MA20219: Analysis 2B

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Contents

Preface

The content of this unit naturally splits into two parts, which are loosely connected.

We first study functions defined on $\mathbb{R}^n$ or subsets thereof. We refer to these as functions of several real variables. The structure of $\mathbb{R}^n$ as a metric space was examined in MA20218, and you have also seen what it means for a function $f\colon U \to \mathbb{R}$, defined on an open set $U \subseteq \mathbb{R}^n$, to be continuous. We now study what it means for such a function to be *differentiable*. We will see that we have several notions of a derivative in this situation, and we will examine the relationships between them.

In the second part, we will study functions $f\colon U \to \mathbb{C}$, defined on a set $U \subseteq \mathbb{C}$. As you know, the complex plane $\mathbb{C}$ can be identified with $\mathbb{R}^2$ and has the same structure as a metric space. This situation may therefore superficially look similar to the case of functions of two real variables. But as you also know, the complex numbers have additional *algebraic* structure that is not present in $\mathbb{R}^2$. Thus we should rather think of a function of one *complex* variable.

If we adapt the ideas seen in previous analysis units, and in particular the idea of a derivative, to complex variables, then we find a remarkably beautiful theory. In particular, a lot of the complications coming from limited differentiability or from various notions of convergence fall away. Of course, this comes at a price, which is that complex differentiability is much more restrictive than the real counterpart. So we will have a very powerful theory, but it will apply only to a relatively small set of functions (which still includes rational, exponential, and trigonometric functions and much more).

Chapter 1

Partial and directional derivatives

For most of this chapter, we assume that $U \subseteq \mathbb{R}^n$ is an open set and we consider functions $f\colon U \to \mathbb{R}$. What does it mean to differentiate such a function?

For a function of *one* real variable, say $g\colon (a,b) \to \mathbb{R}$, we can define the derivative $g'(x)$ at a point $x \in (a,b)$ by the limit of the difference quotient,

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}, \tag{1.1}$$

provided that the limit exists. But such a formula is no longer meaningful if instead of a number $h \in \mathbb{R}$ we have a vector in $\mathbb{R}^n$, as we can't divide by vectors. Hence we need to do something else in this situation. We begin with a concept that is still similar to (1.1).

Throughout the first few chapters, we consider functions defined on a subset of $\mathbb{R}^n$. We think of elements of $\mathbb{R}^n$ as column vectors, but sometimes, in order to avoid cumbersome notation, we write them as row vectors, especially when they are not subject to any vector operations. In particular, we typically write $x = (x_1, \dots, x_n)$ for a generic point in $\mathbb{R}^n$. Furthermore, we write e_j for the j-th standard unit vector in $\mathbb{R}^n$ for $j = 1, \dots, n$. Given $a \in \mathbb{R}^n$ and $r > 0$, we use the notation $B_r(a) = \{x \in \mathbb{R}^n \colon \|x - a\| < r\}$ for the open ball of radius r centred at a.

1.1 Partial derivatives

The underlying idea here is to pretend that we're still dealing with functions of one variable. This is effectively the case if we fix the values of all but one of the variables.

Definition 1.1 (Partial derivative)**.** Let $U \subseteq \mathbb{R}^n$ be an open set and $x \in U$. Suppose that $f \colon U \to \mathbb{R}$ is a given function and $j \in \{1, \dots, n\}$. Then

$$\begin{aligned}\frac{\partial f}{\partial x_j}(x) &= \lim_{h \to 0} \frac{f(x + he_j) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{1}{h}\big(f(x_1, \dots, x_{j-1}, x_j + h, x_{j+1}, \dots, x_n) - f(x_1, \dots, x_n)\big),\end{aligned}$$

if it exists, is the *partial derivative* of f with respect to x_j at the point x.

Example 1.2. Consider $f \colon \mathbb{R}^3 \to \mathbb{R}$ given by $f(x) = x_1^2 x_2 + x_3$. Then

$$\frac{\partial f}{\partial x_1}(x) = 2x_1x_2, \quad \frac{\partial f}{\partial x_2}(x) = x_1^2, \quad \frac{\partial f}{\partial x_3}(x) = 1.$$

Thus finding partial derivatives is no different in principle from differentiating a function of one variable. We have the following notation for the vector comprising all partial derivatives.

Definition 1.3 (Gradient)**.** Let $U \subseteq \mathbb{R}^n$ be open and $f \colon U \to \mathbb{R}$ a function that has partial derivatives at a point $x \in U$. Then the vector

$$\nabla f(x) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(x) \\ \vdots \\ \frac{\partial f}{\partial x_n}(x) \end{pmatrix}$$

is called the *gradient* of f at x.

Example 1.4. For the function in Example 1.2, we have the gradient

$$\nabla f(x) = \begin{pmatrix} 2x_1x_2 \\ x_1^2 \\ 1 \end{pmatrix}.$$

1.2 Directional derivatives

The partial derivatives of f can be seen as the derivatives of a function of the form $t \mapsto f(x + te_j)$ for some $j \in \{1, \dots, n\}$. They give us information about how the function behaves along the line $\{x + te_j : r \in \mathbb{R}\}$ (or the intersection of that line with the domain of f). We may apply the same ideas for any vector $v \in \mathbb{R}^n$ instead of e_j. This gives rise to the notion of a directional derivative.

Definition 1.5 (Directional derivative)**.** Suppose that $U \subseteq \mathbb{R}^n$ is an open set and $f\colon U \to \mathbb{R}$ is a function. Let $v \in \mathbb{R}^n$ and $x \in U$. Then

$$D_v f(x) = \lim_{h\to 0} \frac{f(x+hv) - f(x)}{h},$$

if it exists, is the *directional derivative* of f at x in the direction v.

The partial derivatives are examples of directional derivatives, namely $\frac{\partial f}{\partial x_j} = D_{e_j} f$. You will verify in an exercise that $D_v f(x) = \left(\frac{d}{dt} f(x+tv)\right)\big|_{t=0}$.

Example 1.6. Consider the function $f(x) = x_1^2 x_2 + x_3$ from Example 1.2. Let $x = (1,1,0)$ and $v = (1,2,-1)$. Then

$$f(x+tv) = f(1+t, 1+2t, -t) = (1+t)^2(1+2t) - t.$$

Differentiating at $t = 0$, we obtain $D_v f(x) = 3$.

In this particular example, we can check that

$$D_v f(x) = v \cdot \nabla f(x).$$

The same equation is true for a lot of other examples, but not always. We will see later how this relationship comes about under certain assumptions.

Example 1.7. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \frac{x_1|x_2|}{\sqrt{x_1^2+x_2^2}} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then we see that $f(x_1, 0) = 0$ for all $x_1 \in \mathbb{R}$ and $f(0, x_2) = 0$ for all $x_2 \in \mathbb{R}$. Therefore, we conclude that $\nabla f(0) = 0$.

Now we consider the directional derivatives $D_v f(0)$ for vectors $v \in \mathbb{R}^2 \setminus \{0\}$. We compute

$$f(tv) = \frac{tv_1|v_2|}{\sqrt{v_1^2+v_2^2}}.$$

(This is also true for $t = 0$, as then both sides vanish.) Hence

$$D_v f(0,0) = \frac{v_1|v_2|}{\sqrt{v_1^2+v_2^2}}.$$

1.3 The Jacobian matrix

If we have a vector-valued function $f\colon U \to \mathbb{R}^m$ (for some $m \in \mathbb{N}$), then we can represent it in terms of its component functions $f_1, \dots, f_m\colon U \to \mathbb{R}$. Thus

$$f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix}.$$

Then we can apply the above concepts to the component functions.

Definition 1.8 (Jacobian matrix)**.** Let $U \subseteq \mathbb{R}^n$ be an open set and $f\colon U \to \mathbb{R}^m$ a vector-valued function. Suppose that $x \in U$. If the component functions $f_1, \dots, f_m$ have partial derivatives at x, then

$$Jf(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \cdots & \frac{\partial f_m}{\partial x_n}(x) \end{pmatrix}$$

is the *Jacobian matrix* (or simply *Jacobian*) of f at x.

Example 1.9. Consider the $\mathbb{R}^2$-valued function $f\colon \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^2$ given by

$$f(x) = \frac{1}{\|x\|^2} \begin{pmatrix} x_1 \\ x_2 x_3 \end{pmatrix} = \frac{1}{x_1^2 + x_2^2 + x_3^2} \begin{pmatrix} x_1 \\ x_2 x_3 \end{pmatrix}, \quad x \neq 0.$$

Then we compute

$$Jf(x) = \frac{1}{\|x\|^4} \begin{pmatrix} -x_1^2 + x_2^2 + x_3^2 & -2x_1x_2 & -2x_1x_3 \\ -2x_1x_2x_3 & x_3(x_1^2 - x_2^2 + x_3^2) & x_2(x_1^2 + x_2^2 - x_3^2) \end{pmatrix}.$$

Chapter 2

Differentiable functions

Partial and directional derivatives are easy to compute, but they give limited information about the behaviour of a function. We now consider a somewhat different approach to differentiation. This is based on the idea that the derivative provides a linear approximation of a function. For a differentiable function $g\colon (a,b) \to \mathbb{R}$ of *one* variable, we know that for $x \in (a,b)$, the derivative $g'(x)$ is the slope of the tangent line to the graph of g at the point $(x, g(x))$ (see Figure 2.1). It therefore gives the best linear approximation to the function g near x.

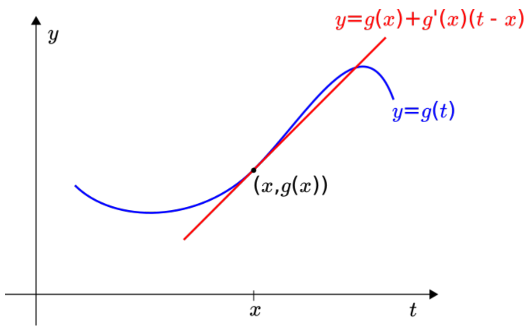

Figure 2.1: Tangent line to the graph of a function

2.1 The derivative

For functions of several variables, we can use a similar idea. The derivative then corresponds to a linear map which approximates the given function. This is no different if we have vector-valued functions $f\colon U \to \mathbb{R}^m$ for some $m \in \mathbb{N}$, so we formulate the concept for this situation.

Definition 2.1 (Derivative)**.** Let $U \subseteq \mathbb{R}^n$ be an open set and $f\colon U \to \mathbb{R}^m$ a vector-valued function. Let $x \in U$. We say that f is *differentiable* at x if there exists a linear map $A\colon \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{v \to 0} \frac{f(x+v) - f(x) - Av}{\|v\|} = 0.$$

If so, then we write $df(x) = A$. This is then called the *derivative* (or *Fréchet derivative*) of f at x.

This condition implies that the linear map $A = df(x)$ approximates f near x in the sense that

$$f(x+v) \approx f(x) + df(x)v$$

when $\|v\|$ is small. Indeed, the difference between the left-hand and the right-hand sides tends to 0 faster than $\|v\|$ as $v \to 0$. This formula may remind you of Taylor's theorem for functions of one variable, and is indeed its first order counterpart for functions of several variables.

Similarly to functions of one variable, differentiability implies continuity.

Proposition 2.2. *Let $U \subseteq \mathbb{R}^n$ be open and $f\colon U \to \mathbb{R}^m$ a vector-valued function that is differentiable at a point $x \in U$. Then f is continuous at x.*

Proof. Let $A = df(x)$ and define the function

$$R(y) = \frac{f(y) - f(x) - A(y-x)}{\|y-x\|}, \quad y \in U \setminus \{x\}.$$

Then the condition from Definition 2.1 (where $v = y - x$) implies that $\lim_{y\to x} R(y) = 0$. Moreover,

$$f(y) = f(x) + A(y-x) + \|y-x\|R(y), \quad y \in U \setminus \{x\},$$

by the definition of R. The map $A\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear and therefore automatically continuous. Therefore, $\lim_{y\to x} f(y) = f(x)$, as required. □

The derivative from Definition 2.1 is related to the Jacobian matrix and therefore to partial derivatives.

Theorem 2.3. *Suppose that $U \subseteq \mathbb{R}^n$ is an open set and $f\colon U \to \mathbb{R}^m$ is a function. Let $x \in U$ such that f is differentiable at x. Then the Jacobian matrix $Jf(x)$ exists (i.e., all the partial derivatives of all the component functions exist at x). Moreover, the linear map $df(x)$ is represented by the matrix $Jf(x)$ with respect to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$.*

In particular, the derivative $df(x)$ is unique.

Proof. Let $j \in \{1, \dots, n\}$. By the condition of Definition 2.1,

$$\frac{f(x + he_j) - f(x)}{h} - df(x)e_j = \frac{h}{|h|} \frac{f(x + he_j) - f(x) - df(x)(he_j)}{\|he_j\|} \to 0$$

as $h \to 0$. Hence

$$\lim_{h \to 0} \frac{f(x + he_j) - f(x)}{h} = df(x)e_j.$$

Looking at the components of this equation, we conclude that the partial derivatives exist. Furthermore, the left-hand side is the j-th column of the matrix $Jf(x)$. Therefore, $Jf(x)$ is the matrix that represents $df(x)$. □

The converse is false: if we have a function such that the Jacobian matrix exists, then it does *not* follow that the function is differentiable.

Example 2.4. The following is a variant of Example 1.7. Define $f \colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{x_1 x_2}{x_1^2 + x_2^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then $\frac{\partial f}{\partial x_j}(0) = 0$ for $j = 1, 2$. But $f(h, h) = \frac{1}{2}$ for every $h \neq 0$. Thus f is not continuous at 0, so by Proposition 2.2 it cannot be differentiable at 0.

But if the partial derivatives are continuous, then differentiability follows.

Theorem 2.5. *Let $U \subseteq \mathbb{R}^n$ be an open set and $f \colon U \to \mathbb{R}^m$ a function. If there exists an open set $V \subseteq U$ with $x \in V$ such that all the partial derivatives of all the components of f exist everywhere in V and are continuous at x, then f is differentiable at x.*

Proof. Let $r > 0$ such that $B_{2r}(x) \subseteq V$. Choose $\ell \in \{1, \dots, m\}$. Given $v \in B_r(0)$, we write

$$\begin{aligned} &f_\ell(x + v) - f_\ell(x) \\ &\quad = f_\ell(x_1 + v_1, \dots, x_n + v_n) - f_\ell(x_1, x_2 + v_2, \dots, x_n + v_n) \\ &\qquad + f_\ell(x_1, x_2 + v_2, \dots, x_n + v_n) - f_\ell(x_1, x_2, x_3 + v_3, \dots, x_n + v_n) \\ &\qquad + \cdots + \\ &\qquad + f_\ell(x_1, \dots, x_{n-1}, x_n + v_n) - f_\ell(x_1, \dots, x_n). \end{aligned} \tag{2.1}$$

For $j = 1, \dots, n$, we consider the functions

$$g_j(t) = f_\ell(x_1, \dots, x_{j-1}, x_j + t, x_{j+1} + v_{j+1}, \dots, x_n + v_n), \quad -r < t < r.$$

We compute

$$g_j'(t) = \frac{\partial f_\ell}{\partial x_j}(x_1, \ldots, x_{j-1}, x_j + t, x_{j+1} + v_{j+1}, \ldots, x_n + v_n)$$

for $-r < t < r$. By the mean value theorem, there exist numbers $s_1, \ldots, s_n$ such that s_j is between 0 and v_j and $g_j(v_j) - g_j(0) = v_j g_j'(s_j)$. That is,

$$\begin{aligned} f_\ell(x_1, \ldots, x_{j-1}, x_j + v_j, \ldots, x_n + v_n) - f_\ell(x_1, \ldots, x_j, x_{j+1} + v_{j+1}, \ldots, x_n + v_n) \\ = v_j \frac{\partial f_\ell}{\partial x_j}(x_1, \ldots, x_{j-1}, x_j + s_j, x_{j+1} + v_{j+1}, \ldots, x_n + v_n) \end{aligned}$$

for $j = 1, \ldots, n$. Hence

$$f_\ell(x + v) - f_\ell(x) = \sum_{j=1}^{n} v_j \frac{\partial f_\ell}{\partial x_j}(x_1, \ldots, x_{j-1}, x_j + s_j, x_{j+1} + v_{j+1}, \ldots, x_n + v_n).$$

Now set

$$\psi_{j\ell} = \frac{\partial f_\ell}{\partial x_j}(x_1, \ldots, x_{j-1}, x_j + s_j, x_{j+1} + v_{j+1} \ldots, x_n + v_n) - \frac{\partial f_\ell}{\partial x_j}(x)$$

for $j = 1, \ldots, n$ and $\ell = 1, \ldots, m$. Then we can write the above equation as

$$f_\ell(x + v) - f_\ell(x) = \sum_{j=1}^{n} v_j \frac{\partial f_\ell}{\partial x_j}(x) + \sum_{j=1}^{n} v_j \psi_{j\ell}.$$

If we let $v \to 0$, then $s_j \to 0$ for $j = 1, \ldots, n$, because s_j is between 0 and v_j. By the continuity of the partial derivatives, we then conclude that $\psi_{j\ell} \to 0$ as well. The quantities $v_j / \|v\|$, on the other hand, stay bounded. Hence

$$\lim_{v \to 0} \frac{f_\ell(x + v) - f_\ell(x) - Jf_\ell(x)v}{\|v\|} = \lim_{v \to 0} \sum_{j=1}^{n} \frac{v_j}{\|v\|} \psi_{j\ell} = 0.$$

Since this is true for all the components, we conclude that

$$\lim_{v \to 0} \frac{f(x + v) - f(x) - Jf(x)v}{\|v\|} = 0.$$

Hence f is differentiable at x, and its derivative is the linear map $v \mapsto Jf(x)v$. $\square$

2.2 The chain rule

We have the following formula for the derivatives of composite functions.

