \cdot (-1) + 5 \cdot 1}{15} = 2.6 \qquad \varepsilon_1^{(1)} = \frac{\left|x_1^{(1)} - x_1^{(0)}\right|}{\left|x_1^{(1)}\right|} = \frac{|2.6-2|}{|2.6|} = 0.2307$$

$$x_2^{(1)} = \frac{-3 - 5x_1^{(1)} - 10x_3^{(0)}}{20} = \frac{-3 - 5 \cdot (2.6) - 10 \cdot 1}{20} = -1.3 \qquad \varepsilon_2^{(1)} = \frac{\left|x_2^{(1)} - x_2^{(0)}\right|}{\left|x_2^{(1)}\right|} = \frac{|-1.3+1|}{|-1.3|} = 0.2307$$

$$x_3^{(1)} = \frac{-7 + 5x_1^{(1)} - 5x_2^{(1)}}{15} = \frac{-7 + 5 \cdot (2.6) - 5 \cdot (-1.3)}{15} = 0.8333 \qquad \varepsilon_3^{(1)} = \frac{\left|x_3^{(1)} - x_3^{(0)}\right|}{\left|x_3^{(1)}\right|} = \frac{|-0.8333-1|}{|-0.8333|} = 0.2000$$

$$\max_i \varepsilon_i^{(1)} = 0.2307 \not< 0.01$$

olduğundan iterasyonlara devam edilir.

2. iterasyon.

$$x_1^{(2)} = \frac{29 - 5x_2^{(1)} + 5x_3^{(1)}}{15} = \frac{29 - 5 \times (-1.3) + 5 \times 0.8333}{15} = 2.6444 \qquad \varepsilon_1^{(2)} = \frac{\left|x_1^{(2)} - x_1^{(1)}\right|}{\left|x_1^{(2)}\right|} = \frac{|2.6444-2.6|}{|2.6444|} = 0.0167$$

$$x_2^{(2)} = \frac{-3 - 5x_1^{(2)} - 10x_3^{(1)}}{20} = \frac{-3 - 5 \times 2.6444 - 10 \times 0.8333}{20} = -1.2277 \qquad \varepsilon_2^{(2)} = \frac{\left|x_2^{(2)} - x_2^{(1)}\right|}{\left|x_2^{(2)}\right|} = \frac{|-1.2277+1.3|}{|-1.2277|} = 0.0588$$

$$x_3^{(2)} = \frac{-7 + 5x_1^{(2)} - 5x_2^{(2)}}{15} = \frac{-7 + 5 \times 2.6444 - 5 \times (-1.2277)}{15} = 0.8240 \qquad \varepsilon_3^{(2)} = \frac{\left|x_3^{(2)} - x_3^{(1)}\right|}{\left|x_3^{(2)}\right|} = \frac{|0.8240-0.8333|}{|0.8240|} = 0.0112$$

$$\max_i \varepsilon_i^{(2)} = 0.0588 \not< 0.01$$

olduğundan iterasyonlara devam edilir.

3. iterasyon.

$$\begin{aligned} x_1^{(3)} &= 2.6172 \qquad & \varepsilon_1^{(3)} &= 0.0103 \\ x_2^{(3)} &= -1.2163 \qquad & \varepsilon_2^{(3)} &= 0.0093 \\ x_3^{(3)} &= 0.8111 \qquad & \varepsilon_3^{(3)} &= 0.0159 \end{aligned}$$

$$\max_i \varepsilon_i^{(3)} = 0.0159 \not< 0.01$$

olduğundan iterasyonlara devam edilir.

4. iterasyon.

$x_1^{(4)} = 2.6091 \qquad \varepsilon_1^{(4)} = 0.0031$
$x_2^{(4)} = -1.2078 \qquad \varepsilon_2^{(4)} = 0.0070$
$x_3^{(4)} = 0.8056 \qquad \varepsilon_3^{(4)} = 0.0068$

$$\max_i \varepsilon_i^{(4)} = 0.0070 < 0.01$$

Denklem sisteminin 4 iterasyon sonundaki yaklaşık çözümü

$$(2.6091, -1.2078, 0.8056).$$

Örnek 2

$$\begin{aligned} 2x_1 + x_2 &= 1 \\ -x_1 + 2x_2 &= 17 \end{aligned}$$

denklem sistemini Gauss-Seidel iterasyon metodunu kullanarak, $\mathbf{x}^{(0)} = (0,0)$ *başlangıç noktası ve* $\delta = 10^{-2}$ *hata sınırı ile yaklaşık olarak çözünüz.*

Çözüm

$$\begin{aligned} |a_{11}| &= 2 > 1 = |a_{12}| \\ |a_{22}| &= 2 > 1 = |a_{21}| \end{aligned}$$

olduğundan Gauss-Seidel yöntemi yakınsaktır.

$$x_1 = \frac{1 - x_2}{2}$$
$$x_2 = \frac{17 + x_1}{2}$$

$\mathbf{x}^{(0)} = (0,0)$ *noktası ile başlayalım.*

1. iterasyon.

