SDL: Relativity

2023/2024 PH20076 Notes + Problems

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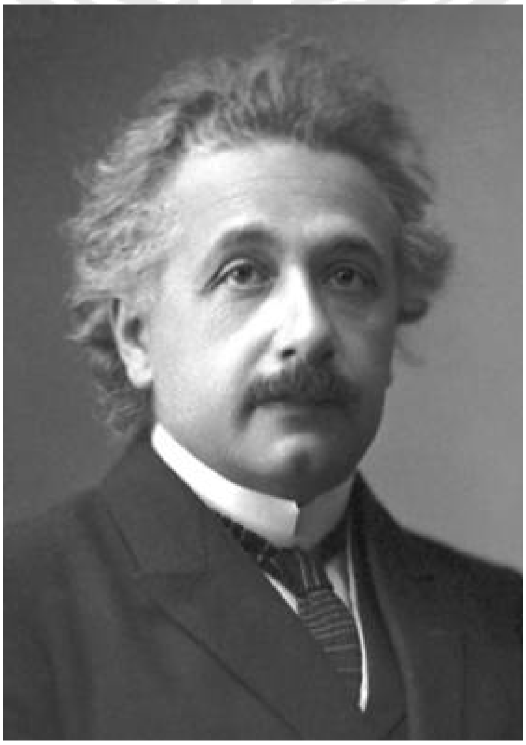


Albert Einstein, 1879-1955 [1]

The following will appear in the exam in this exact format.

For all the following Special Relativity questions, take the speed of light to be $c = 3 \times 10^8$ ms^{-1}.

You may want to make use of the following equations where the symbols have their usual meaning.

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \gamma(v \ll c) \approx 1 + \frac{v^2}{2c^2}$$

$$x' = \gamma\left(x - Vt\right) \qquad {v_x}' = \frac{(v_x - V)}{\left(1 - \frac{V}{c^2}v_x\right)}$$

$$y' = y \qquad {v_y}' = \frac{v_y}{\gamma\left(1 - \frac{V}{c^2}v_x\right)}$$

$$z' = z \qquad {v_z}' = \frac{v_z}{\gamma\left(1 - \frac{V}{c^2}v_x\right)}$$

$$t' = \gamma\left(t - \frac{V}{c^2}x\right) \qquad \beta = \frac{v}{c}$$

$$f'_R = \frac{f_S}{\gamma} \qquad f'_R = f_S\sqrt{\frac{1-\beta}{1+\beta}}$$

$$\Delta s^2 = (c\Delta t)^2 - \Delta x^2 \qquad \vec{p} = \gamma m\vec{v}$$

$$E_{\text{total}} = \gamma mc^2 \qquad E_{\text{rest}} = mc^2$$

$$K = (\gamma - 1)mc^2 \qquad \frac{cp}{E} = \frac{v}{c}$$

$$E_{\text{photon}} = cp \qquad E^2_{\text{total}} = m^2c^4 + p^2c^2$$

Contents

Part A: Introduction to relativity

Section 1

Events in time and space [Self-Study]

Physics is concerned with measurements of phenomena in space and time. Any measurement can be considered as an event $\vec{E}$ which has a particular position in space and time and can be written in 1 dimensional space as the coordinates

$$\vec{E} = (x, t) \tag{1.1a}$$

and in 3 dimensional space as

$$\vec{E} = (x, y, z, t) \tag{1.1b}$$

as measured from a coordinate system by an observer in a *particular* reference frame. Every coordinate system has

- an origin,
- a scale
- and an orientation.

We can make transformation between different reference frames of,

- coordinates,
- velocities,
- and accelerations.

1.1 References frames

Reference frames are physical objects against which events are measured. Any coordinate system must be defined as belonging to (being attached to) a particular reference frame. Relativity is concerned with the comparison of measurements of the same event made in different reference frames, i.e., reference frames that are moving relative to each other. Events do not belong to a particular reference frame, so they can be measured by any observer in any reference frame.

1.2 Making measurements

How do we make a measurement of the time and position of an event in a reference frame when the observer is (by convention) at the origin? We consider that throughout the whole coordinate system we have a system of meter sticks and clocks, see figure 1.1. The sticks and clocks are all at rest relative to the observer. To synchronise the clocks a pulse of light is sent out from the

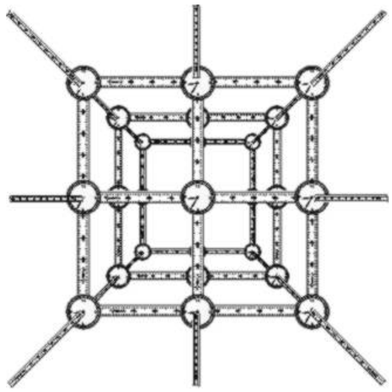


Figure 1.1: Measuring rods and clocks throughout a coordinate system [2].

origin. Each clock can record the time it receives this pulse, and knowing its distance from the origin, correct for the time the light takes to travel. Thus we have an infinite array of synchronised clocks.

If an event occurs, in this thought experiment, it leaves a mark on the meter stick and stops the clock at that position. Any observer can just walk over counting the number of sticks they pass and then measure the time on the stopped clock. (Note, as we'll show much later on, in Special Relativity what we can't do is synchronise all the clocks at the origin and then send them out onto the coordinate grid.)

For example figure 1.2 on the 5th of November Frank, observer O, stands at his garden gate. Penny, observer O', cycles past at constant speed (along a very long and straight road) and just as they pass each other they both start stopwatches. Some while later a firework explodes, let's call this event $\vec{E}$ then we'll have two sets of experimental results for the coordinate of the explosion,

$$\vec{E} = (x, y, z, t) = (x', y', z', t') \,. \tag{1.2}$$

The questions we're going to address is: what is the relationship between these two measurements of the *same* event? But first we have to think about the two distinct types of reference frame: inertial and non-inertial.

- Here the prime on O' tells us we have a IRF that is different to O. This prime has nothing to do with derivatives.
- To make a measurement of the length of an object, you must compare the space coordinates taken of the end of the object measured at the same time - that is they must be simultaneous measurements.

1.3 Inertial Reference Frames (IRF)

These are reference frames undergoing no acceleration. For example, in deep-space. Here, Newton's law hold. Every inertial reference frame is stationary or in *constant* motion relative to

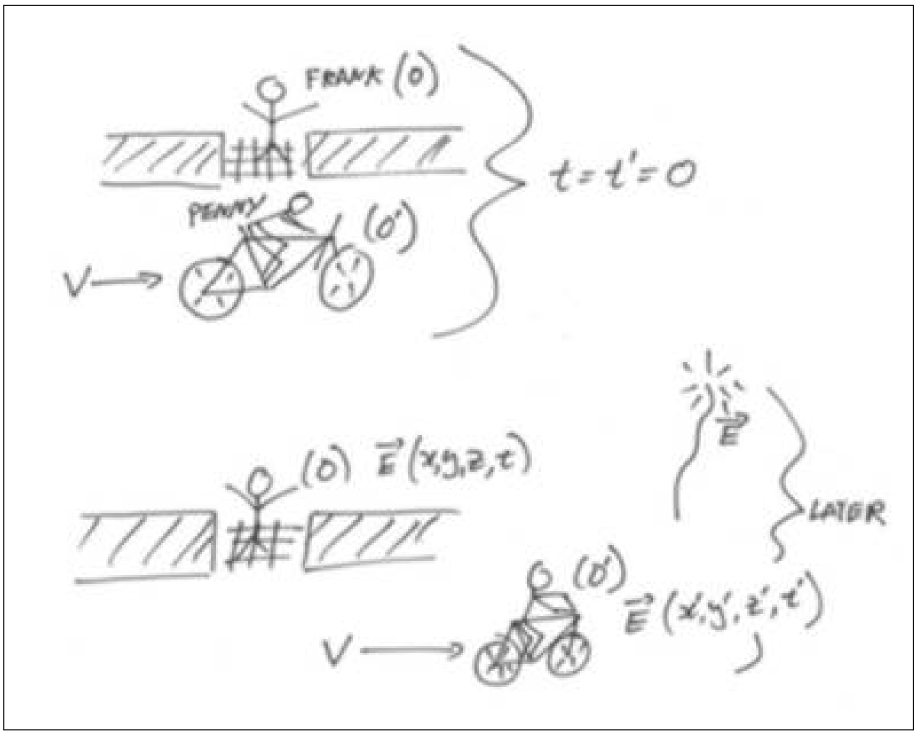


Figure 1.2: 5th November for Frank and Penny.

any other inertial reference frame.

- Non-accelerated
- Law of inertia holds
- In uniform motion or stationary relative to any other inertial reference frame
- All equivalent

1.4 Non-inertial reference frames

Non-inertial reference frames are accelerating. For example, your bike as you swoosh around a corner, or when you accelerate away from traffic lights. Here, Newton's law that the velocity of a body remains constant if no external force acts, fails as such a body, as measured from the perspective of the non-inertial reference frame, will appear to change velocity. Non-inertial reference frames are always in acceleration in comparison to any inertial reference frame. They are described by *General Relativity* and will be introduced in a book you can read

- Accelerated
- Law of inertia fails
- In acceleration relative to any inertial reference frame

Section 2

Classical Galilean/Newton relativity [Self-Study]

One of the strongest principles we have inherited from Galileo is the principle of relativity:

> The laws of physics are the same in all inertial reference frames.

This is usually referred to as Galilean/Newtonian relativity, as it refers to only mechanical laws of motion. This means that the physical processes you have measured in the lab here in Bath give the same results if you would have measured them in Bristol, or Paris, or the moon, or in a boat cruising with constant speed, etcetera. Without this principle the game is up and we may as well pack up our pots and pans and go home.

2.1 Galilean coordinate transformation

Consider again Frank, observer O, in his garden, Penny observer O' on her bike moving at constant velocity $\vec{V}$ in the x-direction $\vec{V} = (V, 0, 0)$, and the firework event $\vec{E}$. The figure 2.1 schematically shows these two reference frames. Let's dispense with names. As measured by O, event $\vec{E}$ occurs at time t and has Cartesian co-ordinates (x, y, z) . What are the corresponding co-ordinates as measured by O'? Let's first state that the origin's of O and O' coincide at time

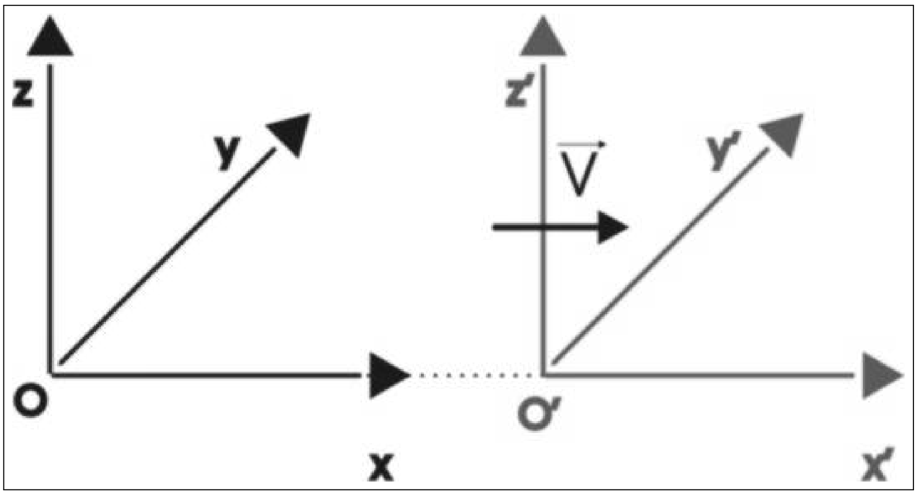


Figure 2.1: Two inertial reference frames in relative motion.

$t = t' = 0$. Since there is no movement in either y or z, the y and z coordinates are the same for both frames,

$$y = y' \tag{2.1a}$$
$$z = z'. \tag{2.1b}$$

Common sense says that x and x' are related through

$$x' = x - Vt. \tag{2.2}$$

We can therefore write down the Galilean coordinate transformations for $O \to O'$

$$x' = x - Vt \tag{2.3a}$$
$$y' = y \tag{2.3b}$$
$$z' = z. \tag{2.3c}$$

And conversely for $O' \to O$

$$x = x' + Vt' \tag{2.4a}$$
$$y = y' \tag{2.4b}$$
$$z = z'. \tag{2.4c}$$

2.2 Galilean time transformation

The time coordinate in both frames must be the same since if

$$x' = x - Vt \tag{2.5a}$$
$$\text{and } x = x' + Vt', \tag{2.5b}$$
$$\text{thus } t = t'. \tag{2.5c}$$

- In Galilean/Newtonian relativity time is absolute.
- The time of an event t, or time duration Δt, is identical for frames O and any frame O'.

2.3 Galilean distance transformation

A distance travelled in O along the x-direction during a period Δt is given by,

$$\Delta x = x_2 - x_1 \tag{2.6a}$$

where x_1 and x_2 are the initial and final x-coordinates respectively. The corresponding distance as measured in O' is

$$\Delta x' = x'_2 - x'_1 \tag{2.6b}$$
$$= (x_2 - Vt) - (x_1 - Vt) \tag{2.6c}$$
$$= x_2 - x_1 \tag{2.6d}$$
$$\Delta x' = \Delta x \tag{2.6e}$$

- Distances are invariant under Galilean transformations.

The distance you measure while walking between carriages of a uniformly moving train is the same as that measured from someone observing you from a station. This can also be expressed as: the length of the ruler used by the train passenger is measured to be the same by both the train passenger and the observer on the platform.

2.4 Galilean velocity transformation

Velocity is the time derivative of position and $t = t'$ so, for $O \to O'$

$$v'_x = \frac{dx'}{dt'} = \frac{d\,(x - Vt)}{dt} = \frac{dx}{dt} - V = v_x - V \tag{2.7a}$$

$$v'_y = \frac{dy'}{dt'} = \frac{dy}{dt} = v_y \tag{2.7b}$$

$$v'_z = \frac{dz'}{dt'} = \frac{dz}{dt} = v_z \tag{2.7c}$$

and conversely for $O' \to O$

$$v_x = \frac{dx}{dt} = \frac{d\,(x' + Vt')}{dt'} = \frac{dx'}{dt'} + V = v'_x + V \tag{2.8a}$$

$$v_y = \frac{dy}{dt} = \frac{dy'}{dt'} = v'_y \tag{2.8b}$$

$$v_z = \frac{dz}{dt} = \frac{dz'}{dt'} = v'_z \tag{2.8c}$$

giving in final vector form

$$\vec{v}\,' = \vec{v} - \vec{V} \tag{2.9a}$$

$$\vec{v} = \vec{v}\,' + \vec{V} \tag{2.9b}$$

where $\vec{v}$ and $\vec{v}\,'$ are the velocity of the same object as measured by O and O'.

For example, observer O' sits on a moving train and measures the velocity $\vec{v}\,'$ of a fellow passenger who is making their way to the buffet car. Observer O in the station measures the velocity of the same buffet bound passenger to be $\vec{v}$. Now we know how to relate these two measurements using Galilean Relativity.

