Digital Signal Processing Notes

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Conventions

$*$ denotes the discrete convolution operation.
$\circledast$ denotes the discrete time periodic convolution operation.
$\circ$ denotes the discrete time circular convolution operation.

Preliminary Notes

This document is still in development and updated continually every other weekend. The content in this document is heavily influenced by EC ENGR 113 (Winter Quarter 2022) which was taught by Professor Achuta Kadambi.

1 Types of Convolution

Challenge Questions

1) How does circular convolution relate to linear convolution?
2) What can be done to ensure circular convolution result obtained using the DFT method matches the linear convolution result?

1.1 Linear Convolution

Definition 1.1 (Convolution). Suppose $x \in \mathbb{R}^d, h \in \mathbb{R}^m$. The **convolution** of x with h, $y \in \mathbb{R}^{(m+d-1)}$, is:

$$y[n] = x[n] * h[n] := \sum_{k=-\infty}^{k=\infty} x[k]h[n-k]$$

Proposition 1.2. The convolution operation is commutative (i.e. $h[n] * x[n] = x[n] * h[n]$)

Proof. We have $x[n] * h[n] = \sum_{k=-\infty}^{k=\infty} x[k]h[n-k]$. Now, define $m = n - k$. Then, $m|_{n=-\infty} = -\infty - k = -\infty$ and $m|_{n=\infty} = \infty - k = \infty$. So, $x[n] * h[n] = \sum_{m=-\infty}^{m=\infty} x[n-m]h[m] = \sum_{m=-\infty}^{m=\infty} h[m]x[n-m] = h[n] * x[n]$. Thus, it follows that $x[n] * h[n] = h[n] * x[n]$. $\square$

Proposition 1.3. The convolution operation is distributive (i.e. $x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$)

Proof. We have $x[n] * (h_1[n] + h_2[n]) = \sum_{k=-\infty}^{k=\infty} x[k](h_1[n-k] + h_2[n-k]) = \sum_{k=-\infty}^{k=\infty} (x[k]h_1[n-k] + x[k]h_2[n-k])$ $= \sum_{k=-\infty}^{k=\infty} x[k]h_1[n-k] + \sum_{k=-\infty}^{k=\infty} x[k]h_2[n-k] = x[n] * h_1[n] + x[n] * h_2[n]$. Thus, it follows that $x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$ $\square$

Proposition 1.4. The convolution operation exhibits the identity property (i.e. $x[n] * \delta[n-m] = x[n-m]$)

Proof. We have $x[n] * \delta[n-m] = \sum_{k=-\infty}^{k=\infty} x[k]\delta[(n-m)-k]$. Since $\delta[(n-m)-k] = \left\{ \begin{array}{ll} 1, & \text{if } k = (n-m) \\ 0, & \text{if } k \neq (n-m) \end{array} \right\}$, we also have $\sum_{k=-\infty}^{k=\infty} x[k]\delta[(n-m)-k] = 0 \times x[0] + ... + 1 \times x[n-m] + ... + 0 \times x[d-1] = x[n-m]$. Thus, it follows that $x[n] * \delta[n-m] = x[n-m]$ $\square$

Exercise. Given $x \in \mathbb{R}^d$ where $x[n] = 0$ for $n < N_1$ or $n \geq (N_1 + d)$ and $h \in \mathbb{R}^m$ where $h[n] = 0$ for $n < N_2$ or $n \geq (N_2 + m)$, show that the convolution of x with h, $y \in \mathbb{R}^{(m+d-1)}$, can be expressed as:

