Task 1

Исследовать функцию на условный экстремум

$U = 3 - 8x + 6y$, если $x^2 + y^2 = 36$

$L(\lambda, x, y) = 3 - 8x + 6y + \lambda(x^2 + y^2 - 36)$

$$\begin{cases} L'_x = -8 + 2x\lambda = 0 \\ L'_y = 6 + 2y\lambda = 0 \\ L'_\lambda = x^2 + y^2 - 36 = 0 \end{cases}$$

$$\begin{cases} x = \frac{4}{\lambda} \\ y = -\frac{3}{\lambda} \\ (\frac{4}{\lambda})^2 + (-\frac{3}{\lambda})^2 - 36 = 0 \end{cases}$$

$\lambda = \pm\frac{\sqrt{21}}{6}$,
$(\frac{\sqrt{21}}{6}, \frac{8\sqrt{21}}{7}, -\frac{6\sqrt{21}}{7}), (-\frac{\sqrt{21}}{6}, -\frac{8\sqrt{21}}{7}, \frac{6\sqrt{21}}{7})$

$L''_{xx} = 2\lambda, L''_{yy} = 2\lambda, L''_{\lambda\lambda} = 0$
$L''_{xy} = 0, L''_{x\lambda} = 2x, L''_{y\lambda} = 2y$

$$\begin{vmatrix} L''_{\lambda\lambda} & L''_{\lambda x} & L''_{\lambda y} \\ L''_{x\lambda} & L''_{xx} & L''_{xy} \\ L''_{y\lambda} & L''_{yx} & L''_{yy} \end{vmatrix} = \begin{vmatrix} 0 & 2x & 2y \\ 2x & 2\lambda & 0 \\ 2y & 0 & 0 \end{vmatrix} = 2y \cdot \begin{vmatrix} 2x & 2y \\ 2\lambda & 0 \end{vmatrix} = -8y^2\lambda$$

$-8 \cdot (-\frac{6\sqrt{21}}{7})^2 \cdot \frac{\sqrt{21}}{6} = -24$

$-8 \cdot (\frac{6\sqrt{21}}{7})^2 \cdot -\frac{\sqrt{21}}{6} = 24$

при $(\frac{\sqrt{21}}{6}, \frac{8\sqrt{21}}{7}, -\frac{6\sqrt{21}}{7})$ будет минимум, а при $(-\frac{\sqrt{21}}{6}, -\frac{8\sqrt{21}}{7}, \frac{6\sqrt{21}}{7})$ - максимум

Task 2

$U = 2x^2 + 12xy + 32y^2 + 15$, если $x^2 + 16y^2 = 64$

$L(\lambda, x, y) = 2x^2 + 12xy + 32y^2 + 15 + \lambda(x^2 + 16y^2 - 64)$

$$\begin{cases} L'_x = 4x + 12y + 2x\lambda = 0 \\ L'_y = 12x + 64y + 32y\lambda = 0 \\ L'_\lambda = x^2 + 16y^2 - 64 = 0 \end{cases}$$

$4x + 12y + 2x\lambda = x(4 + 2\lambda) + 12y = 0$
$x = -\frac{12y}{4+2\lambda}$

$12x + 64y + 32y\lambda = 12x + y(64 + 32\lambda) = 16y(4 + 2\lambda) - \frac{144y}{4+2\lambda} = 0$

$\frac{16y(4+2\lambda)^2 - 144y}{4+2\lambda} = 0, \frac{16y(16+16\lambda+4\lambda^2) - 144y}{4+2\lambda} = 0$

$\frac{256y - 144y + 256y\lambda + 64y\lambda^2}{4+2\lambda} = \frac{64y\lambda^2 + 256y\lambda + 112y}{4+2\lambda} = 0$

$\frac{64y((2+\lambda)^2 - 4) + 112y}{2(2+\lambda)} = \frac{64y(2+\lambda)^2 - 144y}{2(2+\lambda)} = 0$

$\frac{2y(32(2+\lambda)^2 - 72)}{2(2+\lambda)} = \frac{32y(2+\lambda)^2}{2+\lambda} - \frac{72y}{2+\lambda} = 0$

$32y(2 + \lambda) = \frac{72y}{2+\lambda}, (2 + \lambda)^2 = \frac{72}{32}$

$\lambda^2 + 4\lambda + \frac{7}{4} = 0$

$D = 16 - 4\frac{7}{4} = 9, \lambda_{1,2} = \frac{-4\pm3}{2} = (-7/2, -1/2)$

$x_1 = -\frac{12y}{4-2\frac{7}{2}} = 4y, x_2 = -\frac{12y}{4-2\frac{1}{2}} = -4y$