Section 3

The problem with light

Figure 3.1: James Clerk Maxwell, 1831-1879. At school he was at first regarded as shy and rather dull. He made no friendships and spent his occasional holidays in reading old ballads, drawing curious diagrams and making crude mechanical models. This absorption in such pursuits, totally unintelligible to his schoolfellows, procured him a not very complimentary nickname. [3]

3.1 Maxwell versus Galileo/Newton

Numerous physical phenomena have confirmed the Galilean transformations of coordinates, time, length and velocity. They are also intuitively satisfying. But in 1865 the Scottish scientist James Clerk Maxwell, see figure 3.1, proposed a beautiful and simple theory of light. Within Maxwell's equations, and with a little hindsight, the velocity of electromagnetic radiation is finite and in vacuum it has the constant value of c irrespective of the frame in which the propagation is measured,

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 2.99792458 \times 10^8 ms^{-1} \approx 3 \times 10^8 ms^{-1}. \tag{3.1}$$

(This is an exact number as the meter is defined as the length light travels in a vacuum in $\frac{1}{299792458}$ of a second.) It's also quite fast over 600 million mph. We cannot extend our Galilean principle of mechanical relativity to encompass light: Galilean transformations are inconsistent with Maxwell's laws governing the propagation of light.

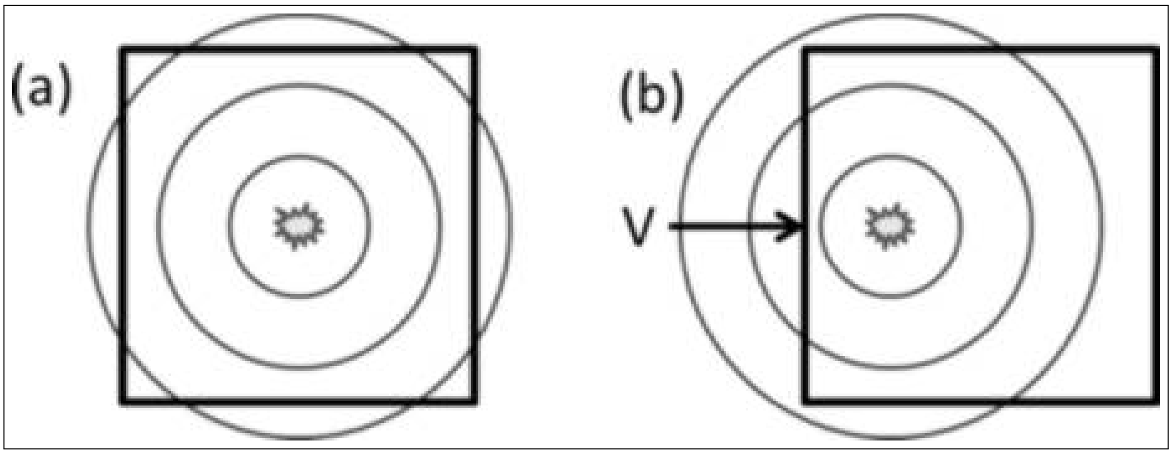


Figure 3.2: Galilean relativity: Flash of light and emitted wavefronts as measured (a) at rest with respect to the light source and (b) in a moving frame of reference.

3.2 Gedankenexperiment (thought experiment)

Consider a flash of light radiating isotropically in all directions of space as shown in figure 3.2. In Galilean relativity, if a flash occurs from a source that is stationary with respect to an observer's reference frame, figure 3.2 then

- the observer will detect spherical wavefronts, that is, the same speed of light in all directions.
- If the observer is now moving with respect to the light source, figure 3.2 (b), then the measured velocity of the emitted wavefront will depend on the position of the measurement, measurements on the left hand side of the box will give a faster speed of light than on the right hand side of the box.
- But Maxwell's theory of electromagnetism states that the velocity of the radiation is constant in all directions and so the flash should produce spherical wave fronts regardless of the frame where it occurs.

Something is wrong.

3.3 Possible solutions

We need to do some experiments to decide if:

1. Therelativity principle holds for all the known laws of physics, including electromagnetism and that electromagnetism laws as given by Maxwell are incorrect.
2. The relativity principle holds for all the laws of physics, including electromagnetism and that Newton's laws of mechanics are incorrect.

3. The relativity principle exists for mechanics, but not for electromagnetism (i.e. light). In the case of electromagnetism there would be a preferred inertial frame that is the frame of the "ether" which would be the medium through which the light-wave propagates at speed c. This would satisfy the majority of classical physics because it was well established that light is a wave and so it is natural to have a medium in which the disturbance (light oscillations) propagates in a similar manner as acoustic waves through air or water waves through water.

Section 4

The Michelson and Morley experiment

A. A. Michelson, later joined by E. W. Morley, set up an experiment to measure the effect of the hypothetical ether motion using an optical interferometer, see figure 4.1. The principle of the experiment is to measure the "time of flight" difference of two optical beams travelling in different directions induced by the apparatus' motion through the Ether. By recombining the beams we should observe an interference pattern caused by this time difference. You can also visit a web site with a interactive Michelson Morley experiment simulation that includes the effect an ether would have.

4.1 Experimental

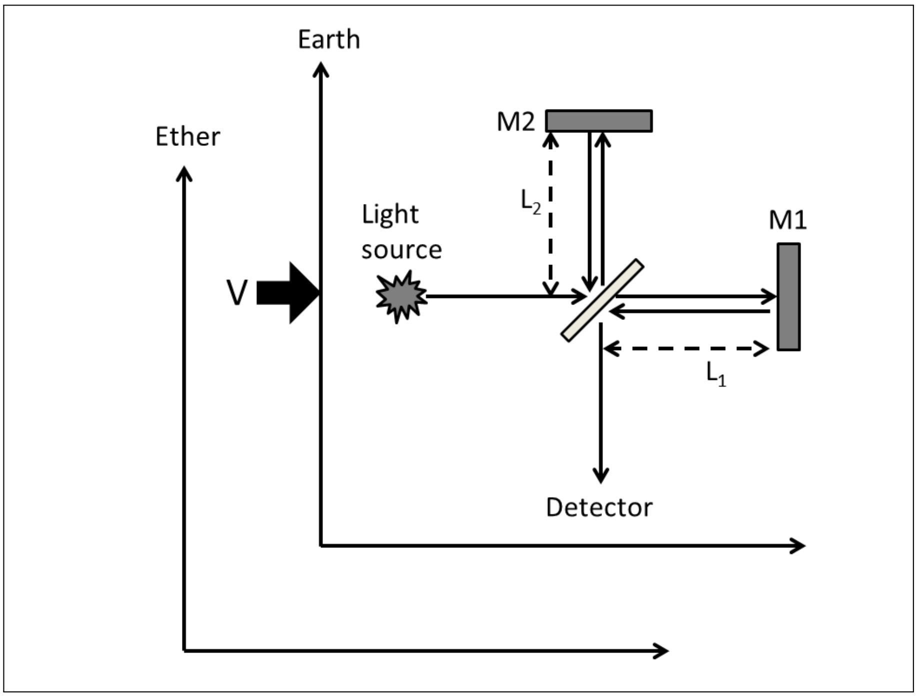


Figure 4.1: Schematic of the Michelson and Morley experimental apparatus.

The apparatus consists of a light source directed toward a beam splitter. Half of the light goes straight through Arm#1 and half is reflected at right angles into Arm#2. These two beams are reflected by two mirrors M1 and M2. After reflection, the beams pass through the splitter again and some fraction of each travels down towards the detector. The detector consists of a telescope to visualise interference fringes. These are created by the difference in the phase between the two recombined beams, which is in turn caused by a difference in the time-of-flight of each arm, which is in turn is caused by the apparatus' motion through the Ether. Let us now compute this time difference.

4.2 Theory

4.2.1 Time for Arm#1 round trip

When the light beam travels in Arm#1 toward mirror M1, its relative speed with respect to the interferometer (i.e. planet Earth) is $c - V$, as it is moving against the ether motion, and so the time for the light to reach M1 on this outward trip is

$$t_1^{out} = \frac{L_1}{c - V}. \tag{4.1a}$$

On the way back, the light will "see" the beam splitter (i.e. interferometer and Earth) moving toward it; the light travels with the ether motion and consequently its relative speed with respect to the interferometer is $c + V$. We have then for the return trip,

$$t_1^{return} = \frac{L_1}{c + V}. \tag{4.1b}$$

And so the total time to travel out and back through Arm-1 is,

$$t_1^{trip} = \frac{L_1}{c - V} + \frac{L_1}{c + V} \tag{4.1c}$$

$$= \left(\frac{2L_1}{c}\right) \frac{1}{1 - \frac{V^2}{c^2}} \tag{4.1d}$$

4.2.2 Time for Arm#2 round trip

Using figure 4.2 we can see that the time for the outward trip is given by the right angle triangle and Pythagoras,

$$\left(ct_2^{out}\right)^2 = (L_2)^2 + \left(Vt_2^{out}\right)^2 \tag{4.2a}$$

$$\therefore t_2^{out} = \left(\frac{L_2}{c}\right) \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}. \tag{4.2b}$$

By symmetry the return path takes the same time giving the total trip time as

$$t_2^{trip} = \left(\frac{2L_2}{c}\right) \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}. \tag{4.2c}$$

4.2.3 Time difference due to the Ether

Combining equations (4.1d) and (4.2c) and setting $L_1 = L_2 = L$ we get

$$\Delta t^{trip} = t_1^{trip} - t_2^{trip} \tag{4.3a}$$

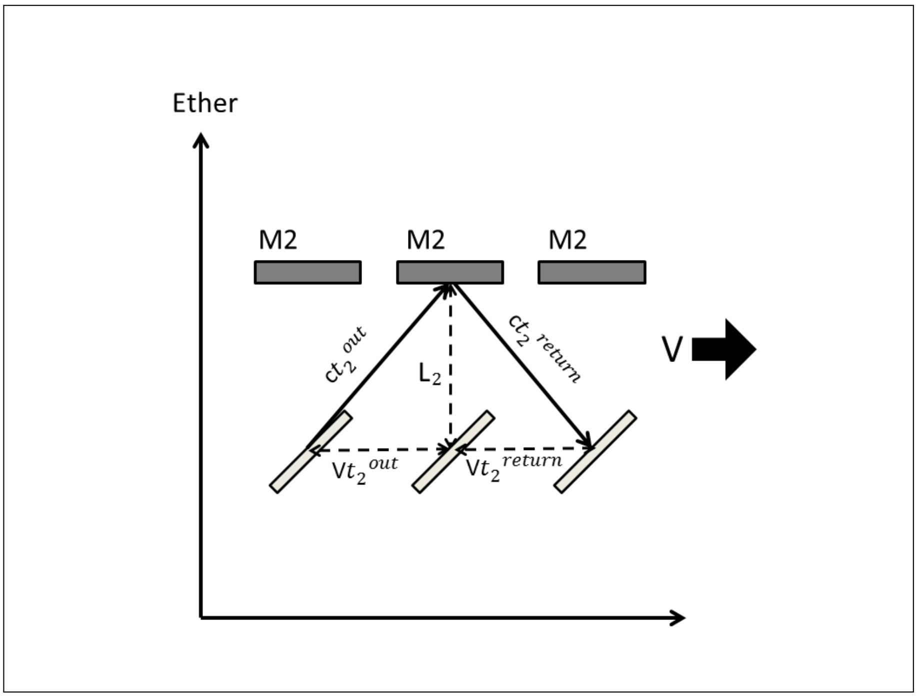


Figure 4.2: Schematic of the Michelson and Morley experimental apparatus showing the path the light beam takes along Arm#2 from the perspective of the reference frame of the ether.

$$= \left(\frac{2L}{c}\right)\frac{1}{1-\frac{V^2}{c^2}} - \left(\frac{2L}{c}\right)\frac{1}{\sqrt{1-\frac{V^2}{c^2}}}. \tag{4.3b}$$

Now we introduce the the following convenient conventions

$$\beta \equiv \frac{V}{c} \tag{4.3c}$$

$$\gamma \equiv \frac{1}{\sqrt{1-\frac{V^2}{c^2}}} = \frac{1}{\sqrt{1-\beta^2}} \tag{4.3d}$$

and if $V \ll c$ then $\beta \ll 1$ and we can use the MacLaurin figure 4.3 expansion for γ

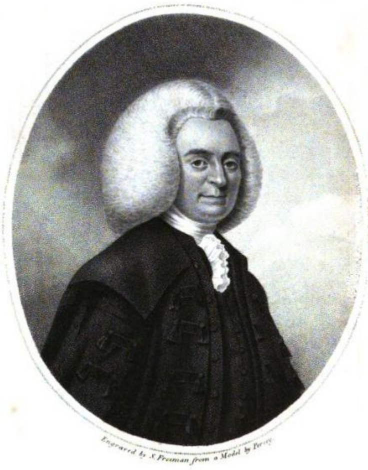


Figure 4.3: Scottish mathematician Colin Maclaurin 1698-1746. Maclaurin also played an active role in the defence of Edinburgh during the Jacobite rebellion of 1745.[4]

$$f(\Delta x) = f(0) + \frac{\Delta x}{1!}\left.\frac{df(x)}{dx}\right|_{x=0} + \frac{\Delta x^2}{2!}\left.\frac{d^2 f(x)}{dx^2}\right|_{x=0} + \ldots \tag{4.4a}$$

$$\therefore \gamma(\beta \ll 1) \approx 1 + \frac{\beta^2}{2} \tag{4.4b}$$

which simplifies equation (4.3b) to

$$\Delta t^{trip} = \frac{2L}{c}\left(\gamma^2 - \gamma\right) \tag{4.4c}$$
$$= \frac{L}{c}\beta^2 \tag{4.4d}$$

and can be converted to a path difference

$$\Delta l^{trip} = c\Delta t^{trip} \tag{4.4e}$$
$$= L\beta^2 \tag{4.4f}$$

and finally to the number of wavelengths N of light this corresponds to

$$N = \frac{\Delta l^{trip}}{\lambda} \tag{4.4g}$$
$$= \frac{L\beta^2}{\lambda} \tag{4.4h}$$
$$N = \frac{L}{\lambda}\left(\frac{V}{c}\right)^2 \tag{4.4i}$$

Phew. What does this mean? Depending on the exact value of N will should get constructive or destructive interference at the detector. Now for the clever part, we rotate the whole experimental apparatus through $45°$, meaning that each arm is now equivalent so $\Delta t^{trip} \to 0$ and $N \to 0$. So as the detector is rotated would should observe N *bright-dark* transitions whilst looking through the detector microscope.

4.3 Results

No measurement has ever found a time difference between the two arms. Therefore

- The speed of light is constant in all inertial reference frames
  - RIGHT
- or that the Ether is somehow attached to the experiment's IRF (ie., the rotating Earth).
  - WRONG

This latter conclusion is philosophically uncomfortable as it puts the Earth spookily back at the centre of things. The former conclusion suggests that either Maxwell or Newton was wrong.

Section 5

Problems for part A

P1: M and M fringes [application, medium]

For each of the following calculate the the number of fringes observed for a 10 m long Michelson and Morley interferometer that uses blue laser light ($\lambda = 475$ nm) if the rest frame of the ether is:

(a) Attached to the Earth but doesn't rotate. [2]

(b) Attached to the Sun. [2]

(c) Attached to the centre of the galaxy. [2]

Some useful numbers: Radius of the Earth is 6.4$\times 10^3$ km, Earth orbit radius is 150 $\times 10^6$ km, Radius of solar system from the galactic centre $\sim 30 \times 10^3$ light-years with a period of rotation of 250 $\times 10^6$ years. [Hint: Consider circular motion ...]