$$y[n] = x[n] * h[n] = \sum_{k=k_1}^{k=k_2} x[k]h[n-k]$$

where $k_2 = min(N_1 + d - 1, n - N_2)$ and $k_1 = max(N_1, n - N_2 - m + 1)$

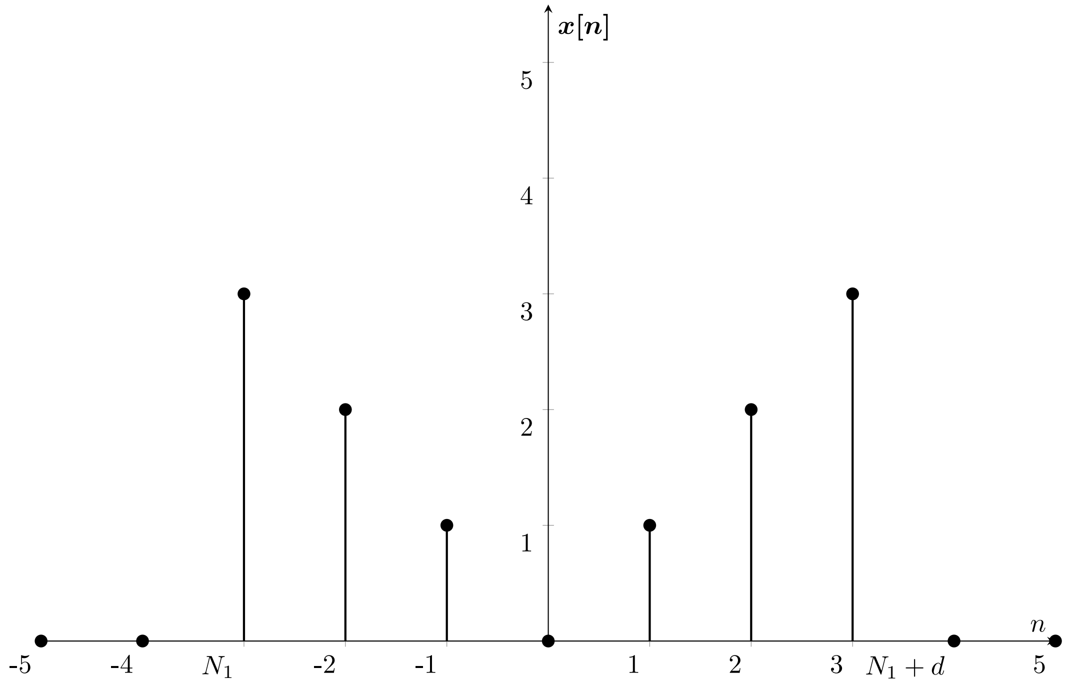

Figure 1: Example signal for exercise 1

Proof. $x[n] * h[n] = \sum_{-\infty}^{\infty} x[k]h[n-k]$. However, notice that $x[k]h[n-k]$ only contributes to the resultant sum if both $x[k]$ and $h[n-k]$ are non-zero simultaneously. We know that $x[k]$ is non-zero for $N_1 \leq k < (N_1 + d)$ and $h[n-k]$ is non-zero for $n - (N_2 + m) < n - k \leq n - N_2$ (note the order of the bounds since we are subtracting k from n). Now, since these bounds are not equal in general we want **the greatest lower bound** and **the least upper bound** which gives that $x[k]h[n-k]$ is nonzero for $max(N_1, n - (N_2 + m) + 1) \leq k \leq min(n - N_2, (N_1 + d) - 1)$. Note that the extra -1 and +1 are both due to our original non-inclusive bounds. This allows us to express the convolution as:

$$y[n] = x[n] * h[n] = \sum_{k=k_1}^{k=k_2} x[k]h[n-k]$$

where $k_2 = min(n - N_2, N_1 + d - 1)$ and $k_1 = max(N_1, n - N_2 - m + 1)$, thus completing the proof. $\square$

1.2 Interpretations of Linear Convolution

An equivalent definition for the convolution operator that may be more intuitive for some people is as follows:

Definition 1.5 (Convolution)**.** Suppose $a \in \mathbb{R}^d, b \in \mathbb{R}^m$. The **convolution** of a with b, $c \in \mathbb{R}^{(m+d-1)}$, is:

$$c[n] = a[n] * b[n] := \sum_{\substack{\text{all } i \text{ and } j \text{ with} \\ i+j=n}} a[i]b[j]$$

This formulation gives way to multiple interpretations. A probabilistic interpretation is as follows: A and B are two discrete random variables such that $P(A = i) = a[i]$ and $P(B = j) = b[j]$ for all i and j. Then, the convolution, $c = a * b$ is equivalent to adding two random together such that $c[n] = P(A + B = n)$. Let's set up a theoretical experiment where is A the value of weighted die 1 and B is the value of weighted die 2 after they are both thrown. We can visualize this scenario in a probability distribution function table for a weighted die.

