1)

(a) 3^{n+1} vs 3^n

① $f(n)$ is $O(g(n))$

prove: if there exist a constant $c>0$, $n_0>0$ for all $n \geq n_0$, $f(n) \leq c \cdot g(n)$, $f(n) = O(g(n))$;

$3^{n+1} < 3^{n+2}$ for any $n_1 > 0$

$3^{n+1} < 3^{n+2} = 3^n \cdot 3^2 = 3^n \cdot 9 \Rightarrow c_1 = 9$ & $n_1 = 1$ work

② $f(n)$ is $\Omega(g(n))$

prove: if there exist a constant $c>0$ $n_0>0$ for all $n \geq n_0$, $f(n) \geq c \cdot g(n)$, then $f(n) = \Omega(g(n))$;

$3^{n+1} > 3^{n-1}$ for any $n_1 > 0$

$3^{n+1} > 3^{n-1} = 3^n \cdot 3^{-1} = 3^n \cdot \frac{1}{3} \Rightarrow c_1 = \frac{1}{3}$ & $n_1 = 1$ wor

(b) 2^{2n} vs 2^n

① $f(n)$ is not $O(g(n))$

prove: if there exist a constant $c>0$, $n_0>0$ for all $n \geq n_0$, $f(n) \leq c \cdot g(n)$, $f(n) = O(g(n))$;

if it were true, $2^{2n} \leq c \cdot 2^n$ for $n \geq n_0$

$$\frac{\cancel{2^{2n}}\ 2^n}{\cancel{2^n}} \leq \frac{c \cdot \cancel{2^n}}{\cancel{2^n}}$$

$$2^n \leq c$$

$$\log_2 2^n \leq \log_2 c$$

$$n \leq \log_2 c$$

whatever c chosen, n will never surpass, invalid

so $f(n)$ is not $O(g(n))$

② $f(n)$ is $\Omega(g(n))$

prove: if there exist a constant $c>0$ $n_0>0$ for all $n \geq n_0$, $f(n) \geq c \cdot g(n)$, then $f(n) = \Omega(g(n))$;

We can see for $n \geq 1$ and $c = 1$, $2^{2n} > 2^n \cdot c$ is trivaially true.

(c) 4^n vs 2^{2n}

① $f(n)$ is $O(g(n))$

prove: if there exist a constant $c>0$, $n_0>0$ for all $n \geq n_0$, $f(n) \leq c \cdot g(n)$, $f(n) = O(g(n))$;

$2^{2n} = (2^2)^n = 4^n$, so $f(n) = g(n)$

as a result, $4^n \leq c \cdot 2^2 n$ when $n>0$ and $c>1$, $n=1$ and $c=2$ work

② $f(n)$ is $\Omega(g(n))$

prove: if there exist a constant $c>0$ $n_0>0$ for all $n \geq n_0$, $f(n) \geq c \cdot g(n)$, then $f(n) = \Omega(g(n))$;

Again, because $2^{2n} = 4^n$, $4^n \geq c \cdot 2^2 n$ when $n>0$ and $c<1$, $n=1$ and $c=\frac{1}{2}$ work

(d) n^2 vs $n^{2.01}$

① $f(n)$ is $O(g(n))$

prove: if there exist a constant $c>0$, $n_0>0$
for all $n \geq n_0$, $f(n) \leq c \cdot g(n)$, $f(n) = O(g(n))$;

Trivially $n^2 \leq n^{2.01}$ when $n \geq 1$ is true

so when $c_0 = 1$, $n_0 = 1$, $n^2 \leq cn^{2.01}$, $f(n) = O(g(n))$.

② $f(n)$ is not $\Omega(g(n))$

prove: if there exist a constant $c>0$ $n_0>0$
for all $n \geq n_0$, $f(n) \geq c \cdot g(n)$, then $f(n) = \Omega(g(n))$;

if satisfied $f(n) \geq c \cdot g(n)$

$n^2 \geq c \cdot n^{2.01}$

divide n^2 from both side: $1 \geq c \cdot n^{0.01}$

$n^{0.01} \leq \frac{1}{c} \Rightarrow \underbrace{n \leq \sqrt[0.01]{\frac{1}{c}}}$

