Persisting Through The Convexity: Convex Modules for the Commutative Grid

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To all my friends, who make my life better every day, and especially my roommates who put up with me every day.

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Abstract

Topological data analysis uses persistence modules to try to distinguish the real geometric features of a data set from noise. These persistence modules are exactly representations of posets. While many different posets appear, $\mathbb{R} \times \mathbb{R}$ and other commutative grids play a key role. Since representations of these posets are *wild*, much effort has gone into attaching useful invariants to their representations. One recent invariant, the *virtual barcode* conveys information about the nice summands of a representation. When the determinant of the *rank matrix* is ± 1, the virtual barcode interprets the representation as a difference of two nice representations. Thus, we investigate for which finite posets the rank matrix has this property.

In this thesis, we focus on finite posets which correspond to a closed interval in $\mathbb{Z} \times \mathbb{Z}$. These posets are particularly important because their representations are the discrete analogue of two-dimensional persistence modules. For a certain family of these posets, we answer the question completely. We also show that unfortunately, for some of these posets the rank matrix has large determinant.

Chapter 1

Introduction

1.1 Background

Topological data analysis (TDA) is a field where one studies the *shape* of a data set. For example, one might wish to decide whether holes or clusters in a point cloud of data are actual features of the data set, or should instead be interpreted as noise. Though relatively new, TDA has been used recently to help study COVID-19 [6], [8], detect financial bubbles [1], look at the brain white matter in maltreated children [2], and analyze the shape of flight data sets [7]. In this thesis, we focus on a problem in pure math related to persistent homology, which is one tool used in TDA. In a sense, persistent homology can be thought of as a two-step process. We first associate our data set to a geometric object. Then, we use tools from topology to analyze this geometric object in order to determine the (topological) features of the data set. One of the key ingredients in this analysis is persistence modules, which are exactly representations of a certain poset. We briefly describe the set up in persistent homology, for more details (see [10]). The reader should note that this introduction to the program in persistent homology is **not required** to understand what follows in this thesis. It is merely added to give context to our problem. It would be perfectly permissible for the reader to skip the remainder of this chapter and begin with Section 2.1.

1.1.1 TDA

Our tool for endowing our data set with a topological structure is the *Vietoris-Rips complex*, which uses *simplices* to give our data geometry. This is important, since a finite set necessarily has discrete topology.

Definition 1.1.1. *An* ***n-dimensional simplex****, is the convex hull of a collection of $n + 1$ points. A* ***simplicial complex*** *K is a collection of simplexes such that*

1. *Every face of K is a simplex in K, and*

2. *The intersection of any two simplexes in K is a face for each of those simplexes.*

(a) 0-simplex, or a point. (b) 1-simplex, or a line. (c) 2-simplex, or a triangle. (d) 3-simplex, or a tetrahedron.

Figure 1.1: Four example simplexes, with all edges visible for clarity.

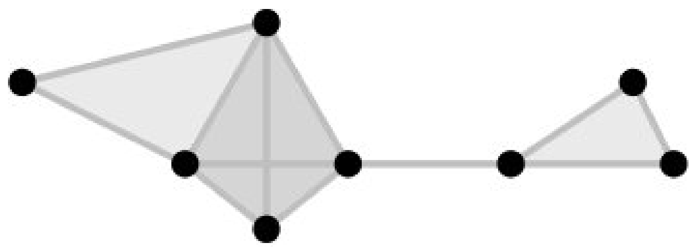


Figure 1.2: An example simplicial complex. Note there are 8 0-simplexes, 12 1-simplexes, 6 2-simplexes, and 1 3-simplex. All edges are visible for clarity.

We'll illustrate how persistent homology works with an example. Suppose we wish to decide whether a data set $X \subseteq \mathbb{R}^3$ should be more correctly interpreted as a sphere or a solid ball. In order to decide between the two candidates, we calculate the *homology* of the Vietoris-Rips complex $(C_\epsilon)_{\epsilon>0}$. Specifically, for each $\epsilon \geq 0$, we let C_ϵ be the abstract simplicial complex whose k-simplices are determined by data points $x_1, x_2, ...x_{k+1} \in X$ where

$$d(x_i, x_j) \leq 2\epsilon \text{ for all } 1 \leq i, j \leq k+1.$$

That is, there is a k-simplex for every $k+1$ points with pairwise distance less than 2ϵ. When $\sigma \leq \tau$ in $[0, \infty)$, there is an inclusion of simplicial complexes $C_\sigma \hookrightarrow C_\tau$, thus we obtain a filtration of simplicial complexes indexed by $[0, \infty)$. The assignment $f : \epsilon \to C_\epsilon$ is a representation of the poset $P = [0, \infty)$ or $\mathbb{R}$ taking values in the collection of simplicial complexes. Since we wish to distinguish between a sphere and a ball, we apply the second homology functor $H_2(-, \mathbb{C})$ to f to obtain the representation of P given by,

$$\epsilon \to H_2(C_\epsilon, \mathbb{C}) = H_2(-, \mathbb{C}) \circ f.$$

As ϵ increases, generators for homology are born and die, as cycles appear and become boundaries. In persistent homology, one takes the viewpoint that the real topological features of the data set can be distinguished from noise by looking for generators which "persist" for long periods of time. Informally, one keeps the indecomposable summands of $H_2(-, K) \circ f$ that correspond to wide intervals. Conversely, cycles which disappear quickly after their appearance (narrow intervals) are interpreted as noise and disregarded.

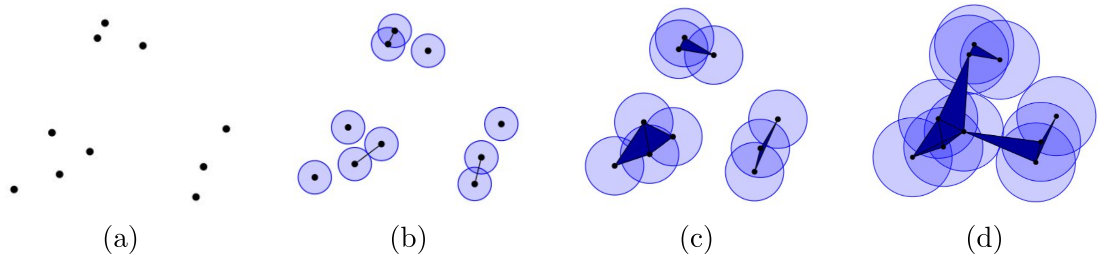


Figure 1.3: Example data set with snapshots of various ϵ.

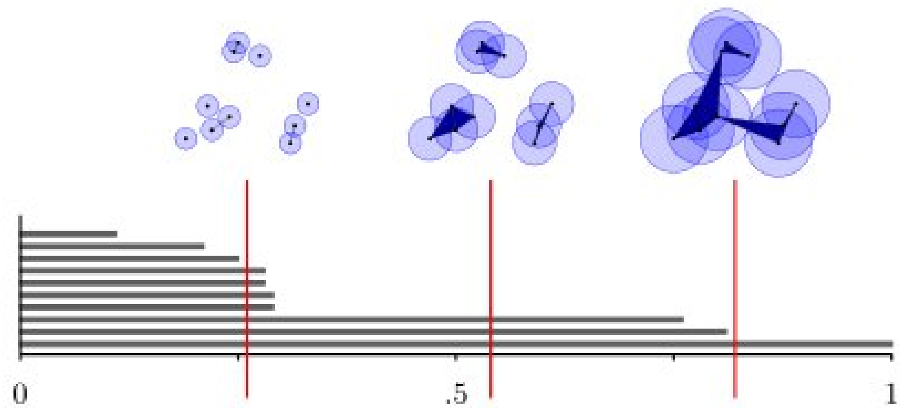


Figure 1.4: For the data in Figure 1.3, we see the 0th homology calculated across ϵ.

This technique has been widely successful in topological data analysis (see, for example [1], [2], [6], [7], [8]). Moreover, since only finitely many values of the real parameter ϵ come from an actual distance in X, we may *discretize* instead to work instead with representations of a finite totally ordered set $\mathbb{A}_n = \{1 < 2 < \cdots < n\}$.

From this description, it's clear that representations of totally ordered sets like $\mathbb{R}$ and $\mathbb{A}_n$ are useful in TDA. On the other hand, representations of many other posets are critical to the analysis as well. For example, in order to resolve issues related to the addition or removal of outliers, the program described above is modified in a way that produces representations of $\mathbb{R} \times \mathbb{R}$ or *two-parameter persistence modules.* In fact, the finite posets we study in this thesis are of interest, exactly because they are the discrete analogue of $\mathbb{R} \times \mathbb{R}$.