$$x_1^{(1)} = \frac{1 - x_2^{(0)}}{2} = \frac{1-0}{2} = 0.5 \qquad \varepsilon_1^{(1)} = \frac{\left|x_1^{(1)} - x_1^{(0)}\right|}{\left|x_1^{(1)}\right|} = \frac{|0.5-0|}{|0.5|} = 1$$
$$x_2^{(1)} = \frac{17 + x_1^{(1)}}{2} = \frac{17+0.5}{2} = 8.75 \qquad \varepsilon_2^{(1)} = \frac{\left|x_2^{(1)} - x_2^{(0)}\right|}{\left|x_2^{(1)}\right|} = \frac{|8.75-0|}{|8.75|} = 1$$

$$\max_i \varepsilon_i^{(1)} = 1 \not< 0.01$$

2. iterasyon.

$$x_1^{(2)} = \frac{1 - x_2^{(1)}}{2} = \frac{1-8.75}{2} = -3.875 \qquad \varepsilon_1^{(2)} = \frac{\left|x_1^{(2)} - x_1^{(1)}\right|}{\left|x_1^{(2)}\right|} = \frac{|-3.875+0.5|}{|-3.875|} = 0.8709$$
$$x_2^{(2)} = \frac{17 + x_1^{(2)}}{2} = \frac{17-3.875}{2} = 6.5625 \qquad \varepsilon_2^{(2)} = \frac{\left|x_2^{(2)} - x_2^{(1)}\right|}{\left|x_2^{(2)}\right|} = \frac{|6.5625-8.75|}{|6.5625|} = 0.3333$$

$$\max_i \varepsilon_i^{(2)} = 0.8709 \not< 0.01$$

3. iterasyon.

$$x_1^{(3)} = \frac{1 - x_2^{(2)}}{2} = \frac{1-6.5625}{2} = -2.7812 \qquad \varepsilon_1^{(3)} = \frac{\left|x_1^{(3)} - x_1^{(2)}\right|}{\left|x_1^{(3)}\right|} = \frac{|-2.7812--3.875|}{|-2.7812|} = 0.3932$$
$$x_2^{(3)} = \frac{17 + x_1^{(3)}}{2} = \frac{17-2.7812}{2} = 7.1094 \qquad \varepsilon_2^{(3)} = \frac{\left|x_2^{(3)} - x_2^{(2)}\right|}{\left|x_2^{(3)}\right|} = \frac{|7.1094-6.5625|}{|7.1094|} = 0.0769$$

$$\max_i \varepsilon_i^{(3)} = 0.3932 \not< 0.01$$

4. iterasyon.

$$x_1^{(4)} = \frac{1 - x_2^{(3)}}{2} = \frac{1 - 7.1094}{2} = -3.0547 \qquad \varepsilon_1^{(4)} = \frac{\left|x_1^{(4)} - x_1^{(3)}\right|}{\left|x_1^{(4)}\right|} = \frac{|-3.0547+2.7812|}{|-3.0547|} = 0.0895$$

$$x_2^{(4)} = \frac{17 + x_1^{(4)}}{2} = \frac{17 - 3.0547}{2} = 6.9726 \qquad \varepsilon_2^{(4)} = \frac{\left|x_2^{(4)} - x_2^{(3)}\right|}{\left|x_2^{(4)}\right|} = \frac{|6.9726-7.1094|}{|6.9726|} = 0.0196$$

$$\max_i \varepsilon_i^{(4)} = 0.0895 \not< 0.01$$

5. iterasyon.

$$x_1^{(5)} = \frac{1 - x_2^{(4)}}{2} = \frac{1 - 6.9726}{2} = -2.9863 \qquad \varepsilon_1^{(5)} = \frac{\left|x_1^{(5)} - x_1^{(4)}\right|}{\left|x_1^{(5)}\right|} = \frac{|-2.9863+3.0547|}{|-2.9863|} = 0.0229$$

$$x_2^{(5)} = \frac{17 + x_1^{(5)}}{2} = \frac{17 - 2.9863}{2} = 7.0068 \qquad \varepsilon_2^{(5)} = \frac{\left|x_2^{(5)} - x_2^{(4)}\right|}{\left|x_2^{(5)}\right|} = \frac{|7.0068-6.9726|}{|7.0068|} = 0.0048$$

$$\max_i \varepsilon_i^{(5)} = 0.0229 \not< 0.01$$

6. iterasyon.

$$x_1^{(6)} = \frac{1 - x_2^{(5)}}{2} = \frac{1 - 7.0068}{2} = -3.0034 \qquad \varepsilon_1^{(6)} = \frac{\left|x_1^{(6)} - x_1^{(5)}\right|}{\left|x_1^{(6)}\right|} = \frac{|-3.0034+2.9863|}{|-3.0034|} = 0.0056$$

$$x_2^{(6)} = \frac{17 + x_1^{(6)}}{2} = \frac{17 - 3.0034}{2} = 6.9983 \qquad \varepsilon_2^{(6)} = \frac{\left|x_2^{(6)} - x_2^{(5)}\right|}{\left|x_2^{(6)}\right|} = \frac{|6.9983-7.0068|}{|6.9983|} = 0.0012$$

$$\max_i \varepsilon_i^{(6)} = 0.0056 < 0.01$$

Denklem sisteminin 6 iterasyon sonundaki yaklaşık çözümü

$$(-3.0034, 6.9983)\,.$$

(Gerçek çözüm ise $(-3,7)$*)*