P2: M and M and Galilean relativity [analysis, medium]

Figure 5.1 shows a Michelson interferometer. Let the distance to each mirror from the inclined mirror at P be L. Assume that there is a stationary medium, the ether, through which the apparatus moves with velocity V in the direction of the mirror M_2. Assume also that Newtonian physics is valid, so that Galilean transforms apply.

(a) If the speed of light is c, what is the speed of light relative to M_2 as it approaches M_2 from P?

(b) How long does it take the light to travel from P to M_2?

(c) What is the speed of light relative to P on the return trip?

(d) How long does it take the light to travel from M_2 to P?

(e) What is the total time for the round trip from P to M_2 and back to P?

(f) Show that the speed of light relative to M_1 is $\sqrt{c^2 - V^2}$ and that this is also the speed relative to P on the return trip.

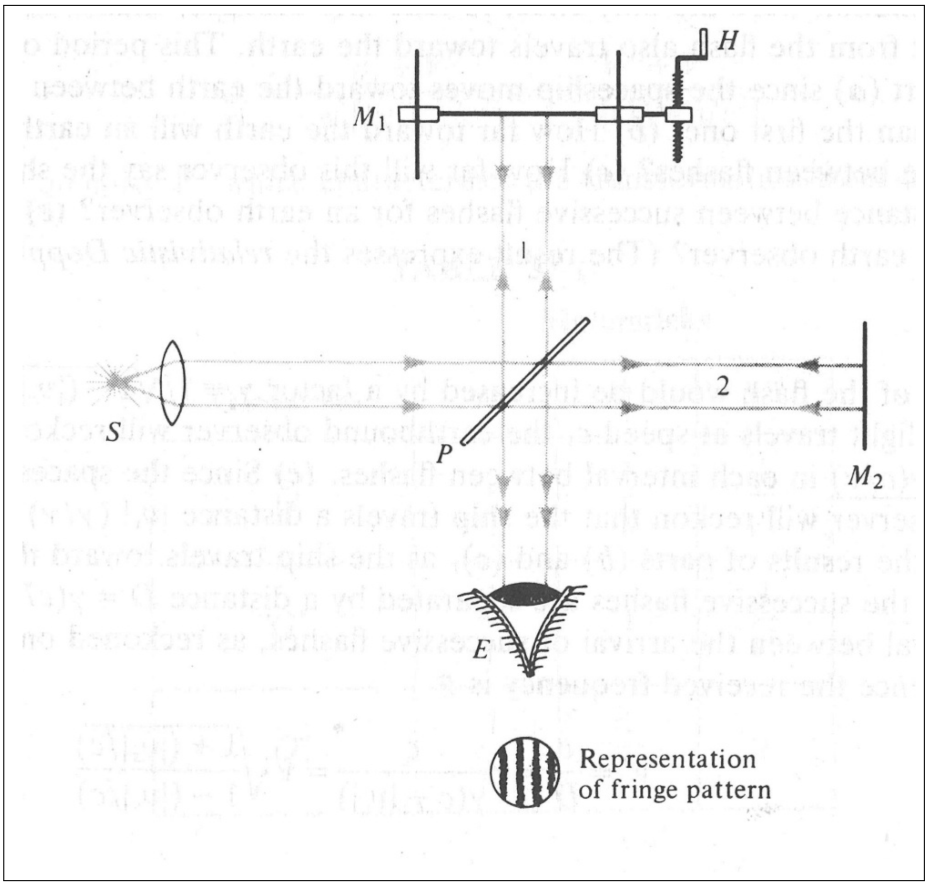


Figure 5.1: M and M interferometer

(g) Calculate the total time for the round trip to the mirror M_1 and back to P.

(h) if $|V|/c \ll 1$ show that the difference in time between the round trip to M_2 and the round trip to M_1 is equal to LV^2/c^3.

(i) What distance does the light travel in this time?

(j) What fraction of wavelength λ of light does this correspond to?

(k) Calculate this fraction given: $L = 10$ m, $\lambda = 600$ nm, $|V| = 3 \times 10^4$ m/s.

P3: Space and time diagrams [synthesis, hard]

(a) Consider the situation shown in figure 1.2. Sketch neatly and to scale the situation according to Frank, taking the x-axis as the distance along the road, and the y-axis as time. You can take $t = 0$ just as Frank and Penny pass each other and that Frank stands at $x = 0$. Take Penny to ride at 10 m/s and the firework is at a distance 100 m and it goes off after 5 s. Draw Frank's position as a function of time, draw Penny's and draw the fireworks, and draw the event of the firework going off. What distance does Frank measure for the distance to the firework and what time it went off? Do the same analysis but from Penny's point of view where she says she's at rest at x=0 and the others are moving at speed -10 m/s. What distance does Penny measure for the distance to the firework and what time it went off? Do they agree or disagree.

(b) Consider the situation of the flash-and-box of figure 3.2. Draw neatly and with the same scale on both axes, taking the x-axis to be distance and the y-axes to be the product ct (speed of light times time) the scenario from the IRF of the box (therefore it doesn't move). The centre of the box is is at $x = 0$ with its edges at ± 1 m. The light is flashed at $x = t = 0$. Plot or draw the (x,ct) coordinates of (a) the box edges (b) the light travelling at speed c to the right and (c) the light travelling at speed c to the left. Indicate when the light strikes the edges of the box - what do you conclude?

(c) Consider the situation of the flash-and-box of figure 3.2. Draw neatly and with the same scale on both axes, taking the x-axis to be distance and the y-axes to be the product ct (speed of light times time) the scenario from the IRF of the lab where the box travels at 0.25 c to the right. Still take the width of the box to be 2 m. At $x = t = 0$ the centre of the box is at $x = 0$ and the light is flashed. Plot or draw the (x,ct) coordinates of (a) the box edges (b) the light travelling to the right at speed c and (c) the light travelling to the left at speed c. Indicate when the light strikes the edges of the box - what do you conclude?

(d) Consider an arm of the MM-interferometer. The two mirrors lie along that x-axis a distance 1 m apart. At $t = 0$ light if flashed from the first mirror at $x = 0$ to the other that is at coordinate $(x = 1, t = 0)$. Assuming the interferometer is stationary in the lab, draw neatly and with the same scale on both axes, taking the x-axis to be distance and the y-axes to be the product ct (speed of light times time) one round trip of the light. Draw the light, and the two mirrors. How long was the trip? Now imagine the same, but the interferometer travels to the $+x$ with speed 0.2 c. Again just at $t = 0$ one mirror is at $x = 0$ and the other at $x = 1$ m and the light is flashed. Does that trip out take longer? Does the round trip take longer?

Section 6

Ideas and Projects for part A

Some extra ideas to play with. These are for you to talk and discuss with your friends and perhaps tutor. They are not going to have solutions, they are for you to play with and see if you can make sense of relativity. They have purposely been written as sort of mini-projects that if you tackled a few you would learn a lot along the way. This year, 2021/2022, is the first year I've included them. So the rest of the course is identical to the past year's course, these are just interesting extras. Please feel free to use moodle or the google-doc to discuss with other students.

- At your desk design and perform a simple experiment to test if you are in an IRF or not. Share your experimental findings, logic and conclusion with a friend. Do they agree? Do you agree with their experiment? You may want to google Einstein's Equivalence principle.
- Analyse a MM interferometer of figure 4.1, but that is rotated 45. What would you see? Would be able to detect the Ether? Share your findings, logic and conclusion with a friend. Do they agree? Do you agree with their workings?
- Plot gamma in python. Also plot the MacLaurin expansion of gamma to zeroth order (constant), first order (constant + linear term) and second order. See if you can see where each approximation starts to deviate from the underlying function gamma.
- Read: Relativity : the Special and General Theory by Albert Einstein, Part I, chapter I to VII.

Part B: An invariant speed of light

Section 7

Einstein's Special Relativity Postulates

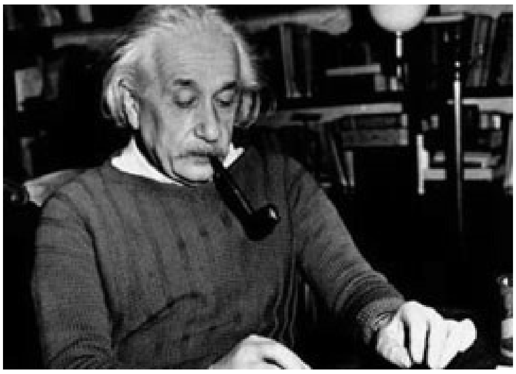


Figure 7.1: 'I believe that pipe smoking contributes to a somewhat calm and objective judgment in all human affairs.' [5]

In 1905 Einstein solved this Maxwell/Newton problem based on pure theoretical physics and thought experiment by creating a self consistent theory, the "Special Theory of Relativity" derived from two postulates:

1. All physical laws are the same in all IRF. This is the extension of the Galilean/Newtonian principal of relativity to encompass all laws of physics, not just mechanics.

2. The invariance of velocity of light in all IRF. In every inertial frame the speed of light in empty space has the same value c independent of the direction of propagation. Any observer in any IRF measuring the velocity of light relative to themselves will find the same result, $c \approx 3 \times 10^{8}\ \mathrm{ms}^{-1}$, despite their relative motions.

This relativity theory is called special because it is limited to inertial reference frames. Extending it to accelerating frames required Einstein more than 10 years of "perspiration" culminating in the general theory of relativity.

Section 8

The relativity of simultaneity

Here we are going to analyse and interpret the observations made of moving objects from the standpoint of two observers in relative motion to each other and with the speed of light constant. We'll show that whether or not two events are simultaneous depends on the velocity of the observer relative to the events, that the time between two events also depends on the observer and the distance between two event depends on the observer.

Einstein's lodgings in Bern was near the main railway station. So all the thought experiments are to do with trains.

8.1 Question

Consider the situation as shown in figure 8.1 where two trains AC and PR move with equal speeds in opposite directions to each other relative to us located at O. Observers B and Q stand at the mid points of their trains. What are the relative lengths of the two trains? We can decided by knowing where the trains' end points are at the same time.

8.2 Gedankenexperiment

- When ends A and P coincide, a flash of light $E1$ is emitted at that position of A and P, see figure 8.2a.

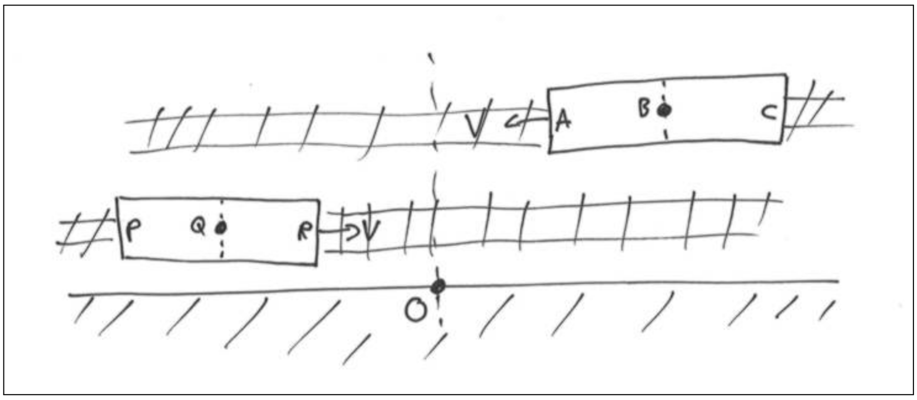
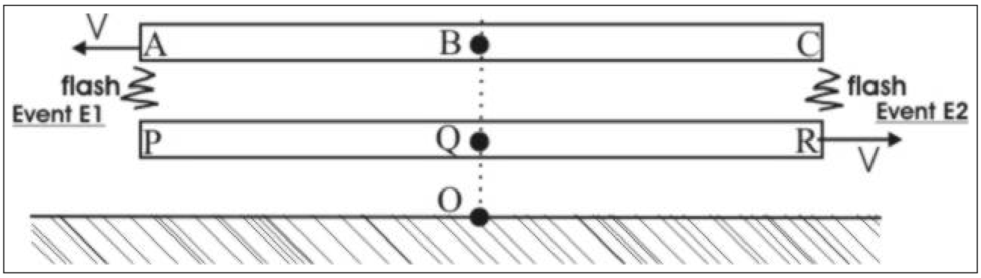
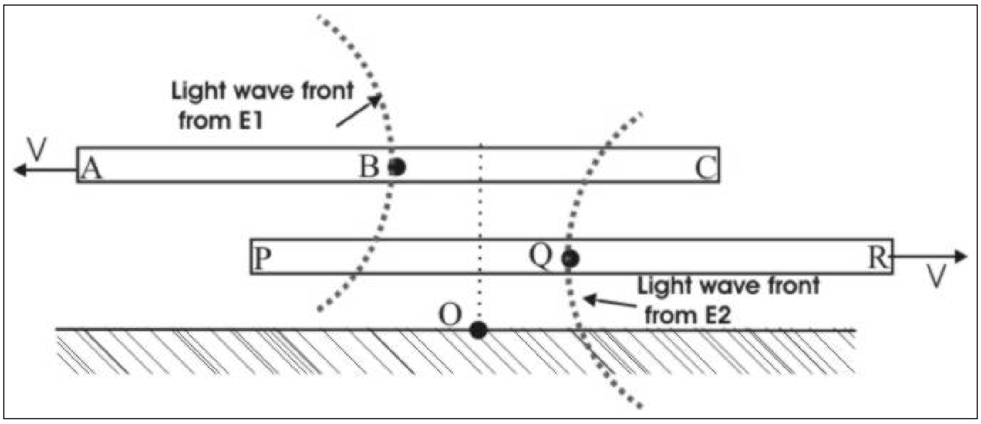


Figure 8.1: How long are the trains?



(a) Trains at time t_0 as deduced by O.



(b) Trains at time $t_0 + \Delta t$ as deduced by O.

Figure 8.2: Trains as deduced by O

- When ends C and R coincide, a flash of light $E2$ is emitted at that position of C and R, see figure 8.2a.

If we can deduce whether the event $E1$ and $E2$ were simultaneous and, if not, the order in which they occurred, we will be able to deduce the relative lengths of AC and PR.

8.3 From observer O

We, at O, see the two sparks at the same time. We deduce that

$$t^O_{E1} = t^O_{E2} = t_0 \tag{8.1}$$

At that time t_0 we saw that both mid points B and Q pass by us, i.e., B, Q and O all coincide. We deduce that the trains are of equal length as shown in figure 8.2a,

$$L^O_{AC} = L^O_{PR}. \tag{8.2}$$

Let's run the clock forward a bit from t_0 and consider the time $t_0 + \Delta t$ as shown in figure 8.2b. From the symmetry of the situation, as deduced by us at O, we find that

- The light from $E1$ reaches B at the same time the light from $E2$ reaches Q.

Whether B and Q will agree that these two beams reached them respectively at the same time is difficult to predict, since the receptions occurred at different places. However, it is clear that all three observers, we at O, B and Q *must* agree on the following points:

- B sees $E1$ before seeing $E2$.
- Q sees $E2$ before seeing $E1$.
- We at O see $E1$ and $E2$ simultaneously.