x	$P(A = x), a[x]$
1	1/16
2	2/16
3	3/16
4	4/16
5	4/16
6	2/16

x	$P(B = x), b[x]$
1	1/16
2	4/16
3	2/16
4	3/16
5	3/16
6	3/16

$P(A) \times P(B)$	$P(B=1)$	$P(B=2)$	$P(B=3)$	$P(B=4)$	$P(B=5)$	$P(B=6)$
$P(A=1)$	1/256	4/256	2/256	3/256	3/256	3/256
$P(A=2)$	2/256	8/256	4/256	6/256	6/256	6/256
$P(A=3)$	3/256	12/256	6/256	9/256	9/256	9/256
$P(A=4)$	4/256	16/256	8/256	12/256	12/256	12/256
$P(A=5)$	4/256	16/256	8/256	12/256	12/256	12/256
$P(A=6)$	2/256	8/256	4/256	6/256	6/256	6/256

Now, we are able to see a pattern confirming our intuition:

$c[0] = 0 = P(A+B=0)$

$c[1] = 0 = P(A+B=1)$

$c[2] = a[1]b[1] = \frac{1}{256} = P(A+B=2)$

$c[3] = a[2]b[1] + a[1]b[2] = \frac{6}{256} = P(A+B=3)$

$c[4] = a[3]b[1] + a[2]b[2] + a[1]b[3] = \frac{13}{256} = P(A+B=4)$

$c[5] = a[4]b[1] + a[3]b[2] + a[2]b[3] + a[1]b[4] = \frac{23}{256} = P(A+B=5)$

$$\vdots$$

Another perspective we can take is viewing the convolution as polynomial multiplication. If a and b are the coefficients of polynomials $p(x) = a[0] + a[1] \cdot x + ... + a[d-1] \cdot x^{d-1}$ and $q(x) = b[0] + b[1] \cdot x + ... + b[m-1] \cdot x^{m-1}$, then $c = a * b$ gives the coefficients of the product polynomial $p(x)q(x) = c[0] + c[1] \cdot x + ... + c[m+d-2] \cdot x^{m+d-2}$.

Proof. We know that $p(x)q(x) = (a[0]+...+a[d-1]\cdot x^{d-1})(b[0]+...+b[m-1]\cdot x^{m-1})$. But instead of writing out each term, lets make a product table of the coefficients of a and b indexed by the order of x plus one. The n^{th} anti-diagonal of this table will give us all the terms of order $n-1$, which when summed, gives us the $(n-1)^{th}$ coefficient of $p(x)q(x)$.

order of x	$\mathbf{a[0]}$	$\mathbf{a[1]}$	$\mathbf{a[2]}$	...	$\mathbf{a[d-2]}$	$\mathbf{a[d-1]}$
$\mathbf{b[0]}$	$a[0]b[0]$	$a[1]b[0]$	$a[2]b[0]$	...	$a[d-2]b[0]$	$a[d-1]b[0]$
$\mathbf{b[1]}$	$a[0]b[1]$	$a[1]b[1]$	$a[2]b[1]$	...	$a[d-2]b[1]$	$a[d-1]b[1]$
$\mathbf{b[2]}$	$a[0]b[2]$	$a[1]b[2]$	$a[2]b[2]$	...	$a[d-2]b[2]$	$a[d-1]b[2]$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$\mathbf{b[m-2]}$	$a[0]b[m-2]$	$a[1]b[m-2]$	$a[2]b[m-2]$	...	$a[d-2]b[m-2]$	$a[d-1]b[m-2]$
$\mathbf{b[m-1]}$	$a[0]b[m-1]$	$a[1]b[m-1]$	$a[2]b[m-1]$	...	$a[d-2]b[m-1]$	$a[d-1]b[m-1]$

Now, using this table, we can see that the 2^{nd} anti-diagonal (line from **b[2]** to **a[2]**) which corresponds to the 1^{st} order coefficients of $p(x)q(x)$ gives:

1^{st} order terms $= a[0]b[1] + b[1]a[0] = c[1]$

It is left as an exercise to the reader to confirm that higher order terms, and the zeroth order term matches that of the problem statement. other terms lead to the same conclusion. For rigor, we give an algebraic proof below:

$p(x)q(x) = (a[0] + ... + a[d-1] \cdot x^{d-1})(b[0] + ... + b[m-1] \cdot x^{m-1}) = (\sum_{i=0}^{i=d-1} a[i]x^i)(\sum_{j=0}^{j=m-1} b[j]x^j) = \sum_{i=0}^{i=d-1} \sum_{j=0}^{j=m-1} a[i]x^i b[j]x^j = \sum_{i=0}^{i=d-1} \sum_{j=0}^{j=m-1} a[i]b[j]x^{i+j}$. Now, to simplify the problem, assume $i+j=n$ for some constant $n \in \mathbb{N}$. Then, the n^{th} order term of the product polynomial is $\sum_{i=0}^{i=d-1} \sum_{j=n-i} a[i]b[n-i]x^n = \sum_{i=1}^{i=d} a[i]b[n-i]x^n = (a[n] * b[n])x^n = c[n]x^n$ where we eliminated the sum $\sum_{j=n-i}$ because the summand doesn't depend on j. Note that

