hw 2

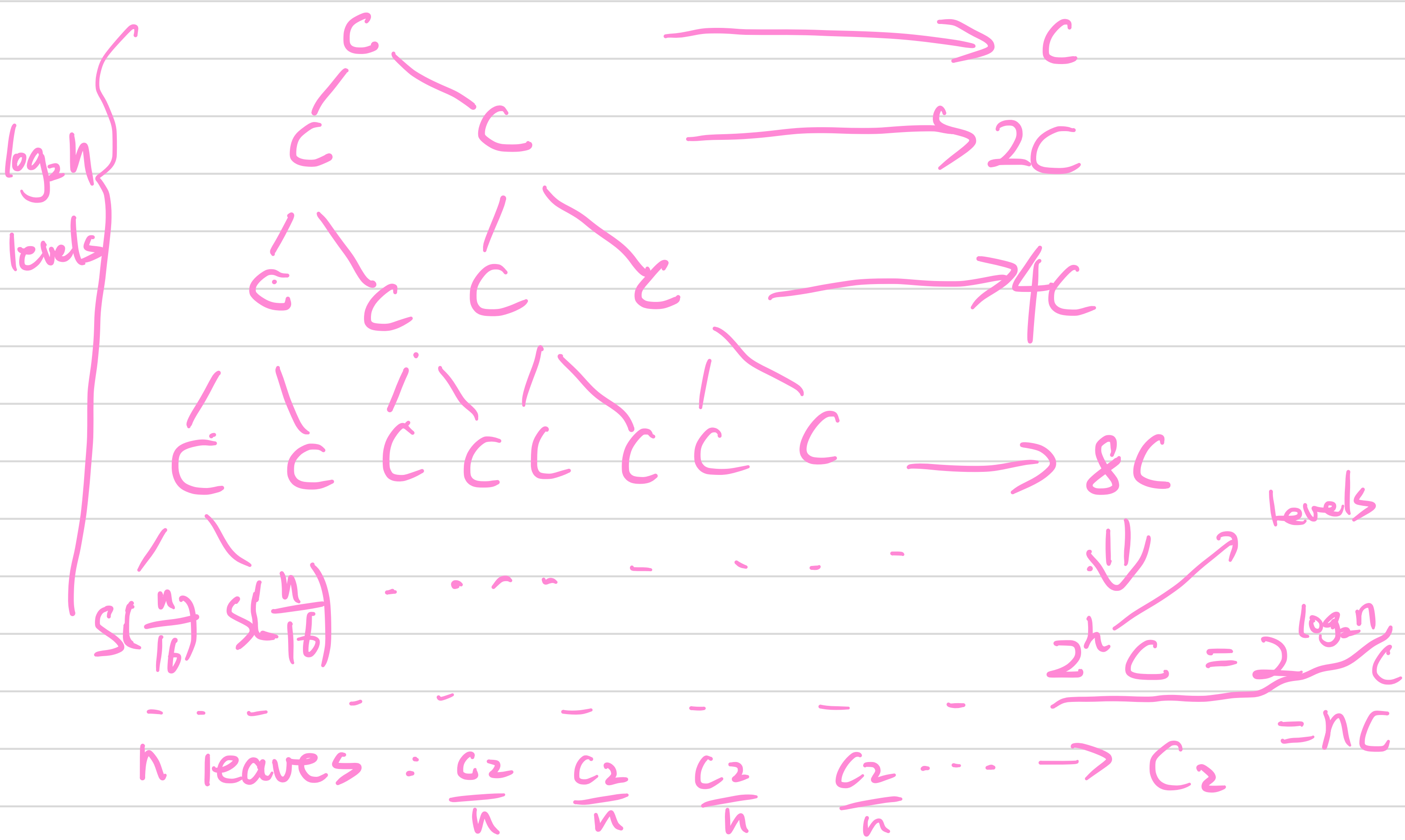
1. $S(n) = 2S(\frac{n}{2}) + \underbrace{\theta(1)}$

$\boxed{S(1) = \theta(1) = C_2}$

$\Downarrow$

$S(n) = 2S(\frac{n}{2}) + c \cdot 1$

$\Downarrow$



Sum: $\sum_{h=0}^{\log_2 n} n = 2n = O(n)$

(b) $S(n) = 2S\left(\frac{n}{2}\right) + \underbrace{\Theta(1)}$

$\hookrightarrow S(n) = 2S\left(\frac{n}{2}\right) + c \cdot 1$

$S(n) \geq dn \rightarrow \Omega(n)$

1) Inductive hypothesis : for all $k < n$, $S(k) \geq d \cdot k$

2) substitute : $S(n) \geq 2 \cdot d \frac{n}{2} + c$ $\left(k = \frac{n}{2}\right)$