8.4 The disagreement

But disagreement will arise in B's and Q's deduction from these facts, since each assumes with equal validity that

1. They are at rest and the others are moving because they are inertial.
2. That light travels at the same speed in either direction relative to them.

8.4.1 B says: figure 8.3

I am stationary and midway between event $E1$ and $E2$. I saw $E1$ before $E2$ therefore:

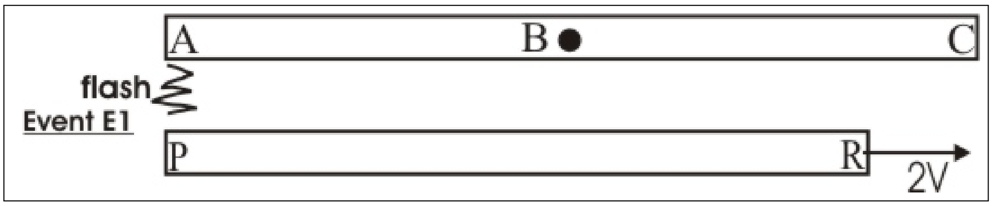


Figure 8.3: B's deduction

$$t^B_{E1} < t^B_{E2} \tag{8.3a}$$
$$L^B_{PR} < L^B_{AC} \tag{8.3b}$$

8.4.2 Q says: figure 8.4

I am stationary and midway between event $E1$ and $E2$. I saw $E2$ before $E1$ therefore:

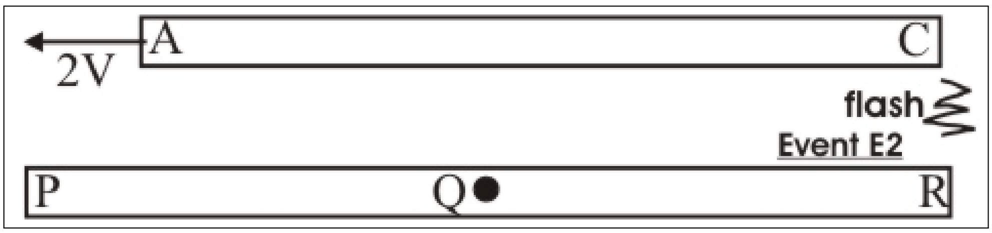


Figure 8.4: Q's deduction

$$t^Q_{E1} > t^Q_{E2} \tag{8.4a}$$
$$L^Q_{PR} > L^Q_{AC} \tag{8.4b}$$

8.5 Who is right?

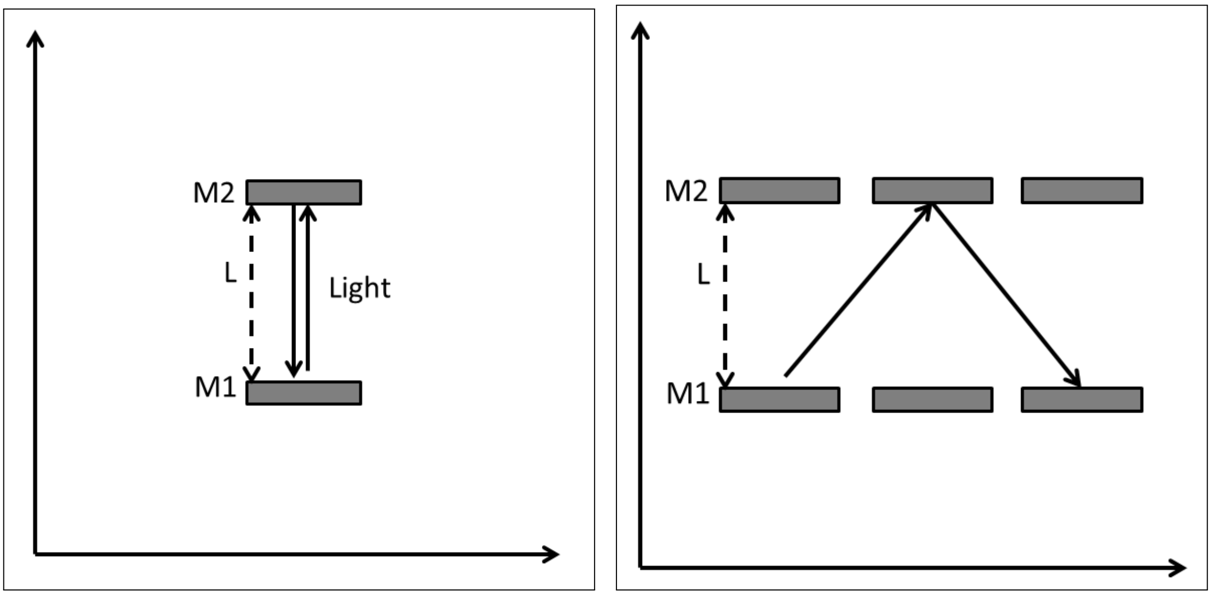
We have that

$$L^B_{PR} < L^B_{AC} \tag{8.5a}$$
$$L^O_{PR} = L^O_{AC} \tag{8.5b}$$
$$L^Q_{PR} > L^Q_{AC} \tag{8.5c}$$

Each interpretation is equally valid. They are all right within their own inertial reference frames! The concepts of length and simultaneity are not absolute, they depend on the reference frame. The conclusions is that simultaneity is relative. Events at different places which are simultaneous for one observer are not simultaneous for another who is moving relative to the first.

Section 9

The relativity of time



(a) Rest frame of light clock.

(b) A light clock in motion.

Figure 9.1: Light clocks

We can define time as a distance divided by a velocity. The figure 9.1a shows a particular type of clock that you'll see in every relativity textbook, a light clock. A pulse of light is directed directly upwards to a mirror and then detected as it returns to the lower mirror. One 'tick' therefore corresponds to two events

E1 Pulse emitted.

E2 Pulse returns.

Light clocks are useful as they are inherently relativistic. One could derive the same results using any other clock but the analysis and maths would be longer and tedious. See movie.

9.1 Clock at rest relative at an observer

From the frame of reference of the clock, the clock is at rest and the two events occur at the same point in space, see figure 9.1a. The time between emission and return

$$\Delta t_0 = \frac{2L}{c} \tag{9.1}$$

is called a proper time and designated Δt_0.

Proper time: the time between events that occur at the same location in an IRF.

9.2 Clock in relative motion to an observer

Now we consider the same clock but as observed from an inertial frame of reference that is in motion relative to the clock. The light will be observed to follow a longer path as shown in figure 9.1b.

What is the time Δt observed between the same two events, pulse emitted and pulse returning, as measured from this moving frame of reference? The path sweeps out two right angle triangles and gives (see section 4.2.2 for derivation)

$$\Delta t = \left(\frac{2L}{c}\right)\frac{1}{\sqrt{1-\frac{V^2}{c^2}}} \tag{9.2a}$$

$$= \gamma\frac{2L}{c}. \tag{9.2b}$$

9.3 Time dilation

This leads us the relationship between the time between ticks of a clock at rest relative to an observer and the time between the ticks of the same clock as observed by an observer in relative motion,

$$\Delta t = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\Delta t_0 \tag{9.3a}$$

$$\text{or}\quad \Delta t = \frac{1}{\sqrt{1-\beta^2}}\Delta t_0 \tag{9.3b}$$

$$\text{or}\quad \Delta t = \gamma\Delta t_0. \tag{9.3c}$$

The time between ticks for a clock at rest is Δt_0. But the time that we measure between ticks of a clock moving past us is $\gamma\Delta t_0$, which is longer than Δt_0. The conclusion is:

- The time between two events for an observer moving relative to those events is longer than for an observer at rest relative to those event.

Moving clocks run slow.

- That is, clocks in an IRF appear to run slow when viewed by an observer in another IRF moving relative to the clocks.

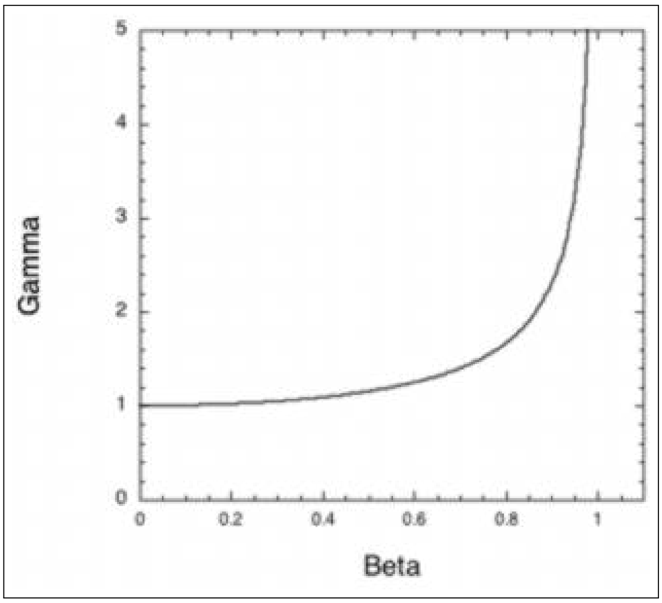


Figure 9.2: Graph of γ as a function of β.

9.4 Time conclusions

This remarkable result is known as time dilation: moving clocks run slow. It violates our intuition that time is absolute. The effect happens for all clocks, it's not a clever trick of light clocks, it's just easier to see the physics and derive the result for a light clock. If there are two clocks in two frames moving relative to each other then in the first frame they measure the other's clock to run slow. But in the other frame they'll see the first clock as running slow! Both observations are equally valid as time is, as we now know, relative.

Now, consider two twins on Earth. One sets of to a distant star in a rocket at near the speed of light and then returns. Who is the younger when they meet back on Earth?

9.5 Scale of relativistic time dilation

The figure 9.2 shows how γ varies as function of β. At zero speed,

$$v = 0 \tag{9.4a}$$
$$\therefore \beta = 0 \tag{9.4b}$$
$$\therefore \gamma = 1 \tag{9.4c}$$
$$\therefore \Delta t = \Delta t_0. \tag{9.4d}$$

So our time dilation equation reduced to the result we thought was correct for a clock at rest. At slow speed

$$v \ll c \tag{9.5a}$$
$$\therefore \beta \ll 1 \tag{9.5b}$$
$$\therefore \gamma \approx 1 + \frac{\beta^2}{2} \tag{9.5c}$$
$$\therefore \Delta t \approx \left(1 + \frac{\beta^2}{2}\right) \Delta t_0. \tag{9.5d}$$

9.6 Speeds approaching c

And as we approach c,

$$\therefore \beta \to 1 \tag{9.6a}$$
$$\therefore \gamma \to \infty \tag{9.6b}$$
$$\therefore \Delta t \to \infty. \tag{9.6c}$$

So the time we measure between ticks of the fast-as-light moving clock approaches infinity. Time in the moving frame of reference stops relative to us!

Section 10

The relativity of length

Let us turn our light clocks on their side and affix the mirrors to either end of a meter stick and set the whole system in motion with speed V, see figure 10.1. We now know the time for a light round trip as measured by us is time dilated to give

$$\Delta t = \gamma \Delta t_0 \tag{10.1a}$$

$$= \gamma \left(\frac{2L_0}{c} \right) \tag{10.1b}$$

due to time dilation. Here we have called the length as measured in the frame of reference of the light clock itself the proper length L_0.

A proper length is that of an object at rest in an IRF.

In section 4 we found the time taken for a round trip of light in a moving clock (i.e., Arm#1) was

$$\Delta t = \gamma^2 \left(\frac{2L}{c} \right) \tag{10.2a}$$

and so the length of the meter stick as measured in the frame moving relative to the stick is

$$\gamma \left(\frac{2L_0}{c} \right) = \gamma^2 \left(\frac{2L}{c} \right) \tag{10.2b}$$

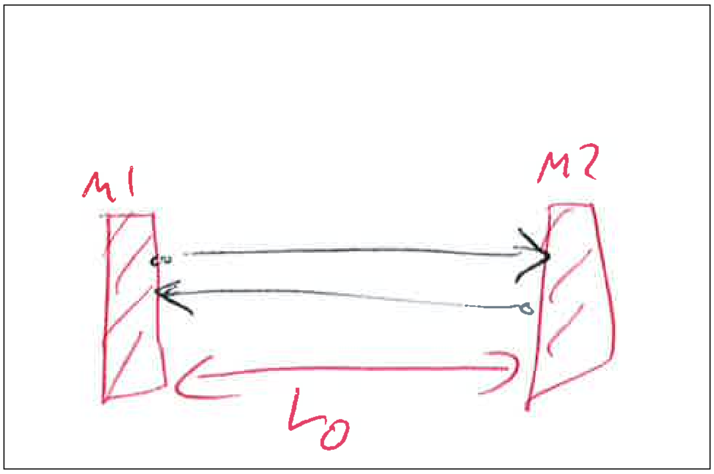


Figure 10.1: A horizontal light clock.

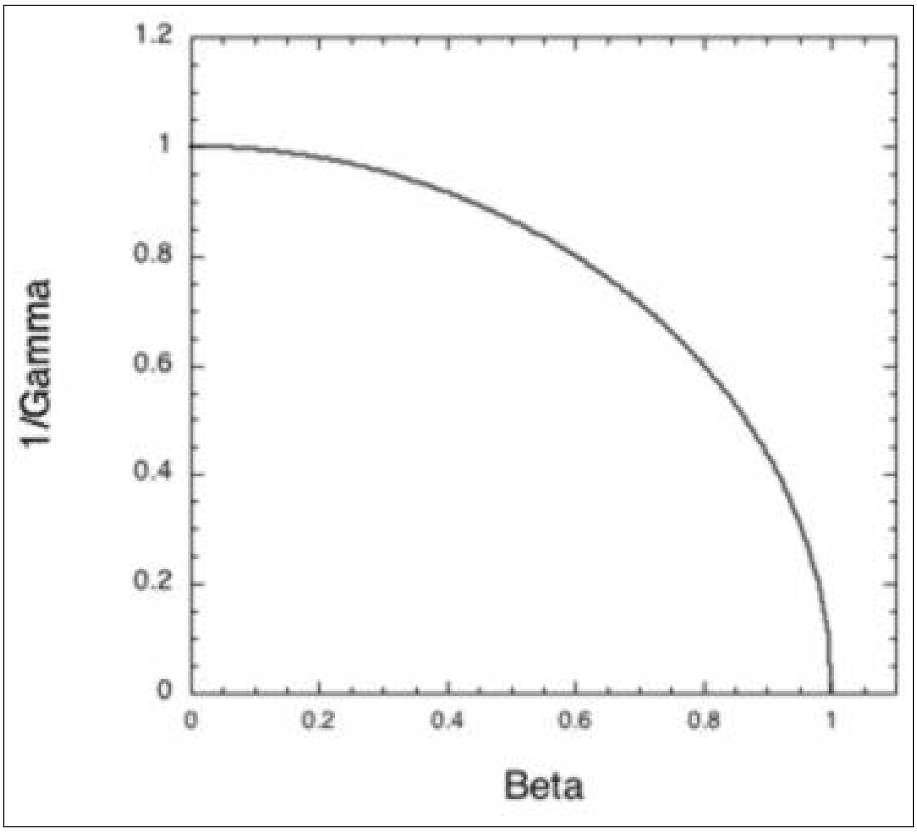


Figure 10.2: Graph of $\frac{1}{\gamma}$ as a function of β.

$$L = \frac{L_0}{\gamma}. \tag{10.2c}$$

This is remarkable! again!