1.1.2 Rank Characters

It's well-known that representations of $\mathbb{R} \times \mathbb{R}$, or finite commutative grids are much wilder than those of totally ordered sets. Indeed, these posets are of *infinite representation type*, so classifying their representations completely is a hopeless task. On the other hand, much work has been dedicated to attaching useful invariants to their representations (see [3], [4] and [5]). Because they're completely determined by their support, it's generally agreed that convex (or interval) modules are the representations that provide the right generalization of the intervals in persistent homology. Because

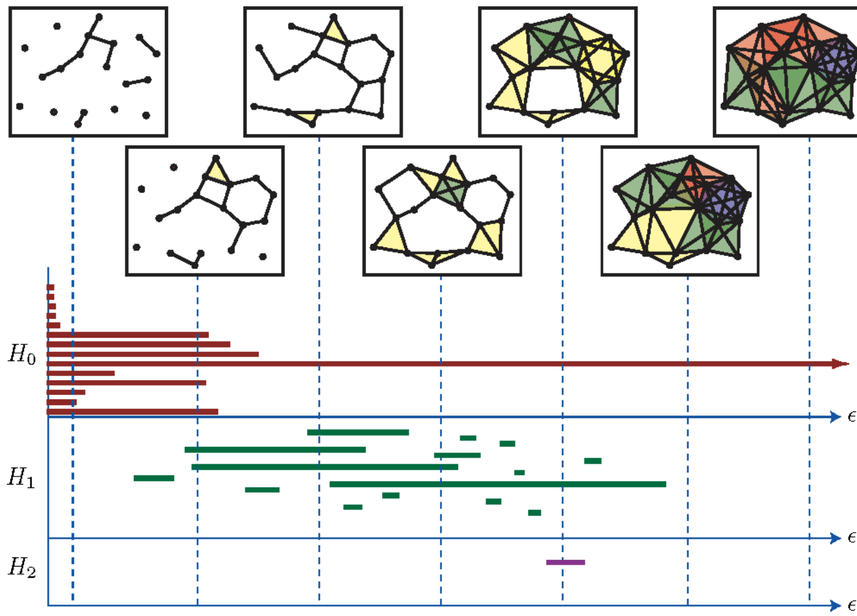


Figure 1.5: A picture of the Vietorus-Rips complex for a data set, along with some barcodes.

of this, associating to an arbitrary representation a finite set of convex modules is desirable. Recently, *rank characters* and *virtual barcodes* [9] have been suggested as an invariant which conveys information about the *convex* summands of a representation. The rank character of a representation, $\chi(M)$ uses both convex modules, and the internal linear transformations of a representation in its construction. When R is the rank matrix of a poset, the virtual barcode of a representation is given by $R^{-1} \cdot \chi(M)$. For any finite poset, we have the following theorem (see [9]).

Theorem 1.1.1. *Let P be a finite, connected poset, let Π be the set of convex subsets of P and let R be the rank matrix for P*

1. *If M and N are direct sums of convex modules, then $\chi(M) = \chi(N) \iff M \cong N$.*

2. *If M is a direct sum of convex modules, then $R^{-1} \cdot \chi(M)$ is the barcode of M*

This theorem says that rank characters are invariants for the representations of P that are rich enough to distinguish between direct sums of convex modules. In this sense, they are analogous to the (ordinary) characters of a group. This also provides a one-sided test which can sometimes be used to determine that a representation *isn't* a direct sum of convex modules, since it follows easily from this theorem that if the virtual barcode $R^{-1} \cdot \chi(M)$ is not a vector with nonnegative integer entries, M is necessarily not a direct sum of convex modules. On the other hand, when M is a direct sum of convex modules, the virtual barcode and barcode agree.

One nice property that the poset P may have is that its rank matrix R might have determinant ± 1. When this is the case, the virtual barcode for **every** representation of P is a vector over $\mathbb{Z}$ (as opposed to $\mathbb{Q}$). In this situation, by interpreting the nonzero entries as multiplicities, the virtual barcode gives us a *virtual decomposition*

$$M \sim \bigoplus_{i=1}^{m} S_i - \bigoplus_{j=1}^{n} T_j \tag{1.1}$$

with $\chi(M) = \chi(\oplus S_i) - \chi(\oplus T_j))$. This decomposition is good up to isomorphism class, and agrees with the Krull-Schmidt decomposition when M is a direct sum of convex modules (in which case $n = 0$). The full repercussions of Equation 1.1 are yet to be analyzed, but the identity suggests interpreting M as a *difference* of two representations, each of which is a direct sum of only convex modules. This is noteworthy, since the isomorphism class of M is arbitrary in that indecomposable summands of M are completely unconstrained.

1.2 Statement of the Problem

With an eye on two-parameter persistence, we focus on those posets P that are the discrete analogue of $\mathbb{R} \times \mathbb{R}$, and investigate exactly when it's the case that the rank matrix has determinant ± 1. When this is the case, the virtual barcode of any representation will allow us to interpret it as a difference of two nice modules.

Thus, let P be a poset given by a closed interval in $\mathbb{Z} \times \mathbb{Z}$. We ask the question whether the rank matrix for P has determinant one.

Chapter 2

Preliminaries

2.1 Representations of Posets

We start with some basic definitions.

Definition 2.1.1. *A* ***poset*** $(P, \leq)$ *is a nonempty set* P*, together with a relation* $\leq$ *on* P *such that for all* $p, q, s \in P$*:*

1. $p \leq p$

2. *If* $p \leq q$ *and* $q \leq p$*, then* $p = q$

3. *If* $p \leq q$ *and* $q \leq s$*, then* $p \leq s$

That is, a poset is a nonempty set with a binary operation that is reflexive, antisymmetric and transitive.

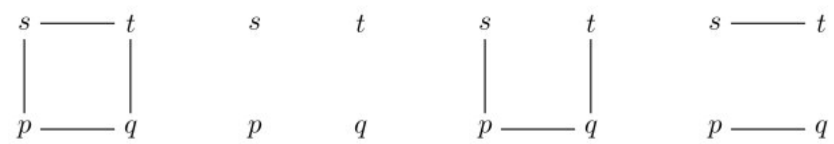
Example 2.1.1. $\mathbb{R}, \mathbb{Z}$ *and* $\mathbb{Q}$ *are posets with* $\leq$*.*

We point out that if P and Q are posets, then the Cartesian product $P \times Q$ is a poset with the relation given by $(p_1, q_1) \leq (p_2, q_2) \iff p_1 \leq p_2$ and $q_1 \leq q_2$. Thus, in particular $\mathbb{Z} \times \mathbb{Z}$ and $\mathbb{R} \times \mathbb{R}$ are posets. It's often useful to represent finite posets pictorally. See for example, Figures 2.1 and 2.2.

When P is a poset with $p \leq q$, we write $[p, q]$ for the set $\{s \in P | p \leq s \leq q\} \subseteq P$. We call $[p, q]$ the **interval** from p to q. For example, in $\mathbb{Z}$, the interval $[1, 5] = \{1, 2, 3, 4, 5\}$.

Example 2.1.2. *In Figure 2.1a we have the interval* $[p, t] = \{p, q, s, t\}$*. In Figure 2.1b we have the interval* $[p, p] = \{p\}$*. In Figure 2.1c we have the intervals* $[p, t] = \{p, q, t\}$*. In Figure 2.1d we have the intervals* $[s, t] = \{s, t\}$*.*

We say the interval $[p, q] \subseteq P$ is **maximal**, if for all other intervals $[p', q']$ in P with $[p, q] \subseteq [p', q']$, it must be the case that $p = p'$ and $q = q'$. For example, in Figure 2.1a $[p, t]$ is a maximal interval. In Figure 2.1c we have the maximal intervals $[p, s]$ and $[p, t]$. In Figure 2.1d we have the maximal intervals $[p, q]$ and $[s, t]$.



(a) (b) (c) (d)

Figure 2.1: Four different examples of posets on the set $\{p, q, s, t\}$. Here we use the convention that a point x is less than or equal to y when there is a north/east oriented path from x to y. For example, in (d) the nontrivial relations are $p \leq q$ and $s \leq t$.

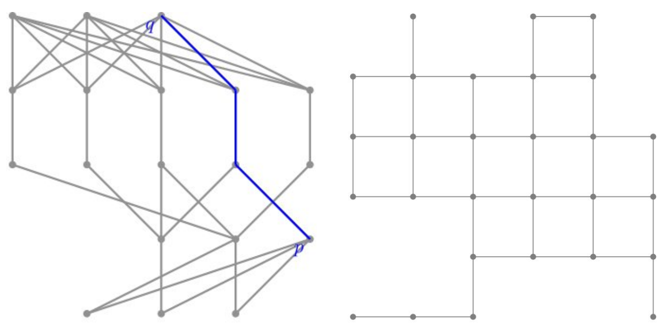
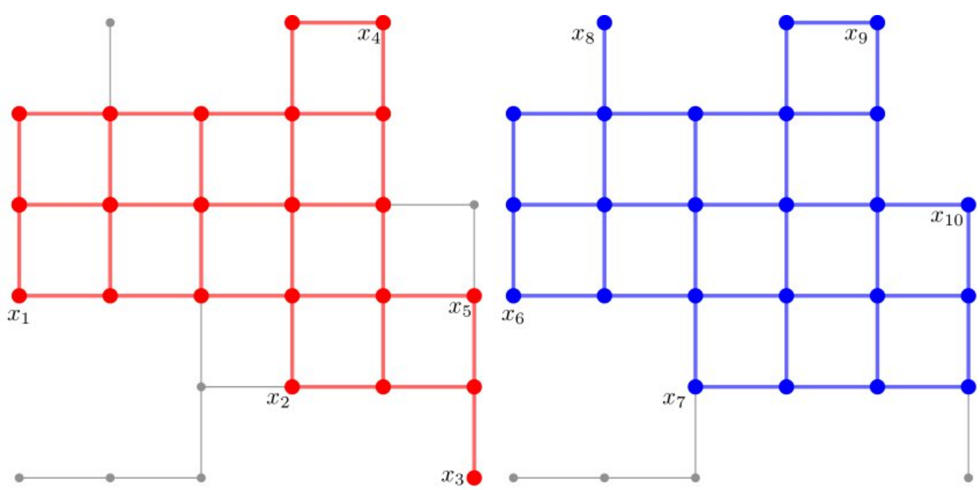


Figure 2.2: The points in gray on the left are the poset P_1. Here, $p \leq q$ if $p = q$ or there is an upward oriented path from p to q. Two related points are highlighted for emphasis. On the right is the commutative grid P_2. Here we use the convention that a point x is less than or equal to y when there is a north or east oriented path from x to y.



(a) The module S_1 (b) The module S_2

Figure 2.3: Both are on the poset P_2 in Figure 2.2 and are convex.

We're now ready to define a representation of a poset.