3) Algebra $\left\{ = dn + c \right.$

so when $c \geq 0$, $S(n) \geq dn$

Prove:

2. $T(n) = 18T\left(\frac{n}{3}\right) + \theta(n^2) = O(n^3)$ $\quad \underline{T(1)=1}$

$\hookrightarrow 18T\left(\frac{n}{3}\right) + c\cdot n^2$

$T(n) \le d\cdot n^3 \Rightarrow O(n^3)$

1. Inductive hypothesis : for all $k<n$, $T(k) \le d\cdot k^3$

2. Substitute: $T(n) = 18\cdot dk^3 + c\cdot n^2$

$= 18\cdot d\left(\frac{n}{3}\right)^3 + c\cdot n^2 \quad \left(k=\frac{n}{3}\right)$

3. Algebra: $= 18\cdot d\frac{n^3}{27} + c\cdot n^2$

$= n^3\cdot\frac{18}{27}d + cn^2$

$= \frac{2}{3}dn^3 + cn^2$

$= dn^3 - \left(\frac{1}{3}dn^3 - cn^2\right)$

$n^2\left(\frac{1}{3}dn - c\right)$

$\le dn^3$ when $\boxed{\begin{array}{l} d>3c \\ \text{when } n\ge 0\end{array}}$

$\Downarrow$

$T(n) \le cn^3$

Base case: $T(1) = 1 \le c\cdot 1^3$ ✓

(2) $T(n) = 18T(\frac{n}{3}) + \theta(n^2)$

This satified the requirment of master Theorem, which is:

$$\begin{cases} a \geq 1 & (a = 18) \\ b > 1 & (b = 3) \\ f(n) > 0 \text{ for } n > n_0 & (f(n) = c \cdot n^2) \end{cases}$$

$\log_b a = \log_3 18 \approx 2.63$ (> 2 but < 3)

so $n^{\log_b a} > n^2$

$f(n) = O(n^{(\log_b a) - \varepsilon})$ ($\varepsilon > 0$) $\quad \varepsilon$: $2.63 - 2$

leaves level dominates polynomially

$T(n) = \theta(n^{\log_b a})$

so $T(n) = \theta(n^{\log_3 18}) = \theta(n^{2.63})$

$\therefore$ As the result $O(n^3)$ is not the best upper bond for $T(n)$

(a)

3. $T(n) = 10 \cdot T(\frac{n}{3}) + \theta(n^2 \log^5 n)$

$a = 10, \ b = 3, \ c = ?$ $f(n) = c \cdot n^2 \log^5 n$

$\log_b a \approx 2.095$

$n^{2.095} \ ? \ c \cdot n^2 \cdot \log^5 n$

$\cancel{n^2} \cdot n^{0.095}$ $c \cdot \cancel{n^2} \cdot \log^5 n$

$n^{0.095}$ vs $c \cdot \log^5 n$

$n^{0.01} < n^{0.095}$

$n^{0.01}$ vs $c \cdot \log^5 n$ L'Hopital rule

$$\lim_{n \to \infty} \frac{n^a}{c \cdot \log^5 n} = \lim_{n \to \infty} \frac{a \cdot n^{a-1}}{\frac{5}{x}} = \lim_{n \to \infty} a \cdot n^a = \infty \ (a > 0)$$

so $n^{0.01} > c \cdot \log^5 n$, $n^{\log_b a} > f(n)$

so case 1 applies; leave level dominates

$T(n) = \theta(n^{\log_b a}) = \theta(n^{\log_3 10}) \approx \theta(n^{2.095})$

1b) $T(n) = T\left(\frac{19n}{72}\right) + \theta(n^2) \longrightarrow f(n) = c \cdot n^2$