$n \in \{0, m+d-2\}$. Thus, it follows that $p(x)q(x) = c[0] + c[1] \cdot x + ... + c[m+d-2] \cdot x^{m+d-2}$.

$\square$

Lastly, the simplest interpretation of convolution is viewing it as a moving average of $b \in \mathbb{R}^m$, for the special case that our filer $a \in \mathbb{R}^d$ is a signal such that $\sum_{i=0}^{d-1} a[i] = 1$. For example, let $a = [\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]$, b $= [0, 2, 5]$, and $c = a * b$. Then,

$$c = [\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}] * [0,2,5] = [(0 \times \tfrac{1}{3}), (0 \times \tfrac{1}{3} + 2 \times \tfrac{1}{3}), (0 \times \tfrac{1}{3} + 2 \times \tfrac{1}{3} + 5 \times \tfrac{1}{3}), (2 \times \tfrac{1}{3} + 5 \times \tfrac{1}{3}), (5 \times \tfrac{1}{3})] = [0, \tfrac{2}{3}, \tfrac{7}{3}, \tfrac{7}{3}, \tfrac{5}{3}].$$

Notice that $\sum_{i=0}^{2} b[i] = \sum_{i=0}^{4} c[i] = 7$.

1.3 Periodic Convolution

Definition 1.6 (Periodic Convolution). Suppose $\tilde{x}$ and $\tilde{h}$ are periodic signals with a period of N. The **periodic convolution** of $\tilde{x}$ with $\tilde{h}$, $\tilde{y}$, is:

$$\tilde{y}[n] = \tilde{x}[n] \circledast \tilde{h}[n] := \sum_{k=0}^{k=N-1} \tilde{x}[k]\tilde{h}[n-k]$$

1.4 Circular Convolution

Definition 1.7 (Circular Convolution). Suppose x and h are signals of length N. The **periodic convolution** of x with h, y, is:

$$y[n] = x[n] \circ h[n] := \sum_{k=0}^{k=N-1} x[k]h[n-k]_{mod(N)}$$

for $n = 0, ..., N-1$

1.5 Challenge Questions

2 Lecture 1

A discrete time signal is a **function** defined only at time instants that are integer multiples of a fixed time increment T, where $t = nT$ for some $n \in \mathbb{Z}$. As a result, the independent variable n is an integer, and is referred to as the *sample index.*

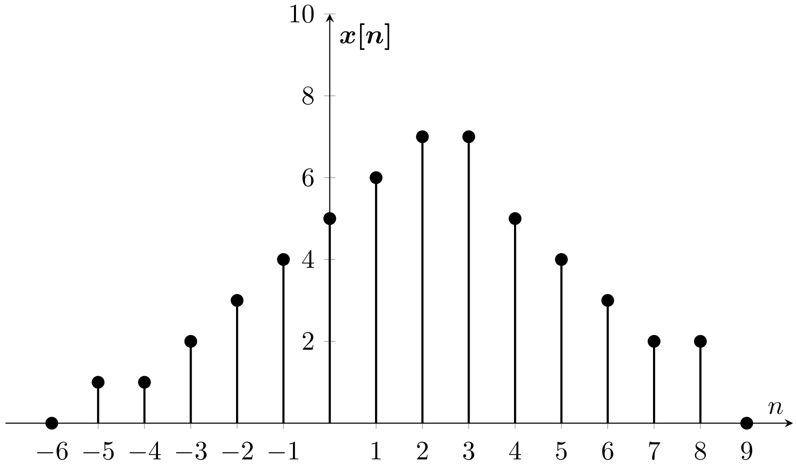

Figure 1: An example of a discrete signal

2.1 Arithmetic Operations

Constant offset $g[n] = x[n] + A$
Gain factor $g[n] = Bx[n]$
Signal addition $g[n] = x_1[n] + x_2[n]$
Signal multiplication $g[n] = x_1[n]x_2[n]$
Time reversal $g[n] = x[-n]$