Definition 2.1.2. *Let P be a poset. A* ***representation****, M, of P is given by:*

1. *an assignment of complex vector spaces $M(p)$ for each $p \in P$, and*
2. *an assignment of $\mathbb{C}$-linear transformations $M(p \leq q) : M(p) \to M(q)$ for each $p, q \in P$ with $p \leq q$*

with the additional property that $M(p \leq s) = M(q \leq s) \circ M(p \leq q)$ for every $p, q, s \in P$ with $p \leq q \leq s$.

That is, a representation is an assignment of vector spaces and linear transformations that respects transitivity of $\leq$.

Example 2.1.3. *Let M be defined by*

$$M(x) = \begin{cases} \mathbb{C} \text{ for } x \in [0,1) \\ 0 \text{ for } x \notin [0,1) \end{cases} \qquad \text{and } M(x \leq y) = (1) \text{ for } x, y \in [0,1), x \leq y$$

Then M is a representation of the poset $[0, \infty)$.

In fact, any interval in the poset $\mathbb{R}$ gives rise to a representation in a similar way.

We are particularly interested in representations similar to the one above. These are the convex modules.

Definition 2.1.3. *Let $P, \leq$ be a poset. A nonempty subset $S \subseteq P$ is* **convex** *if it's edge-connected, and if $x, y \in S$ with $x \leq y$, then $[x, y] \subseteq S$.*

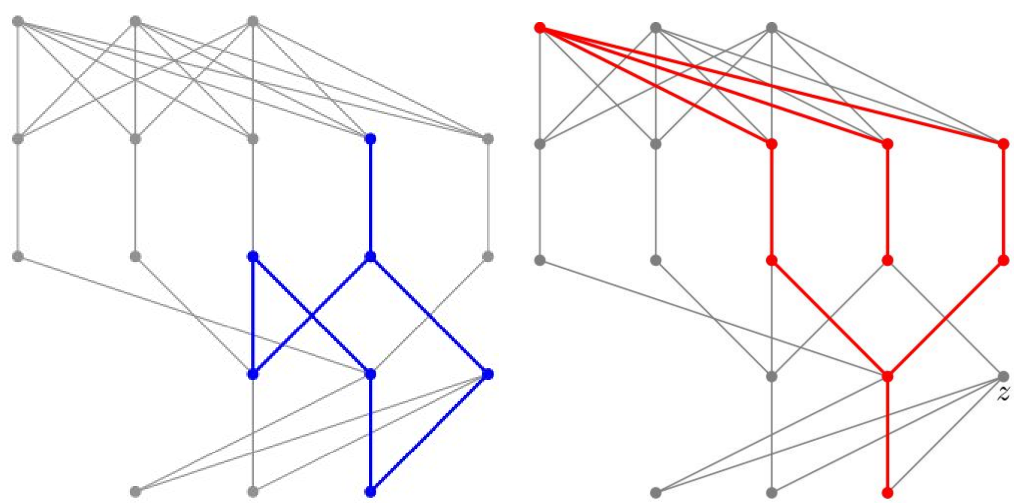


Figure 2.4: The set S_4 is comprised of the points in blue on the poset P_1 in Figure 2.2 and is convex, but the red collection is not convex because it needs the point z.

Lemma 2.1.1. *Let S be a convex subset of P. Let M_S be given by*

$$M_S(p) := \begin{cases} \mathbb{C} & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases} \qquad \text{and} \qquad M_S(x \leq y) := \begin{cases} Id & \text{if } \{x,y\} \subseteq S \\ 0 & \text{if } \{x,y\} \not\subseteq S. \end{cases}$$

Then, M_S is a well-defined representation of P.

Proof. Since M_S assigns a complex vector space to every element of P, we need only show that the linear transformations respect transitivity in P. Let $x \leq y \leq z$ in P. First, say $M_S(x \leq z) = Id$. Then, it must be the case that $x, z \in S$. Since $x \leq y \leq z$, $y \in S$ since S is convex. Then, $M_S(y \leq z) = Id$ and $M_S(x \leq y) = Id$, so

$$M_S(y \leq z) \circ M_S(x \leq y) = Id \circ Id = Id = M_S(x \leq z)$$

as required. On the other hand, if $M_S(x \leq z) = 0$, then $x \notin S$ or $y \notin S$. In either case, $M_S(x \leq y) = 0$ or $M_S(y \leq z) = 0$. Thus, their composition is zero, so $M_S(x \leq z) = M_S(y \leq z) \circ M_S(x \leq y)$. Thus, M_S is a well-defined representation of P. □

When S is convex, we say the module M_S is the *convex module* corresponding to the set S. Thus, convex modules can be identified with their support, which is necessarily a convex set. Because of this, in what follows we will often not distinguish between convex modules and their support. Since convex subsets contain the interval given by any two of its comparable elements, every convex subset S is the union of those intervals in P maximal with respect to being in S. Thus we have the following definitions:

Definition 2.1.4. *Let M be a convex module. The* ***minimals*** *in M is the set $min(M) = \{x \in M | \forall y \in M, \text{ if } y \leq x \text{ then } y = x\}$. Similarly, we define the* ***maximals*** *in M as $max(M) = \{y \in M | \forall x \in M, \text{ if } y \leq x \text{ then } x = y\}$.*

It's clear from the definition that minimals are necessarily not related to each other. Likewise for maximals. When M is convex, all of the maximal intervals will be of the form $[x, y]$, where $x \in \min(M), y \in \max(M)$ and $x \leq y$.

From the sets S_1 and S_2 from Figure 2.3 we find: For S_1, we have $\min(S_1) = \{x_1, x_2, x_3\}$ and $\max(S_1) = \{x_4, x_5\}$. For S_2, we have $\min(S_2) = \{x_6, x_7\}$ and $\max(S_2) = \{x_8, x_9, x_{10}\}$.

2.2 Rank Matrix

Now that we understand convex modules, we introduce the other central idea of the thesis, the rank matrix. Since the entries of this matrix will be indexed by convex modules, we will now frequently use lower case letters for convex modules.

Definition 2.2.1. *Let P be a poset, Π be the set of all convex modules in P. For modules $i, j \in \Pi$ then the matrix $\mathbf{M_{j,i}}$ is the $|max(i)|$ by $|min(i)|$ matrix with integer coefficients given by the following. If $min(i) = \{x_1, ..., x_\ell\}$ and $max(i) = \{y_1, ..., y_k\}$, the x, y entry of $M_{j,i}$ is given by*

$$(M_{j,i})_{y,x} = \begin{cases} 1 & \text{if } x \leq y \text{ and } [x,y] \in j \\ 0 & \text{otherwise.} \end{cases}$$

For $j, i \in \Pi$, when we consider the matrix $M_{j,i}$, we refer to j as the comparison module, and i as the input module.

Now this is a complicated definition, so we will unpack with some examples. From the sets S_1 and S_2 from Figure 2.3 we can calculate both M_{S_1,S_2} and M_{S_2,S_1}.

To calculate M_{S_1,S_2} we first note that S_2 has two minimals, x_6 and x_7 so our matrix we take the rank of will have two columns. Next note that there are three maximals, x_8, x_9, and x_{10} giving us three rows. Now, $x_6 \leq x_8$, so $[x_6, x_8]$ is an interval but x_8 is not in S_2 so $[x_6, x_8]$ is not in S_1, so we get a zero in this entry of the matrix. Next, $x_6 \leq x_9$, so $[x_6, x_9]$ is an interval and $[x_6, x_9] \subseteq S_1$ so we have a one in this entry of the matrix. Next, $x_6 \leq x_{10}$, so $[x_6, x_{10}]$ is an interval but x_{10} is not in S_2 so $[x_6, x_{10}]$ is not in S_1, so we get a zero in this entry of the matrix. Next, $x_7 \not\leq x_8$, so $[x_7, x_8]$ is not an interval, so we get a zero in this entry of the matrix. Next, $x_7 \leq x_9$, so $[x_7, x_9]$ is an interval but x_7 is not in S_2 so $[x_7, x_9]$ is not in S_1, so we get a zero in this entry of the matrix. Next, $x_7 \leq x_{10}$, so $[x_7, x_{10}]$ is an interval but x_7 is not in S_2 so $[x_7, x_{10}]$ is not in S_1, so we get a zero in this entry of the matrix. Thus we end by calculating

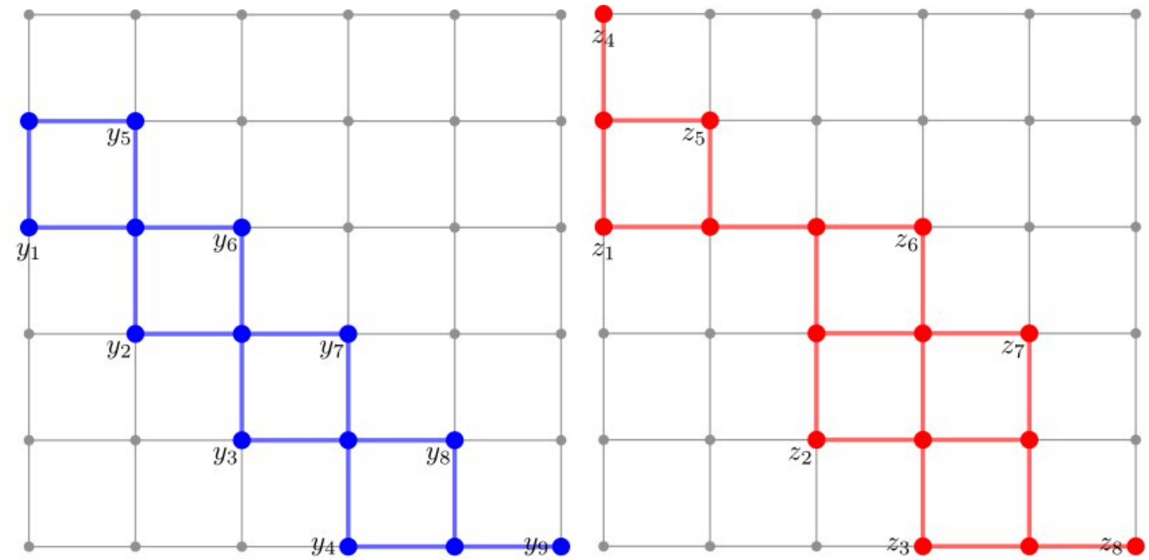
$$M_{S_1,S_2} = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \tag{2.1}$$