$(\pm4y)^2 + 16y^2 - 64 = 0, y = \pm\sqrt{2}$

$y_1 = \pm\sqrt{2}, y_2 = \pm\sqrt{2}$

Получаем следующие точки:

$(-3.5, 4\sqrt{2}, \sqrt{2})$
$(-3.5, -4\sqrt{2}, -\sqrt{2})$
$(-0.5, -4\sqrt{2}, \sqrt{2})$
$(-0.5, 4\sqrt{2}, -\sqrt{2})$

$$\begin{cases} L'_x = 4x + 12y + 2x\lambda = 0 \\ L'_y = 12x + 64y + 32y\lambda = 0 \\ L'_\lambda = x^2 + 16y^2 - 64 = 0 \end{cases}$$

$L''_{xx} = 4 + 2\lambda, L''_{yy} = 64 + 32\lambda, L''_{\lambda\lambda} = 0$
$L''_{xy} = 12, L''_{x\lambda} = 2x, L''_{y\lambda} = 32y$

$$\begin{vmatrix} L''_{\lambda\lambda} & L''_{\lambda x} & L''_{\lambda y} \\ L''_{x\lambda} & L''_{xx} & L''_{xy} \\ L''_{y\lambda} & L''_{yx} & L''_{yy} \end{vmatrix} = \begin{vmatrix} 0 & 2x & 32y \\ 2x & 4+2\lambda & 12 \\ 32y & 12 & 0 \end{vmatrix} = -2x \cdot \begin{vmatrix} 2x & 12 \\ 32y & 0 \end{vmatrix} + 32y \cdot \begin{vmatrix} 2x & 4+2\lambda \\ 32y & 12 \end{vmatrix} =$$

$768xy + 32y(24x - 32y(4+2\lambda)) = 768xy + 768xy - 1024y^2(4+2\lambda) =$

$1536xy - 4096y^2 - 2048\lambda = 512(3xy - 8y^2 - 4\lambda)$

$(-3.5, 4\sqrt{2}, \sqrt{2}) : \Delta = 512(24 - 16 - 4(-3.5)) = 11264$, максимум

$(-3.5, -4\sqrt{2}, -\sqrt{2}) : \Delta = 512(24 - 16 - 4(-3.5)) = 11264$, максимум

$(-0.5, -4\sqrt{2}, \sqrt{2}) : \Delta = 512(-24 - 16 - 4(-0.5)) = -19456$, минимум

$(-0.5, 4\sqrt{2}, -\sqrt{2}) : \Delta = 512(-24 - 16 + 2) = -19456$, минимум

Task 3

Найти производную функции $U = x^2 + y^2 + z^2$ по направлению вектора $\vec{c}(-9, 8, -12)$ в точку M(8,-12,9);

$U'_x = 2x$
$U'_y = 2y$
$U'_z = 2z$

$|c| = \sqrt{(-9)^2 + 8^2 + (-12)^2} = 17$

$gradU(8, -12, 9) = (16, -22, 18)$

$\overrightarrow{b_0} = (-\frac{9}{17}, \frac{8}{17}, -\frac{12}{17})$

$U'_{\vec{b}}(8, -12, 9) = -16 \cdot \frac{9}{17} - 22 \cdot \frac{8}{17} - 18 \cdot \frac{12}{17} = -\frac{526}{17} = -31.53$

Task 4

Найти производную функции $U = e^{x^2+y^2+z^2}$ по направлению вектора $\vec{d}(4, -13, -16)$ в точку L(-16,4,-13);

$U'_x = 2xe^{x^2+y^2+z^2}$
$U'_y = 2ye^{x^2+y^2+z^2}$
$U'_z = 2ze^{x^2+y^2+z^2}$

$|d| = \sqrt{4^2 + (-13)^2 + (-16)^2} = 21$

$gradU(-16, 4, -13) = (-32e^{441}, 8e^{441}, -26e^{441})$

$\overrightarrow{d_0} = (4/21, -13/21, -16/21)$

$U'_{\vec{d}}(-16, 4, -13) = \frac{-4\cdot 32e^{441} - 13\cdot 8e^{441} + 26\cdot 16e^{441}}{21} = \frac{184}{21}e^{441} = 8.76e^{441}$