Theorem 2.6 (Chain rule)**.** *Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open sets. Suppose that $f\colon U \to \mathbb{R}^m$ and $g\colon V \to \mathbb{R}^k$ are two functions such that $f(U) \subseteq V$. Let $x \in U$. If f is differentiable at x and g is differentiable at $f(x)$, then the composition $g \circ f$ is differentiable at x with*

$$d(g \circ f)(x) = dg(f(x)) \circ df(x).$$

In terms of the Jacobian matrices, we can express the chain rule by the formula

$$J(g \circ f)(x) = Jg(f(x))\, Jf(x).$$

Proof. Write $y = f(x)$ and define

$$\phi(v) = f(x+v) - f(x) - df(x)v$$

for $v \in U - x$ and

$$\psi(w) = g(y+w) - g(y) - dg(y)w$$

for $w \in V - y$. Then

$$\lim_{v \to 0} \frac{\phi(v)}{\|v\|} = 0 \quad \text{and} \quad \lim_{w \to 0} \frac{\psi(w)}{\|w\|} = 0 \tag{2.2}$$

by the definition of the derivative.

Let $v \in \mathbb{R}^n \setminus \{0\}$ such that $x + v \in U$ and set $w = f(x+v) - f(x)$. If $w = 0$, then obviously $g(f(x+v)) - g(f(x)) = 0$. Otherwise,

$$\begin{aligned} g(f(x+v)) - g(f(x)) &= dg(y)(f(x+v) - f(x)) + \psi(w) \\ &= dg(y)\left(df(x)v + \phi(v)\right) + \frac{\psi(w)}{\|w\|}\|f(x+v) - f(x)\| \\ &= dg(y)df(x)v + dg(y)\phi(v) + \frac{\psi(w)}{\|w\|}\|df(x)v + \phi(v)\|. \end{aligned}$$

Hence

$$\begin{aligned} &\frac{g(f(x+v)) - g(f(x)) - dg(y)df(x)v}{\|v\|} \\ &\qquad\qquad = dg(y)\frac{\phi(v)}{\|v\|} + \frac{\psi(w)}{\|w\|}\left\|df(x)\frac{v}{\|v\|} + \frac{\phi(v)}{\|v\|}\right\|. \end{aligned}$$

If $\|v\| \to 0$, then $\|w\| \to 0$ as well by the continuity of f at x. Moreover, if $\|\cdot\|_{\text{op}}$ denotes the operator norm on linear maps $\mathbb{R}^n \to \mathbb{R}^m$, then

$$\left\| df(x) \frac{v}{\|v\|} \right\| \leq \|df(x)\|_{\text{op}}.$$

Together with (2.2), this implies that

$$\lim_{v \to 0} \frac{g(f(x+v)) - g(f(x)) - dg(y)df(x)v}{\|v\|} = 0.$$

Hence $dg(y) \circ df(x)$ is the derivative of $g \circ f$ at x. $\square$

Example 2.7. Let $f\colon \mathbb{R} \to \mathbb{R}^2$ and $g\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $f(t) = \left(\begin{smallmatrix} t \\ t^2 \end{smallmatrix}\right)$ and $g(x) = x_2 \sin x_1$. Then

$$Jf(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix}, \quad Jg(x) = (x_2 \cos x_1, \sin x_1).$$

So $Jg(f(t)) = (t^2 \cos t, \sin t)$ and

$$J(g \circ f)(t) = Jg(f(t))Jf(t) = (t^2 \cos t, \sin t) \begin{pmatrix} 1 \\ 2t \end{pmatrix} = t^2 \cos t + 2t \sin t.$$

We can check that this is true by computing the derivative of $(g \circ f)(t) = t^2 \sin t$.

Chapter 3

Second order derivatives

If the partial derivatives of a function $f\colon U \to \mathbb{R}$ exist at every $x \in U$, then we obtain new functions $\frac{\partial f}{\partial x_j}\colon U \to \mathbb{R}$ for $j = 1, \ldots, n$. They may have partial derivatives of their own. If so, this gives rise to *second order partial derivatives*. We define

$$\frac{\partial^2 f}{\partial x_j \partial x_k} = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_k}\right) \quad \text{and} \quad \frac{\partial^2 f}{\partial x_j^2} = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_j}\right)$$

for $j, k = 1, \ldots, n$. Partial derivatives of even higher order are defined similarly.

3.1 The Hessian matrix

We now have a closer look at the second order derivatives.

Example 3.1. Consider the function $f\colon \mathbb{R}^3 \to \mathbb{R}$ given by $f(x) = x_1^2 x_2 + x_1 x_3^2$. Then

$$\begin{aligned}
\frac{\partial^2 f}{\partial x_1^2}(x) &= 2x_2, & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) &= 2x_1, & \frac{\partial^2 f}{\partial x_1 \partial x_3}(x) &= 2x_3,\\
\frac{\partial^2 f}{\partial x_2 \partial x_1}(x) &= 2x_1, & \frac{\partial^2 f}{\partial x_2^2}(x) &= 0, & \frac{\partial^2 f}{\partial x_2 \partial x_3}(x) &= 0,\\
\frac{\partial^2 f}{\partial x_3 \partial x_1}(x) &= 2x_3, & \frac{\partial^2 f}{\partial x_3 \partial x_2}(x) &= 0, & \frac{\partial^2 f}{\partial x_3^2}(x) &= 2x_1.
\end{aligned}$$

It is no accident that some of these second order partial derivatives coincide. We will see shortly that

$$\frac{\partial^2 f}{\partial x_j \partial x_k} = \frac{\partial^2 f}{\partial x_k \partial x_j} \tag{3.1}$$

holds for $j, k = 1, \ldots, n$ under relatively modest assumptions on f.

Definition 3.2 (Hessian matrix)**.** Let $U \subseteq \mathbb{R}^n$ be open. Suppose that $f\colon U \to \mathbb{R}$ is a function that has second order partial derivatives at a point $x \in U$. Then the $(n \times n)$-matrix

$$Hf(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(x) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(x) & \cdots & \frac{\partial^2 f}{\partial x_n^2}(x) \end{pmatrix}$$

is called the *Hessian matrix* (or simply *Hessian*) of f at x.

Theorem 3.3 (Symmetry of the Hessian matrix)**.** *Suppose that $U \subseteq \mathbb{R}^n$ is open and $f\colon U \to \mathbb{R}$ is a function such that all the first and second derivatives exists and are continuous in U. Then $(Hf(x))^T = Hf(x)$ for all $x \in U$.*

Here $(Hf(x))^T$ denotes the transposed Hessian matrix. In other words, under the assumptions of the theorem, equation (3.1) holds for $j, k = 1, \ldots, n$. The assumption on the continuity of the partial derivatives is required, as the statement is not true otherwise.

Proof. Fix $x \in U$. As U is open, there exists $r > 0$ such that $B_{2r}(x) \subseteq U$. Choose $h, \tilde{h} \in (-r, r)$ and consider the function (of one variable) $\phi\colon (-r, r) \to \mathbb{R}$ given by

$$\phi(s) = f(x + se_j + \tilde{h}e_k) - f(x + se_j).$$

Then ϕ is differentiable with

$$\phi'(s) = \frac{\partial f}{\partial x_j}(x + se_j + \tilde{h}e_k) - \frac{\partial f}{\partial x_j}(x + se_j).$$

By the mean value theorem, there exists a number s_1 between 0 and h such that $\phi(h) - \phi(0) = h\phi'(s_1)$. That is,

$$\begin{aligned} f(x + he_j + \tilde{h}e_k) &- f(x + he_j) - f(x + \tilde{h}e_k) + f(x) \\ &= h\left(\frac{\partial f}{\partial x_j}(x + s_1 e_j + \tilde{h}e_k) - \frac{\partial f}{\partial x_j}(x + s_1 e_j)\right). \end{aligned} \tag{3.2}$$

We now apply the mean value theorem in a similar way to the function $\psi\colon (-r, r) \to \mathbb{R}$ given by

$$\psi(t) = \frac{\partial f}{\partial x_j}(x + s_1 e_j + te_k).$$

This yields a number t_1 between 0 and $\tilde{h}$ such that

$$\frac{\partial f}{\partial x_j}(x + s_1 e_j + \tilde{h} e_k) - \frac{\partial f}{\partial x_j}(x + s_1 e_j) = \tilde{h} \frac{\partial^2 f}{\partial x_k \partial x_j}(x + s_1 e_j + t_1 e_k).$$

Inserting this into the previous equation (3.2), we obtain

$$f(x + h e_j + \tilde{h} e_k) - f(x + h e_j) - f(x + \tilde{h} e_k) + f(x) = h\tilde{h} \frac{\partial^2 f}{\partial x_k \partial x_j}(x + s_1 e_j + t_1 e_k).$$

Repeating the same procedure with j and k exchanged will give two numbers s_2 and t_2, with s_2 between 0 and h and t_2 between 0 and $\tilde{h}$, such that

$$f(x + h e_j + \tilde{h} e_k) - f(x + h e_j) - f(x + \tilde{h} e_k) + f(x) = h\tilde{h} \frac{\partial^2 f}{\partial x_j \partial x_k}(x + s_2 e_j + t_2 e_k).$$

Hence

$$\frac{\partial^2 f}{\partial x_k \partial x_j}(x + s_1 e_j + t_1 e_k) = \frac{\partial^2 f}{\partial x_j \partial x_k}(x + s_2 e_j + t_2 e_k).$$

Finally we use the continuity of the partial derivatives. As we let $h, \tilde{h} \to 0$, the corresponding numbers s_1, t_1, s_2, t_2 will tend to 0 as well. In the limit, we obtain the equation

$$\frac{\partial^2 f}{\partial x_k \partial x_j}(x) = \frac{\partial^2 f}{\partial x_j \partial x_k}(x).$$

As this holds true for every pair of indices j, k, this proves the symmetry of $Hf(x)$. □

3.2 Taylor's theorem

Recall Taylor's theorem for functions $g\colon (a,b) \to \mathbb{R}$ of one variable: if g is k times continuously differentiable in (a,b), then for any $x, y \in (a,b)$ there exists t between x and y such that

$$g(y) = g(x) + g'(x)(y - x) + \cdots + \frac{g^{(k-1)}(x)}{(k-1)!}(y - x)^{k-1} + \frac{g^{(k)}(t)}{k!}(y - x)^k.$$

It follows that there is a function $R\colon (a,b) \to \mathbb{R}$ such that $\lim_{y \to x} R(y) = 0$ and

$$g(y) = g(x) + g'(x)(y - x) + \cdots + \frac{g^{(k)}(x)}{k!}(y - x)^k + R(y)(y - x)^k.$$

For functions of several variables, we have similar statements. However, while the basic ideas are similar to the above, the formulation becomes more complicated. For this reason, we only give Taylor's theorem for second order derivatives here. The corresponding first order formula is a direct consequence of Theorem 2.3 and is discussed in the exercises.

Theorem 3.4 (Taylor; second order version)**.** *Let $U \subseteq \mathbb{R}^n$ be open and $f\colon U \to \mathbb{R}$ a function with continuous partial derivatives up to second order. Let $x \in U$.*

1. *Suppose that $y \in U$ and consider the line segment $L = \{sy + (1 - s)x \colon s \in [0,1]\}$ between x and y. If $L \subseteq U$, then there exists $z \in L$ such that*
$$f(y) = f(x) + df(x)(y-x) + \frac{1}{2}(y-x)\cdot Hf(z)(y-x). \tag{3.3}$$
2. *Suppose that $r > 0$ is such that $B_r(x) \subseteq U$. Then there exists a function $R\colon B_r(x) \to \mathbb{R}$ such that $\lim_{y\to x} R(y) = 0$ and*
$$f(y) = f(x)+df(x)(y-x)+\frac{1}{2}(y-x)\cdot Hf(x)(y-x)+R(y)\|y-x\|^2 \tag{3.4}$$
for all $y \in B_r(x)$.

Proof. The idea is to apply Taylor's theorem for functions of one variable to
$$g(s) = f(sy + (1-s)x).$$
If we differentiate with respect to s, we obtain the formula
$$g'(s) = df(sy+(1-s)x)(y-x) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(sy+(1-s)x)(y_j - x_j)$$
by the chain rule (Theorem 2.6) and Theorem 2.3. Similarly,
$$\begin{aligned} g''(s) &= \sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_k}(sy+(1-s)x)(y_j-x_j)(y_k-x_k) \\ &= (y-x)\cdot Hf(sy+(1-s)x)(y-x). \end{aligned}$$

If $L \subseteq U$, then these calculations are valid at least in an interval $I \subseteq \mathbb{R}$ with $[0,1] \subseteq I$ and the second derivative is continuous. Hence we may apply Taylor's theorem for functions of one variable, which gives the first statement.

If $B_r(x) \subseteq U$, then the conclusion from the first statement applies to any $y \in B_r(x)$. Hence for every $y \in B_r(x)$, there exists $z_y \in \{sy + (1-s)x \colon s \in [0,1]\}$ such that (3.3) holds true for z_y instead of z. Define

$$R(y) = \frac{1}{2\|y-x\|^2}(y-x) \cdot \big(Hf(z_y) - Hf(x)\big)(y-x)$$

for $y \in B_r(x) \setminus \{x\}$ and $R(x) = 0$. Then (3.4) holds true. As the second partial derivatives of f are continuous, it also follows that

$$|R(y)| \leq \|Hf(z_y) - Hf(x)\|_{\text{op}} \to 0$$

as $y \to x$. This proves the second statement. □

Chapter 4

Minima and maxima

We now study how we can use the derivatives to find the local minima and maxima of a function. More precisely, we are interested in the following concepts.

Definition 4.1 (Local minima and maxima)**.** Let $U \subseteq \mathbb{R}^n$ be an open set and $f\colon U \to \mathbb{R}$ a function. A point $x \in U$ is called a *local minimum* of f if there exists $r > 0$ such that $B_r(x) \subseteq U$ and $f(x) \leq f(y)$ for all $y \in B_r(x)$. It is called a *local maximum* if there exists $r > 0$ such that $B_r(x) \subseteq U$ and $f(x) \geq f(y)$ for all $y \in B_r(x)$.

4.1 A necessary condition

If we want to find the local minima and maxima of a differentiable function, then the following is helpful.

Theorem 4.2. *Suppose that $U \subseteq \mathbb{R}^n$ is open and $f\colon U \to \mathbb{R}$ is a function. If $x \in U$ is a local minimum or local maximum and f is differentiable at x, then $df(x) = 0$.*

Proof. Let $j \in \{1, \ldots, n\}$. If x is a local minimum of f and $r > 0$ is as in Definition 4.1, then 0 is a local minimum of the function $g\colon (-r, r) \to \mathbb{R}$ defined by $g(t) = f(x + te_j)$. We know that g is differentiable at 0 with

$$g'(0) = \frac{\partial f}{\partial x_j}(x).$$

Hence $\frac{\partial f}{\partial x_j}(x) = 0$.

As this is true for every j and the derivative $df(x)$ is determined by the partial derivatives by Theorem 2.3, it follows that $df(x) = 0$.

The arguments for local maxima are the same. $\square$

Example 4.3. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ with $f(x) = x_1^2 - 3x_1x_2 + 4x_2^2 + x_1 - 8x_2$. What are its local minima/maxima?

We compute $Jf(x) = (2x_1 - 3x_2 + 1, -3x_1 + 8x_2 - 8)$. Any local minimum or maximum has to satisfy the equations

$$\begin{aligned} 2x_1 - 3x_2 + 1 &= 0 \\ -3x_1 + 8x_2 - 8 &= 0 \end{aligned}$$

simultaneously. The unique solution is $x = (16/7, 13/7)$. At the moment we don't have any general tools to determine whether it is indeed a local minimum or maximum, but it is our only candidate.

Because the solutions of the equation $df(x) = 0$ have specific relevance for this problem, they have a name.

Definition 4.4 (Critical point/saddle point)**.** Let $U \subseteq \mathbb{R}^n$ be open and $f\colon U \to \mathbb{R}$ a function. If $x \in U$ satisfies $df(x) = 0$, then it is called a *critical point* of f.

A critical point that is neither a local minimum nor a local maximum is called a *saddle point* of f.

4.2 Sufficient conditions

The first derivative on its own doesn't tell us whether a critical point is a local minimum, a local maximum, or a saddle point. With the help of the Hessian matrix, however, we can often find out.

We recall a few facts from linear algebra. Consider a symmetric $(n \times n)$-matrix A with real entries. Then there exists a change of coordinates such that A becomes diagonal. In particular, the following notions depend only on the eigenvalues.

1. We say that A is *positive definite* if $v \cdot Av > 0$ for all $v \in \mathbb{R}^n \setminus \{0\}$. This is the case if, and only if, all eigenvalues of A are positive. Indeed, if $\lambda_1 > 0$ is the smallest eigenvalue, then $v \cdot Av \geq \lambda_1 \|v\|^2$ for all $v \in \mathbb{R}^n$.