To calculate M_{S_2,S_1} we first note that S_2 has three minimals, x_1, x_2, and x_3 so our matrix we take the rank of will have three columns. Next note that there are two maximals, x_4 and x_5 giving us two rows. Now, $x_1 \leq x_4$, so $[x_1, x_4]$ is an interval and $[x_1, x_4] \subseteq S_2$, so we get a one in this entry of the matrix. Next, $x_1 \leq x_5$, so $[x_1, x_5]$ is an interval and $[x_1, x_5] \subseteq S_2$ so we have a one in this entry of the matrix. Next, $x_2 \leq x_4$, so $[x_2, x_4]$ is an interval and $[x_2, x_4] \subseteq S_2$, so we get a one in this entry of the matrix. Next, $x_2 \leq x_5$, so $[x_2, x_5]$ is an interval, so we get $[x_2, x_5] \subseteq S_2$, so we get a one in this entry of the matrix. Next, $x_3 \not\leq x_5$, so $[x_3, x_5]$ is not an interval, so we get a zero in this entry of the matrix. Next, $x_3 \leq x_5$, so $[x_3, x_5]$ is an interval but x_3 is not in S_2 so $[x_3, x_5]$ is not in S_2, so we get a zero in this entry of the matrix. Thus we end by calculating

$$M_{S_2,S_1} = \left(\begin{smallmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{smallmatrix}\right) \tag{2.2}$$

Example 2.2.1. *Using the set S_6 and S_7 from Figure 2.5, we can carefully calculate M_{S_6,S_7} and M_{S_7,S_6}. We can see that $min(S_6) = \{q_1, q_2, q_3, q_4\}$ and $max(S_6) = \{q_5, q_6, q_7, q_8, q_9\}$. We can see that $min(S_7) = \{s_1, s_2, s_3\}$ and $max(S_7) = \{s_4, s_5, s_6, s_7, s_8\}$.*

We can calculate M_{S_6,S_7} as follows. (Note that our table below is actually a matrix put into table form to emphasize the minimals and maximals)



(a) The module S_6 (b) The module S_7

Figure 2.5: Both modules are on the poset of the 6x6 grid of points. Here we use the convention that a point x is less than or equal to y when there is a north or east oriented path from x to y.

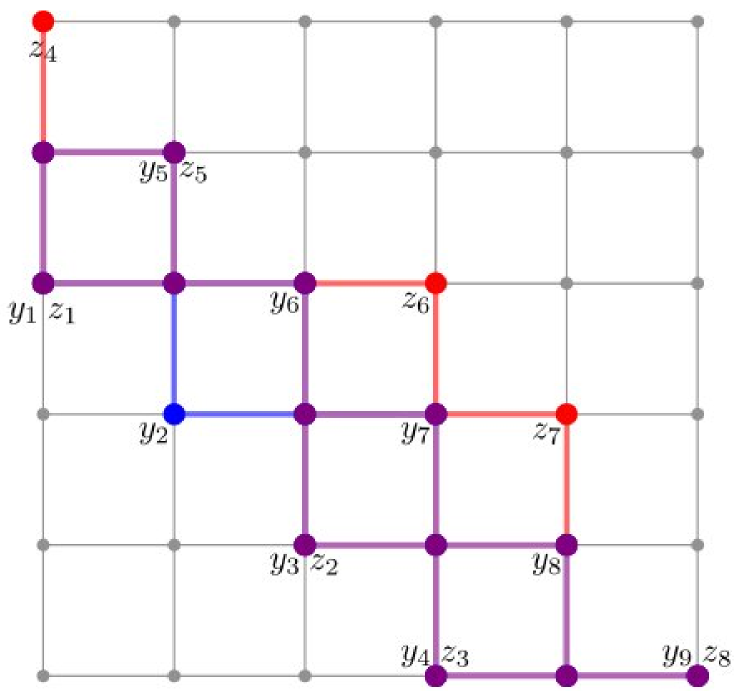


Figure 2.6: The module in Subfigure 2.5a and Subfigure 2.5b overlapped, with the intersection in purple.

$(M_{S_6,S_7})_{y,x}$	z_1	z_2	z_3
z_4	*0*	*0*	*0*
z_5	*1*	*0*	*0*
z_6	*0*	*0*	*0*
z_7	*0*	*0*	*0*
z_8	*0*	*0*	*1*

So we get $M_{S_6,S_7} = \begin{pmatrix} 0&0&0\\ 1&0&0\\ 0&0&0\\ 0&0&0\\ 0&0&1 \end{pmatrix}$

We can calculate M_{S_7,S_6} *as follows:*

$(M_{S_7,S_6})_{y,x}$	y_1	y_2	y_3	y_4
y_5	*1*	*0*	*0*	*0*
y_6	*1*	*0*	*1*	*0*
y_7	*0*	*0*	*1*	*1*
y_8	*0*	*0*	*1*	*1*
y_9	*0*	*0*	*0*	*1*

So we get $M_{S_7,S_6} = \begin{pmatrix} 1&0&0&0\\ 1&0&1&0\\ 0&0&1&1\\ 0&0&1&1\\ 0&0&0&1 \end{pmatrix}$

Definition 2.2.2. *Let P be a poset, Π be the set of all convex modules in P. Define the function*

$$o : \Pi \times \Pi \to \mathbb{Z}$$

by $o(j,i) = rank(M_{j,i})$. Then we define the ***rank matrix*** *as the $|\Pi| \times |\Pi|$ matrix with the jth row of the rank matrix as the vector $(rank(M_{j,i}))_{i\in\Pi}$.*

Now we revisit our previous examples, this time computing the entries in the rank matrix. We see that in Equation 2.1, $o(S_1, S_2) = rank\left(\begin{pmatrix} 0&0\\ 1&0\\ 0&0 \end{pmatrix}\right) = 1$ and in Equation 2.2, $o(S_2, S_1) = rank\left(\begin{pmatrix} 1&1&0\\ 1&1&0 \end{pmatrix}\right) = 1$. In Example 2.2.1, we can calculate $o(S_6, S_7) = rank\left(\begin{pmatrix} 0&0&0\\ 1&0&0\\ 0&0&0\\ 0&0&0\\ 0&0&1 \end{pmatrix}\right) = 2$ and $o(S_7, S_6) = rank\left(\begin{pmatrix} 1&0&0&0\\ 1&0&1&0\\ 0&0&1&1\\ 0&0&1&1\\ 0&0&0&1 \end{pmatrix}\right) = 3$.

Example 2.2.2. *For the poset in Figure 2.1a, we see the entire rank matrix calculated below. Again, rows are comparison modules (with their associated module at the top, described by their maximal intervals) and columns are the input module (with their assosiated modules in the same order as the rows).*

$$\begin{array}{r} [p,p] \\ [q,q] \\ [s,s] \\ [t,t] \\ [p,q] \\ [s,t] \\ [p,s] \\ [q,t] \\ [p,t] \\ [p,q]\cup[p,s] \\ [q,t]\cup[s,t] \end{array} \begin{pmatrix} 1&0&0&0&1&1&0&0&1&0&1\\ 0&1&0&0&1&0&1&0&1&1&1\\ 0&0&1&0&0&1&0&1&1&1&1\\ 0&0&0&1&0&0&1&1&0&1&1\\ 0&0&0&0&1&0&0&0&1&0&1\\ 0&0&0&0&0&1&0&0&1&0&1\\ 0&0&0&0&0&0&1&0&0&1&1\\ 0&0&0&0&0&0&0&1&0&1&1\\ 0&0&0&0&1&1&0&0&1&0&1\\ 0&0&0&0&0&0&1&1&0&1&1\\ 0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix}$$

Chapter 3

Convex Modules for the Commutative Grid

3.1 The Commutative Grid and its Consequences

We now restrict our attention to posets that are given by closed intervals in $\mathbb{Z} \times \mathbb{Z}$. This class of posets contains the finite totally ordered posets $\mathbb{A}_n$. Moreover, they have special interest in TDA themselves (see Section 1.1), since they are to two-dimensional persistence modules exactly what equioriented $\mathbb{A}_n$ is to (one-dimensional) persistence modules. Thus, we now define the posets whose rank matrix we will study.

Definition 3.1.1. *Let P be the interval $[(1,1),(g,h)] \subseteq \mathbb{Z} \times \mathbb{Z}$. That is, $P = \{s_{p,q} := (p,q) \in \mathbb{Z} \times \mathbb{Z} | 1 \leq p \leq g, 1 \leq q \leq h\}$ where $s_{p,q} \leq s_{p',q'}$, if and only if $p \leq p'$ and $q \leq q'$. We call P the* $\mathbf{g} \times \mathbf{h}$ ***commutative grid****.*

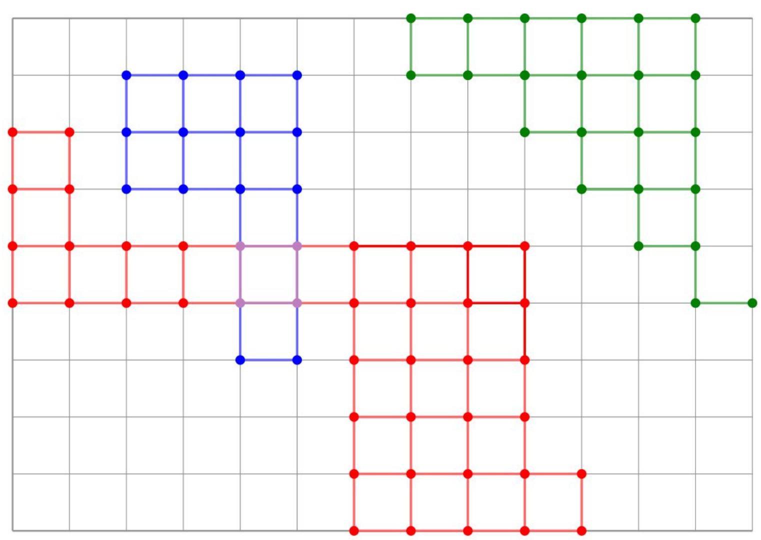


Figure 3.1: The 15×11 commutative grid, with three examples of convex modules. There is a green module, a red module, and a blue module with the intersection of the red and blue modules in purple.