Moving objects shrink along their direction of travel as measured by an observer in an IRF.

The length of the meter stick is measured to be shorter than a meter by an observer who is moving relative to that stick. Moving objects contract. The figure 10.2 shows how $\frac{1}{\gamma}$ varies as function of β. At low speed $\beta \approx 0$ so $\frac{1}{\gamma} \approx 1$. But as $V \to c$, $\beta \to 1$ and so $\frac{1}{\gamma} \to 0$ that means the measured length of a object moving at near the speed of light approaches zero (gulp).

Section 11

Muon decay

Muons are elementary particles similar to electrons but are 207 times more massive. They are unstable and decay into an electron and a pair of neutrinos within a characteristic lifetime. For a muon at rest this lifetime τ is 2.2 μs. When cosmic rays impact on the Earth's upper atmosphere highly energetic muons are created at an altitude of 20 km that travel at close to the speed of light. They can be detected at the Earth's surface. In classical physics this is impossible since the distance a muon can travel during its lifetime is $c\tau = 660$ m. [6].

11.1 Time dilation

But we know that as we see it, the internal clock of the speeding muon will run slow so instead we have

$$u\gamma_u\tau_0 > 20\text{km} \tag{11.1}$$

$$\frac{u}{\sqrt{1-\frac{u^2}{c^2}}} > \frac{20\text{km}}{2.2\mu s} \tag{11.2}$$

which gives the speed of the muon as $u > 0.9995c$.

11.2 Length contraction

From the rest reference frame of the muon it has a life time of τ is 2.2 μs and the Earth is speeding towards it at $0.9995c$. But now the height of the atmosphere undergoes length contraction and becomes

$$L = \frac{L_0}{\gamma_u} \tag{11.3}$$

$$L = \frac{20km}{\gamma(u = 0.9995c)} = 630m < 660m \tag{11.4}$$

and so the muon can reach the surface of the Earth.

Section 12

Problems for part B

P4: Time Dilation [application, easy]

A spaceship flies past the Earth at $0.93c$. A lecture on the Earth takes 50 mins, how long did the lecture take according to someone in the spaceship? [2]

P5: Length Contraction [application, easy]

A spaceship flies past the Earth at $0.990c$. A crew member on the ship measures the ship to be 400 m long. How long is the ship as measured by an observer on Earth? [2]

P6: Flashing lights [analysis, medium]

A spaceship, at great height, flies past the Earth. You see its light go on for a very short time 0.190s. On board the spaceship the Captain had set the pulse of light to be only 12.0 ms. How fast must the spacecraft be moving relative to the Earth? [2]

P7: Meter stick [application, easy]

How long is a 1 m rod according to an observer moving at speed $0.95c$ in a direction parallel to the rod? [2]

P8: A galaxy far far away [evaluation, hard]

The distance to the nearest star in our galaxy is or the order of 10^5 light years.

(a) Is it possible, in principle, for a human to make a round trip to this star? [2]

(b) How fast would they have to travel? [4]

(c) How much more time would elapse on Earth than during their trip? [2]

(d) From the perspective of the stay-at-home humans, the traveller has zoomed off and come back. So who is the younger upon the glorious arrival of the traveller back on Earth? [2]

P9: Breeding germs [application, easy]

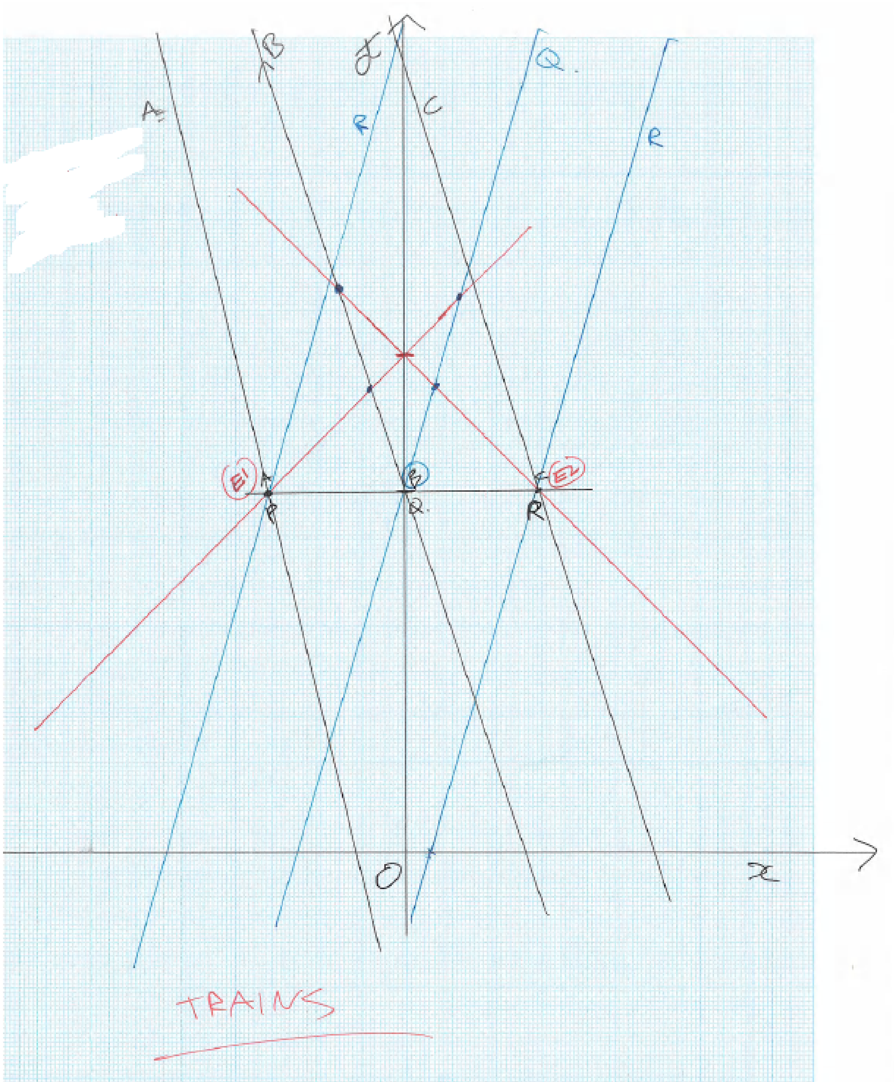
A certain strain of bacteria doubles in number each 20 days, Two of these bacteria are placed on a spaceship and sent away from the Earth for 1000 Earth days. During this time, the speed of the ship was $0.9950c$. How many bacteria would be aboard when the ship lands back on the Earth? [Ignore acceleration and Gravitation effects.]

P10: Point of view [knowledge, medium]

Cosmic rays that strike the upper atmosphere can generate muon particles that travel with a speed near c. Classically these muons should decay before they have had time to travel down to the surface of the Earth, yet we can detect them. Without calculation explain this result from the point of view of the Earth observer and from the muon. [2]

P11: Make sense of this space-time diagram

Consider the Gedankenexperiment of section 8.2 from the IRF (ignoring gravity as motion is in $x - y$ place) and therefore make sense of the following diagram. Explain your thoughts to your friend. Can you draw an similar diagram from the IRF of the two trains?



A
B
C
P
Q
R
E1
E2
O
TRAINS

Section 13

Ideas and Projects for part B

Some extra ideas to play with. These are for you to talk and discuss with your friends and perhaps tutor. They are not going to have solutions, they are for you to play with and see if you can make sense of relativity. They have purposely been written as sort of mini-projects that if you tackled a few you would learn a lot along the way. This year, 2021/2022, is the first year I've included them. So the rest of the course is identical to the past year's course, these are just interesting extras. Please feel free to use moodle or the google-doc to discuss with other students.

- Plot $1/\gamma$ in python. Also plot the MacLaurin expansion of $1/\gamma$ to zeroth order (constant), first order (constant + linear term) and second order. See if you can see where each approximation starts to deviate from the underlying function $1/\gamma$.
- Read: Relativity : the Special and General Theory by Albert Einstein, Part I, chapter VIII to X.

Part C: Space-Time

Section 14

Lorentz coordinate transforms

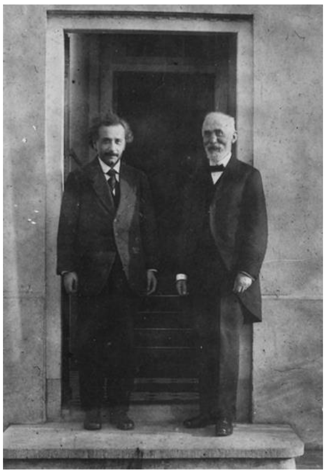


Figure 14.1: Einstein and Hendrik Lorentz (1853-1928) [7].

Here we will develop a general way to transform the (x, y, z, t) coordinates of an event between IRFs, the Lorentz transformations. We'll show how from these we can recover the time-dilation and length-contraction results of the previous section and that we can recover the Galilean transforms at low speed.

14.1 Derivation of Lorentz coordinate transformations

This derivation follows that of Appendix I in "Relativity" by A. Einstein. Consider again the two reference frames O and O' as in figure 14.2 where O' move to the right hand side with velocity V relative to O. Since the direction of motion is parallel to the x and x' axis we can say that

$$y = y' \tag{14.1}$$
$$z = z'. \tag{14.2}$$

We are going to build a relationship between O and O' based on things we can measure in O. A light pulse emitted at $t = t' = 0, \quad x = x' = 0$ travelling along $+x$ obeys the equation

$$x = ct \tag{14.3}$$
$$\therefore x - ct = 0 \tag{14.4}$$

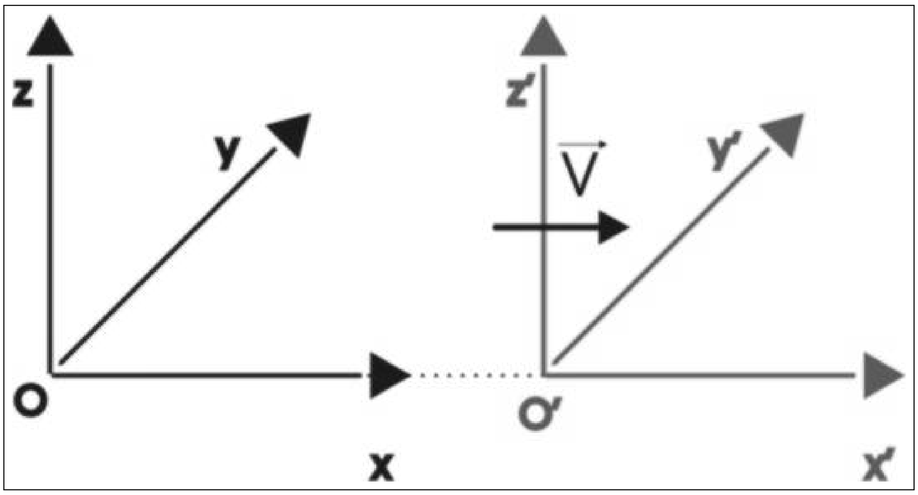


Figure 14.2: Two inertial reference frames in relative motion.

The same pulse is also travelling along $+x'$ and obeys the equation

$$x' = ct' \tag{14.5}$$
$$\therefore x' - ct' = 0 \tag{14.6}$$

The space-time coordinates, which can be thought of as events, must satisfy equation (14.4) and must also satisfy (14.6) since they are the same event.

$$\therefore (x' - ct') = \lambda (x - ct) \tag{14.7}$$

where λ is some constant. In a similar fashion we can construct for light travelling in the $-x$ and $-x'$ direction

$$(x' + ct') = \mu (x + ct) . \tag{14.8}$$

For convenience we can write,

$$a = \frac{\lambda + \mu}{2} \tag{14.9}$$
$$b = \frac{\lambda - \mu}{2} \tag{14.10}$$

and adding, or subtracting, equation (14.7) from equation (14.8) we have that

$$x' = ax - bct \tag{14.11}$$
$$ct' = act - bx. \tag{14.12}$$

Now the origin of O' is at $x' = 0$ and according to O moves with speed V

$$0 = ax - bct \tag{14.13}$$
$$\therefore x = \frac{bc}{a} t \tag{14.14}$$
$$x = Vt \tag{14.15}$$
$$\therefore V = \frac{bc}{a}. \tag{14.16}$$

And so we can write that

$$x' = a\,(x - Vt) \tag{14.17}$$
$$t' = a\left(t - \frac{V}{c^2}x\right). \tag{14.18}$$

Now, the principle of relativity says that a meter stick at rest in O will be measured by O' to have the same length as a meter stick at rest in O' has as measured by O. These are entirely equivalent circumstances. Let's take a snapshot of the meter stick in O' from Os reference frame at time $t = 0$ then, if one end of the stick is at the origin the other end of the meter stick obeys,

$$x' = ax \tag{14.19}$$
$$\& \;\; x' = 1 \tag{14.20}$$
$$\therefore \Delta x = \frac{1}{a} \tag{14.21}$$

Where

$$\Delta x = \text{ Length of O's meter stick as measured by O.} \tag{14.22}$$

Now do the opposite and take a snap-shot of the meter stick in O from O' at $t' = 0$. Rearrange equations (14.17) and (14.18) to get the the relationship

$$x' = a\left(1 - \frac{V^2}{c^2}\right)x \tag{14.23}$$

and we know in O that $x = 1$

$$\therefore \Delta x' = a\left(1 - \frac{V^2}{c^2}\right). \tag{14.24}$$

where

$$\Delta x' = \text{ Length of Os meter stick as measured by O'.} \tag{14.25}$$

The length of the meter stick must be the same for O measuring O's stick and vice versa.

$$\Delta x = \Delta x' \tag{14.26}$$
$$\therefore a = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} = \gamma. \tag{14.27}$$

Thus we arrive at the Lorentz coordinate transforms for $O \to O'$

$$x' = \gamma\,(x - Vt) \tag{14.28a}$$
$$y' = y \tag{14.28b}$$
$$z' = z \tag{14.28c}$$
$$t' = \gamma\left(t - \frac{V}{c^2}x\right) \tag{14.28d}$$

and for $O' \to O$

$$x = \gamma\left(x' + Vt'\right) \tag{14.29a}$$
$$y = y' \tag{14.29b}$$
$$z = z' \tag{14.29c}$$
$$t = \gamma\left(t' + \frac{V}{c^2}x'\right) \tag{14.29d}$$

We stated in section 1 that we wanted a way to relating the measurement of and event $\vec{E}$ between two different IRFs that is how

$$(x, y, z, t) \Leftrightarrow (x', y', z', t') \tag{14.30}$$

and we've done it.

14.2 4-vector notation [Non-examinable]

In the literature of relativity, space-time coordinates are often expressed in four-vector form. They are defined so that the length of a four-vector is invariant under a coordinate transformation. This invariance is associated with physical ideas. The invariance of the space-time four-vector is associated with the fact that the speed of light is a constant. (see hyperphysics for more info)

We define

$$\vec{R} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \tag{14.31}$$

but unlike normal matrix maths here the inner product, to ensure Lorentz invariance, is given by

$$\vec{R} \cdot \vec{R} = c^2t^2 - x^2 - y^2 - z^2 \tag{14.32}$$

we'll see this 'space-time' measure in the next section. And we can write that Lorentz coordinate transforms as

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \tag{14.33}$$

which you may prefer. Now when you see 4-vector notation you'll know what's what. This notation is used in General Relativity should you ever venture that way.