2. We say that A is *negative definite* if $v \cdot Av < 0$ for all $v \in \mathbb{R}^n \setminus \{0\}$. This is the case if, and only if, all eigenvalues of A are negative. Indeed, if $\lambda_n < 0$ is the greatest eigenvalue, then $v \cdot Av \leq \lambda_n \|v\|^2$ for all $v \in \mathbb{R}^n$.

3. We say that A is *indefinite* if there exist $v_1, v_2 \in \mathbb{R}^n$ such that $v_1 \cdot Av_1 > 0$ and $v_2 \cdot Av_2 < 0$. This is the case if, and only if, A has positive and negative eigenvalues.

Note that these three categories do not cover all possible cases, as we may have 0 as an eigenvalue.

Theorem 4.5 (Second derivative test)**.** *Let $U \subseteq \mathbb{R}^n$ be an open set and $f\colon U \to \mathbb{R}$ a function with continuous partial derivatives up to second order. Suppose that $x \in U$ is a critical point of f.*

1. *If $Hf(x)$ is positive definite, then x is a local minimum of f.*
2. *If $Hf(x)$ is negative definite, then x is a local maximum of f.*
3. *If $Hf(x)$ is indefinite, then x is a saddle point of f.*

If $Hf(x)$ is neither positive definite nor negative define nor indefinite, then this test is inconclusive.

Proof. Let $r > 0$ such that $B_r(x) \subseteq U$. By Taylor's theorem (Theorem 3.4), there exists a function $R\colon B_r(x) \to \mathbb{R}$ such that

$$f(y) = f(x) + \frac{1}{2}(y - x) \cdot Hf(x)(y - x) + R(y)\|y - x\|^2 \tag{4.1}$$

for all $y \in B_r(x)$ and $\lim_{y \to x} R(y) = 0$.

If $Hf(x)$ is positive definite, then choose $c > 0$ such that $v \cdot Hf(x)v \geq c\|v\|^2$ for all $v \in \mathbb{R}^n$. (For example, we may choose the smallest eigenvalue of $Hf(x)$.) Since $R(y) \to 0$ as $y \to x$, we may choose $s \in (0, r]$ such that $|R(y)| \leq c/2$ for all $y \in B_s(x)$. Then

$$\begin{aligned} f(y) &= f(x) + \frac{1}{2}(y - x) \cdot Hf(x)(y - x) + R(y)\|y - x\|^2 \\ &\geq f(x) + \frac{c}{2}\|y - x\|^2 - \frac{c}{2}\|y - x\|^2 \\ &\geq f(x) \end{aligned}$$

for all $y \in B_s(x)$. Therefore, we have a local minimum at x.

If $Hf(x)$ is negative definite, we use the same arguments to conclude that we have a local maximum.

If $Hf(x)$ is indefinite, choose two eigenvectors $v_1, v_2 \in \mathbb{R}^n$ of $Hf(x)$ that belong to eigenvalues $\lambda_1 < 0$ and $\lambda_2 > 0$, respectively. Let $c = \min\{-\lambda_1, \lambda_2\}$ and choose $s \in (0, r]$ such that $|R(y)| \leq c/4$ for all $y \in B_s(x)$.

Now insert $y = x + tv_1$ into (4.1). This gives

$$f(y) = f(x) + \frac{1}{2}t^2 \left(v_1 \cdot Hf(x)v_1 + 2R(y)\|v_1\|^2\right).$$

If $|t|$ is chosen so small that $y \in B_s(x)$, then

$$v_1 \cdot Hf(x)v_1 + 2R(y)\|v_1\|^2 \leq \lambda_1\|v_1\|^2 + \frac{c}{2}\|v_1\|^2 \leq -\frac{c}{2}\|v_1\|^2 < 0.$$

Hence $f(x + tv_1) < f(x)$ whenever $|t|$ is sufficiently small. So x cannot be a local minimum.

Similarly, using v_2 instead of v_1, we show that x is no local maximum. $\square$

Example 4.6. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ with $f(x) = x_1^2 - 3x_1x_2 + 4x_2^2 + x_1 - 8x_2$ from Example 4.3. We already know that there is a unique critical point at $x = (16/7, 13/7)$. Now we compute

$$Hf(x) = \begin{pmatrix} 2 & -3 \\ -3 & 8 \end{pmatrix}.$$

The eigenvalues are $5 \pm 3\sqrt{2}$, which are both positive. Hence we have a local minimum.

Example 4.7. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x) = \cos(x_1)e^{x_2^2}$. We compute

$$Jf(x) = e^{x_2^2}(-\sin(x_1), 2x_2\cos(x_1))$$

and

$$Hf(x) = e^{x_2^2}\begin{pmatrix} -\cos(x_1) & -2x_2\sin(x_1) \\ -2x_2\sin(x_1) & (4x_2^2 + 2)\cos(x_1) \end{pmatrix}.$$

The critical points of f will satisfy the equations $-e^{x_2^2}\sin(x_1) = 0$ and $2e^{x_2^2}x_2\cos(x_1) = 0$ simultaneously. As the exponential function has no zeroes and the functions sin and cos have no common zeroes, we must have $\sin(x_1) = 0$ and $x_2 = 0$. The solutions are of the form $(\ell\pi, 0)$ for $\ell \in \mathbb{Z}$.

At these critical points, we find that

$$Hf(\ell\pi, 0) = (-1)^\ell\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}.$$

The matrix $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 2 \end{smallmatrix}\right)$ has a positive eigenvalue (namely 2) and a negative one (namely -1). Hence it is indefinite, and the same applies when we multiply by -1. Therefore, all of the critical points are saddle points.

Chapter 5

Complex differentiation

We now consider functions of one *complex* variable with complex values. There is a bijective map $\mathbb{R}^2 \to \mathbb{C}$ with $(x, y) \mapsto x + iy$, and the complex plane inherits the metric space structure from $\mathbb{R}^2$ from this identification. The notions of balls (also called disks in this context), open/closed sets, or convergence in $\mathbb{C}$ come from this structure. For example, for $z = x + iy \in \mathbb{C}$ and $r > 0$, we have the disk $B_r(z) = \{w \in \mathbb{C} \colon |w - z| < r\}$.

5.1 The complex derivative

The definition of the derivative for functions of a complex variable is quite similar to functions of one real variable.

Definition 5.1 (Complex derivative)**.** Let $U \subseteq \mathbb{C}$ be an open set and $f \colon U \to \mathbb{C}$ a function. For $z \in U$, we say that f is *complex differentiable* (or *differentiable* for short when there is no danger of confusion with real differentiability) at z if

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists. If so, then $f'(z)$ is called the *complex derivative* of f at z.

Note that in this limit, the number h is complex.

Example 5.2. Consider the function $f \colon \mathbb{C} \to \mathbb{C}$ with $f(z) = z$. Then f is differentiable at every point $z \in \mathbb{C}$, as

$$f'(z) = \lim_{h \to 0} \frac{z + h - z}{h} = 1.$$

Example 5.3. Recall the complex conjugate, defined by $\overline{x+iy} = x - iy$. If $f : \mathbb{C} \to \mathbb{C}$ is defined by $f(z) = \bar{z}$, then

$$\frac{f(z+h) - f(z)}{h} = \frac{\bar{z} + \bar{h} - \bar{z}}{h} = \frac{\bar{h}}{h}.$$

We claim that there is no limit as $h \to 0$. In order to see why, we first consider $h \in \mathbb{R}$. Then $\bar{h}/h = 1$. But for $h \in i\mathbb{R}$, we find that $\bar{h}/h = -1$, regardless of the size of $|h|$. Therefore, there is no limit as $h \to 0$, and the function is *not* complex differentiable anywhere.

Since this definition is so similar to the derivative of functions of one real variable, it comes as no surprise that the familiar differentiation rules still apply.

Theorem 5.4 (Algebra of derivatives)**.** *Let $U \subseteq \mathbb{C}$ be an open set and $f, g : U \to \mathbb{C}$ two functions. Suppose that $z \in U$ such that f and g are differentiable at z. Then*

1. *$f + g$ is differentiable at z with $(f+g)'(z) = f'(z) + g'(z)$,*
2. *fg is differentiable at z with $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$, and*
3. *if $g(z) \neq 0$, then f/g is differentiable at z with*

$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2}.$$

Proof. This follows with the same arguments as for the real derivative. □

Theorem 5.5 (Chain rule)**.** *Let $U, V \subseteq \mathbb{C}$ be two open sets and $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ two functions such that $f(U) \subseteq V$. Suppose that $z \in U$ is such that f is differentiable at z and g is differentiable at $f(z)$. Then $g \circ f$ is differentiable at z with $(g \circ f)'(z) = g'(f(z))f'(z)$.*

Proof. Again it suffices to adapt the proof for the real derivative. □

The following is common terminology for a function that has a complex derivative everywhere in its domain.

Definition 5.6 (Holomorphic function)**.** Let $U \subseteq \mathbb{C}$ be open. If a function $f : U \to \mathbb{C}$ is complex differentiable at every $z \in U$, then it is called *holomorphic* in U.

5.2 The Cauchy-Riemann equations

With the usual identification of $\mathbb{C}$ and $\mathbb{R}^2$, a function of one complex variable gives rise to a function of two real variables. Although the complex derivative is different from the derivative for functions of two real variables, there is still a connection.

Theorem 5.7 (Cauchy-Riemann)**.** *Let $U \subseteq \mathbb{C}$ be an open set and define $V = \{(x, y) \in \mathbb{R}^2 \colon x + iy \in U\}$. Suppose that $f\colon U \to \mathbb{C}$ and $u, v\colon V \to \mathbb{R}$ are functions such that*

$$f(x + iy) = u(x, y) + iv(x, y), \quad (x, y) \in V.$$

Then for $z = x + iy \in U$, the following are equivalent.

1. *The function f is complex differentiable at z.*
2. *Both u and v are real differentiable at (x, y) and satisfy the* Cauchy-Riemann equations

$$\frac{\partial u}{\partial x}(x, y) = \frac{\partial v}{\partial y}(x, y), \quad \frac{\partial u}{\partial y}(x, y) = -\frac{\partial v}{\partial x}(x, y).$$

If these conditions are satisfied, then

$$f'(z) = \frac{\partial u}{\partial x}(x, y) - i\frac{\partial u}{\partial y}(x, y). \tag{5.1}$$

Another way to formulate the Cauchy-Riemann equations is to consider the vector-valued function $F\colon V \to \mathbb{R}^2$ with $F = \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)$. Then the equations are satisfied if, and only if, the Jacobian matrix of F at (x, y) is of the form

$$JF(x, y) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

for some $a, b \in \mathbb{R}$.

Proof. First assume that f is complex differentiable at $z = x + iy \in U$ with $f'(z) = a + ib$. Write $h = k + i\ell$ and define

$$R(h) = \frac{f(z + h) - f(z)}{h} - f'(z), \quad h \in (U - z) \setminus \{0\}.$$

Then $\lim_{h\to 0} R(h) = 0$ and

$$f(z + h) = f(z) + hf'(z) + hR(h). \tag{5.2}$$

In terms of u and v, we can write equation (5.2) as follows (splitting it into the real and the imaginary parts):

$$
\begin{aligned}
u(x+k, y+\ell) &= u(x,y) + ak - b\ell + \operatorname{Re}(hR(h)),\\
v(x+k, y+\ell) &= v(x,y) + bk + a\ell + \operatorname{Im}(hR(h)).
\end{aligned}
$$

Thus for the linear operators $A, B\colon \mathbb{R}^2 \to \mathbb{R}$, given by $A(k,\ell) = ak - b\ell$ and $B(k,\ell) = bk + a\ell$, we find that

$$
\begin{aligned}
\frac{u(x+k, y+\ell) - u(x,y) - A(k,\ell)}{\|(k,\ell)\|} &= \operatorname{Re}\left(\frac{h}{|h|}R(h)\right) \to 0\\
\frac{v(x+k, y+\ell) - v(x,y) - B(k,\ell)}{\|(k,\ell)\|} &= \operatorname{Im}\left(\frac{h}{|h|}R(h)\right) \to 0
\end{aligned}
$$

as $(k,\ell) \to 0$. This means that $A = du(x,y)$ and $B = dv(x,y)$. In terms of partial derivatives, this is

$$
\frac{\partial u}{\partial x}(x,y) = a, \qquad \frac{\partial u}{\partial y}(x,y) = -b, \tag{5.3}
$$

$$
\frac{\partial v}{\partial x}(x,y) = b, \qquad \frac{\partial v}{\partial y}(x,y) = a. \tag{5.4}
$$

Comparing these, we obtain the Cauchy-Riemann equations. We also see that (5.1) holds true.

Conversely, suppose that u, v are real differentiable and the Cauchy-Riemann equations hold at (x,y). Then there exist $a, b \in \mathbb{R}$ such that (5.3) and (5.4) hold true. Reversing the above steps, we can then find a function R such that (5.2) holds true and $\lim_{h\to 0} R(h) = 0$. This means that f is complex differentiable at $z = x + iy$. $\square$

Example 5.8. Consider the exponential function $\exp\colon \mathbb{C} \to \mathbb{C}$ defined by

$$
\exp(z) = e^z = e^{x+iy} = e^x(\cos y + i \sin y), \quad z = x + iy \in \mathbb{C}.
$$

We write $u(x,y) = e^x \cos y$ and $v(x,y) = e^x \sin y$. We compute

$$
\begin{aligned}
\frac{\partial u}{\partial x}(x,y) &= e^x \cos y, & \frac{\partial u}{\partial y}(x,y) &= -e^x \sin y\\
\frac{\partial v}{\partial x}(x,y) &= e^x \sin y, & \frac{\partial v}{\partial y}(x,y) &= e^x \cos y.
\end{aligned}
$$

The Cauchy-Riemann equations are satisfied everywhere, so this function is holomorphic in $\mathbb{C}$. We also see that $\exp'(z) = \exp(z)$.

We may now also define $\cosh z = \frac{1}{2}(e^z + e^{-z})$ and $\sinh z = \frac{1}{2}(e^z - e^{-z})$. The algebra of derivatives then gives $\cosh'(z) = \sinh(z)$ and $\sinh'(z) = \cosh(z)$.

Finally, we define $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$ and $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$. These functions are now defined for every $z \in \mathbb{C}$. If restricted to $\mathbb{R}$, they coincide with the familiar cosine and sine. By the algebra of derivatives, they are holomorphic with $\cos'(z) = -\sin(z)$ and $\sin'(z) = \cos(z)$.

Chapter 6

Complex integration

We know from the theory of functions of a real variable that differentiation and integration complement each other, as shown by the fundamental theorem of calculus. We will see that something similar is true for complex variables as well. But we first have to discuss what we mean by integration in $\mathbb{C}$. If we want to integrate a complex-valued function $f\colon [a,b] \to \mathbb{C}$ of a *real* variable, then we simply integrate the real and imaginary parts separately:

$$\int_a^b f(t)\,dt = \int_a^b \operatorname{Re} f(t)\,dt + i \int_a^b \operatorname{Im} f(t)\,dt.$$

But as we mostly work with complex variables, we will need a more general notion of integration.

6.1 Contour integrals

Definition 6.1 (Contour)**.** A *contour* is a function $\gamma\colon [a,b] \to \mathbb{C}$, for $a, b \in \mathbb{R}$ with $a < b$, that is piecewise continuously differentiable in the following sense: γ is continuous on $[a,b]$ and there exist $t_0, t_1, \ldots, t_K \in [a,b]$ such that $a = t_0 < t_1 < \cdots < t_K = b$ and γ is continuously differentiable in (t_{k-1}, t_k) and γ' has a continuous extension to $[t_{k-1}, t_k]$ for every $k = 1, \ldots, K$.

The point $\gamma(a)$ is called the *initial point* and $\gamma(b)$ is called the *terminal point* of the contour. We say that the contour is *closed* if $\gamma(a) = \gamma(b)$. For a set $S \subseteq \mathbb{C}$, we speak of a contour *in* S if $\gamma([a,b]) \subseteq S$.

The following notation will be convenient. Given $z, w \in \mathbb{C}$, we write $[z,w]$ for the contour $[0,1] \to \mathbb{C}, t \mapsto tw + (1-t)z$. This contour has initial point z and terminal point w and takes values on the line segment between them.

Example 6.2. Let $\gamma\colon [-1, 1+\pi] \to \mathbb{C}$ be defined by

$$\gamma(t) = \begin{cases} t & \text{if } -1 \leq t \leq 1, \\ e^{i(t-1)} & \text{if } 1 < t \leq 1+\pi. \end{cases}$$

Then γ is a closed contour as $\gamma(1+\pi) = e^{i\pi} = -1 = \gamma(-1)$.

Contour integrals are related to, but not quite the same as the line integrals you have seen in MA10236.

Definition 6.3 (Contour integral)**.** Let $U \subseteq \mathbb{C}$ be an open set and $f\colon U \to \mathbb{C}$ a continuous function. Given a contour $\gamma\colon [a, b] \to U$, the *contour integral* of f along γ is

$$\int_\gamma f(z)\, dz = \int_a^b f(\gamma(t))\gamma'(t)\, dt.$$

Since γ is only piecewise continuously differentiable, the derivative γ' does not exist everywhere in $[a, b]$ in general. There are only finitely many points, however, where it does not exist, and they can be ignored for the purpose of integration. More precisely, we can choose a function $g\colon [a, b] \to \mathbb{C}$ such that $g(t) = \gamma'(t)$ at every $t \in [a, b]$ where the derivative exists, and then we can replace γ' by g in the above integral. The function g is not unique, but any two choices will differ at only finitely many points and will give rise to the same value.