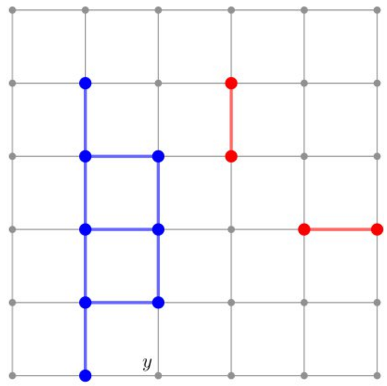


Figure 3.2: The red set is not edge connected so it is not a convex module. The point $y = s_{3,1}$ is greater than a point in the blue module and less than another, so it should be in the module. But it's not so the blue set is not a convex module.

Now that we're working in the commutative grid, we have a new property of convex modules that will be useful:

Definition 3.1.2. *Let m be a convex module for P. Then, the* ***bounding box*** *of m is the smallest closed interval $k \subseteq P$ with $m \subseteq k$. When k is the bounding box of m we write $b_m = k$.*

Example 3.1.1. *In Figure 3.1, the red convex module's bounding box is $[s_{1,1}, s_{11,9}]$. The blue convex module's bounding box is $[s_{3,4}, s_{6,9}]$. The green convex module's bounding box is $[s_{7,2}, s_{13,7}]$.*

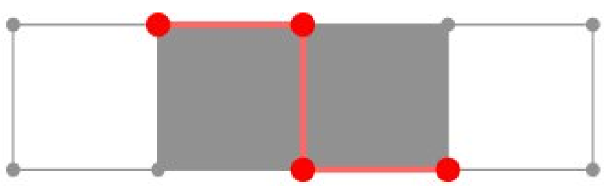


Figure 3.3: In the 6x2 commutative grid, we see the convex module in red and its bounding box shaded in grey.

Note that the $g \times h$ commutative grid and the $h \times g$ commutative grid are isomorphic as posets, since the bijection $f : g \times h \to h \times g$ given by

$$f(s_{p,q}) = s_{q,p}$$

is order preserving. Since isomorphic posets have the same representation theory, we need only consider $g \times h$, where $g \leq h$. Thus, we begin with $g = 1$, that is, finite totally ordered sets.

Next, we will deal with a simple case for the commutative grid:

Proposition 3.1.1. *If P is the $1 \times h$ commutative grid, then the determinant of the rank matrix is one.*

Proof. First, since $g = 1$, every convex module contains just one maximal interval. Consider an ordering on Π so that all modules with the same vertical length are grouped together, with the shortest group first. Then, if $i > j$, one of two things happen. Either i is longer than j, in which case $i \not\subseteq j$ so $\text{rank}(M_{j,i}) = \text{rank}((0)) = 0$. Otherwise, i and j have the same vertical length, but are not equal. Thus, again we have that $i \not\subseteq j$, thus $\text{rank}(M_{j,i}) = \text{rank}((0)) = 0$. So the matrix will be lower triangular. And if $j = i$, then $\text{rank}(M_{j,i}) = \text{rank}((1)) = 1$ so the matrix is lower triangular with 1's on the diagonal, thus the matrix has determinant 1. $\square$

Since the $1 \times n$ case is trivial, in this thesis we focus on $2 \times h$ commutative grid, $h \geq 1$, though the arbitrary $g \times h$ case can be similarly studied. Some code I wrote seems to suggest that when $g < 4$ or $h < 4$, the determinant of the rank matrix was always 1. However, whenever $g \geq 4$ and $h \geq 4$, the determinant of the rank matrix was not always in $\{\pm 1\}$.

3.2 Narrowing the Scope

In all that follows, unless explicitly state otherwise we always consider $P = 2 \times h$ for some $h \geq 1$.

Definition 3.2.1. *Let m be a convex module with corresponding convex set $\{m_1 \dots m_k\}$. Recall, that each $m_\ell = s_{p_\ell, q_\ell}$. If every m_ℓ has p coordinate equal to 1 or 2, we say the module is* ***one wide*** *(and on the left). Otherwise, we say the module is* ***two wide****.*

That is, a module is on the left if all the p values equal 1, and on the right if all the p values equal 2. It's easy to see that other widths are not possible for $P = 2 \times h$. Moreover, because every module will be at most three maximal intervals. Thus we make the following definitions.

Definition 3.2.2. *Let m be a convex module. Let $\mathbf{m_1}$ denote the interval (if it exists) where each p coordinate is 1. Let $\mathbf{m_2}$ be the maximal interval (if it exists) that is two wide, so the interval whose element's p values have both 1's and 2's. Similarly, let $\mathbf{m_3}$ denote the interval (if it exists) where each p coordinate is 2.*

Notice m_1 is on the left and m_3 is on the right.

Example 3.2.1. *In Figure 3.4a, letting it be m, the figure has one interval, $m_1 = [s_{1,3}, s_{1,6}]$. In Figure 3.4b, letting it be m', the figure has three intervals, $m'_1 = [s_{1,2}, s_{1,6}], m'_2 = [s_{1,2}, s_{2,4}]$, and $m'_3 = [s_{2,1}, s_{2,4}]$. In Figure 3.4c, letting it be m'', the figure has one interval, $m''_3 = [s_{2,5}, s_{2,5}]$.*

Now that we have names for the intervals themselves, we can calculate $o(j, i)$ more efficiently through the following method:

Proposition 3.2.1. *When calculating $o(j, i)$, if no maximal intervals of i are in j, then $o(j, i) = 0$. If $i_1 \subseteq j$ and $i_3 \subseteq j$, then $o(j, i) = 2$. Else, $o(j, i) = 1$.*

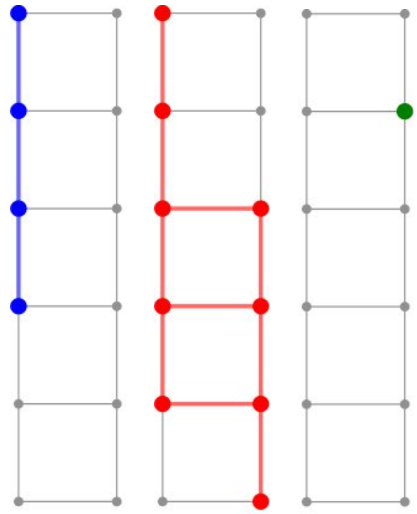


Figure 3.4: In the 2×6 commutative grid, Subfigure 3.4a is one wide (on the left), Subfigure 3.4b is two wide and Subfigure 3.4c is one wide (and on the right).

3.3 Parametrizing Convex Modules for the $2 \times$ h Grid (The Ordering)

The reader may notice that up until now, the ordering on Π was not specified. We now choose a convenient ordering on the convex modules for the commutative grid.

Definition 3.3.1. *Let m be a convex module. We define the five characteristics of m we will use to order Π. For $m \in \Pi$, let $s_{1,q}$ be the element with the largest q value in m (with $p = 1$), and similarly for $s_{2,q'}$. Let $s_{1,q'''}$ be the element with the lowest q value in m (with $p = 1$), and similarly for $s_{2,q'''}$*

1. $\mathbf{a_m} = q - q''' + 1$.
2. $\mathbf{w_m}$ *is the width of the bounding box (one wide corresponds to $w_m = 1$ and two wide corresponds with $w_m = 2$).*
3. $\mathbf{c_m} = q'''$.

Then, if m is one wide let:

1. $\mathbf{e_m}$ *equal 1 if the module is on the left and 2 if the module is on the right.*
2. $\mathbf{x_m} = 1$.

If m is two wide let:

1. $\mathbf{e_m} = q'' - q''' + 1$.

2. $\mathbf{x_m} = q - q' + 1$.

The way to visualize the different letters is as follows. The parameter a_m represents the range that the convex module spans vertically and w_m represents the range horizontally. The parameter c_m represents how high up in the grid the bottom of the convex module is, whereas e_m represents the side of the commutative grid that the convex module is on when $w_m = 1$. When $w_m = 2$, it represents how much higher the lower left point is that the bottom right point, where $e_m = 1$ means their heights are identical. The parameter x_m has no physical interpretation when $w_m = 1$, but it represents how much lower the top right point is than the the top left point when $w_m = 2$. Again $x_m = 1$ means their heights are identical. Crucially, $e_m > 1$ iff m_3 exists and $x_m > 1$ iff m_1 exists.

In Subfigure 3.4a, the blue module y has the following values: $a_y = 4, w_y = 1, c_y = 3, e_y = 1, x_y = 1$. In Subfigure 3.4b, the red module y' has the following values: $a_{y'} = 6, w_{y'} = 2, c_{y'} = 1, e_{y'} = 2, x_{y'} = 3$. In Figure 3.4c, the green module y'' has the following values: $a_{y''} = 1, w_{y''} = 1, c_{y''} = 5, e_{y''} = 2, x_{y''} = 1$.

Note that the function from $\Pi \to \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ given by

$$m \to (a_m, w_m, c_m, e_m, x_m)$$

is injective, thus each convex module is uniquely described by the tuple $(a_m, w_m, c_m, e_m, x_m)$. Thus, these quantities parameterize Π, and will give us our ordering.