14.3 Recovery of Galilean transforms at $V \ll c$

We should check that we recover the "common sense" Galilean transforms at low speed. Since $\gamma \approx 1 + \frac{\beta^2}{2}$ at $V \ll c$ we can immediately write for $O \to O'$

$$x' \approx \left(1 + \underbrace{\frac{V^2}{2c^2}}_{\text{small}}\right)(x - Vt) \tag{14.34a}$$

$$\therefore x' = x - Vt \tag{14.34b}$$

and for

$$y' = y \tag{14.34c}$$
$$z' = z \tag{14.34d}$$

lastly

$$t' \approx \left(1 + \frac{V^2}{2c^2}\right)\left(t - \frac{V}{c^2}x\right) \tag{14.34e}$$
$$\approx t - \frac{Vx}{c^2} + \frac{tV^2}{2c^2} - \frac{xV^3}{2c^4} \tag{14.34f}$$

where

$$x = Vt \tag{14.34g}$$

therefore

$$\approx t - \frac{tV^2}{2c^2} - \frac{tV^4}{2c^4} \tag{14.34h}$$
$$\approx t\left(1 - \underbrace{\frac{1}{2}\frac{v^2}{c^2}}_{\text{small}} - \underbrace{\frac{v^4}{c^4}}_{\text{f. small}}\right) \tag{14.34i}$$

Now for $V \ll c$ we can write $c \to \infty$ thus

$$t' \approx t \tag{14.34j}$$

and for $O' \to O$

$$x \approx x' + Vt' \tag{14.34k}$$
$$y = y' \tag{14.34l}$$
$$z = z' \tag{14.34m}$$
$$t \approx t'. \tag{14.34n}$$

Which are indeed the Galilean coordinate transforms.

14.4 Recovery of Lorentz time dilation

From the Lorentz coordinate transforms we can recover time dilation. Consider two events that occur at the same place in O' and that has coordinates (x'_0, t'_1) and (x'_0, t'_2). The time between events is a proper time and is

$$\Delta t'_0 = t'_2 - t'_1. \tag{14.35a}$$

we can now write the corresponding times as measured in O

$$t_1 = \gamma\left(t'_1 + \frac{V}{c^2}x'_0\right) \tag{14.35b}$$

$$t_2 = \gamma \left(t'_2 + \frac{V}{c^2} x'_0 \right) \tag{14.35c}$$

hence we can write

$$\Delta t = t_2 - t_1 \tag{14.35d}$$
$$= \gamma \left(t'_2 + \frac{V}{c^2} x'_0 \right) - \gamma \left(t'_1 + \frac{V}{c^2} x'_0 \right) \tag{14.35e}$$
$$= \gamma \left(t'_2 - t'_1 \right) \tag{14.35f}$$
$$\Delta t = \gamma \Delta t'_0. \tag{14.35g}$$

Excellent.

14.5 Recovery of Lorentz length contraction

We'll do this in section 17.4 after we've looked at space-time .

Section 15

The relativistic Doppler effect [Self-study]

[This section is examinable and is left to the students to study so will not be 'lectured'. The content should be within your grasp, but is not trivial. The purpose of this self-study section is to give you a flavour of partly unguided physics where it will be up to you to figure a few things out. You may well find this is the best form of learning.]

The Doppler effect for sound states that a sound wave with frequency f_S emitted by a sound source (e.g. car police siren) would be detected by a receiver in motion with respect to the source at a different frequency. The frequency of the sound wave as measured by the receiver f_R is related to the frequency as measured in reference frame at rest with the source, to the velocity of the sound in air v_{air}, and to the respective velocities of the source and the receiver by

$$f_R = f_S \left(\frac{V_{air} - V_R}{V_{air} - V_S} \right) \tag{15.1}$$

with the receiver and source separating. The difference between f_s and f_R is the Doppler shift. But note,

- All the velocities are relative to the speed of sound in air. The medium that carries the sound is a common reference to the receiver and source.
- For light there is no medium, so what are V_R and V_S?

Consider a light source emitting short pulses with a period T_S, i.e., with a frequency $f_S = \frac{1}{T_S}$ as measured in the frame of the source O. The problem is to find the period (or frequency) of the emitted pulses as measured by a receiver in a moving frame O' with respect to the source.

15.1 Transverse Doppler shift

In the normal Doppler effect at the point that the police car drives past you the frequency you hear is unshifted, since at that point the car's velocity is perpendicular (transverse) to you. The same for light, except now we know that moving sources have slow clocks so there will be a shift.

If the frame of the receiver is moving perpendicularly to the direction of propagation of the light pulses (or vice versa), see figure 15.1. Then since the transverse coordinates are not affected by the Lorentz transformation ($y = y'$, $z = z'$) then the motion of the receiver (or source) is immaterial and the only effect is normal (!) time dilation. The time between two pulses will be

$$T'_R = t'_2 - t'_1 \tag{15.2a}$$

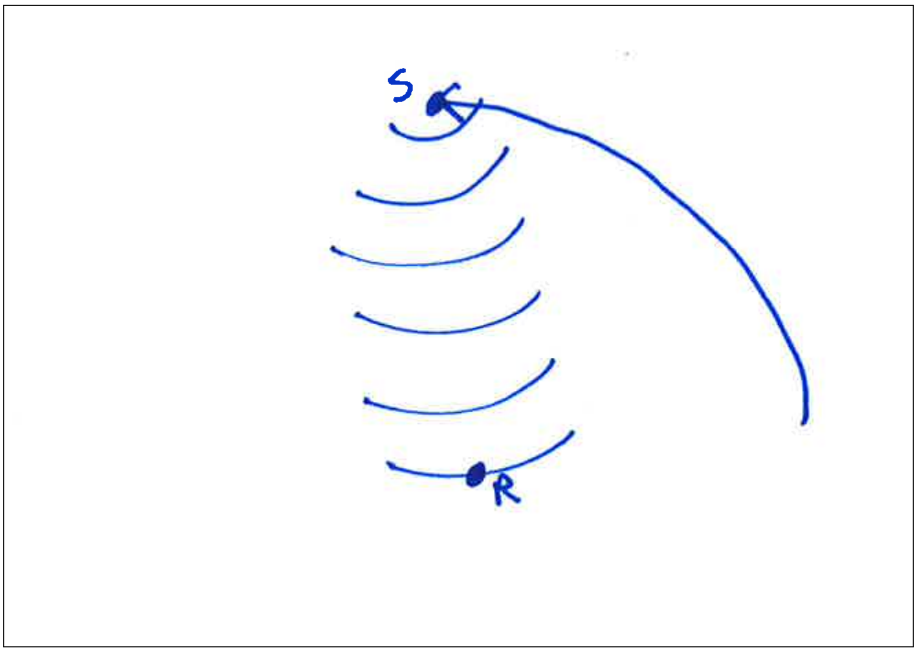


Figure 15.1: Sputnik.

$$= \gamma T_S \tag{15.2b}$$

where T_S is a proper time,

$$\therefore f'_R = \frac{f_S}{\gamma} \tag{15.2c}$$

15.2 Longitudinal Doppler shift

Now let's have our light pulse gun stationary in IRF O. We can write the coordinates to two pulses as:

- Event 1: Emission of first pulse: $x_1 = 0$, $t_1 = t$
- Event 2: Emission of the second pulse: $x_2 = 0$, $t_2 = t + T_S$

so the pulses occur at the same point in O and spaced one proper period apart. Now if frame O' is moving away with speed V along the x-axis (as usual) then we can determine the coordinates as measured in O' as

$$x'_1 = \gamma (x_1 - V t_1) \tag{15.3a}$$
$$= -\gamma V t \tag{15.3b}$$
$$t'_1 = \gamma \left(t_1 - \frac{V}{c^2} x_1 \right) \tag{15.3c}$$
$$= \gamma t \tag{15.3d}$$

and

$$x'_2 = \gamma (x_2 - V t_2) \tag{15.3e}$$
$$= -\gamma V (t + T_S) \tag{15.3f}$$

$$t_2' = \gamma \left(t_2 - \frac{V}{c^2} x_2 \right) \tag{15.3g}$$

$$= \gamma \left(t + T_S \right) \tag{15.3h}$$

So the pulses are separated in both time and space as measured by O'. Therefore the time separating the arrival of the two pulses at the receding receiver is the sum to two contributions, the separation in time as measured by O'

$$t_2' - t_1' = \gamma T_S \tag{15.4a}$$

and the time taken for the extra distanced travelled by the second pulse (the "normal" bit of the Doppler effect),

$$\frac{|x_2' - x_1'|}{c} = \frac{|-\gamma V T_S|}{c} \tag{15.4b}$$

$$= \gamma \beta T_S \tag{15.4c}$$

Therefore the time between arrival of pulses at the receiver is the sum of these two contributions

$$T_R' = \gamma T_S + \gamma \beta T_S \tag{15.4d}$$

$$= \frac{1+\beta}{\sqrt{1-\beta^2}} T_S \tag{15.4e}$$

writing

$$\frac{1+\beta}{\sqrt{1-\beta^2}} = \frac{\sqrt{1+\beta}\sqrt{1+\beta}}{\sqrt{1-\beta^2}} \tag{15.4f}$$

$$= \sqrt{1+\beta}\sqrt{\frac{1+\beta}{(1+\beta)(1-\beta)}} \tag{15.4g}$$

$$= \sqrt{\frac{1+\beta}{1-\beta}} \tag{15.4h}$$

hence (finally)

$$T_R' = T_S \sqrt{\frac{1+\beta}{1-\beta}}. \tag{15.4i}$$

And finally we can invert to get the frequency for receiver-source separating

$$f_R' = f_S \sqrt{\frac{1-\beta}{1+\beta}}. \tag{15.4j}$$

and receiver-source approaching

$$f_R' = f_S \sqrt{\frac{1+\beta}{1-\beta}}. \tag{15.4k}$$

Section 16

Space-time, causality the future and the past

Here examine how space and time are really two components of a Lorentz invariant measure called "space-time". We'll consider what we can deduce about two events given their separation in space-time and determine if two event can have a cause-and-effect relationship or not. We saw in section 7 that the time ordering of two events depended on the frame of reference from which they were measured. Here we will firm up this assertion.

16.1 Light spheres

Imagine a light flash occurring at an origin of your IRF. You will observe a spherical light front at time t with radius

$$r^2 = c^2t^2 \tag{16.1a}$$
$$= x^2 + y^2 + z^2 \tag{16.1b}$$

which we can re-write as

$$0 = c^2t^2 - x^2 - y^2 - z^2. \tag{16.1c}$$

Another observer of the same flash passing by the origin (in the x-direction) so that (as ever) $x = x' = 0$, $y = y' = 0$, $z = z' = 0$ at $t = t' = 0$. They also see light travelling at c and so they see a spherical light front with radius

$$r'^2 = c^2t'^2 \tag{16.2a}$$
$$= x'^2 + y'^2 + z'^2 \tag{16.2b}$$

which we can re-write as

$$0 = c^2t'^2 - x'^2 - y'^2 - z'^2. \tag{16.2c}$$

So we can write

$$c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2 \tag{16.3a}$$

Is this true for all x, y, z and t, and all x', y', z' and t'? Let's Lorentz transform one side to see if it matches. Since only relative x-velocity

$$y^2 = y'^2 \tag{16.3b}$$
$$z^2 = z'^2 \tag{16.3c}$$
$$\therefore c^2t^2 - x^2 = c^2t'^2 - x'^2 \tag{16.3d}$$

Now use Lorentz coordinate transforms on the RHS

$$c^2t^2 - x^2 = c^2\left(\gamma\left(t - \frac{Vx}{c^2}\right)\right)^2 - \left(\gamma\left(x - Vt\right)\right)^2 \tag{16.3e}$$
$$= \gamma^2c^2\left(t^2 - \frac{2tVx}{c^2} + \frac{V^2x^2}{c^4}\right) - \gamma^2\left(x^2 - 2Vtx + V^2t^2\right) \tag{16.3f}$$

the middle two terms cancel, and then we collect time terms and space terms

$$= \gamma^2c^2\left(t^2 - \frac{V^2t^2}{c^2}\right) - \gamma^2\left(x^2 - \frac{V^2x^2}{c^2}\right) \tag{16.3g}$$
$$= \gamma^2c^2t^2\underbrace{\left(1 - \frac{V^2}{c^2}\right)}_{=\frac{1}{\gamma^2}} - \gamma^2x^2\underbrace{\left(1 - \frac{V^2}{c^2}\right)}_{=\frac{1}{\gamma^2}} \tag{16.3h}$$
$$\therefore = c^2t^2 - x^2 \tag{16.3i}$$
$$= (ct)^2 - x^2 \tag{16.3j}$$

So this measurement of time and space is the same (zero in this case) for both IRFs observers of the expanding light sphere. This measure is know as the "space-time separation" and it is Lorentz invariant: all observes agree on the space-time separation of two events.

16.2 Lorentz invariant space-time Δs^2

We have seen that for light events (the spherical light front) the space-time separation is the same for two observers. In that particular case the separation was zero. In general we can write (in 2D) that

$$(c\Delta t)^2 - \Delta x^2 = (c\Delta t')^2 - \Delta x'^2 \tag{16.4a}$$

$$= \Delta s^2 \tag{16.4b}$$

where Δs is the space-time between the two events.

This a profound finding. Lorentz essentially says that space and time are linked, they are interchangeable, they can change into each other, they are the same thing. But at our pedestrian speeds we don't notice the space-time nature of the Universe. It's only at fractions of c that we begin to mix space and time together. We'll also see in sections 22 and 23 that energy and mass have a similar relationship and are in fact interchangeable (boom!).

16.2.1 $\Delta s^2 > 0$ Time-like separation

If not the square of a quantity, assume everything below refers to the magnitude of the quantity.