Example 6.4. Let $f\colon \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = z + 1$. The contour $\gamma = [1, 2+i]$ is given by $\gamma(t) = (1+i)t + 1$ and satisfies $\gamma'(t) = 1 + i$ for $t \in [0, 1]$. Hence

$$\int_{[1,2+i]} (z+1)\, dz = \int_0^1 ((1+i)t + 2)(1+i)\, dt = \int_0^1 (2it + 2 + 2i)\, dt = 2 + 3i.$$

Example 6.5. Consider the closed contour $\gamma\colon [0, 2\pi] \to \mathbb{C}$ given by $\gamma(t) = e^{it}$, describing the unit circle in $\mathbb{C}$. Also consider the function $f\colon \mathbb{C} \to \mathbb{C}$ with $f(z) = z^2 + 2\bar{z}$ for $z \in \mathbb{C}$. Then $\gamma'(t) = ie^{it}$ and

$$\begin{aligned} \int_\gamma f(z)\, dz &= \int_0^{2\pi} f(e^{it}) ie^{it}\, dt \\ &= i\int_0^{2\pi} (e^{2it} + 2e^{-it})e^{it}\, dt \\ &= i\int_0^{2\pi} (e^{3it} + 2)\, dt \\ &= i\int_0^{2\pi} (\cos(3t) + 2)\, dt - \int_0^{2\pi} \sin(3t)\, dt = 4\pi i. \end{aligned}$$

Definition 6.6 (Contour connected)**.** A set $S \subseteq \mathbb{C}$ is called *contour connected* if for any two points $w, z \in S$, there exists a contour in S with initial point w and terminal point z.

6.2 Reparametrisation and concatenation

A contour $\gamma\colon [a,b] \to \mathbb{C}$ gives rise to a curve $\Gamma = \gamma([a,b])$ in $\mathbb{C}$. The same Γ arises, however, from many different contours, which can be thought of as different parametrisations of Γ.

While the contour integral is defined in terms of the contour, not the corresponding curve, it actually depends on the specific parametrisation only in a very weak way. If we know the curve and the orientation of the parametrisation, then the integral is determined. This allows us to think geometrically and illustrate contours with pictures.

In the following, suppose that $U \subseteq \mathbb{C}$ is an open set and $f\colon U \to \mathbb{C}$ a continuous function. If we have a contour $\gamma\colon [a,b] \to U$ and a continuously differentiable, strictly increasing function $\phi\colon [\tilde{a}, \tilde{b}] \to [a,b]$ (for some $\tilde{a}, \tilde{b} \in \mathbb{R}$ with $\tilde{a} < \tilde{b}$), then $\tilde{\gamma} = \gamma \circ \phi$ is also a contour in U. If $\phi(\tilde{a}) = a$ and $\phi(\tilde{b}) = b$, then

$$\int_{\tilde{\gamma}} f(z)\,dz = \int_\gamma f(z)\,dz.$$

(You will verify this formula in an exercise.)

Reparametrisations that reverse the orientation do not have this property. Given a contour $\gamma\colon [a,b] \to U$, we define $-\gamma\colon [-b,-a] \to U$ by $-\gamma(t) = \gamma(-t)$. Then it is easy to check that

$$\int_{-\gamma} f(z)\,dz = -\int_\gamma f(z)\,dz.$$

Often it is convenient to concatenate different contours as follows. Suppose that we have the contours $\gamma_1\colon [a_1,b_1] \to U$ and $\gamma_2\colon [a_2,b_2] \to U$. If $\gamma_1(b_1) = \gamma_2(a_2)$ (the terminal point of γ_1 coincides with the initial point of γ_2), then we can define $\hat{\gamma}\colon [a_1+a_2, b_1+b_2] \to U$ as follows:

$$\hat{\gamma}(t) = \begin{cases} \gamma_1(t-a_2) & \text{if } a_1+a_2 \le t \le b_1+a_2, \\ \gamma_2(t-b_1) & \text{if } b_1+a_2 < t \le b_1+b_2. \end{cases}$$

This is then also a contour in U, which first traces the curve given by γ_1 and then the curve given by γ_2. In this case, we can check that

$$\int_{\hat{\gamma}} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz.$$

We use the notation $\hat{\gamma} = \gamma_1 + \gamma_2$ for this concatenation.

It follows in particular that

$$\int_{\gamma-\gamma} f(z)\,dz = 0$$

for any contour γ in U.

Despite the notation, the symbols $+$ and $-$ do *not* correspond to a group operation here. In particular, we cannot combine any two contours this way, but only if their initial and terminal points coincide as described.

Because in order to evaluate the contour integral, it's enough to know the curve $\Gamma = \gamma([a,b])$ and the orientation of the parametrisation, we sometimes write

$$\int_\Gamma f(z)\,dz,$$

provided that it's clear from the context what the orientation is.

6.3 Estimates and convergence

In order to prove certain statements about integrals, we need some inequalities. We begin with an estimate that you have already seen for real integrands and is now extended to complex integrands.

Lemma 6.7. *Let $a, b \in \mathbb{R}$ with $a < b$ and let $f\colon [a,b] \to \mathbb{C}$ be continuous. Then*

$$\left|\int_a^b f(t)\,dt\right| \leq \int_a^b |f(t)|\,dt.$$

Proof. Define

$$I = \int_a^b f(t)\,dt.$$

If I happens to be real, then the inequality follows from simple facts about real integrals. In general, the idea is to multiply everything by $\bar{I}$ first, thus making it real.

We compute

$$|I|^2 = \bar{I}I = \bar{I}\int_a^b f(t)\,dt = \int_a^b \bar{I}f(t)\,dt = \operatorname{Re}\int_a^b \bar{I}f(t)\,dt,$$

because the expression on the left-hand side is obviously real. Hence

$$|I|^2 = \int_a^b \operatorname{Re}(\bar{I}f(t))\,dt \leq \int_a^b |\bar{I}f(t)|\,dt = \int_a^b |I||f(t)|\,dt = |I|\int_a^b |f(t)|\,dt.$$

If $I = 0$, then the desired inequality is obvious. Otherwise, it now suffices to divide by $|I|$ on both sides. $\square$

Definition 6.8 (Length of a contour)**.** Let $\gamma\colon [a,b] \to \mathbb{C}$ be a contour. Then its *length* is

$$L(\gamma) = \int_a^b |\gamma'(t)|\,dt.$$

Lemma 6.9. *Let $U \subseteq \mathbb{C}$ be an open set and $f\colon U \to \mathbb{C}$ a continuous function. Suppose that $\gamma\colon [a,b] \to U$ is a contour and set $\Gamma = \gamma([a,b])$. Then*

$$\left|\int_\gamma f(z)\,dz\right| \leq L(\gamma) \max_{z\in\Gamma} |f(z)|.$$

Proof. Define $M = \max_{z\in\Gamma} |f(z)|$. Then

$$\begin{aligned}\left|\int_\gamma f(z)\,dz\right| &= \left|\int_a^b f(\gamma(t))\gamma'(t)\,dt\right| \\ &\leq \int_a^b |f(\gamma(t))||\gamma'(t)|\,dt \\ &\leq M\int_a^b |\gamma'(t)|\,dt = ML(\gamma)\end{aligned}$$

by Lemma 6.7. $\square$

Next we study what happens when we integrate the members of a uniformly convergent sequence of functions.

Proposition 6.10. *Let $U \subseteq \mathbb{C}$ be an open set and suppose that $\gamma\colon [a,b] \to U$ is a contour. Set $\Gamma = \gamma([a,b])$. Let $(f_k)_{k\in\mathbb{N}}$ be a sequence of continuous functions $f_k\colon U \to \mathbb{C}$ and let $f\colon U \to \mathbb{C}$ be another continuous function. If $f_k \to f$ uniformly on Γ as $k \to \infty$, then*

$$\int_\gamma f(z)\,dz = \lim_{k\to\infty} \int_\gamma f_k(z)\,dz.$$

Proof. Define $M_k = \max_{z\in\Gamma} |f_k(z) - f(z)|$. Then by the uniform convergence, we know that $\lim_{k\to\infty} M_k = 0$. Lemma 6.9 then implies that

$$\left|\int_\gamma f_k(z)\,dt - \int_\gamma f(z)\,dz\right| = \left|\int_\gamma (f_k(z) - f(z))\,dz\right| \leq M_k L(\gamma) \to 0$$

as $k \to \infty$. $\square$

The following is an obvious consequence of Proposition 6.10.

Corollary 6.11. *Let* $U \subseteq \mathbb{C}$ *be an open set and suppose that* $\gamma\colon [a,b] \to U$ *is a contour. Set* $\Gamma = \gamma([a,b])$. *Let* $(f_k)_{k\in\mathbb{N}}$ *be a sequence of continuous functions* $f_k\colon U \to \mathbb{C}$ *and let* $f\colon U \to \mathbb{C}$ *be another continuous function. If* $f = \sum_{k=1}^{\infty} f_k$ *uniformly on* Γ, *then*

$$\int_\gamma f(z)\,dz = \sum_{k=1}^{\infty} \int_\gamma f_k(z)\,dz.$$

Chapter 7

Primitives

Similarly to integrals in real analysis, contour integrals are easily evaluated if a primitive of the integrand is known. But whereas for functions of a real variable, continuity is enough to guarantee the existence of a primitive, the situation is more complicated for complex variables.

In this chapter, we first discuss the complex counterpart to the fundamental theorem of calculus. Then we will see some results on the existence of primitives.

7.1 The fundamental theorem of complex integration

Definition 7.1 (Primitive)**.** Let $U \subseteq \mathbb{C}$ be an open set and $f\colon U \to \mathbb{C}$ a function. Another function $F\colon U \to \mathbb{C}$ is called a *primitive* of f if $F'(z) = f(z)$ for all $z \in U$.

We will see a connection between primitives and contour integrals. For the proof of the corresponding statement, we will need the following lemma.

Lemma 7.2. *Let $z_0 \in \mathbb{C}$ and $r > 0$. Suppose that $f, F\colon B_r(z_0) \to \mathbb{C}$ are two functions such that f is continuous and*

$$F(z) = F(z_0) + \int_{[z_0,z]} f(w)\,dw$$

for all $z \in B_r(z_0)$. Then F is differentiable at z_0 and $F'(z_0) = f(z_0)$.

Proof. For $h \in B_r(0)$, note that

$$\int_{[z_0,z_0+h]} f(z_0)\,dw = hf(z_0).$$

Hence

$$\begin{aligned}\left|\frac{F(z_0+h)-F(z_0)}{h}-f(z_0)\right| &= \left|\frac{1}{h}\int_{[z_0,z_0+h]}(f(w)-f(z_0))\,dw\right| \\ &\leq \frac{L([z_0,z_0+h])}{|h|}\max_{w\in[z_0,z_0+h]}|f(w)-f(z_0)|\end{aligned}$$

by Lemma 6.9. It is clear that $L([z_0, z_0+h]) = |h|$. By the continuity of f,

$$\max_{w\in[z_0,z_0+h]}|f(w)-f(z_0)| \leq \max_{|w-z_0|\leq|h|}|f(w)-f(z_0)| \to 0$$

as $h \to 0$. Therefore,

$$\left|\frac{F(z_0+h)-f(z_0)}{h}-f(z_0)\right| \to 0$$

as $h \to 0$, which implies that $F'(z_0) = f(z_0)$. □

Theorem 7.3 (Fundamental theorem of complex integration)**.** *Suppose that $U \subseteq \mathbb{C}$ is open and contour connected and $f\colon U \to \mathbb{C}$ is continuous.*

1. *If f has a primitive $F\colon U \to \mathbb{C}$, then for any contour $\gamma\colon [a,b] \to U$,*

$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a)). \tag{7.1}$$

2. *If*

$$\int_\gamma f(z)\,dz = 0$$

for every closed contour $\gamma\colon [a,b] \to U$, then f has a primitive.

Proof. Suppose that F is a primitive of f. By the chain rule, we compute

$$\frac{d}{dt}F(\gamma(t)) = F'(\gamma(t))\gamma'(t) = f(\gamma(t))\gamma'(t)$$

at any $t \in [a,b]$ where γ is differentiable. Hence

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt = \int_a^b \frac{d}{dt}F(\gamma(t))\,dt = F(\gamma(b)) - F(\gamma(a)).$$

This proves the first statement.

For the second statement, we first note that if we are to find a primitive, it must satisfy (7.1). The idea is therefore to use this formula to construct F.

To this end, fix an arbitrary point $z_0 \in U$. Given $z \in U$, we then pick a contour $\gamma\colon [a,b] \to U$ with initial point z_0 and terminal point z and set

$$F(z) = \int_\gamma f(w)\,dw.$$

But we must show that

- F is well-defined, i.e., it does not depend on the choice of γ, and
- it is actually a primitive, i.e., $F'(z) = f(z)$ for all $z \in U$.

We first show that F is well-defined. Suppose that we have two contours $\gamma_1\colon [a_1,b_1] \to U$ and $\gamma_2\colon [a_2,b_2] \to U$, both with initial point z_0 and terminal point z. Then $\gamma_1 - \gamma_2$ is a closed contour in U. So by what we know from Section 6.2 and the assumptions of the theorem,

$$\int_{\gamma_1} f(w)\,dw - \int_{\gamma_2} f(w)\,dw = \int_{\gamma_1-\gamma_2} f(z)\,dz = 0.$$

It follows that F does not depend on the choice of the contour.

Finally, for $z \in U$, we want to show that $F'(z) = f(z)$. To this end, choose $r > 0$ such that $B_r(z) \subseteq U$. If we choose a contour γ in U with initial point z_0 and terminal point z, then

$$F(z) = \int_\gamma f(w)\,dw$$

by construction of F. For any $\tilde{z} \in B_r(z)$, the contour $\gamma + [z,\tilde{z}]$ is in U and has initial point z_0 and terminal point $\tilde{z}$. Therefore,

$$F(\tilde{z}) = \int_{\gamma+[z,\tilde{z}]} f(w)\,dw.$$

Hence

$$F(\tilde{z}) - F(z) = \int_{\gamma+[z,\tilde{z}]} f(w)\,dw - \int_\gamma f(w)\,dw = \int_{[z,\tilde{z}]} f(w)\,dw.$$

That is, we are in the situation of Lemma 7.2. It follows that $F'(z) = f(z)$. $\square$

Example 7.4. Consider $\gamma\colon [0,1] \to \mathbb{C}$ with $\gamma(t) = 2 + t + \pi i t^5$. What is

$$\int_\gamma e^z\, dz?$$

We note that $\gamma(0) = 2$ and $\gamma(1) = 3 + \pi i$. The function $z \mapsto e^z$ has the primitive $z \mapsto e^z$. Hence by Theorem 7.3,

$$\int_\gamma e^z\, dz = e^{3+\pi i} - e^2 = -e^2(e+1).$$

The fundamental theorem of complex integration has the following consequences.

Corollary 7.5. *Suppose that $U \subseteq \mathbb{C}$ is a contour connected, open set. If $f\colon U \to \mathbb{C}$ is holomorphic with $f'(z) = 0$ for every $z \in U$, then f is constant.*

Proof. Given any two points $z_1, z_2 \in U$, we choose a contour γ in U with initial point z_1 and terminal point z_2. Then by Theorem 7.3,

$$f(z_2) - f(z_1) = \int_\gamma f'(z)\, dz = 0,$$

which means that f is constant. $\square$

Corollary 7.6. *Let $U \subseteq \mathbb{C}$ be an open set. Suppose that $f_k\colon U \to \mathbb{C}$, for $k \in \mathbb{N}$, are holomorphic functions that converge pointwise to $f\colon U \to \mathbb{C}$. Suppose further that the derivatives f_k' converge uniformly to a function $g\colon U \to \mathbb{C}$. Then f is holomorphic and $g = f'$.*

Proof. Let $z_0 \in U$ and choose $r > 0$ with $B_r(z_0) \subseteq U$. Then by Theorem 7.3,

$$f_k(z) - f_k(z_0) = \int_{[z_0,z]} f_k'(w)\, dw$$

for all $k \in \mathbb{N}$ and all $z \in B_r(z_0)$. By the pointwise convergence of f_k and the uniform convergence of f_k', we conclude that

$$f(z) - f(z_0) = \lim_{k\to\infty} (f_k(z) - f_k(z_0)) = \lim_{k\to\infty} \int_{[z_0,z]} f_k'(w)\, dw = \int_{[z_0,z]} g(w)\, dw.$$

Now Lemma 7.2 implies that $f'(z_0) = g(z_0)$. $\square$

7.2 Goursat's theorem

If we want to show that a given function has a primitive, Theorem 7.3 tells us that we should examine integrals over closed contours.

We first restrict our attention to contours following the boundaries of triangles. When speaking of a triangle T, we mean a shape in $\mathbb{C}$ enclosed by three line segments, including the interior as well as the boundary. The boundary of T is denoted by ∂T. If represented by a contour, we always assume that the orientation is anticlockwise (as in Figure 7.1).