Definition 3.3.2. *To fix the rank matrix, we order* Π *with the lexicographical ordering of* (a, w, c, e, x).

For the commutative grid 2×2, we can see the entire ordering of Π in Example 2.2.2. While there is a lot to digest here, there are three main properties of this ordering that we care about.

Proposition 3.3.1. *Let* $m, n \in \Pi$.

1. *if* $2a_{b_m} + w_{b_m} < 2a_{b_n} + w_{b_n}$ *then* $m < n$
2. *if* $m \leq n$ *then* $2a_{b_m} + w_{b_m} \leq 2a_{b_n} + w_{b_n}$
3. *if* $b_m = b_n$, $w_m = w_n = 2$, *and* $m < n$, *then* $m_2 \not\subseteq n$

Note 1. and 2. are really just the statement that we care about the lexicographical ordering of specifically a and w.

Proof. The first two properties are just resulting from the definition of lexicographical ordering, noting that w_m is always 1 or 2.

For the third property we first introduce some simple machinery. Let $s_{2,q}$ be the point in m with the highest q value of points in m and $s_{2,q'}$ be the point in n with the highest q value of points in n. Now suppose we had $b_m = b_n$, $w_m = w_n = 2$, and $m < n$. Because $b_m = b_n$, $a_m = a_n$, $w_m = w_n$, and $c_m = c_n$ so either $e_m = e_n$ and $x_m < x_n$, or $e_m < e_n$. Assume that the first case was true. Then, focusing just on the fact that $x_m < x_n$, then the interval m_2 contains the point s_{2_q}, but by the assumption, $q > q'$, so s_{2_q} is not in n, so $m_2 \subseteq n$. The second case is true by a similar argument.

□

While not entirely necessary for the main proof, something useful that I thought about while doing the project was the following.

Proposition 3.3.2. *For a convex module $m \in \Pi$, $1 \leq c_m$ and $a_m + c_m - 1 \leq h$. If m is one wide, then $1 \leq a_m$. If m is two wide then $a_m \geq e_m + x_m - 1$, $e_m \geq 1$, and $x_m \geq 1$.*

The proof is simple, and left as an exercise for the reader.

Corollary 3.3.2.1. *A map from the set of one wide modules in the $2 \times h$ commutative grid to $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, with an element of $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ denoted as (a, w, c, e, x), the map will be bijective if we restrict $1 \leq c$, $w = 1$, $1 \leq a$, and $a + c - 1 \leq h$.*

A map from the set of two wide modules in the $2 \times h$ commutative grid to $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, with an element of $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ denoted as (a, w, c, e, x), the map will be bijective if we restrict $1 \leq c$, $w = 2$, $a + c - 1 \leq h$, $e_x - 1 \leq a$, $1 \leq e$ and $1 \leq x$.

At the url [https://github.com/Selenotropism/Thesis-Files] both the rank matrix and ordering for 2×5 and 2×9 are posted, since the matrices are too large to include. The convex module orderings are in the associated convex module storage file. Note that a convex module will have 0 on points in the poset that are not in the module, and X on points in the poset, with lines between points. For the rank matrix, the numbering is found in the associated rank matrix file. Red lines separate a change in a or w. Green lines separate a change in c. Yellow lines separate a change in e. No lines separate a change in x, however, the lack of line can be treated as such. The reader could look at the convex module ordering for 2×5 for a good intuitive description of what is happening.

Chapter 4

Proof of Main Result

In this chapter we prove our main results. Namely, we prove the following theorem.

Theorem 4.0.1. *Main result*
The determinant of the rank matrix for $P = 2 \times h$ is one.

Before proving our main result, we discuss the strategy for the proof. Our strategy is to use elementary row operations to turn the rank matrix into a different matrix whose entries we label $r(j,i)$. That is, our row operations send the entry $o(j,i)$ to $r(j,i)$, where the resulting matrix is upper triangular with plus or minus one on the main diagonal. Then, we will show that the number of minus ones is even, so the result follows by the nature of the used row operations.

4.1 The Algorithm

We now describe the row operations we will use to make the rank matrix upper diagonal. We begin with the rank matrix for a $2 \times h$ commutative grid, with the ordering described in Definition 3.3.2. Then we will generate the resulting matrix $(r(j,i))_{j,i\in\Pi}$ through the following row operations.

Let $j \in \Pi$, and let $o(j,i)$ be the value in the rank matrix. Then, we define $r(j,i)$, the entry in the resulting matrix recursively as follows. Let K be the set of intervals for the commutative grid. Let $j \in \Pi$ and let $U_j = \{j \cap k | k \in K, \emptyset \neq j \cap k \neq j\}$. Then,

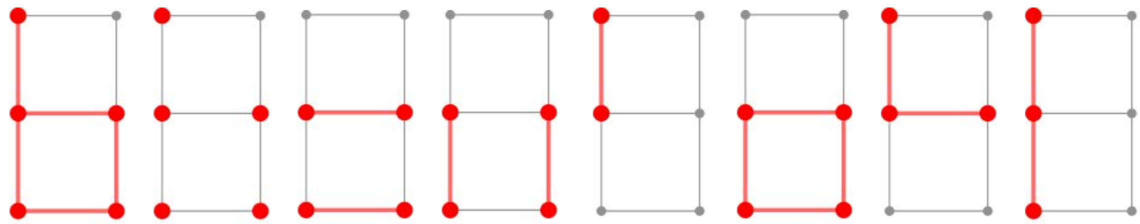
$$r(j,i) = o(j,i) - \sum_{u \in U_j} r(u,i) \tag{4.1}$$

That is, for $\Pi = \{j_1, ... j_{|\Pi|}\}$ (in our ordering),

$$r_{j_1} = o_{j_1} - \sum_{u \in U_{j_1}} r_u$$

$$r_{j_2} = o_{j_2} - \sum_{u \in U_{j_2}} r_u$$

$$\dots$$



(a) (b) (c) (d) (e) (f) (g) (h)

Figure 4.1: If Subfigure 4.1a is our row j, then our algorithm tells us to subtract the 13 rows associated with the modules in Subfigures 4.1b through 4.1h. Note that in Subfigure 4.1b there are 5 modules represented in this image. Similarly in Subfigure 4.1c and Subfigure 4.1d, each image has two modules in them.

$$r_{j_{|\Pi|}} = o_{j_{|\Pi|}} - \sum_{u \in U_{j_{|\Pi|}}} r_u$$

For example, in Subfigure 4.1b, these convex modules are rows subtracted from the row in Subfigure 4.1a. The intervals $k_1, ...k_5$ that are intersected with Subfigure 4.1a are $[s_{1,1}s_{1,1}]$,...,$[s_{1,3}s_{1,3}]$. In Subfigure 4.1c, these convex modules are rows subtracted from the row in Subfigure 4.1a. The intervals k_1, k_2 that are intersected with Subfigure 4.1a are $[s_{1,1}s_{2,1}]$ and $[s_{1,2}s_{2,2}]$. In Subfigure 4.1d, we find these by intervals $[s_{1,1}s_{1,2}]$ and $[s_{2,1}s_{2,2}]$. In Subfigure 4.1e, this module is the intersection of the module in Subfigure 4.1a with the interval $k =[s_{1,2}, s_{1,3}]$. In Subfigure 4.1f, this module is the intersection of the module in Subfigure 4.1a with the interval $k =[s_{1,1}, s_{2,2}]$. In Subfigure 4.1g, this module is the intersection of the module in Subfigure 4.1a with the interval $k =[s_{1,2}, s_{2,3}]$. In Subfigure 4.1h, this module is the intersection of the module in Subfigure 4.1a with the interval $k =[s_{1,1}, s_{1,3}]$.

Note in Figure 4.1 some rows have multiple different intervals in k intersected with j that equals that row, but we still only subtract those rows once.

4.2 The entry r(j,i)

In this section we will prove that the determinant of the rank matrix can be read off of the resulting matrix $(r(j,i))_{i,j\in\Pi}$, and that it has a nice formula.

Proposition 4.2.1. *Let $P = 2 \times h$ and $(o(j,i))_{j,i\in\Pi}$ be the rank matrix for P in our ordering. Then, after performing the row operations described in Equation 4.1, we obtain, $(r(j,i))_{j,i\in\Pi}$, where $r(j,i)$ is given below.*

Let I be the set of maximal intervals in i.

$$r(j,i) = \begin{cases} (-1)^{|I'|+1} & \textit{if } \exists I' \in 2^I \textit{ such that } u := \bigcup_{x\in I'} x \in \Pi \textit{ and } j \supseteq u, b_j = b_u \\ 0 & \textit{otherwise.} \end{cases}$$

We'll prove Proposition 4.2.1 by using strong induction on our ordering on Π. Specifically, after dealing with the base case, we assume that for each $j' < j$ in Π, $o_{j'}$ has been replaced by $r_{j'}$. We then need only show that the row operations in the algorithm transform o_j to r_j.

We start with the base case, or showing that the algorithm converts o_{j_1} to r_{j_1}. Note that the row 1 is the single point at the bottom left of the poset, and thus we cannot have any $k \in K$ with $k \cap j_1 \neq j_1$. Thus, nothing will be subtracted from row 1 in the rank matrix, so we expect row 1 to be the same in both matrices. Note that in both the rank matrix and the resulting matrix, by the inductive hypothesis, only when $i = j = 1$, will $r(j,i) = 1 = o(j,i)$, as expected. In all other cases we can see that $r(j,i) = 0 = o(j,i)$, also as expected.

Before we can tackle the general case, since lots of rows are subtracted, we determine which row operations matter for a fixed column i. That is, which operations can potentially affect i. Said differently, which of the used row operations have nonzero entries in i. (Note the following Lemma and Corollary are assuming the inductive hypothesis).