$\Downarrow$

whatever c we choose,
n will never surpass it

(e) $n^{0.9}$ vs 0.9^n

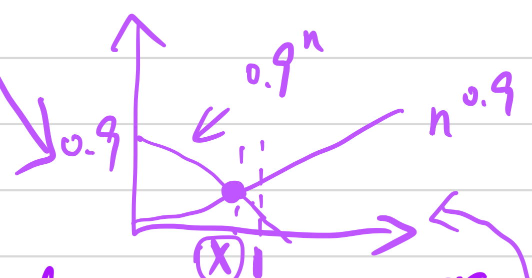
$f(n)$ is not $O(g(n))$ but $f(n)$ is $\Omega(g(n))$

prove: if there exist a constant $c>0$, $n_0>0$ for all $n \geq n_0$, $f(n) \leq c \cdot g(n)$, $f(n) = O(g(n))$;

for big-Ω, if there exist a constant $c>0$ $n_0>0$ for all $n \geq n_0$, $f(n) \geq c \cdot g(n)$, then $f(n) = \Omega(g(n))$;

$n^{0.9}$ is an increasing function when n increases but 0.9^n is a decreasing function when n increases two functions intersects at:

$n^{0.9} = 0.9^n$,



for $n>0$, $n = \sqrt[0.9]{0.9^n}$, 0.9^n for $n>0$, never larger than 1, so

As a result, $\Leftarrow$ $\sqrt[0.9]{0.9^n}$ will no larger than 1, too. so the result is, at some point $n<1$, two intersects, let's say point $\boxed{X}$

when $n=1$, $c=1$,

$f(n) \geq c \cdot g(n)$

so $f(n)$ is $\Omega(g(n))$

But for big-O, $f(n) \leq c_1 \cdot g(n)$, $n^{0.9} \leq c \cdot 0.9^n$, whatever c we choose, n will eventually surpass it

n needs to be less than $\frac{X}{c_1}$

(a) $\log^c n$ vs $\log n$

$\Rightarrow (\log n)^c$ vs $\log n$

$f(n)$ is not $O(g(n))$ but $\Omega(g(n))$

prove: if $f(n)$ is $O(g(n))$:

$$(\log n)^c \le c_1 \cdot \log n$$

devide $\log n$ from both side: $(\log n)^{c-1} \le c_1$

whatever constant c we choose, n will eventually surpass it

for $f(n) = \Omega(g(n))$, $f(n) = c_1 \cdot g(n)$.

because $\log^c n = (\log n)^c$, and $c > 1$,

so $\log^c n > \log n$ when $c_1 = 1$, $n \ge 1$

so $f(n) = O(g(n))$

(b) $\log n^c$ vs $\log n$ $\quad c = \theta 1$

$\log n^c = c \log n$

$f(n)$ is $O(g(n))$ and $f(n)$ is $\Omega(g(n))$

prove:

$f(n) \le c_1 \cdot g(n)$ when $c \le c_1$

$f(n) \ge c_1 \cdot g(n)$ when $c \ge c_1$

so for this question, $n = 1$ and $c_1 = c$ work

(c) $\log(c \cdot n)$ vs $\log n$ $\quad c = \Theta(1)$

$\log(c \cdot n) = \log c + \log n$

$f(n)$ is $O(g(n))$ and $f(n)$ is $\Omega(g(n))$

prove:

Because $\log(c \cdot n) = \log c + \log n, \leq 2\log n$

so when $c_1 = 2$, $n = 1$, $\log n^c \leq c_1 \cdot \log n$ work;

$f(n) \leq c_1 \cdot g(n)$,

$f(n)$ is $O(g(n))$

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for Big Ω

(c)

for big-Ω

① c is larger than 1

when c is larger than 1, log c is positive,

take $c_2 = 1$, $n = 1$, $\log(n \cdot c) \geq c_2 \cdot \log n$,

$(\log c + \log n)$

$f(n) = \Omega(g(n))$;

② c is smaller than 1 grater than 0

when c is smaller than 1, log c is negative,

$\log c \cdot n = \log c + \underline{\log n} \geq c_3 \cdot \cancel{\log n}$

$$c_3 \leq \frac{(\log n + \log c)}{\log n} = 1 + \frac{\log c}{\log n}$$

for $n_0 > 1$, $n > n_0$, $\frac{\log c}{\log n}$ is negative

$-1 < \frac{\log c}{\log n} < 0$, so there is a c_3

larger than 0 exisits.