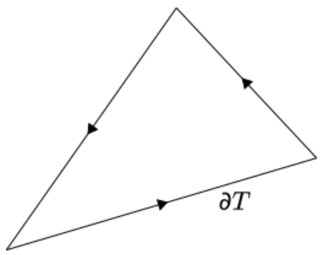

Figure 7.1: Oriented boundary of a triangle

Theorem 7.7 (Goursat). *Let $U \subseteq \mathbb{C}$ be an open set and $f\colon U \to \mathbb{C}$ a holomorphic function. Then for any triangle $T \subseteq U$,*

$$\int_{\partial T} f(z)\, dz = 0. \tag{7.2}$$

Proof. Set $T_0 = T$. We subdivide the triangle by drawing line segments between any two midpoints of the sides of T (as in Figure 7.2). This gives four smaller triangles, which we denote by T_{01}, T_{02}, T_{03}, and T_{04}. It is clear that

$$\int_{\partial T_0} f(z)\, dz = \sum_{j=1}^{4} \int_{\partial T_{0j}} f(z)\, dz,$$

as the contributions from the interior line segments cancel each other out. Consequently, by the triangle inequality,

$$\left| \int_{\partial T_0} f(z)\, dz \right| \le \sum_{j=1}^{4} \left| \int_{\partial T_{0j}} f(z)\, dz \right|,$$

and there exists one triangle among $T_{01}, \ldots, T_{04}$, denoted T_1, such that

$$\left| \int_{\partial T_0} f(z)\, dz \right| \le 4 \left| \int_{\partial T_1} f(z)\, dz \right|.$$

We subdivide T_1 the same way and find T_2 such that

$$\left|\int_{\partial T_1} f(z)\,dz\right| \leq 4\left|\int_{\partial T_2} f(z)\,dz\right|.$$

Moreover, T_2 is contained in T_1 and similar to T_1, and the diameters satisfy

$$\operatorname{diam} T_2 = \frac{1}{2}\operatorname{diam} T_1.$$

Continuing the same process indefinitely, we obtain a sequence of triangles $T_0, T_1, T_2, \ldots$ such that $T_k \subseteq T_{k-1}$ for every $k \in \mathbb{N}$ and

$$\left|\int_{\partial T} f(z)\,dz\right| \leq 4^k\left|\int_{\partial T_k} f(z)\,dz\right|. \tag{7.3}$$

Define $r_k = \operatorname{diam} T_k$. Then we also conclude that

$$r_k = \frac{r_0}{2^k} \quad \text{and} \quad L(\partial T_k) = \frac{L(\partial T)}{2^k}.$$

From MA20218, we know that a nested sequence of compact sets has a non-empty intersection. This applies to the sequence $(T_k)_{k\in\mathbb{N}_0}$. So there exists $z_0 \in \bigcap_{k=0}^{\infty} T_k$. As f is holomorphic, it is differentiable at z_0. Therefore, there exists a function $R\colon U \to \mathbb{C}$ with $\lim_{z\to z_0} R(z) = 0$ such that

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + R(z)(z - z_0)$$

for every $z \in U$. Integrating over ∂T_k, we conclude that

$$\int_{\partial T_k} f(z)\,dz = \int_{\partial T_k} (f(z_0) + f'(z_0)(z - z_0))\,dz + \int_{\partial T_k} R(z)(z - z_0)\,dz. \tag{7.4}$$

The function $z \mapsto f(z_0) + f'(z_0)(z - z_0)$ is a polynomial and has a primitive (namely $z \mapsto f(z_0)(z - z_0) + \frac{1}{2}f'(z_0)(z - z_0)^2$). By Theorem 7.3, this means that

$$\int_{\partial T_k} (f(z_0) + f'(z_0)(z - z_0))\,dz = 0. \tag{7.5}$$

Furthermore, by Lemma 6.9,

$$\left|\int_{\partial T_k} R(z)(z - z_0)\,dz\right| \leq L(\partial T_k) \max_{z\in\partial T_k}\big(|R(z)||z - z_0|\big). \tag{7.6}$$

Recall that

$$L(\partial T_k) = 2^{-k} L(\partial T). \tag{7.7}$$

Since $z_0 \in T_k$, we can estimate

$$\begin{aligned} \max_{z\in\partial T_k}\big(|R(z)||z-z_0|\big) &\le \max_{z\in B_{r_k}(z_0)}\big(|R(z)||z-z_0|\big) \\ &\le r_k \max_{z\in B_{r_k}(z_0)} |R(z)| \\ &= 2^{-k} r_0 \max_{z\in B_{2^{-k}r_0}(z_0)} |R(z)|. \end{aligned} \tag{7.8}$$

Combining (7.4)–(7.8), we conclude that

$$\left|\int_{\partial T_k} f(z)\,dz\right| \le 4^{-k} r_0 L(\partial T) \max_{z\in B_{2^{-k}r_0}(z_0)} |R(z)|.$$

Now (7.3) implies that

$$\left|\int_{\partial T} f(z)\,dz\right| \le r_0 L(\partial T) \max_{z\in B_{2^{-k}r_0}(z_0)} |R(z)|.$$

Since $\lim_{z\to z_0} R(z) = 0$, the right-hand side converges to 0 as $k\to\infty$. Hence we have shown that (7.2) is true. $\square$

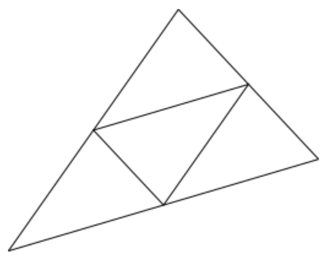

Figure 7.2: Subdivision of the triangle

7.3 The local Cauchy theorem

With Goursat's theorem in hand, we can prove that holomorphic functions have a primitive, provided that the domain has a suitable geometry.

Definition 7.8 (Star-shaped)**.** A set $S \subseteq \mathbb{C}$ is called *star-shaped* if there exists $z_0 \in S$ with the property that for any $z \in S$, the line segment $[z_0, z]$ is contained in S.

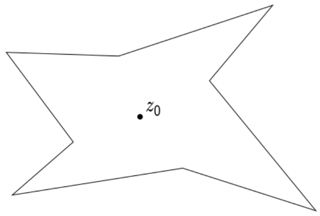

Figure 7.3: A star-shaped set

Example 7.9. The set shown in Figure 7.3 is star-shaped, but $\mathbb{C} \setminus \{0\}$ is not.

Theorem 7.10 (Local Cauchy theorem)**.** *Suppose that $U \subseteq \mathbb{C}$ is open and star-shaped. If $f \colon U \to \mathbb{C}$ is holomorphic, then it has a primitive.*

Proof. This proof is similar to the arguments for the second statement in Theorem 7.3.

Choose $z_0 \in U$ with the property as in Definition 7.8. Then we define $F \colon U \to \mathbb{C}$ by the formula

$$F(z) = \int_{[z_0,z]} f(w)\, dw.$$

Given $z \in U$, we may choose $r > 0$ such that $B_r(z) \subseteq U$. For $h \in B_r(0)$, we find that

$$F(z+h) - F(z) = \int_{[z_0,z+h]} f(w)\, dw - \int_{[z_0,z]} f(w)\, dw = \int_{[z,z_0]+[z_0,z+h]} f(w)\, dw.$$

Now note that the triangle T with corners z, z_0, and $z+h$ is contained in U. By Goursat's theorem (Theorem 7.7),

$$\int_{\partial T} f(w)\, dw = 0.$$

This can be rewritten as

$$\int_{[z,z_0]} f(w)\, dw + \int_{[z_0,z+h]} f(w)\, dw + \int_{[z+h,z]} f(w)\, dw = 0.$$

Therefore,

$$F(z+h) - F(z) = \int_{[z,z+h]} f(w)\, dw.$$

By Lemma 7.2, it follows that $F'(z) = f(z)$. Hence F is a primitive of f. $\square$

Example 7.11. Given a closed contour $\gamma\colon [a,b] \to \mathbb{C}$, consider the integral

$$\int_\gamma \cos(z^2)\, dz.$$

Without the local Cauchy theorem, it's not obvious that there is a primitive. But the function $z \mapsto \cos(z^2)$ clearly has a derivative everywhere in $\mathbb{C}$. So it is holomorphic. Moreover, the set $\mathbb{C}$ is star-shaped. Therefore, the function has a primitive and the integral vanishes.

Theorem 7.10 is formulated for star-shaped sets. This condition can be relaxed, but some assumptions on the geometry of U are necessary. Roughly speaking, the statement becomes false if U has holes.

Example 7.12. Let $z_0 \in \mathbb{C}$ and $r > 0$. For $U = \mathbb{C} \setminus \{z_0\}$ and $k \in \mathbb{N}$, consider the function $f_k\colon U \to \mathbb{C}$ given by $f_k(z) = 1/(z - z_0)^k$. For the contour $\gamma\colon [0, 2\pi] \to U$ with $\gamma(t) = z_0 + re^{it}$, we compute

$$\int_\gamma f_k(z)\, dz = \int_0^{2\pi} \frac{rie^{it}}{r^k e^{ikt}}\, dt = r^{1-k} i \int_0^{2\pi} e^{i(1-k)t}\, dt.$$

For $k = 1$, this gives

$$\int_\gamma f_1(z)\, dz = 2\pi i.$$

It follows that f_1 does not have a primitive in U. Whenever we restrict the domain to a star-shaped subset $V \subset U$, however, Theorem 7.10 applies and gives a local primitive in V.

For $k \geq 2$, we compute

$$\int_\gamma f_k(z)\, dz = 0.$$

For these functions, we do in fact have primitives. Setting

$$g_k(z) = \frac{1}{(1-k)(z - z_0)^{k-1}}$$

for $k \geq 2$, we see that $g_k'(z) = f_k(z)$.

Chapter 8

The Cauchy integral theorem

In this chapter, we want to extend the local Cauchy theorem to more general sets U. We will not only remove the assumption that U is star-shaped, but also allow sets with holes. Example 7.12 shows, however, that we cannot expect primitives in general in this case even for holomorphic functions. Instead, we look for conditions such that

$$\int_{\gamma} f(z)\, dz = 0$$

for closed contours γ, or equivalently, such that

$$\int_{\gamma_1} f(z)\, dz = \int_{\gamma_2} f(z)\, dz$$

for a pair of contours γ_1 and γ_2 with identical initial and terminal points.

8.1 Homotopy

The idea of deforming one contour into another is at the centre of this chapter. This can be formalised as follows.

Definition 8.1 (Homotopy)**.** Let $U \subseteq \mathbb{C}$ be an open set. Two contours $\gamma_1, \gamma_2 \colon [a, b] \to U$ with $\gamma_1(a) = \gamma_2(a)$ and $\gamma_2(b) = \gamma_2(b)$ are called *homotopic* in U if there exists a continuous map $\psi \colon [a, b] \times [0, 1] \to U$ such that

- $\psi(t, 0) = \gamma_1(t)$ and $\psi(t, 1) = \gamma_2(t)$ for all $t \in [a, b]$,
- $\psi(a, s) = \gamma_1(a) = \gamma_2(a)$ and $\psi(b, s) = \gamma_1(b) = \gamma_2(b)$ for all $s \in [0, 1]$, and

- $\psi(\cdot, s)$ is a contour for every $s \in [0,1]$ and $\psi(t, \cdot)$ is a contour for every $t \in [a,b]$.

The idea behind this definition is that γ_1 can be deformed in a piecewise continuously differentiable way into γ_2, while leaving the initial and terminal points the same. We require that both contours are defined on the same interval here, but this is no significant restriction, because they can always be reparametrised accordingly.

The third condition on ψ is not usually assumed in this context, but this extra assumption will make our life a bit easier. The resulting definition is still equivalent to the standard one.

Example 8.2. Consider the contours $\gamma_1, \gamma_2 \colon [0,\pi] \to \mathbb{C}$ with $\gamma_1(t) = e^{it}$ and $\gamma_2(t) = e^{-it}$ for $t \in [0,\pi]$. Then the map $\phi\colon [0,\pi] \times [0,1] \to \mathbb{C}$ with $\psi(t,s) = \cos t + i(1-2s)\sin t$ has the properties from Definition 8.1. Hence these two contours are homotopic in $\mathbb{C}$.

For closed contours, we relax the condition about the initial and terminal points.

Definition 8.3 (Homotopy of closed contours)**.** Let $U \subseteq \mathbb{C}$ be an open set. Two closed contours $\gamma_1, \gamma_2 \colon [a,b] \to U$ are called *homotopic* in U if there exists a continuous map $\psi\colon [a,b] \times [0,1] \to U$ such that

- $\psi(t,0) = \gamma_1(t)$ and $\psi(t,1) = \gamma_2(t)$ for all $t \in [a,b]$ and
- $\psi(\cdot, s)$ is a closed contour for every $s \in [0,1]$ and $\psi(t,\cdot)$ is a contour for every $t \in [a,b]$.

A closed contour $\gamma\colon [a,b] \to U$ is called *null-homotopic* in U if it is homotopic in U to a constant contour $\gamma_0\colon [a,b] \to U$, i.e., a contour with $\gamma_0(t) = \gamma_0(a)$ for all $t \in [a,b]$.

Example 8.4. The contour $\gamma\colon [0,2\pi] \to \mathbb{C}$, $t \mapsto e^{it}$, is null-homotopic in $\mathbb{C}$. In order to see this, consider $\psi(t,s) = (1-s)e^{it}$.

8.2 The Cauchy integral theorem

We can now extend Theorem 7.10 as follows.

Theorem 8.5 (Cauchy integral theorem)**.** *Suppose that $U \subseteq \mathbb{C}$ is an open set and $f\colon U \to \mathbb{C}$ a holomorphic function. If $\gamma_1, \gamma_2\colon [a,b] \to U$ are two* *contours that are homotopic in U, in the sense of either Definition 8.1 or Definition 8.3, then*

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz. \tag{8.1}$$

If $\gamma\colon [a,b] \to U$ is a closed contour that is null-homotopic in U, then

$$\int_{\gamma} f(z)\,dz = 0.$$

Proof. The second statement is an immediate consequence of the first one, so it suffices to prove (8.1).

Suppose first that γ_1 and γ_2 satisfy the conditions of Definition 8.1. Let $\psi\colon [a,b]\times[0,1] \to U$ be the corresponding homotopy. Set $C = \psi([a,b]\times[0,1])$. As ψ is continuous, this is a compact set in U. Therefore, there exists $r > 0$ such that $B_r(x) \subseteq U$ for every $x \in C$.

Note that ψ is a continuous map on the compact domain $[a,b] \times [0,1]$. Therefore, it is in fact uniformly continuous. In particular, there exists $\delta > 0$ such that whenever we have $t_1, t_2 \in [a,b]$ and $s_1, s_2 \in [0,1]$ with $(t_2 - t_1)^2 + (s_2 - s_1)^2 < 2\delta^2$, then $|\psi(t_2, s_2) - \psi(t_1, s_1)| < r$.

Subdivide the interval $[a,b]$ by choosing $t_0, \ldots, t_K$ with $a = t_0 < \cdots < t_K = b$ and $t_k - t_{k-1} < \delta$ for $k = 1, \ldots, K$. Similarly, subdivide $[0,1]$ by choosing $s_0, \ldots, s_J$ with $0 = s_0 < \cdots < s_J = 1$ and $s_j - s_{j-1} < \delta$ for $j = 1, \ldots, J$. Consider the rectangles $R_{kj} = [t_{k-1}, t_k] \times [s_{j-1}, s_j]$ and their images $Q_{kj} = \psi(R_{kj})$ (see Figure 8.1).

The boundary of R_{kj} consists of four line segments. Restricting ψ to any of these line segments gives rise to a contour in U. The concatenation of all four of them, with the orientation taken anticlockwise when we move around R_{kj}, gives a closed contour in U, which we denote by ∂Q_{kj}. (We keep the orientation given by the boundary of R_{kj}, even if ψ should reverse it.)

By construction, we know that Q_{kj} is contained in the disk $B_r(\psi(t_k, s_j)) \subseteq U$. Hence ∂Q_{kj} is a contour in $B_r(\psi(t_k, s_j))$. This is a star-shaped set and f is holomorphic there, so Theorem 7.10 tells us that

$$\int_{\partial Q_{kj}} f(z)\,dz = 0$$

for $k = 1, \ldots, K$ and $j = 1, \ldots, J$. Hence

$$\sum_{k=1}^{K} \sum_{j=1}^{J} \int_{\partial Q_{kj}} f(z)\,dz = 0. \tag{8.2}$$

Now we note that many of the integrals in this sum partially cancel each other. For example, consider the line segment $\{t_k\} \times [s_{j-1}, s_j]$, which is part of the boundary of R_{kj}. It also forms part of the boundary of $R_{k+1,j}$, but with reversed orientation. Therefore, the corresponding integrals appear with reversed orientation in the above sum and cancel each other out. What is left in the end is exactly

$$\int_{\gamma_1} f(z)\,dz - \int_{\gamma_2} f(z)\,dz,$$

and (8.2) implies that this difference vanishes.