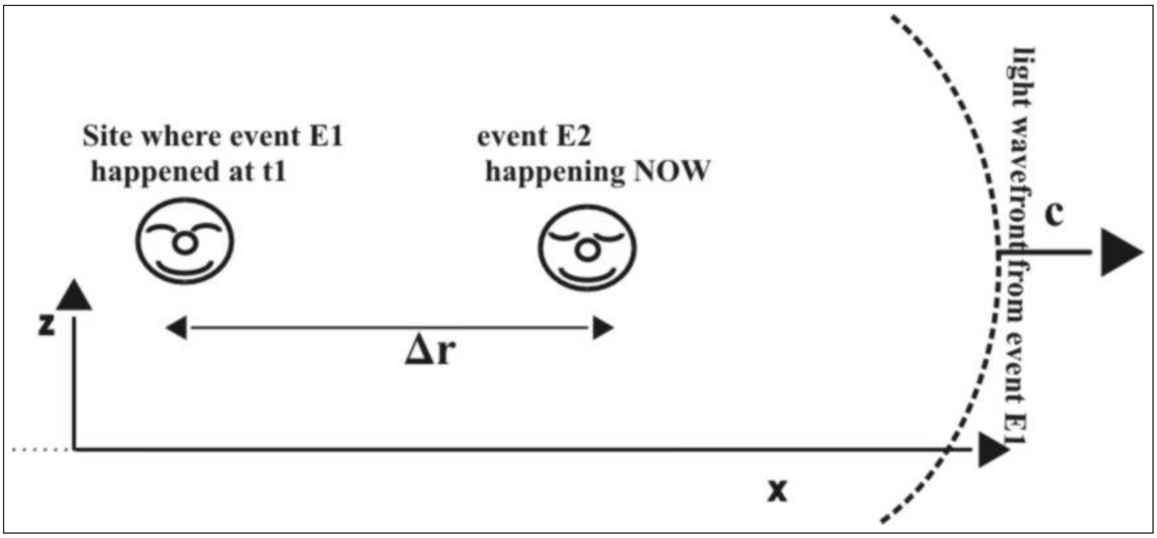
The figure 16.1a shows two events $E1$ that happened at some time t_1 in the past, the wavefront of the light emitted from this event and event $E2$ that is occurring now (in the snapshot time of the graph). You see that the wavefront has passed event $E2$ therefore $|c\Delta t| > |\Delta x|$. We can further say that all observers must agree on the following,

- Light from $E1$ reaches position $E2$ before $E2$ occurs.
- Therefore, it is possible that event $E1$ could have triggered (i.e. caused) $E2$.
- A cause-effect relationship between $E1$ and $E2$ is possible.
- There must be no disagreement about the time order of $E1$ and $E2$; if $E1$ occurred before $E2$ in one frame, it does so in all reference frames. We can safely say $E1$ occurred in the past.
- Δx can be zero for some observers, since $c\Delta t > 0$, therefore all is still OK. In other words, an observer can be present at each event, so as for them the events occur at the same location, without exceeding the speed limit c.
- Δt cannot be zero for any observer. This would contradict $c\Delta t > \Delta x$ even if $\Delta x = 0$. Thus all observers agree the events are separated in time. This reinforces the possibility of a causal relationship between $E1$ and $E2$.

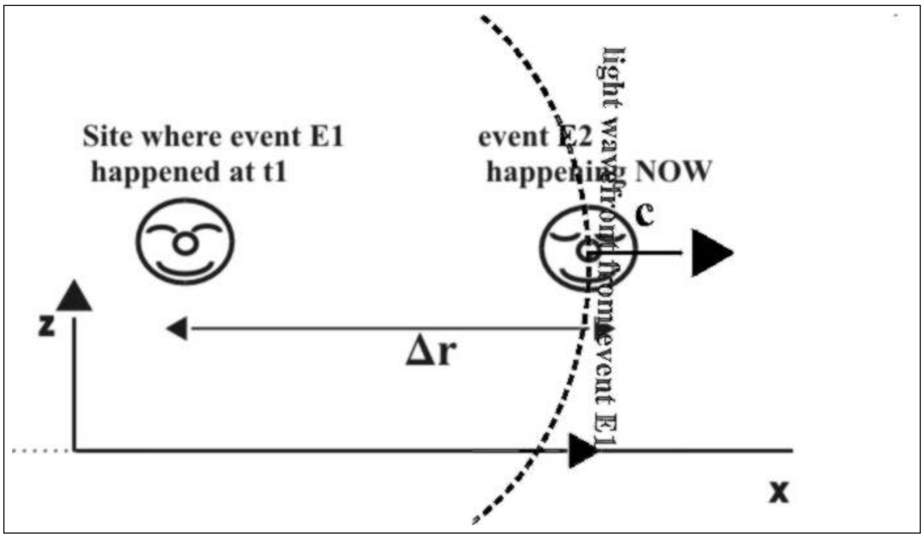
16.2.2 $\Delta s^2 = 0$ Light-like separation

The figure 16.1b shows two events $E1$ that happened at some time t_1 in the past, the wavefront of the light emitted from this event and event $E2$ that is occurring now (in the snapshot time of the graph). You see that the wavefront reaches event $E2$ just as $E2$ occurs, therefore $c\Delta t = \Delta x$ and so $\Delta s^2 = 0$. We can further say that,

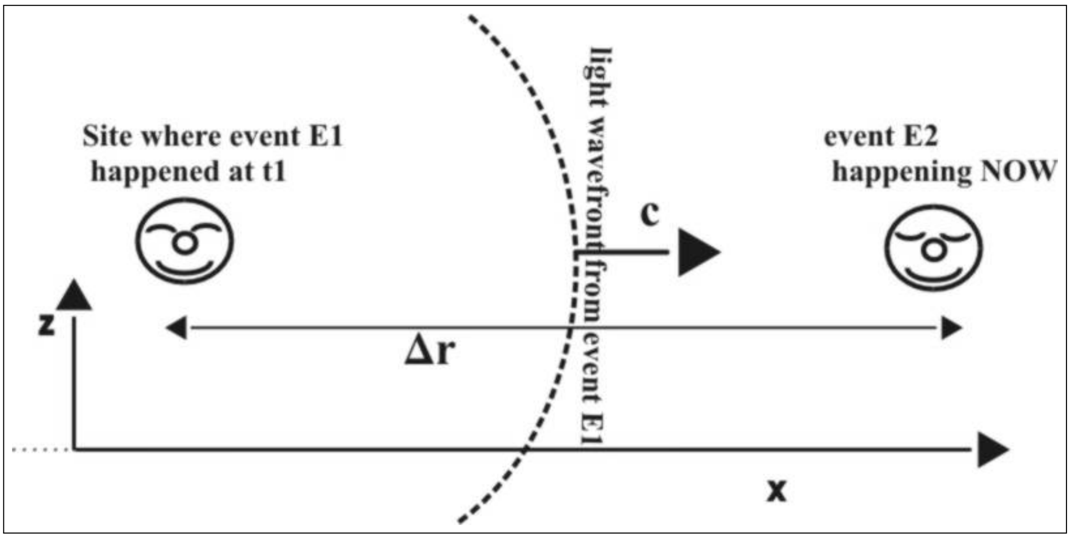
- For a material observer ($m \neq 0$) $E1$ happened before $E2$ since light took some time to travel from $E1$ to $E2$.
- For an observer travelling at c they say that $\Delta x = 0$ and so also $c\Delta t = 0$. The two events, to them, occur at the same place and simultaneously. Recall that if $V = c$ then that observers clock is stopped so all time for them has the same value.



(a) Time-like separation between two events.



(b) Light-like separation between two events.



(c) Space-like separation between two events.

Figure 16.1: Space-time and causality

- A very fast moving observer may say that $\Delta x \sim 0$ and so $c\Delta t \sim 0$.
- We can say that there can be a causal relationship between the two events connected by light.

16.2.3 $\Delta s^2 < 0$ Space-like separation

The figure 16.1c shows two events $E1$ and event $E2$ that is occurring now (in the snapshot time of the graph). You see that the wavefront of event $E1$ has not yet reached $E2$ as it occurs, therefore $|c\Delta t| < |\Delta x|$ and so $\Delta s^2 < 0$. all observers agree on the following,

- Light from either event cannot reach the site of the other before it happens.
- There can be no causal relationship between $E1$ and $E2$.
- Since $c\Delta t < \Delta x$, then $\Delta x \neq 0$. No observer can get from one event to the other to alongside each. You can't travel faster than the speed of light.
- $c\Delta t$ can be zero from some observers.
- $c\Delta t$ can be positive or negative. Some observers see $E1$ before $E2$, some $E2$ before $E1$ and some $E1$ and $E2$ at the same time.
- The concepts of past and future are meaningless since the two events are not connected.

16.3 Causality summary

The space-time separation of any two events can take one of three possible cases:

- $\Delta s^2 > 0$ which means that $|c\Delta t| > |\Delta x|$, this is called time-like separation.
- $\Delta s^2 = 0$ which means that $|c\Delta t| = |\Delta x|$, this is called light-like separation.
- $\Delta s^2 < 0$ which means that $|c\Delta t| < |\Delta x|$, this is called space-like separation.

Section 17

Space-time diagrams, light cones and world-lines

Space-time has 4 dimensions (x, y, z, t) and needs 4-dimensional paper. This, however, is expensive. Let's ignore one spatial dimension (e.g. z), and instead draw in perspective, with ct being in the vertical axis. The figure 17.1 shows a space-time diagram with a light cone and world line.

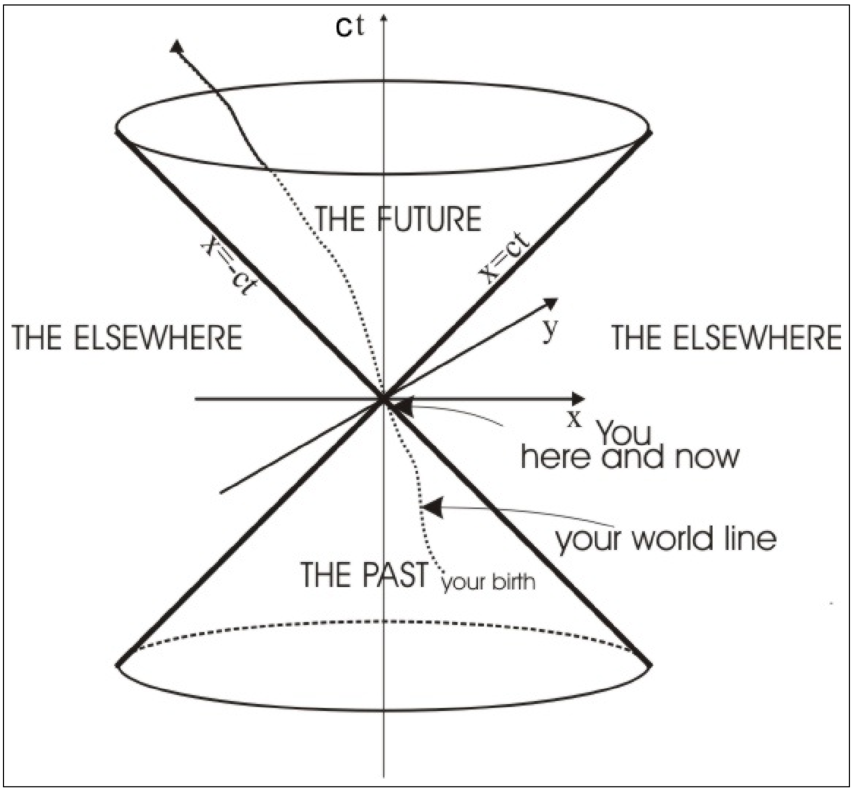


Figure 17.1: Space-time diagram, light-cone and world-line

17.1 Constructing a light-cone

The space-time diagram simply plots ct (vertical axis) to the (in this case) 2D spatial-coordinates (x, y) relative to a particular event.

An event is represented by the origin of the diagram: you here and now!

What we then see is the path that light from this event takes, in this case is describes a cone

$$ct = \sqrt{x^2 + y^2} \tag{17.1}$$

In 1D this would simply be the line

$$x = ct. \tag{17.2}$$

We can now draw three conclusions about any possible future event:

- Any event lying within the light cone
  - lie the future for this event
  - may have a causal relation to the origin's event
  - has time-like separation.
- Any event lying on the boundary of the light cone
  - may have a causal relation to the origin event
  - if causal only through a light connection
  - has light-like separation.
- Any event lying elsewhere
  - cannot have been caused by the event at the origin
  - has space-like separation.

We can also turn this logic around and state that the light cone

$$ct = -\sqrt{x^2 + y^2} \tag{17.3}$$

defines the past of this event. Any event lying within this past light-cone may have caused the event at the origin. Consider figure 17.2. There are three events A, B and C. What can we say?

- Event B lies in the future light cone of A and the future light cone of C. B could be caused by either A or C.
- Event C lies in event A's elsewhere and so A and C are unconnected - no causal relationship.

What does this mean? We have just defined the future and the past. Your space-time diagram of figure 17.1 maps out where you can possible go from this lecture seat, and where you could have possible come from. All else is elsewhere. A light cone presents how the past, future and elsewhere are attached to a single point of the space-time (i.e. an event). Causality between two events and communication between two observers is possible only inside the light cone.

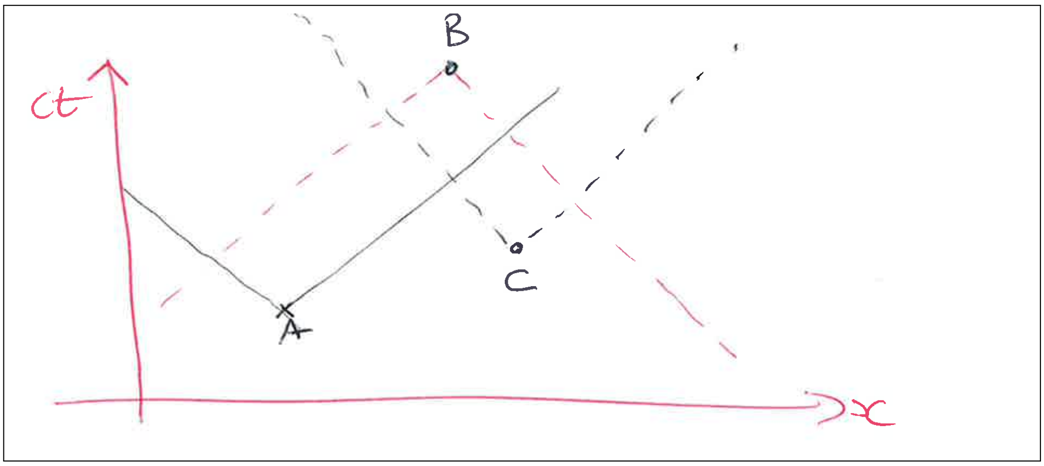


Figure 17.2: Three events in space-time

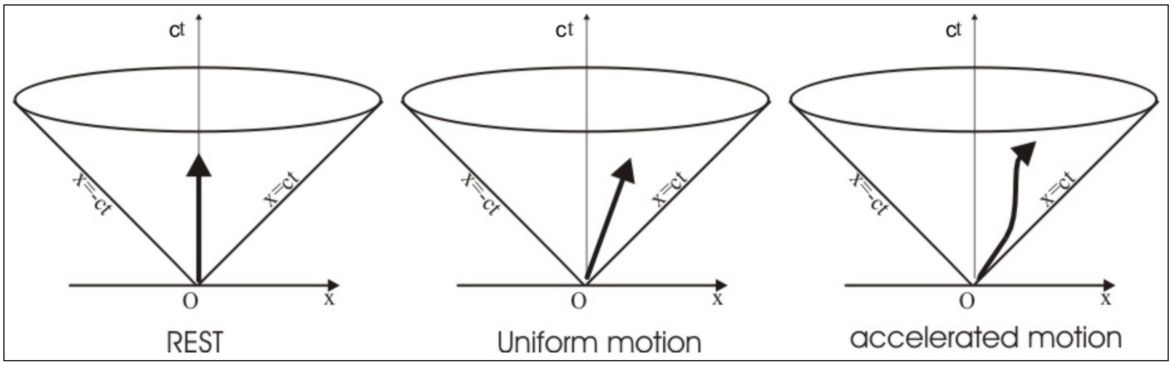


Figure 17.3: World-line for motion: left - stationary, middle - constant motion, right - some acceleration route.