(d) $\log_a n$ vs $\log_b n$

$$\log_b n = \frac{\log_a n}{\log_a b} = \frac{1}{\log_a b} \cdot \log_a n$$

$f(n)$ is $O(g(n))$ and $f(n)$ is $\Omega(g(n))$

prove:

since $\log_b n = \frac{1}{\log_a b} \cdot \log_a n$

$\log_a n \leq \frac{1}{\log_a b} \cdot \log_a n \cdot c_1$, if $\begin{cases} n_1 > 0 \\ c_1 \leq \log_a b \end{cases}$

At the same time, $\log_a n \geq \frac{1}{\log_a b} \cdot \log_a n \cdot C_2$, if $\begin{cases} n_2 > 0 \\ c_2 \geq \log_a b \end{cases}$

As a result $f(n)$ is $\Theta(g(n))$

The restriction on a, b is, because $\log_a n = \frac{\log_b n}{\log_b a}$ and $\log_b n = \frac{\log_a n}{\log_a b}$, $\log_a b / \log_b a$ both shouldn't be zero, if a or b's value $= 1$, then the expresson $\log_a b / \log_b a$ will be invalid as denominators

(3) if, $f(x) = O(x)$, $g(x) = O(x)$

then, $f(x) + c \cdot g(y) = O(x+y)$

Prove: $\because f(x) = O(x)$ $\therefore f(x) \leq c_1 \cdot x$

$\because g(x) = O(x)$ $\therefore g(x) \leq c_2 \cdot x$

$\Downarrow$ $g(y) \leq c_3 \cdot y$

$f(x) + c \cdot g(y) \leq c_1 \cdot x + c \cdot c_3 \cdot y$

let $c_4 = c \cdot c_3$;

let $c_5 = c_1 \cdot c_4$;

$f(x) + c \cdot g(y) \leq c_5 \cdot x + c_5 \cdot y$

$\leq c_5 (x+y)$

By defination $f(n) = O(g(n))$ when $f(n) \leq c \cdot g(n) =$

$f(x) + c \cdot g(y) = O(x+y)$

4. $f(n) = \sum_{x=1}^{n} (\log^3 n \cdot x^{29})$; find $g(n)$ $f(n)=\Theta(g(n))$

$$= \log^3 n \cdot \sum_{x=1}^{n} x^{29}$$

$$\parallel$$

$$(\log n)^3 \cdot ?$$

Answer: $g(n)$ is $\underline{\log^3 n \cdot n^{30}}$

Proof is below:

Big-O:

$$\sum_{x=1}^{n} x^{29} = 1^{29} + 2^{29} + 3^{29} + \dots + \left(\frac{n}{2}\right)^{29} + \dots\dots n^{29}$$

To exaggerate: n^{29} n^{29} n^{29} .... n^{29} n^{29}

$$\Rightarrow n^{29} \cdot n = \boxed{n^{30}} \cdot (\log n)^3, \text{ which } > \sum_{x=1}^{n} x^{29} \cdot n^{30},$$

So for $n_1 = 1$, $c_1 = 1$, $\sum_{x=1}^{n} (\log^3 n \cdot x^{29}) < (\log^3 n \cdot n^{30}) \cdot 1$,

$f(n) \leq c \cdot g(n)$, $f(n)$ is $O(g(n))$;

Big-Ω:

lower every terms to $\left(\frac{n}{2}\right)^{29}$

$$\sum_{x=1}^{n} x^{29} = \underbrace{1^{29} + 2^{29} + 3^{29} + \dots +}_{\frac{n}{2}} \underbrace{\left(\frac{n}{2}\right)^{29} + \dots n^{29}}_{\frac{n}{2}}$$

To underestimate:

leave out this part will smaller than origin function

$$\left(\frac{n}{2}\right)^{29} \cdot \frac{n}{2} = \boxed{\frac{n^{30}}{4}} \cdot (\log n)^3 > \frac{1}{4} (n^{30} (\log n)^3)$$

So for $n_1 = 1$, $c_1 = \frac{1}{4}$, $\sum_{x=1}^{n} (\log^3 n \cdot x^{29}) > \frac{n^{30}}{4} \cdot (\log n)^3 > \frac{1}{4}(n^{30} \cdot \log^3 n)$

$f(n) \geq c \cdot g(n)$, $f(n)$ is $\Omega(g(n))$.