For closed contours that are homotopic in the sense of Definition 8.3, the proof is essentially the same. □

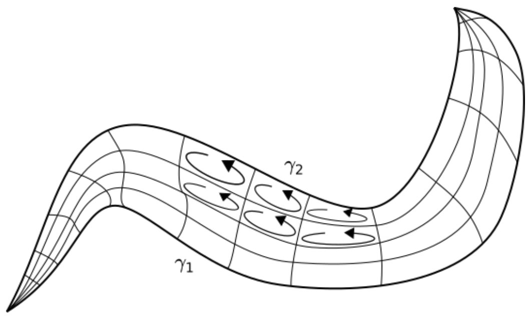

Figure 8.1: The sets Q_{kj}

Example 8.6. For a fixed $z_0 \in \mathbb{C}$, let $U = \mathbb{C} \setminus \{z_0\}$. For the contour $\gamma_1 \colon [0, 2\pi] \to U$ given by $\gamma_1(t) = z_0 + e^{it}$, we know that

$$\int_{\gamma_1} \frac{dz}{z - z_0} = 2\pi i$$

by Example 7.12. This means that for any contour γ_2 homotopic to γ_1 in U, we also find that

$$\int_{\gamma_2} \frac{dz}{z - z_0} = 2\pi i.$$

This applies, for example, to $\gamma_2 \colon [0, 2\pi] \to U$ with $\gamma_2(t) = z_0 + \rho(t)e^{it}$ for any piecewise continuously differentiable function $\rho\colon [0, 2\pi] \to (0, \infty)$ with $\rho(0) = \rho(2\pi)$.

8.3 The local Cauchy formula

One of the consequences of Theorem 8.5 is the following useful formula.

Theorem 8.7 (Local Cauchy formula)**.** *Let $U \subseteq \mathbb{C}$ be an open set and $f\colon U \to \mathbb{C}$ a holomorphic function. Let $z_0 \in U$ and $r > 0$ such that $\overline{B_r(z_0)} \subseteq U$. Consider the contour $\gamma\colon [0, 2\pi] \to U$ parametrising $\partial B_r(z_0)$ by $\gamma(t) = z_0 + re^{it}$. Then*

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z}\, dw$$

for every $z \in B_r(z_0)$.

Proof. Let $z \in B_r(z_0)$. Define $g\colon U \setminus \{z\} \to \mathbb{C}$ by

$$g(w) = \frac{f(w) - f(z)}{w - z}.$$

Then g is holomorphic in $U \setminus \{z\}$. Furthermore, it satisfies

$$\lim_{w \to z} g(w) = f'(z). \tag{8.3}$$

For any $\rho > 0$ with $\rho + |z - z_0| \leq r$, the closed contour γ is homotopic in $U \setminus \{z\}$ to $\gamma_\rho\colon [0, 2\pi] \to U \setminus \{z\}$ given by $\gamma_\rho(t) = z + \rho e^{it}$. (A homotopy is given by $\psi(t, s) = (1 - s)z_0 + sz + ((1 - s)r + s\rho)e^{it}$.) Hence by Theorem 8.5,

$$\int_\gamma g(w)\, dw = \int_{\gamma_\rho} g(w)\, dw.$$

We now consider the behaviour of the right-hand side as $\rho \to 0$. According to Lemma 6.9,

$$\left| \int_{\gamma_\rho} g(w)\, dw \right| \leq 2\pi\rho \max_{w \in \partial B_\rho(z)} |g(w)|.$$

Because of (8.3), we further know that

$$\lim_{\rho \to 0} \max_{w \in \partial B_\rho(z)} |g(w)| = |f'(z)|.$$

Hence

$$\int_\gamma g(w)\, dw = \lim_{\rho \to 0} \int_{\gamma_\rho} g(w)\, dw = 0.$$

That is,

$$0 = \int_\gamma \frac{f(w) - f(z)}{w - z}\, dw = \int_\gamma \frac{f(w)}{w - z}\, dw - f(z) \int_\gamma \frac{dw}{w - z}. \tag{8.4}$$

But then, using Theorem 8.5 again, we see that

$$\int_\gamma \frac{f(w)}{w - z}\, dw = f(z) \int_\gamma \frac{dw}{w - z} = f(z) \int_{\gamma_\rho} \frac{dw}{w - z}.$$

As in Example 7.12, we compute

$$\int_{\gamma_\rho} \frac{dz}{w-z} = 2\pi i.$$

Hence the desired identity follows from (8.4). $\square$

Chapter 9

Analytic functions

Recall Taylor's theorem for functions of one real variable, which states that a k times differentiable function can be approximated by a polynomial of degree k, and the coefficients are given by the derivatives of the function. Some (but not all) functions with derivatives of any order can even be expanded as a power series, called the Taylor series.

We will see that for holomorphic functions, not only do we automatically have derivatives of any order, but we can always represent them locally by power series. But first we discuss complex power series in general.

9.1 Power series

Recall the following facts and definitions from MA20218, the first of which is a useful convergence criterion for series.

Lemma 9.1 (Weierstrass M-test). *For $S \subseteq \mathbb{C}$, let $(f_k)_{k\in\mathbb{N}}$ be a sequence of functions $f_k\colon S \to \mathbb{C}$. Suppose that for every $k \in \mathbb{N}$, there exists $M_k \geq 0$ such that $|f_k(z)| \leq M_k$ for all $z \in S$. If*

$$\sum_{k=1}^{\infty} M_k < \infty,$$

then the series $\sum_{k=1}^{\infty} f_k$ converges uniformly in S.

Proof. You have seen the arguments for real-valued functions $f_k\colon A \to \mathbb{R}$ on a set $A \subseteq \mathbb{R}$ in MA20218. The same arguments apply here. □

A *power series* centred at $z_0 \in \mathbb{C}$ is a series of the form $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ for certain coefficients $a_k \in \mathbb{C}$. Given such a power series, the number

$$R = \frac{1}{\limsup_{k\to\infty} \sqrt[k]{|a_k|}}$$

is called the *radius of convergence* (which may possibly be 0 or ∞). If the limit

$$\lim_{k\to\infty} \frac{|a_k|}{|a_{k+1}|}$$

exists, then it coincides with R.

For any $z \in B_R(z_0)$, the power series $\sum_{k=0}^\infty a_k(z-z_0)^k$ converges absolutely, and for any $z \notin \overline{B_R(z_0)}$, it diverges. (For points on $\partial B_R(z_0)$, we may have convergence or divergence.) Indeed, the following is true.

Proposition 9.2. *Let $\sum_{k=0}^\infty a_k(z-z_0)^k$ be a power series centred at $z_0 \in \mathbb{C}$. Suppose that $R > 0$ is its radius of convergence.*

1. *Then for any $r < R$, the power series converges uniformly in $B_r(z_0)$.*
2. *The power series $\sum_{k=1}^\infty ka_k(z-z_0)^{k-1}$ also has radius of convergence R.*
3. *The function $f\colon B_R(z_0) \to \mathbb{C}$ given by $f(z) = \sum_{k=0}^\infty a_k(z-z_0)^k$ is holomorphic in $B_R(z_0)$ with $f'(z) = \sum_{k=1}^\infty ka_k(z-z_0)^{k-1}$.*

Proof. For the proof of the first statement, fix $r < R$ and set $M_k = |a_k|r^k$. Then $\sum_{k=0}^\infty M_k < \infty$. Since $|a_k(z-z_0)^k| \leq |a_k|r^k = M_k$ for every $z \in B_r(z_0)$, the Weierstrass M-test implies uniform convergence in $B_r(z_0)$.

For the proof of the second statement, we recall that

$$R = \frac{1}{\limsup_{k\to\infty} |a_k|^{1/k}}.$$

Let R' be the radius of convergence of $\sum_{k=1}^\infty ka_k(z-z_0)^{k-1}$. For any $z \neq z_0$, convergence of the power series $\sum_{k=1}^\infty ka_k(z-z_0)^{k-1}$ is equivalent to convergence of $\sum_{k=1}^\infty ka_k(z-z_0)^k$, as every term is multiplied by $z - z_0$. Hence R' coincides with the radius of convergence of the second power series. That is,

$$R' = \frac{1}{\limsup_{k\to\infty} (k|a_k|)^{1/k}}.$$

But as $k^{1/k} = e^{(\log k)/k} \to 1$ as $k \to \infty$, we see that $R' = R$.

Finally, in order to prove the third statement, we use Corollary 7.6. The functions $f_K(z) = \sum_{k=0}^K a_k(z-z_0)^k$ are polynomials, so they are holomorphic and $f_K'(z) = \sum_{k=1}^K ka_k(z-z_0)^{k-1}$. Furthermore, we know that for any $r \in (0, R)$, the functions f_K converge to f and f_K' converge uniformly in $B_r(z_0)$ to $\sum_{k=1}^\infty ka_k(z-z_0)^{k-1}$ as $K \to \infty$. Corollary 7.6 therefore implies that $f'(z) = \sum_{k=1}^\infty ka_k(z-z_0)^{k-1}$ in $B_r(z_0)$. As this is true for any $r < R$, the formula holds in fact in $B_R(z_0)$. $\square$

Definition 9.3 (Analytic function)**.** Let $U \subseteq \mathbb{C}$ be an open set. A function $f\colon U \to \mathbb{C}$ is called *analytic* if for any $z_0 \in U$ there exists $r > 0$ with $B_r(z_0) \subseteq U$ and there exists a power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$, with radius of convergence at least r, such that

$$f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k$$

for all $z \in B_r(z_0)$.

Example 9.4. Let $w_0 \in \mathbb{C}$. Consider the function $f\colon \mathbb{C}\setminus\{w_0\} \to \mathbb{C}$ given by $f(z) = 1/(w_0 - z)$. We want to show that f is analytic. To this end, recall that the geometric series $\sum_{k=0}^{\infty} \alpha^k$ converges for all $\alpha \in \mathbb{C}$ with $|\alpha| < 1$ and

$$\sum_{k=0}^{\infty} \alpha^k = \frac{1}{1-\alpha}. \tag{9.1}$$

Fix $z_0 \in \mathbb{C}$. We are looking for a power series expansion of $f(z)$ centred at z_0. Note that

$$\frac{1}{w_0 - z} = \frac{1}{(w_0 - z_0) - (z - z_0)} = \frac{1}{w_0 - z_0} \frac{1}{1 - \frac{z-z_0}{w_0-z_0}}.$$

Set $\alpha = (z-z_0)/(w_0-z_0)$ and consider (9.1). If $|z - z_0| < |w_0 - z_0|$, then $|\alpha| < 1$, and we conclude that

$$\frac{1}{1 - \frac{z-z_0}{w_0-z_0}} = \sum_{k=0}^{\infty} \left(\frac{z-z_0}{w_0-z_0}\right)^k.$$

Hence

$$\frac{1}{w_0 - z} = \frac{1}{w_0 - z_0} \sum_{k=0}^{\infty} \left(\frac{z-z_0}{w_0-z_0}\right)^k = \sum_{k=0}^{\infty} \frac{(z-z_0)^k}{(w_0-z_0)^{k+1}}$$

for all $z \in B_{|w_0-z_0|}(z_0)$. The right-hand side is a power series centred at z_0, so f is analytic.

9.2 Analyticity of holomorphic functions

We now show that holomorphic functions are analytic, and not only that, but we have a formula for the coefficients of the corresponding power series.

Theorem 9.5. *Let $U \subseteq \mathbb{C}$ be an open set and $f \colon U \to \mathbb{C}$ a holomorphic function. Suppose that $z_0 \in U$ and $r > 0$ are such that $\overline{B_r(z_0)} \subseteq U$. Then f has the power series expansion*

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

in $B_r(z_0)$, where

$$a_k = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(w)}{(w - z_0)^{k+1}} \, dw.$$

These coefficients satisfy $|a_k| \leq r^{-k} \max_{w \in \partial B_r(z_0)} |f(w)|$, which means that the radius of convergence of the power series is at least r.

Proof. By Theorem 8.7, we know that

$$f(z) = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(w)}{w - z} \, dw$$

for all $z \in B_r(z_0)$. As in Example 9.4, we see that

$$\frac{1}{w - z} = \frac{1}{w - z_0} \sum_{k=0}^{\infty} \left(\frac{z - z_0}{w - z_0} \right)^k = \sum_{k=0}^{\infty} \frac{(z - z_0)^k}{(w - z_0)^{k+1}}$$

if $|z - z_0| < |w - z_0|$. So this holds true in particular for all $w \in \partial B_r(z_0)$. Thus

$$f(z) = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \sum_{k=0}^{\infty} \frac{f(w)}{(w - z_0)^{k+1}} (z - z_0)^k \, dw.$$

We want to show that the integral can be exchanged with the infinite sum, so that

$$f(z) = \frac{1}{2\pi i} \sum_{k=0}^{\infty} \int_{\partial B_r(z_0)} \frac{f(w)}{(w - z_0)^{k+1}} \, dw \, (z - z_0)^k = \sum_{k=0}^{\infty} a_k (z - z_0)^k. \quad (9.2)$$

By Corollary 6.11, in order to show that the integral and the infinite sum can indeed be exchanged, it suffices to show that the series

$$\sum_{k=0}^{\infty} \frac{f(w)}{(w - z_0)^{k+1}} (z - z_0)^k$$

converges uniformly on $\partial B_r(z_0)$. To this end, we estimate

$$\left| \frac{f(w)}{(w - z_0)^{k+1}} (z - z_0)^k \right| = \frac{|f(w)|}{r^{k+1}} |z - z_0|^k \leq \frac{|z - z_0|^k}{r^{k+1}} \max_{\tilde{w} \in \partial B_r(z_0)} |f(\tilde{w})|.$$

For any fixed $z \in B_r(z_0)$, define

$$M_k = \frac{|z - z_0|^k}{r^{k+1}} \max_{\tilde{w} \in \partial B_r(z_0)} |f(\tilde{w})|.$$

As $|z - z_0| < r$, we conclude that $\sum_{k=0}^{\infty} M_k < \infty$. By the Weierstrass M-test, we have uniform convergence, and thus (9.2) is proved.

The estimate for the coefficients follows from Lemma 6.9. It is then clear that the radius of convergence is at least r. $\square$

Corollary 9.6. *Suppose that $U \subseteq \mathbb{C}$ is an open set. A function $f\colon U \to \mathbb{C}$ is holomorphic if, and only if, it is analytic.*

Proof. From Proposition 9.2, we conclude that an analytic function is holomorphic. The converse follows from Theorem 9.5. $\square$

Corollary 9.7. *Let $U \subseteq \mathbb{C}$ be an open set and $f\colon U \to \mathbb{C}$ a holomorphic function. Then f has derivatives of any order. For $z_0 \in U$ and $r > 0$ with $\overline{B_r(z_0)} \subseteq U$, the k-th derivative $f^{(k)}$ satisfies*

$$f^{(k)}(z_0) = \frac{k!}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(w)}{(w - z_0)^{k+1}}\, dw. \tag{9.3}$$

Furthermore,

$$|f^{(k)}(z_0)| \leq \frac{k!}{r^k} \max_{w \in \partial B_r(z_0)} |f(w)|.$$

Proof. Theorem 9.5 gives a power series expansion of f in $B_r(z_0)$. It follows from Proposition 9.2 that power series are complex differentiable to any order. We obtain (9.3) by differentiating the power series in Theorem 9.5 k times.

The estimates for $|f^{(k)}(z_0)|$ now follow from Lemma 6.9, using the fact that

$$\left| \frac{f(w)}{(w - z_0)^{k+1}} \right| = \frac{|f(w)|}{r^{k+1}}$$

for $w \in \partial B_r(z_0)$ and $L(\partial B_r(z_0)) = 2\pi r$. $\square$

Corollary 9.8. *Suppose that $U \subseteq \mathbb{C}$ is open. For $k \in \mathbb{N}$, let $f_k\colon U \to \mathbb{C}$ be a holomorphic function. Suppose that the sequence $(f_k)_{k\in\mathbb{N}}$ converges uniformly in any compact set $K \subseteq U$. Then the limit $f = \lim_{k\to\infty} f_k$ is holomorphic.*

Proof. Let z_0 in U. Choose $r > 0$ such that $B_r(z_0) \subseteq U$. Since z_0 was chosen arbitrarily, it suffices to show that f is holomorphic in $B_r(z_0)$.

Let $\gamma\colon [a, b] \to B_r(z_0)$ be a closed contour. Set $\Gamma = \gamma([a, b])$ and note that $\Gamma \subseteq U$ is compact.

The set $B_r(z_0)$ is star-shaped. Therefore, the local Cauchy theorem (Theorem 7.10) implies that every f_k has a primitive in $B_r(z_0)$. The fundamental theorem of complex integration (Theorem 7.3) then tells us that

$$\int_\gamma f_k(z)\,dz = 0$$

for every $k \in \mathbb{N}$. According to Proposition 6.10, it follows that

$$\int_\gamma f(z)\,dz = 0.$$

This holds true for every closed contour γ in $B_r(z_0)$. By the fundamental theorem of complex integration, we therefore have a primitive $F\colon B_r(z_0) \to \mathbb{C}$ of f in $B_r(z_0)$.

Now we apply Corollary 9.7 to F. We conclude that $f = F'$ is holomorphic. □

9.3 Liouville's theorem

The following theorem applies to functions that are holomorphic in all of $\mathbb{C}$.

Definition 9.9. A function $f\colon \mathbb{C} \to \mathbb{C}$ is called *entire* if it is holomorphic in $\mathbb{C}$.

Theorem 9.10 (Liouville)**.** *If an entire function is bounded, then it is constant.*

Proof. If $f\colon \mathbb{C} \to \mathbb{C}$ is entire, then the estimates from Corollary 9.7 can be applied in any disk $B_r(z) \subset \mathbb{C}$. For $k = 1$, they give

$$|f'(z)| \le \frac{1}{r} \max_{w \in \partial B_r(z)} |f(w)|.$$

If f is bounded, then we may choose $M \ge 0$ such that $|f(w)| \le M$ for all $w \in \mathbb{C}$. Then $|f'(z)| \le M/r$ for all $r > 0$. This can only be true if $f'(z) = 0$ for all $z \in \mathbb{C}$. Corollary 7.5 then implies that f is constant. □

Finally, we have an application to algebra.