$a = 1 \quad b = \frac{72}{19} \quad c = 2$

>0

satisfied $a \geq 1$, $b > 1$ and $f(n) > 0$ when $n > n_0$

$n^{\log_b a} \ ? \ c \cdot n^c$

$\log_{\frac{72}{19}} 1 = 0$

$n^0 < cn^2$, $n^{\log_b a} < f(n)$

Case 3 applies, $f(n) = \Omega(n^{(\log_b a + \varepsilon)})$

2

$T(n) = \theta(n^2)$

(c) $T(n) = n \cdot T(\frac{n}{5}) + n^{\log_5 n}$

Answer: Master method cannot apply because $a = n$, which is not a constant; subproblem recursion is infinity;

in every level, the tree node spread to n ← make no sense

$n^{\log_b a}$ doesn't work when $a \to \infty$

(d) $T(n) = 3 \cdot T(\frac{n}{2}) + n^2 \in f(n)$

$a = 3 \quad b = 2 \quad c = 2$

satified $a \geq 1$ $b > 1$, $f(n) > 0$ $(n > 0)$

$\log_b a = \log_2 3$

$n^{\log_2 3} < n^2 \Rightarrow n^{\log_b a} < f(n)$

case 3 applies, $f(n) = \Omega(n^{\log_b a + \varepsilon})$, $\varepsilon = 2 - \log_2 3$

$T(n) = \Theta(n^2)$

(e) $T(n) = T(\frac{n}{n-1}) + 1$

Answer: Master theorem not applicable here

Reason: $b = n-1$, not a constant,

b is the factor by which the subproblem size reduced in each recursive call. It is not making sense to have it $n-1$.

(f) $T(n) = 4 \cdot T(\frac{n}{16}) + \sqrt{n} \cdot \log^4 n \rightarrow f(n)$

$a = 4 \quad b = 16 \quad c = ?$

$\log_b a = \frac{1}{2}$, $\quad f(n) = n^{\frac{1}{2}} \cdot \log^4 n$

$n^{\log_b a} = n^{\frac{1}{2}}$

case 2 is satisfied because

$f(n) = \Theta(n^{\log_b a} \cdot \log^k n) \quad \leftarrow k=4$

$= \Theta(n^{\frac{1}{2}} \cdot \log^4 n)$

all levels ~ same

$T(n) = \Theta(n^{\frac{1}{2}} \cdot \log^4 n \cdot \log n)$

4. (a)

$$T(n) = T(\sqrt{n}) + \log n$$

$k = \log_2 n$

$\log n$	$\log n$
$\downarrow$	$\mid$
$\log n^{\frac{1}{2}}$	$\frac{1}{2}\log n$
$\downarrow$	$\mid$
$\log n^{\frac{1}{4}}$	$\frac{1}{4}\log n$
$\vdots$	$\vdots$
	$\frac{1}{2^k}\log n$

$\rightarrow$ Geometric series

$$\sum_{0}^{\log_2 n} \frac{1}{2^k}\log n = 2\log n$$

$$T(n) = \Theta(\log n)$$

(b) $T(n) = T\sqrt{n} + \log n$

Try: $\theta(\log n)$: $\begin{cases} T(n) \le c \cdot \log n \\ T(n) \ge c \cdot \log n \end{cases}$

big-O : Hypothesis: $T(k) \le c \cdot \log k$ for $k < n$

Substitude: $T(k) \le c \cdot \log(\sqrt{n})$ $(k = \sqrt{n})$

Algebra : $= c \cdot \log n^{\frac{1}{2}}$

$= \frac{1}{2} c \cdot \log n$

$= \underbrace{c \cdot \log n}{} - \underbrace{\frac{1}{2} c \cdot \log n}_{\text{(positive when } n>0)}$

$T(k) \le 1 \cdot \log n \le c \cdot \log n$

$T(k) = O(\log n)$

big Ω:

Hypothesis: $T(k) \geq c \cdot \log k$ for $k < n$

Substitude: $T(k) \geq c \cdot \log(\sqrt{n})$ $(k = \sqrt{n})$

Algebra:

$$= c \cdot \log n^{\frac{1}{2}}$$

$$= \frac{1}{2} c \cdot \log n$$

$$= \frac{1}{4} c \cdot \log n + \underbrace{\frac{1}{4} c \cdot \log n}_{(>0 \text{ when } n>0)}$$

$$T(k) \geq \frac{1}{4} \log n \geq c \cdot \log n$$

$$T(k) = \Omega(\log n)$$

As a result: $T(k) = \Theta(\log n)$

(c)

$n = 2^m$ $\qquad$ $m = \log_2 n$

$T(n) = T(n^{\frac{1}{2}}) + \log n$

$= T(2^m) = T(2^{\frac{1}{2}m}) + \log 2^m$

$= T(2^m) = T(2^{\frac{1}{2}m}) + m$

variable m, log it

$\Downarrow$

$S(m) = S(\frac{m}{2}) + m \to f(m)$

$m^{\log_b a} = 1$, $< m$

Case 3 applies: $f(m) = \Omega(m^{\log_b a})$

$S(m) = \Theta(m)$

$\because m = \log_2 n$

$\therefore T(n) = \Theta(\log n)$