Lemma 4.2.1. *Restriction Lemma*
Suppose we focus on some column i. Only the following rows u, will have the value $r(u,i) \neq 0$.

1. $u = i_1$ *(with $r(u,i) = 1$)*
2. $u = i_2$ *(with $r(u,i) = 1$)*
3. $u = i_3$ *(with $r(u,i) = 1$)*
4. $i_1 \cup i_2 \subseteq u \in \Pi$ *and* $b_u = b_{i_1 \cup i_2}$ *(with $r(u,i) = -1$)*
5. $i_2 \cup i_3 \subseteq u \in \Pi$ *and* $b_u = b_{i_2 \cup i_3}$ *(with $r(u,i) = -1$)*
6. $i_1 \cup i_2 \cup i_3 \subseteq u \in \Pi$ *and* $b_u = b_{i_1 \cup i_2 \cup i_3}$ *(with $r(u,i) = 1$)*

Proof. Suppose we had a module $i, u \in \Pi$. If u was not in any of the forms above, then by the induction hypothesis $r(u,i) = 0$. $\square$

The restriction lemma tells us exactly which row operations matter, when the input module i is fixed. This lemma will prove useful in the proof of our main result. However, we now use the lemma to modify our algorithm into a more useful form.

Corollary 4.2.1.1. *The Modified Algorithm*
Let's fix some $i \in \Pi$, while the algorithm is transforming j. For rows $j' < j$, then those rows will be of the form $r_{j'}$, whereas all other rows j'' are of the form $o_{j''}$. The entry will be $o(j,i)$, which the algorithm is going to change $r(j,i)$. It should be noted that just because a row u needs to be of one of the six forms in the Restriction Lemma to have $r(u,i) \neq 0$, it doesn't mean a row is of one of those forms necessarily is subtracted of from row j. Again, for some interval in the commutative grid k, we need

$u = k \cap j$. *Now we can modify our algorithm to be as follows by defining new variables:*

$$A_1(j,i) = \begin{cases} 1 & \textit{if } \exists k \in K \textit{ with } i_1 = k \cap j \neq j \\ 0 & \textit{otherwise.} \end{cases}$$

$$A_2(j,i) = \begin{cases} 1 & \textit{if } \exists k \in K \textit{ with } i_2 = k \cap j \neq j \\ 0 & \textit{otherwise.} \end{cases}$$

$$A_3(j,i) = \begin{cases} 1 & \textit{if } \exists k \in K \textit{with } i_3 = k \cap j \neq j \\ 0 & \textit{otherwise.} \end{cases}$$

$$A_{1,2}(j,i) = \begin{cases} -1 & \textit{if } \exists k \in K \textit{ with } i_1 \cup i_2 \subseteq k \cap j \neq j \textit{ and } b_{i_1 \cup i_2} = b_{j \cap k} \\ 0 & \textit{otherwise.} \end{cases}$$

$$A_{2,3}(j,i) = \begin{cases} -1 & \textit{if } \exists k \in K \textit{ with } i_2 \cup i_3 \subseteq k \cap j \neq j \textit{ and } b_{i_2 \cup i_3} = b_{j \cap k} \\ 0 & \textit{otherwise.} \end{cases}$$

$$A_{1,2,3}(j,i) = \begin{cases} 1 & \textit{if } \exists k \in K \textit{ with } i_1 \cup i_2 \cup i_3 \subseteq k \cap j \neq j \textit{ and } b_{i_1 \cup i_2 \cup i_3} = b_{j \cap k} \\ 0 & \textit{otherwise.} \end{cases}$$

Now we have the modified algorithm which will be equivalent to the original algorithm, just far easier to use:

$$r(j,i) = o(j,i) - A_1(j,i) - A_2(j,i) - A_3(j,i) - A_{1,2}(j,i) - A_{2,3}(j,i) - A_{1,2,3}(j,i)$$

Now we have the machinery to tackle the inductive step. The following is the proof of the inductive step.

First a reminder on how the inductive step works. In this step we will assume that the algorithm has gotten to row j. This means that for all rows j', $1 \leq j' < j$ then the algorithm will have already modified those rows. So, by the inductive hypothesis, we can assume for any row j' with $j' < j$, that the entry in row j' and column i will have the value $r(j',i)$, and for any row $j'' \geq j$, the entry in row j'' and column i will have the value $o(j'',i)$, as they have not yet been modified. Now we will show that for our row j, the algorithm will convert the entry in column i from $o(j,i)$ to $r(j,i)$. Since casework is unavoidable, we'll list all the cases first. The first four of the cases will relate to the lower condition in Prop 4.2.1, and the last three will relate to the upper condition. Additionally these cases are pairwise mutually exclusive and exhaust all possibilities of the relationship between i and j. Notice, for example, that we can't have j containing maximal intervals with it's bounding box smaller than the bounding box of those maximal intervals. This would contradict the definition of a bounding box.

1. j contains no maximal intervals of i, we should find $r(j,i) = 0$
2. j contains exactly one maximal interval of i but j's bounding box is larger than the maximal interval's bounding box, we should find $r(j,i) = 0$
3. j contains exactly two maximal intervals of i but j's bounding box is larger the maximal intervals' bounding box, we should find $r(j,i) = 0$
4. j contains all three maximal intervals of i but j's bounding box is larger than the maximal intervals' bounding box, we should find $r(j,i) = 0$
5. j contains one maximal interval of i and j's bounding box is equal to the maximal interval's bounding box, we should find $r(j,i) = 1$
6. j contains exactly two maximal intervals of i and j's bounding box is equal to the maximal intervals' bounding box, we should find $r(j,i) = -1$
7. j contains all three maximal intervals of i and j's bounding box is equal to the the maximal intervals' bounding box, we should find $r(j,i) = 1$

Suppose j contains no maximal intervals of i. Thus $o(j,i) = 0$. However because no maximal interval in i is a subset of j, $A_1(j,i) = A_2(j,i) = A_3(j,i) = A_{1,2}(j,i) = A_{2,3}(j,i) = A_{1,2,3}(j,i) = 0$ so $r(j,i) = o(j,i) - A_1(j,i) - A_2(j,i) - A_3(j,i) - A_{1,2}(j,i) - A_{2,3}(j,i) - A_{1,2,3}(j,i) = 0 - 0 - 0 - 0 - 0 - 0 - 0 = 0$, as expected.

Suppose j contains exactly one maximal interval of i. Thus $o(j,i) = 1$. Letting the one maximal interval contained in j be i_x, note for any i_y with $y \neq x$, $i_y \not\subseteq j$ so any subset of j will also not contain i_y. Thus $A_y(j,i) = 0$, $A_{1,2}(j,i) = 0$, $A_{2,3}(j,i) = 0$ and $A_{1,2,3}(j,i) = 0$. Now to calculate $A_x(j,i)$. Note that there must be an interval $k \in K$ with $i_x = k \cap j$ because $k = i_x$ will work as i_x is in j. Now take any $k \in K$ with $i_x = k \cap j$. However if $b_{i_x} = b_j$ then $k \cap j = i_x = j$, so $A_x(j,i) = 0$. Meaning $A_1(j,i) + A_2(j,i) + A_3(j,i) = 0$, thus $r(j,i) = o(j,i) - A_1(j,i) - A_2(j,i) - A_3(j,i) - A_{1,2}(j,i) - A_{2,3}(j,i) - A_{1,2,3}(j,i) = 1 - 0 - 0 - 0 - 0 = 1$, as expected. However if j's bounding box is bigger than i_x, then $k = i_x$ will have the property that $i_x = k \cap j \neq j$, so $A_x(j,i) = 1$. Meaning $A_1(j,i) + A_2(j,i) + A_3(j,i) = 1$, thus $r(j,i) = o(j,i) - A_1(j,i) - A_2(j,i) - A_3(j,i) - A_{1,2}(j,i) - A_{2,3}(j,i) - A_{1,2,3}(j,i) = 1 - 1 - 0 - 0 - 0 = 0$, as expected.

Suppose j contains exactly two maximal intervals of i. It must be the case that j contains just i_1 and i_2 or just i_2 and i_3, as if j contains i_1 and i_3, then it will contain i_2. Assume for now that j contains just i_1 and i_2, but the argument for when j contains just i_2 and i_3 will be same, save for swapping i_1 with i_3 and vice versa. We see that both i_1 and i_2 must exist and are in j and if i_3 exists, it is not a subset of j. We know by the modified algorithm how to calculate $r(j,i)$. Note because just i_1 and i_2 are in j, that $o(j,i) = 1$. Note that if we take $k = i_1$ that $k \cap j = i_1 \neq j$, so $A_1(j,i) = 1$. Similarly, if we take $k = i_2$, then $k \cap j = i_2 \neq j$, so $A_2(j,i) = 1$. However we know if i_3 exists, that it is not a subset of j, so no subset of j could be equal to or contain i_3, so $A_3(j,i) = 0$, $A_{2,3}(j,i) = 0$, and $A_{1,2,3}(j,i) = 0$. Now if $b_j = b_{i_1 \cup i_2}$, then we can't have a $k \in K$ such that $i_1 \cup i_2 \subseteq k \cap j \neq j$, as both i_1 and i_2 are in j. Thus $A_{1,2}(j,i) = 0$.

So thus we see $r(j,i) = o(j,i) - A_1(j,i) - A_2(j,i) - A_3(j,i) - A_{1,2}(j,i) - A_{2,3}(j,i) - A_{1,2,3}(j,i) = 1-1-1-0-0-0-0 = -1$, as expected. However if instead the bounding block of j is bigger than $i_1 \cup i_2$'s bounding block, then we can take the interval k that contains i_1 and i_2, we have $k \in K$ with $j \neq k \cap j \supseteq i_1 \cup i_2$ and $b_{i_1 \cup i_2} = b_{j \cap k}$, because we said that j's bounding box is larger than i's bounding box, so $A_{1,2}(j,i) = -1$. Thus $r(j,i) = o(j,i) - A_1(j,i) - A_2(j,i) - A_3(j,i) - A_{1,2}(j,i) - A_{2,3}(j,i) - A_{1,2,3}(j,i) = 2-1-1-0+1+0-0 = 0$, as expected.