Theorem 9.11 (Fundamental theorem of algebra)**.** *Every non-constant polynomial with complex coefficients has a complex zero.*

Proof. Suppose that $p(z) = a_n z^n + \cdots + a_1 z + a_0$ is a polynomial with coefficients $a_0, \ldots, a_n \in \mathbb{C}$, where $a_n \neq 0$. We argue by contradiction, assuming that p is not constant (i.e., $n > 0$) and there is no zero (i.e., $p(z) \neq 0$ for every $z \in \mathbb{C}$). Then the function $q \colon \mathbb{C} \to \mathbb{C}$ with $q(z) = 1/p(z)$ is an entire function. As

$$\lim_{|z| \to \infty} |p(z)| = \lim_{|z| \to \infty} |z|^n \left| a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right| = \infty,$$

we conclude that

$$\lim_{|z| \to \infty} |q(z)| = 0.$$

In particular, there exists $R > 0$ such that $|q(z)| \leq 1$ whenever $|z| > R$. By the Weierstrass extreme value theorem, the function q is also bounded in $\overline{B_R(0)}$. Therefore, it is bounded in $\mathbb{C}$. Liouville's theorem now states that q is constant, and so is p, in contradiction to the assumption. $\square$

Chapter 10

Isolated singularities

In this chapter, we want to understand functions that are not necessarily differentiable (or even defined) at every point in the set of interest. Typical examples include the functions given by

$$f(z) = \frac{1}{(z - z_0)^k},$$

for fixed $z_0 \in \mathbb{C}$ and $k \in \mathbb{N}$ (or more generally, rational functions, given by the quotients of two polynomials) or

$$f(z) = e^{1/z}.$$

The points where a function fails to be analytic are called *singularities*.

10.1 Laurent series

By definition, a function cannot have a power series expansion at a singularity. But often, power series can be replaced by the following.

Definition 10.1 (Laurent series)**.** Let $z_0 \in \mathbb{C}$. A *Laurent series* centred at z_0 is a series of the form $\sum_{k=-\infty}^{\infty} a_k(z-z_0)^k$. We split a Laurent series into the *power series part* $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ and the *principal part* $\sum_{k=-\infty}^{-1} a_k(z-z_0)^k$.

We say that the Laurent series *converges* at a point $z \in \mathbb{C}$ if both the power series part and the principal part converge at z.

Instead of a single radius of convergence as for power series, convergence for Laurent series is characterised by two radii. You will show in an exercise that there exist $r, R \in [0, \infty) \cup \{\infty\}$ such that the series $\sum_{k=-\infty}^{\infty} a_k(z - z_0)^k$ converges absolutely for all $z \in \mathbb{C}$ with $r < |z - z_0| < R$ and diverges when

$|z - z_0| < r$ or $|z - z_0| > R$. Furthermore, the convergence is uniform in any compact set $C \subset B_R(z_0) \setminus \overline{B_r(z_0)}$.

We have the following expansion by Laurent series of functions that are holomorphic on an annulus.

Theorem 10.2 (Laurent). *Let $U \subseteq \mathbb{C}$ be an open set and $f : U \to \mathbb{C}$ a holomorphic function. For $z_0 \in \mathbb{C}$ and $r, R > 0$ with $r < R$, suppose that $\overline{B_R(z_0)} \setminus B_r(z_0) \subseteq U$. Let $r_0 \in (r, R)$ and consider the contour $\gamma : [0, 2\pi] \to U$ with $\gamma(t) = z_0 + r_0 e^{it}$. For $k \in \mathbb{Z}$, let*

$$a_k = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w - z_0)^{k+1}} \, dw.$$

Then

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k \tag{10.1}$$

for all $z \in B_R(z_0) \setminus \overline{B_r(z_0)}$.

This Laurent series expansion is unique in the following sense: if $f(z) = \sum_{k=-\infty}^{\infty} b_k (z - z_0)^k$ for all $z \in B_R(z_0) \setminus \overline{B_r(z_0)}$, then $b_k = a_k$ for all $k \in \mathbb{Z}$.

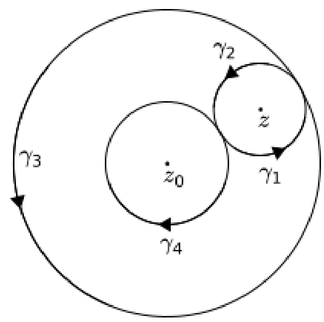

Figure 10.1: Contours used in the proof

Proof. Let $A = B_R(z_0) \setminus \overline{B_r(z_0)}$. Then for any $z \in A$, we may choose $s > 0$ such that $\overline{B_s(z)} \subseteq A$. By Theorem 8.7,

$$f(z) = \frac{1}{2\pi i} \int_{\partial B_s(z)} \frac{f(w)}{w - z} \, dw$$

(where the orientation of $\partial B_s(z)$ is anticlockwise).

Now fix $\phi \in [0, 2\pi)$ and $\rho > 0$ such that $z - z_0 = \rho e^{i\phi}$. Define $\tilde{r} = \rho - s$ and $\tilde{R} = \rho + s$. Consider the following contours $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ (depicted in Figure 10.1):

- $\gamma_1, \gamma_2 \colon [0, \pi] \to U$ with $\gamma_1(t) = z + se^{i(t+\phi-\pi)}$ and $\gamma_2(t) = z + se^{i(t+\phi)}$,
- $\gamma_3, \gamma_4 \colon [0, 2\pi] \to U$ with $\gamma_3(t) = z_0 + \tilde{R}e^{i(t+\phi)}$ and $\gamma_4(t) = z_0 + \tilde{r}e^{i(\phi-t)}$.

Then $\partial B_s(z)$ is parametrised by $\gamma_1 + \gamma_2$. We also consider

$$\gamma_0 = \gamma_1 + \gamma_3 - \gamma_1 + \gamma_4.$$

Then, as demonstrated in Figure 10.2, the contour γ_0 is homotopic to $\gamma_1 + \gamma_2$ in $A \setminus \{z\}$ after a suitable reparametrisation. By Theorem 8.5, this means that

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\gamma_1+\gamma_2} \frac{f(w)}{w-z}\, dw \\ &= \frac{1}{2\pi i} \int_{\gamma_0} \frac{f(w)}{w-z}\, dw \\ &= \frac{1}{2\pi i} \int_{\gamma_3} \frac{f(w)}{w-z}\, dw + \frac{1}{2\pi i} \int_{\gamma_4} \frac{f(w)}{w-z}\, dw. \end{aligned}$$

The term

$$\frac{1}{2\pi i} \int_{\gamma_3} \frac{f(w)}{w-z}\, dw$$

can now be expressed as a power series with the same arguments as in the proof of Theorem 9.5. We recall that

$$\frac{f(w)}{w-z} = \sum_{k=0}^{\infty} \frac{f(w)}{(w-z_0)^{k+1}} (z-z_0)^k$$

when $|z - z_0| < |w - z_0|$, and the convergence is uniform on $\partial B_{\tilde{R}}(z_0)$. Therefore,

$$\frac{1}{2\pi i} \int_{\gamma_3} \frac{f(w)}{w-z}\, dw = \frac{1}{2\pi i} \sum_{k=0}^{\infty} \int_{\gamma_3} \frac{f(w)}{(w-z_0)^{k+1}}\, dw\, (z-z_0)^k.$$

Moreover, the contour γ_3 is homotopic to γ in A. Hence, using Theorem 8.5, we see that

$$\frac{1}{2\pi i} \int_{\gamma_3} \frac{f(w)}{(w-z)^{k+1}}\, dw = a_k$$

for $k \geq 0$.

We use similar arguments for

$$\frac{1}{2\pi i} \int_{\gamma_4} \frac{f(w)}{w-z}\, dw.$$

We observe that

$$\frac{1}{w-z} = -\frac{1}{z-z_0}\frac{1}{1-\frac{w-z_0}{z-z_0}} = -\frac{1}{z-z_0}\sum_{\ell=0}^{\infty}\left(\frac{w-z_0}{z-z_0}\right)^{\ell} = -\sum_{\ell=0}^{\infty}\frac{(w-z_0)^{\ell}}{(z-z_0)^{\ell+1}},$$

as long as $|w-z_0| < |z-z_0|$. The substitution $k = -(\ell+1)$ now gives

$$\frac{1}{w-z} = -\sum_{k=-\infty}^{-1}\frac{(z-z_0)^k}{(w-z_0)^{k+1}}.$$

The convergence is uniform on $\partial B_{\tilde{r}}(z_0)$ by the same arguments as in the proof of Theorem 9.5. Hence

$$\frac{1}{2\pi i}\int_{\gamma_4}\frac{f(w)}{w-z}\,dw = -\frac{1}{2\pi i}\sum_{k=-\infty}^{-1}\int_{\gamma_4}\frac{f(w)}{(w-z_0)^{k+1}}\,dw\,(z-z_0)^k.$$

The contour $-\gamma_4$ is homotopic to γ in A. Hence

$$-\frac{1}{2\pi i}\int_{\gamma_4}\frac{f(w)}{(w-z_0)^{k+1}}\,dw = \frac{1}{2\pi i}\int_{-\gamma_4}\frac{f(w)}{(w-z_0)^{k+1}}\,dw = a_k$$

for $k < 0$. Combining all of these identities, we find the Laurent series expansion (10.1).

It remains to show that this Laurent series expansion is unique. Suppose therefore that

$$f(z) = \sum_{k=-\infty}^{\infty} b_k(z-z_0)^k$$

for all $z \in A$. Substituting this into the formulas in the statement of the theorem, we conclude that

$$a_k = \frac{1}{2\pi i}\sum_{\ell=-\infty}^{\infty} b_\ell\int_\gamma (w-z_0)^{\ell-k-1}\,dw$$

for all $k \in \mathbb{Z}$. By the calculations from Example 7.12, we know that

$$\int_\gamma (w-z_0)^{\ell-k-1}\,dw = 0$$

for $k \neq \ell$; and for $k = \ell$, the integral is $2\pi i$. Therefore, we conclude that $a_k = b_k$. $\square$

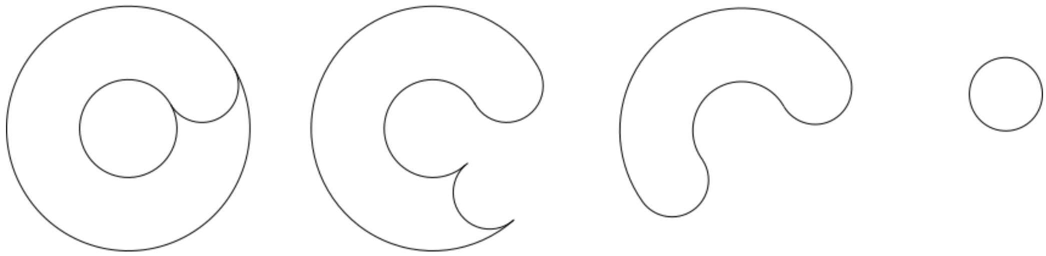

Figure 10.2: Homotopy between γ_0 and $\partial B_s(z)$

10.2 Classification of isolated singularities

Suppose that we have a holomorphic function $f\colon U \to \mathbb{C}$ for an open set $U \subseteq \mathbb{C}$ with a puncture at $z_0 \in \mathbb{C}$. That is, we assume that $z_0 \notin U$, but there exists $r > 0$ such that $B_r(z_0) \setminus \{z_0\} \subseteq U$.

Then Theorem 10.2 can be applied in any annulus $B_S(z_0) \setminus \overline{B_s(z_0)}$ with $0 < s < S < r$. Moreover, by the uniqueness of the Laurent series expansion, we can let $s \to 0$ and $S \to r$ and find that we have unique coefficients $a_k \in \mathbb{C}$ such that

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

for every $z \in B_r(0) \setminus \{z_0\}$. We now study this situation in more detail.

We distinguish between the following cases.

Definition 10.3 (Classification of isolated singularities)**.** Suppose that $U \subseteq \mathbb{C}$ is an open set and there exists $z_0 \in \mathbb{C} \setminus U$ such that $B_r(z_0) \setminus \{z_0\} \subseteq U$ for some $r > 0$. Let $f\colon U \to \mathbb{C}$ be a holomorphic function with the Laurent series expansion $f(z) = \sum_{k=-\infty}^{\infty} a_k(z - z_0)^k$. We say that z_0 is a

- *removable singularity* of f if $a_k = 0$ for all $k < 0$,
- *pole* of order k_0 of f (for $k_0 \in \mathbb{N}$) if $a_{-k_0} \neq 0$ but $a_k = 0$ for all $k < -k_0$,
- *essential singularity* of f if there exist infinitely many $k < 0$ with $a_k \neq 0$.

The following result helps to distinguish between these categories.

Theorem 10.4. *Let $z_0 \in \mathbb{C}$ and $r > 0$. Suppose that $f\colon B_r(z_0) \setminus \{z_0\} \to \mathbb{C}$ is holomorphic. Then the following holds true.*

1. *The singularity at z_0 is removable if, and only if, there exists $s \in (0, r)$ such that f is bounded in $B_s(z_0) \setminus \{z_0\}$. If so, then there exists a holomorphic function $g\colon B_r(z_0) \to \mathbb{C}$ such that $f(z) = g(z)$ for all $z \in B_r(z_0) \setminus \{z_0\}$.*

2. *The singularity at z_0 is a pole if, and only if, $\lim_{z \to z_0} |f(z)| = \infty$.*

Proof. To prove the first statement, assume first that z_0 is a removable singularity of f. Then f has a power series expansion in $B_r(z_0) \setminus \{z_0\}$, and thus it is bounded in $B_s(z_0) \setminus \{z_0\}$ for any $s \in (0, r)$. Moreover, the power series gives rise to a holomorphic function $g\colon B_r(z_0) \to \mathbb{C}$ that extends f.

Conversely, suppose that f is bounded in $B_s(z_0) \setminus \{z_0\}$ for some $s \in (0, r)$. Let $\rho \in (0, s)$. By Theorem 10.2, we have a Laurent series expansion

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

for $z \in B_r(z_0) \setminus \{z_0\}$, where

$$a_k = \frac{1}{2\pi i} \int_{\partial B_\rho(z_0)} \frac{f(w)}{(w - z_0)^{k+1}} \, dw$$

for all $k \in \mathbb{Z}$. We conclude that

$$|a_k| \leq \frac{1}{\rho^k} \max_{z \in B_s(z_0)} |f(z)|$$

by Lemma 6.9. Letting $\rho \to 0$, we find that $a_k = 0$ for all $k < 0$. Hence we have a removable singularity.

Next we consider the second statement of the theorem. Suppose first that z_0 is a pole of f. Then there exists $k_0 \in \mathbb{N}$ such that

$$f(z) = \sum_{k=-k_0}^{\infty} a_k (z - z_0)^k$$

for all $z \in B_r(z_0) \setminus \{z_0\}$ and $a_{-k_0} \neq 0$. Define

$$g(z) = (z - z_0)^{k_0} f(z) = \sum_{k=0}^{\infty} a_{k-k_0} (z - z_0)^k.$$

Then g is continuous at z_0, so $\lim_{z \to z_0} g(z) = a_{-k_0}$. Therefore,

$$\lim_{z \to z_0} |f(z)| = \lim_{z \to z_0} \frac{|g(z)|}{|z - z_0|^{k_0}} = \infty.$$

Conversely, suppose that $\lim_{z\to z_0} |f(z)| = \infty$. Then there exists $s > 0$ such that $|f(z)| \geq 1$ for all $z \in B_s(z_0) \setminus \{z_0\}$. Therefore, the function $g(z) = 1/f(z)$ is well-defined in $B_s(z_0) \setminus \{z_0\}$. Furthermore, it is holomorphic by Theorem 5.4. Since $\lim_{z\to z_0} |g(z)| = \lim_{z\to z_0} 1/|f(z)| = 0$, the singularity of g at z_0 is removable by the first statement. Hence there exists a power series expansion

$$g(z) = \sum_{k=0}^{\infty} b_k (z - z_0)^k$$

for $z \in B_s(z_0) \setminus \{z_0\}$. Since $g(z_0) = \lim_{z\to z_0} g(z) = 0$, we conclude that $b_0 = 0$.

By construction, the function g cannot vanish identically. Hence there exists at least some $k \in \mathbb{N}$ such that $b_k \neq 0$. Let k_0 be the smallest index with this property and define

$$h(z) = \frac{g(z)}{(z - z_0)^{k_0}} = \sum_{k=0}^{\infty} b_{k+k_0} (z - z_0)^k.$$

Then h is analytic, and hence holomorphic, in $B_s(z_0)$. Moreover, $h(z_0) = b_{k_0} \neq 0$. Therefore, there exists $s' > 0$ such that $h(z) \neq 0$ in $B_{s'}(z_0)$. It follows that $1/h(z)$ is also analytic in $B_{s'}(z_0)$. In particular, there exists a power series expansion

$$\frac{1}{h(z)} = \sum_{k=0}^{\infty} c_k (z - z_0)^k,$$

with $c_0 \neq 0$, for $z \in B_{s'}(z_0)$. Thus

$$f(z) = \frac{1}{(z - z_0)^{k_0} h(z)} = \sum_{k=-k_0}^{\infty} c_{k+k_0} (z - z_0)^k.$$

That is, z_0 is a pole of order k_0 of f. $\square$

Chapter 11

The residue theorem

In this chapter we discuss a theorem that makes it quite easy to compute contour integrals of functions with isolated singularities. Remarkably, this theory is helpful even for evaluating certain integrals in real analysis.