Suppose j contains all three maximal intervals of i. It must be the case that j contains i_1, i_2, and i_3. Thus, we see that both i_1, i_2, and i_3 must exist and are in j. We know by the modified algorithm how to calculate $r(j,i)$. Note because just i_1, i_2 and i_3 are in j, that $o(j,i) = 2$. Note that if we take $k = i_1$ that $k \cap j = i_1 \neq j$, so $A_1(j,i) = 1$. Similarly, if we take $k = i_2$, then $k \cap j = i_2 \neq j$, so $A_2(j,i) = 1$. Similarly, if we take $k = i_3$, then $k \cap j = i_3 \neq j$, so $A_3(j,i) = 1$. Now if we take the smallest interval that contains i_1 and i_2, call it k, we see that $j \neq k \cap j \supseteq i_1 \cup i_2$ and $b_{i_1 \cup i_2} = b_{j \cap k}$, so $A_{1,2}(j,i) = -1$. Similarly, if we take the smallest interval k that contains i_2 and i_3, we see that $j \neq k \cap j \supseteq i_2 \cup i_3$ and $b_{i_2 \cup i_3} = b_{j \cap k}$, so $A_{2,3}(j,i) = -1$. Now if $b_j = b_{i_1 \cup i_2 \cup i_3}$, we can't have a $k \in K$ with $j \neq k \cap j \supseteq i_1 \cup i_2 \cup i_3$ and $b_{i_1 \cup i_2 \cup i_3} = b_{j \cap k}$ as the only j that contains i_1, i_2 and i_3 is j itself which we can't have, so $A_{1,2,3}(j,i) = 0$. Thus, $r(j,i) = o(j,i) - A_1(j,i) - A_2(j,i) - A_3(j,i) - A_{1,2}(j,i) - A_{2,3}(j,i) - A_{1,2,3}(j,i) = 2-1-1-1+1+1-0 = 1$, as expected. Instead, if the bounding box of j was larger than the bounding box of $i_1 \cup i_2 \cup i_3$, we can take the interval k that contains i_1, i_2 and i_3, thus we have $k \in K$ with $j \neq k \cap j \supseteq i_1 \cup i_2 \cup i_3$ and $b_{i_1 \cup i_2 \cup i_3} = b_{j \cap k}$, because we said that j's bounding box is larger than i's bounding box, so $A_{1,2,3}(j,i) = 1$. Thus $r(j,i) = o(j,i) - A_1(j,i) - A_2(j,i) - A_3(j,i) - A_{1,2}(j,i) - A_{2,3}(j,i) - A_{1,2,3}(j,i) = 2-1-1-1+1+1-1 = 0$, as expected.

Thus the inductive step is proved, so Proposition 4.2.1 is proved by strong induction.

Corollary 4.2.1.1. *If $i < j$, then $r(j,i) = 0$*

Proof. If i's bounding box is smaller then j's, one of last three cases will take effect and $r(j,i) = 0$. If i and j's bounding box is the same, then either $e_i < e_j$ or $x_i < x_j$ (as they must both be two wide). In either case, we see that $i_2 \not\subseteq j_2$, thus none of the $r(j,i) \neq 0$ cases are possible, so $r(j,i) = 0$. □

4.3 Main Result

We now return to the proof of our main result.

Theorem 4.0.1 *Main result*
The determinant of the rank matrix for $P = 2 \times h$ is one.

Proof. By Corollary 4.2.1.1 is upper triangular. On the main diagonal, if a module i has one or three maximal intervals, then $r(i, i) = 1$. Otherwise, i has two maximal intervals and $r(i, i) = -1$. However, for any module i with this property, there is a different module i' that also has two maximal intervals. Specifically, the module i will have $e_i = 1$ and $x_i > 1$, or it will have $e_i > 1$ and $x_i = i$. So the module i' with $e_{i'} = x_i$ and $x_{i'} = e_i$, which is necessarily different. Thus we have an even number of those modules, which all contribute a -1. Thus elements on the main diagonal multiply to give 1, so the resulting matrix has determinant 1. Since the only row operations we used are determinant preserving, the rank matrix has determinant 1. $\square$

Chapter 5

Conclusion

Thus we've seen that in the case of the $2 \times h$ commutative grid that the determinant is 1. This says that when the program in TDA leads to a representation of $2 \times h$, the virtual barcode for the representation is a vector over $\mathbb{Z}$. Thus we can get a virtual decomposition of our representation, so it can be interpreted as a difference of two direct sums of convex modules. Again, the full implications of this are yet to be studied, but potentially very useful for TDA.

The code suggests for $3 \times h$ that the determinant is 1 as well, and though we don't have a proof at the moment, we can imagine that the ideas in this thesis are close to working. However there are some minor hiccups that need to be dealt with, as this case is more finicky than $2 \times n$. Notably, the inductive hypothesis seems to work perfectly to describe all the modules that can be found in $3 \times n$ except for two types of modules.

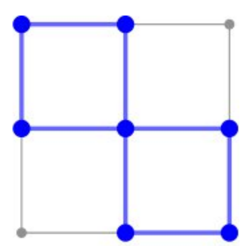
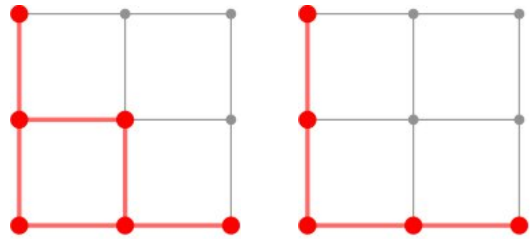


Figure 5.1: In the 3×3 commutative grid

In Figure 5.1, this module has properties that can't happen in $2 \times n$, namely have two minimals and two maximals, but both minimals go to both maximals. The issue with this module is that the inductive hypothesis would suggest that whenever j contains this module that we would get a value of -1, but we always find that we get a value of 1. Luckily this doesn't actually matter for calculations as while it means we have to modify the inductive hypothesis slightly, the proof still works with this modification.



(a) (b)

$r(j,i)$	a	b
a	0	1
b	1	1

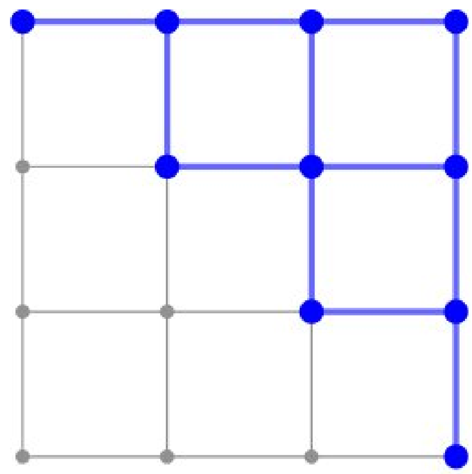
Figure 5.2: In the 3×3 commutative grid, with the resulting matrix, focused on the columns of a and b in Subfigure 5.2a and 5.2b respectively.

In Figure 5.2a, this type of module has properties that can't happen in any $2 \times h$ grid. Specifically, it has one minimal that goes to three different maximals. This poses a problem because it turns out strange if Figure 5.2a is a and Figure 5.2a is b, we find in the resulting matrix in Figure 5.2 that we have a 0 on the main diagonal, and a 1 on the lower left, where everything should be 0.

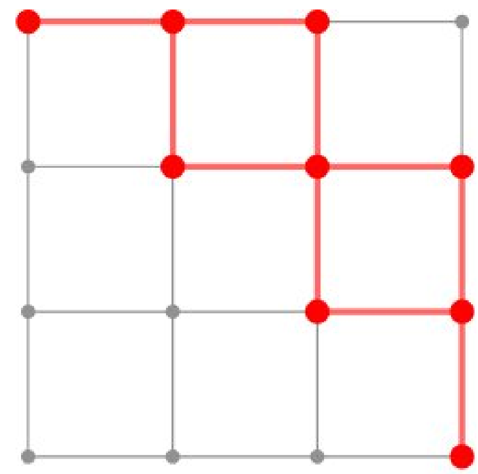
As far as I'm aware, this doesn't end up being that big of a problem as it looks like it is possible to just swap row a with row b and every a and b have their 180 degree rotational partners, so we multiply the determinant by -1 an even amount of times. The only part that makes this problem slightly harder is what happens if we have versions of Figure 5.2a where the left edge is longer, as we can have multiple different types of Figure 5.2a with the middle maximal interval extending up multiple amounts. Lastly, there's the version of Figure 5.2a where you take it then take a second of it rotated 180 degrees and stapled across the horizontal edge, as these modules can be 180 degrees rotationally symmetric.

However, for $h \geq 4$ and $g \geq 4$ we very quickly run into a problem. From my testing, if a commutative grid is $h \times g$ then the determinant will not be one. To give a taste of the resulting determinants, here are some sample calculated values:

h, g	4	5	6	7
4	2	511	35184372088830	$\approx 4.6768 * 10^{49}$
5	511	$\approx 2.9710 * 10^{28}$	$\approx 9.3394 * 10^{165}$	
6	35184372088830	$\approx 9.3394 * 10^{165}$		
7	$\approx 4.6768 * 10^{49}$			



(a) This module has four minimals to one maximal



(b) This module has three minimals to one maximal, twice, overlapping

Figure 5.3: In the 4×4 commutative grid

I also did not have time to look into why 4×4 commutative grids do not have determinant 1, but in the modules in Figure 5.3 are some possible elements that would cause the determinant to not be 1.

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