11.1 Winding numbers

In the following theory, we will need to know how many times a given contour winds around a specific point. This is often clear geometrically, but there is also a more analytic way to determine this number.

Definition 11.1 (Winding number)**.** Suppose that $\gamma\colon [a,b] \to \mathbb{C}$ is a closed contour and set $\Gamma = \gamma([a,b])$. For $z \in \mathbb{C} \setminus \Gamma$, the *winding number* of γ with respect to z is

$$W(\gamma, z) = \frac{1}{2\pi i} \int_\gamma \frac{dw}{w - z}.$$

Example 11.2. Let $z_0 \in \mathbb{C}$. Consider $\gamma\colon [0, 2\pi] \to \mathbb{C}$ with $\gamma(t) = z_0 + re^{it}$, parametrising $\partial B_r(z_0)$. For $z \in \mathbb{C} \setminus \partial B_r(z_0)$, what is $W(\gamma, z)$?

Example 7.12 gives the answer for $z = z_0$, namely $W(\gamma, z_0) = 1$. For all other $z \in \mathbb{C} \setminus \partial B_r(z_0)$, we can use Cauchy's integral theorem (Theorem 7.10) to compute the winding number. If $z \in B_r(z_0)$, then γ is homotopic to $\partial B_1(z)$ in $\mathbb{C} \setminus \{z\}$. Since the function $w \mapsto 1/(w - z)$ is holomorphic in this set, it follows that $W(\gamma, z) = 1$. If $z \in \mathbb{C} \setminus \overline{B_r(z_0)}$, then γ is null-homotopic in $\mathbb{C} \setminus \{z\}$. Hence in this case, we conclude that $W(\gamma, z) = 0$.

The results from this example are consistent with the geometric interpretation, and so is the following statement.

Lemma 11.3. *Suppose that $\gamma\colon [a,b] \to \mathbb{C}$ is a closed contour and $\Gamma = \gamma([a,b])$. If $z \in \mathbb{C} \setminus \Gamma$, then $W(\gamma, z)$ is an integer. Furthermore, the function $W(\gamma, \cdot)\colon \mathbb{C} \setminus \Gamma \to \mathbb{Z}$ is continuous.*

Proof. Since γ is a contour, it is piecewise continuously differentiable. That is, it is continuous and there exist $t_0, \ldots, t_K$ with $a = t_0 < \cdots < t_K = b$ such that γ is continuously differentiable in (t_{k-1}, t_k) and γ' has a continuous extension to $[t_{k-1}, t_k]$ for $k = 1, \ldots, K$.

For $t \in [a,b]$, define

$$F(t) = \int_a^t \frac{\gamma'(s)}{\gamma(s) - z}\, ds.$$

Then $F(b) = 2\pi i W(\gamma, z)$. The function F is continuous, and in each of the intervals (t_{k-1}, t_k), it is differentiable with

$$F'(t) = \frac{\gamma'(t)}{\gamma(t) - z}$$

by the fundamental theorem of calculus. Now define $G(t) = e^{-F(t)}(\gamma(t) - z)$. Then we compute

$$G'(t) = e^{-F(t)}\gamma'(t) - F'(t)e^{-F(t)}(\gamma(t) - z) = 0$$

in (t_{k-1}, t_k). Hence G is constant in these intervals. It is also continuous in $[a,b]$, and therefore it is constant in $[a,b]$. In other words, there exists a constant $c \in \mathbb{C}$ such that

$$e^{-F(t)}(\gamma(t) - z) = c$$

for all $t \in [a,b]$. So $\gamma(t) - z = ce^{F(t)}$. Clearly $c \neq 0$, since $z \notin \Gamma$.

Since γ is a closed contour, it satisfies $\gamma(a) = \gamma(b)$. By the above formula, this means that

$$c = ce^{F(a)} = ce^{F(b)}.$$

As $c \neq 0$, we conclude that $e^{F(b)} = 1$. Therefore, there exists $\ell \in \mathbb{Z}$ such that $F(b) = 2\pi \ell i$. Hence $W(\gamma, z) = \frac{F(b)}{2\pi i} = \ell \in \mathbb{Z}$.

Next we prove continuity of $W(\gamma, \cdot)$ on $\mathbb{C} \setminus \Gamma$. Consider $z_0, z \in \mathbb{C} \setminus \Gamma$. Then

$$\begin{aligned} W(\gamma, z) - W(\gamma, z_0) &= \frac{1}{2\pi i} \int_\gamma \left(\frac{1}{w - z} - \frac{1}{w - z_0} \right) dw \\ &= \frac{1}{2\pi i} \int_\gamma \frac{z - z_0}{(w - z)(w - z_0)}\, dw. \end{aligned}$$

Let $r = \operatorname{dist}(z_0, \Gamma)$. If $|z - z_0| \leq \frac{r}{2}$, then $\operatorname{dist}(z, \Gamma) \geq \frac{r}{2}$ and it follows that

$$\max_{w \in \Gamma} \left| \frac{z - z_0}{(w - z)(w - z_0)} \right| \leq \frac{2}{r^2} |z - z_0|.$$

According to Lemma 6.9, this implies that

$$|W(\gamma, z) - W(\gamma, z_0)| \leq \frac{L(\gamma)}{\pi r^2} |z - z_0|.$$

In particular, if we let $z \to z_0$, then $W(\gamma, z) \to W(\gamma, z_0)$. □

11.2 Residues

If we have a Laurent series $\sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$ and wish to integrate along a contour term by term, then the term with index -1 behaves differently from the rest. This is because it alone does not have a primitive in $\mathbb{C} \setminus \{z_0\}$ (as seen in Example 7.12). The corresponding coefficient a_{-1} therefore has special relevance.

Definition 11.4 (Residue)**.** Suppose that $U \subseteq \mathbb{C}$ is an open set and $z_0 \in \mathbb{C}$ is a point such that $z_0 \notin U$, but there exists $r > 0$ such that $B_r(z_0) \setminus \{z_0\} \subseteq U$. Let $f \colon U \to \mathbb{C}$ be a holomorphic function with Laurent series expansion

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

in $B_r(z_0) \setminus \{z_0\}$. Then a_{-1} is called the *residue* of f at z_0. We write

$$\operatorname{Res}(f, z_0) = a_{-1}.$$

In order to find the residue, we don't necessarily need to calculate the full Laurent series. Instead, we can often use tools such as the following.

Proposition 11.5. *Suppose that $U \subseteq \mathbb{C}$ is an open set and $f, g \colon U \to \mathbb{C}$ are holomorphic functions. Let $z_0 \in U$. If $f(z_0) \neq 0$, $g(z_0) = 0$, and $g'(z_0) \neq 0$, then f/g has a pole of order 1 at z_0 and*

$$\operatorname{Res}\left(\frac{f}{g}, z_0\right) = \frac{f(z_0)}{g'(z_0)}.$$

Proof. Since $g(z_0) = 0$, the power series expansion of g at z_0 has no zero order term. Therefore, there exists a holomorphic function $h \colon U \to \mathbb{C}$ such that

$g(z) = (z - z_0)h(z)$ for all $z \in U$. Differentiating, we find that $g'(z_0) = h(z_0)$. In particular, we conclude that $h(z_0) \neq 0$, and there exists $r > 0$ such that f/h is analytic in $B_r(z_0)$.

Let

$$\frac{f(z)}{h(z)} = \sum_{k=0}^{\infty} b_k (z - z_0)^k$$

be the power series expansion of f/h at z_0. Then $b_0 = f(z_0)/h(z_0) \neq 0$. Moreover,

$$\frac{f(z)}{g(z)} = \sum_{k=0}^{\infty} b_k (z - z_0)^{k-1} = \sum_{k=-1}^{\infty} b_{k+1} (z - z_0)^k.$$

Therefore, we have a pole of order 1. The residue is

$$\operatorname{Res}\left(\frac{f}{g}, z_0\right) = b_0 = \frac{f(z_0)}{h(z_0)} = \frac{f(z_0)}{g'(z_0)}.$$

□

Example 11.6. Consider the function

$$\tan z = \frac{\sin z}{\cos z},$$

which has isolated singularities at $\frac{\pi}{2} + \ell\pi$ for every $\ell \in \mathbb{N}$. With the help of Proposition 11.5, we compute

$$\operatorname{Res}\left(\tan, \frac{\pi}{2} + \ell\pi\right) = -\frac{\sin(\frac{\pi}{2} + \ell\pi)}{\sin(\frac{\pi}{2} + \ell\pi)} = -1.$$

11.3 The residue theorem

Theorem 11.7 (Residue theorem)**.** *Let $U \subseteq \mathbb{C}$ be an open set and $\gamma\colon [a, b] \to U$ a closed contour that is null-homotopic in U. Set $\Gamma = \gamma([a, b])$. Suppose that $z_1, \ldots, z_N \in U \setminus \Gamma$ are finitely many distinct points and $f\colon U \setminus \{z_1, \ldots, z_N\} \to \mathbb{C}$ is holomorphic. Then*

$$\int_\gamma f(z)\, dz = 2\pi i \sum_{n=1}^{N} W(\gamma, z_n) \operatorname{Res}(f, z_n).$$

So in this situation, in order to evaluate the integral, all we need to do is find the relevant winding numbers and residues. The winding numbers are often geometrically obvious, and the residues depend only on the behaviour of f near the singularities.

Proof. For each $n = 1, \dots, N$, consider the Laurent series expansion

$$f(z) = \sum_{k=-\infty}^{\infty} a_{nk}(z - z_n)^k$$

for $z \in B_r(z_n) \setminus \{z_n\}$, where $r > 0$. We are interested above all in the principal parts and define

$$f_n(z) = \sum_{k=-\infty}^{-1} a_{nk}(z - z_n)^k, \quad z \in \mathbb{C} \setminus \{z_n\}.$$

Since by definition, these series converge for $z \in B_r(z_n) \setminus \{z_n\}$, they converge uniformly in $\mathbb{C} \setminus B_s(z_n)$ for any $s > 0$. Hence the functions f_n are well-defined in $\mathbb{C} \setminus \{z_n\}$ and are holomorphic there by Corollary 9.8.

Now define

$$g(z) = f(z) - \sum_{n=1}^{N} f_n(z), \quad z \in U \setminus \{z_1, \dots, z_N\}.$$

In $B_r(z_n) \setminus \{z_n\}$, we have the formula

$$g(z) = \sum_{k=0}^{\infty} a_{nk}(z - z_n)^k - \sum_{m \neq n} f_m(z).$$

Thus if we extend g to z_n by

$$g(z_n) = a_{n0} - \sum_{m \neq n} f_m(z_n), \quad n = 1, \dots, N,$$

then we obtain a holomorphic function $g \colon U \to \mathbb{C}$. As γ is null-homotopic in U, Theorem 8.5 implies that

$$\int_\gamma g(z)\, dz = 0.$$

Therefore,

$$\int_\gamma f(z)\, dz = \sum_{n=1}^{N} \int_\gamma f_n(z)\, dz = \sum_{n=1}^{N} \sum_{k=-\infty}^{-1} \int_\gamma a_{nk}(z - z_n)^k\, dz,$$

the last identity following from the fact that we have uniform convergence of the principal parts of the Laurent series on Γ.

Finally, we note that for every $k \neq -1$, the function $z \mapsto a_{nk}(z - z_n)^{k+1}/(k+1)$ is a primitive of $a_{nk}(z - z_n)^k$. Hence

$$\int_\gamma a_{nk}(z - z_n)^k \, dz = 0$$

for $k \neq -1$. For the remaining terms, we observe that

$$\int_\gamma \frac{a_{n,-1}}{z - z_n} \, dz = 2\pi i a_{n,-1} W(\gamma, z_n) = 2\pi i W(\gamma, z_n) \operatorname{Res}(f, z_n)$$

by the definitions of the winding number and the residue. Now it suffices to sum everything up. □

Example 11.8. Suppose we need to compute

$$\int_{\partial B_2(0)} \tan z \, dz.$$

As we have seen in Example 11.6, we have poles at $\frac{\pi}{2} + \ell\pi$, for $\ell \in \mathbb{Z}$, with residues -1. We can choose $U = B_3(0)$, then $\partial B_2(0)$ is null-homotopic in U. Of all the singularities of tan, only $\pm\frac{\pi}{2}$ belong to U. Clearly $W(\partial B_2(0), \pi/2) = 1$ and $W(\partial B_2(0), -\pi/2) = 1$. Hence

$$\int_{\partial B_2(0)} \tan z \, dz = 2\pi i \operatorname{Res}\left(\tan, \frac{\pi}{2}\right) + 2\pi i \operatorname{Res}\left(\tan, -\frac{\pi}{2}\right) = -4\pi i.$$

11.4 Real integrals

It is not surprising that the residue theorem can be used to evaluate integrals as in Example 11.8. But it is also useful for certain definite integrals of real functions, where the connection to complex analysis is less obvious.

Example 11.9. What is

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + 1} \, dx?$$

By definition of improper Riemann integrals, we know that

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + 1} \, dx = \lim_{R \to \infty} \int_{-R}^{R} \frac{\cos x}{x^2 + 1} \, dx.$$

The idea is now to interpret the right-hand side as the real part of an integral over the line segment $[-R, R]$ in $\mathbb{C}$. Define $\gamma_1 \colon [-R, R] \to \mathbb{C}$ such that $\gamma_1(t) = t$, and consider the function $f(z) = \frac{e^{iz}}{z^2+1}$. Then

$$\int_{-R}^{R} \frac{\cos x}{x^2 + 1} \, dx = \operatorname{Re} \int_{\gamma_1} f(z) \, dz.$$

If we want to evaluate the right-hand side using the residue theorem, then we have to close up the contour. Therefore, we define $\gamma_2\colon [0,\pi] \to \mathbb{C}$, $t \mapsto Re^{it}$. Then $\gamma = \gamma_1 + \gamma_2$ is closed and of course null-homotopic in $\mathbb{C}$.

The function f has isolated singularities at $\pm i$. We can find the residues with Proposition 11.5. In particular:

$$\operatorname{Res}(f, i) = \frac{e^{-1}}{2i}.$$

Hence by the residue theorem,

$$\int_\gamma f(z)\,dz = 2\pi i W(\gamma, i)\operatorname{Res}(f, i) = \pi e^{-1},$$

as long as $R > 1$.

Define $\Gamma_2 = \gamma_2([0,\pi])$. Then by Lemma 6.9,

$$\left|\int_{\gamma_2} f(z)\,dz\right| \le \pi R \max_{z\in\Gamma_2} |f(z)|.$$

Furthermore, for $z = x + iy \in \Gamma_2$, we have the inequality $y \ge 0$. Hence we can estimate

$$|e^{iz}| = |e^{ix-y}| = e^{-y} \le 1.$$

The triangle inequality implies that

$$|z^2 + 1| \ge |z|^2 - 1 = R^2 - 1$$

for $z \in \Gamma_2$. Therefore,

$$\left|\int_{\gamma_2} f(z)\,dz\right| \le \frac{\pi R}{R^2 - 1} \to 0$$

as $R \to \infty$. We now conclude that

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx = \lim_{R\to\infty} \operatorname{Re}\left(\int_\gamma f(z)\,dz - \int_{\gamma_2} f(z)\,dz\right) = \pi e^{-1}.$$

Example 11.10. What is

$$\int_0^{2\pi} \frac{dt}{3 + 2\sin t}?$$

Here we try to write this integral as a contour integral over the contour $\gamma\colon [0, 2\pi] \to \mathbb{C}$ with $\gamma(t) = e^{it}$. Then given a function f, we compute

$$\int_\gamma f(z)\,dz = \int_0^{2\pi} ie^{it} f(e^{it})\,dt.$$

Hence we need to find f such that

$$ie^{it} f(e^{it}) = \frac{1}{3 + 2\sin t}, \quad t \in [0, 2\pi].$$

To this end, we can use the formula

$$\sin t = \frac{1}{2i}(e^{it} - e^{-it}) = \frac{1}{2i}\left(e^{it} - \frac{1}{e^{it}}\right).$$

Hence we solve the equation

$$izf(z) = \frac{1}{3 - i(z - \frac{1}{z})} = \frac{iz}{z^2 + 3iz - 1}$$

for $f(z)$, which gives

$$f(z) = \frac{1}{z^2 + 3iz - 1}.$$

We now note that the function $g(z) = z^2 + 3iz - 1$ has zeros at $\frac{i}{2}(-3 \pm \sqrt{5})$. Since $2 < \sqrt{5} < 3$, one of these is in the disk $B_1(0)$, namely $a = \frac{i}{2}(-3 + \sqrt{5})$. The residue of f at a is

$$\mathrm{Res}(f, a) = \frac{1}{i\sqrt{5}}.$$

Hence

$$\int_0^{2\pi} \frac{dt}{3 + 2\sin t} = \int_\gamma f(z)\, dz = \frac{2\pi}{\sqrt{5}}.$$