17.2 Your world line

Your journey here, many causally connected events from you birth, and your future journey, many causally connected events must lie within this light cone. If we now join all these event up we get your world-line as depicted in figure 17.1. Let's consider three simple cases for world-lines:

- You remain stationary in your seat, figure 17.3 left, no change in your x and y positions.
- You move off at constant velocity. A canted line, figure 17.3 middle, with slope

$$x = vt, t = \frac{x}{v}, ct = \frac{c}{v}x = \frac{1}{\beta}x \tag{17.4}$$

- You move off and change velocity - i.e., accelerate, and so you traverse some wavy line, figure 17.3 right.

It is important to keep in mind the scale of velocities where the space-time diagram is expressive. For example, in our daily encounters the maximum speed is that of jumbo jet 1000kmh$^{-1} < 1000$ ms$^{-1} \ll c$. Consequently all our world-lines are almost a vertical line.

17.3 Space-time in space-time diagrams

We have seen that the space-time separation,

$$\Delta s^2 = (c\Delta t)^2 - \Delta x^2 = (c\Delta t')^2 - \Delta x'^2 \tag{17.5}$$

is invariant to Lorentz transformation, i.e., change of reference frame. So let us take one event to be the origin of a space-time diagram figure 17.4) and the other event A to occur in the future. We know that we can plot a space-time diagram for any IRF and the space-time distance to A

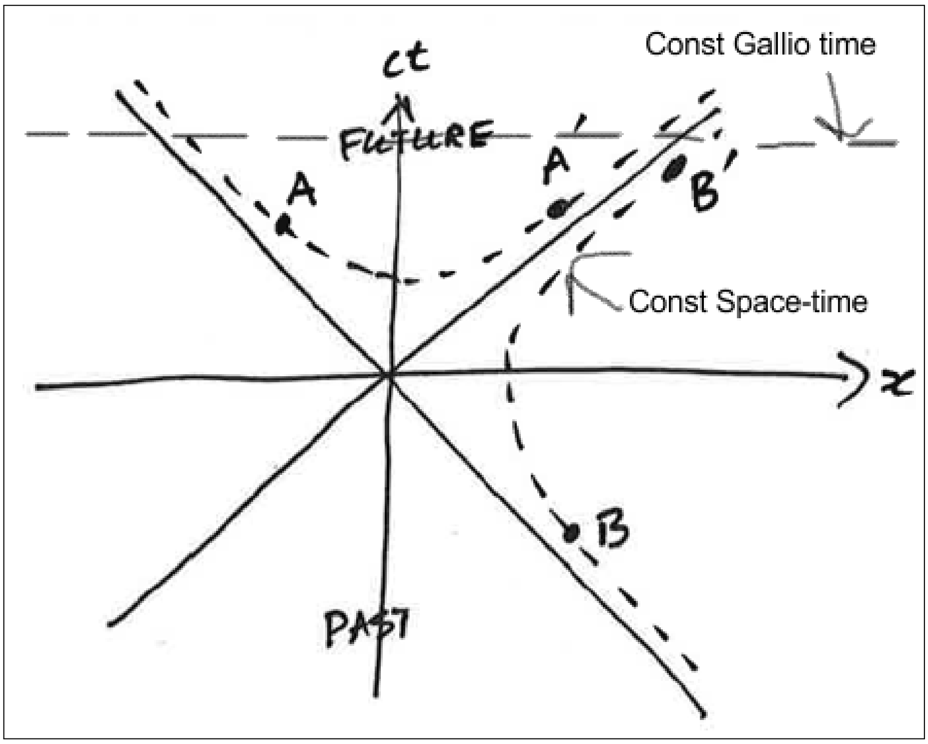


Figure 17.4: Space-time diagram of all possible measures of events A and B.

must be the same for these other frames, mathematically we say that

$$\Delta s_A{}^2 > 0 \tag{17.6a}$$
$$= (c\Delta t)^2 - \Delta x^2 \tag{17.6b}$$
$$\therefore (c\Delta t)^2 = \Delta s_A{}^2 + \Delta x^2 \tag{17.6c}$$
$$c\Delta t = +\sqrt{\Delta s_A{}^2 + \Delta x^2} \tag{17.6d}$$

which we can plot as as one half of a hyperbola in figure 17.4. And as $\Delta x \to \infty$ so $ct \to \Delta x$. So we can graphically see that there is no reference frame that swaps the time ordering of the origin event and event A - causality is preserved for time-like separation, no matter what your speed. This prevents you from time-travelling into the past and doing wicked things.

What about an event that has space-like separation? Event B lies elsewhere and has

$$\Delta s_B{}^2 < 0 \tag{17.7a}$$
$$= (c\Delta t)^2 - \Delta x^2 \tag{17.7b}$$
$$\therefore (c\Delta t)^2 = \Delta s_B{}^2 + \Delta x^2 \tag{17.7c}$$

$$c\Delta t = \pm\sqrt{{\Delta s_B}^2 + \Delta x^2} \tag{17.7d}$$

which we can plot as as one half of a hyperbola in figure 17.4. Now we see that depending on the choice of reference frame B can be measured to occur before, simultaneously with or after the event at the origin. This was the case for our trains in section 8. Since B is space-like separated from the origin this should come as no surprise.

17.4 Recovery of Lorentz length contraction (part 2)

This goes back to section 14.5, can we recover Lorentz Length contraction from the Lorentz Coordinate transforms? This is good time to use space-time diagrams to understand them more and understand their relationship to the coordinate transforms.

We need to more accurately define what we mean by the length of an object. It is, no matter what the reference frame, the distance between the two ends of the object taken from simultaneous measurements.

17.4.1 The object is moving in the initial IRF

An object whizzes past you in your IRF O at velocity V figure 17.5. You simultaneously measure

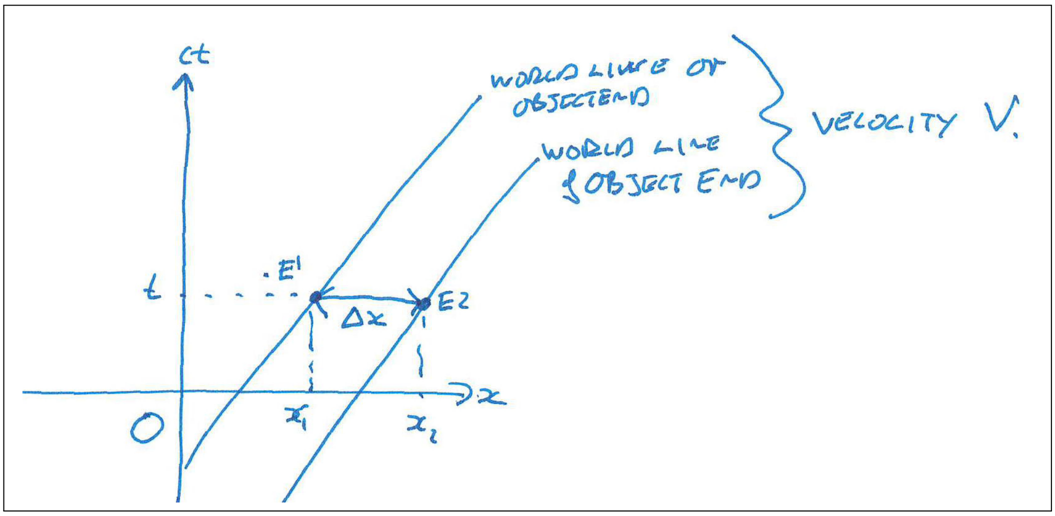


Figure 17.5: Measurement of an object in constant motion.

the position of either end of the object with coordinates

$$E_1 = (x_1, t_1) \tag{17.8}$$
$$E_2 = (x_2, t_2). \tag{17.9}$$

Since they are simultaneous measurements we know

$$t_1 = t_2 \tag{17.10}$$

and we can write the distance between those two measurements as

$$\Delta x = x_2 - x_1. \tag{17.11}$$

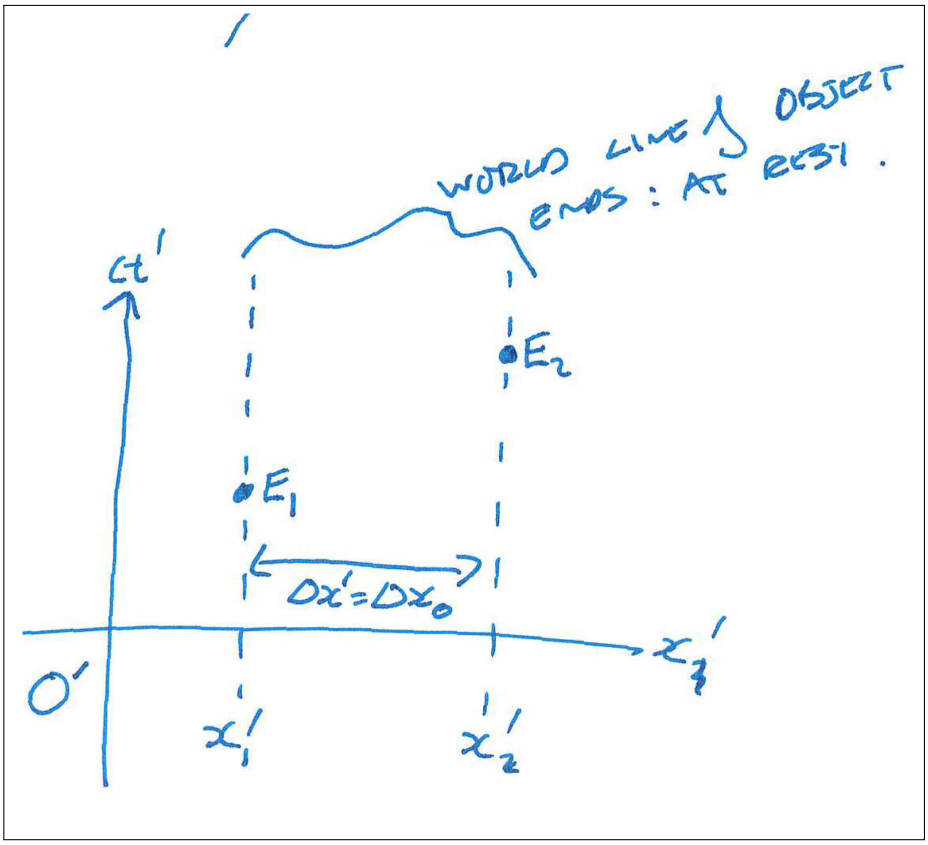


Figure 17.6: Measurement of an object at rest.

Now the clever bit: In another reference frame O' moving with velocity V they will measure a proper length since the object is at rest AND in O' it doesn't matter if the length measurement is simultaneous or not since the object (and its ends) are not moving, figure 17.6. So using Lorentz Coordinate Transforms

$$x'_1 = \gamma\,(x_1 - Vt_1) \tag{17.12}$$
$$x'_2 = \gamma\,(x_2 - Vt_2) \tag{17.13}$$
$$\therefore \Delta x' = x'_2 - x'_1 \tag{17.14}$$
$$= \gamma\,(x_2 - Vt_2) - \gamma\,(x_1 - Vt_1) \tag{17.15}$$
$$= \gamma(x_2 - x_2) \tag{17.16}$$
$$\Delta x' = \gamma\Delta x + \gamma\,(-Vt + Vt)\,. \tag{17.17}$$

Since the object is at rest in O' they measure a proper length hence

$$\Delta x' = \Delta x_0 \tag{17.18}$$

giving us in the end

$$\therefore \Delta x = \frac{\Delta x_0}{\gamma} \tag{17.19}$$

which is our old friend Lorentz Length Contraction.

17.4.2 The object is at rest in the first IRF

An object is at rest in your IRF O. You make a measure of its length using the two events figure 17.7,

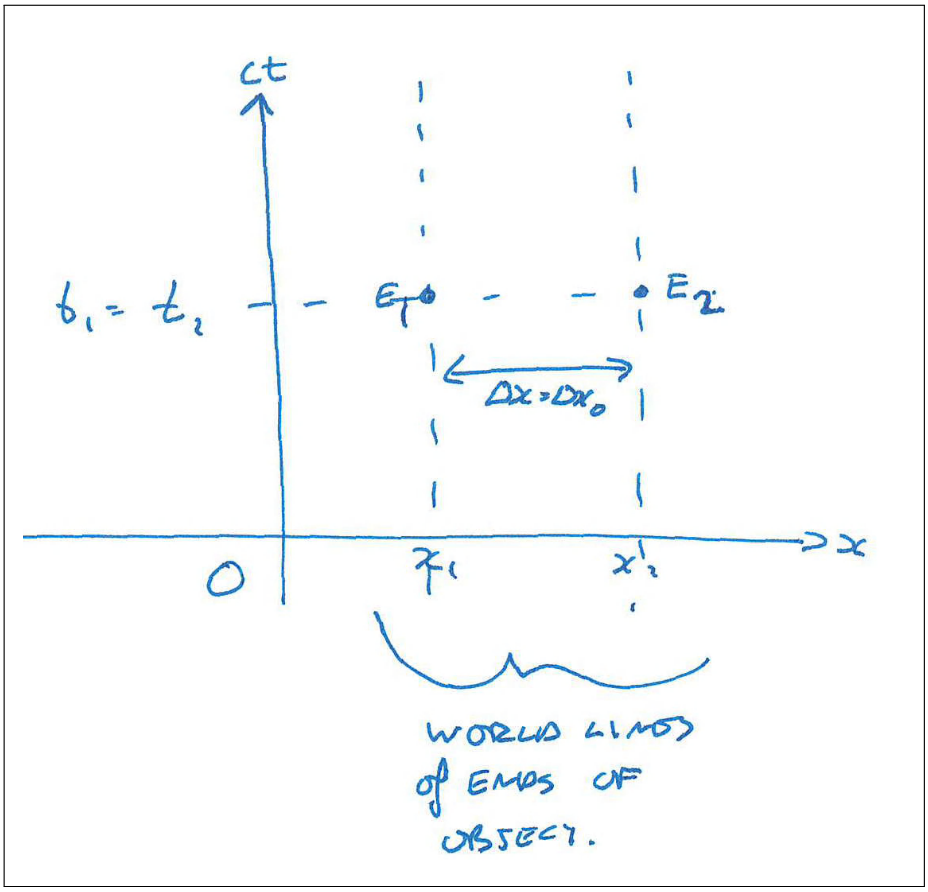


Figure 17.7: Measurement of an object at rest.

$$E_1 = (x_1, t_1) \tag{17.20}$$
$$E_2 = (x_2, t_2). \tag{17.21}$$

Since they are simultaneous measurements we know

$$t_1 = t_2 \tag{17.22}$$

and we can write the distance between those two measurements as

$$\Delta x = x_2 - x_1. \tag{17.23}$$

and since it is a proper length we can write

$$\Delta x_0 = x_2 - x_1. \tag{17.24}$$

Now can we simply Lorentz Transform to the moving frame O' moving with velocity V? Nearly but not quite . . .

$$x'_1 = \gamma(x_1 - Vt_1) \tag{17.25}$$
$$x'_2 = \gamma(x_2 - Vt_2) \tag{17.26}$$
$$\therefore \Delta x'_{12} = \gamma(x_2 - x_1) \tag{17.27}$$
$$\Delta x'_{12} = \gamma \Delta x_0 \tag{17.28}$$

but that equation has the γ factor in seemingly the wrong place? Let's look at a space-time diagram figure 17.8. Lorentz transforming E_1 and E_2 results in two points in O' that are not at

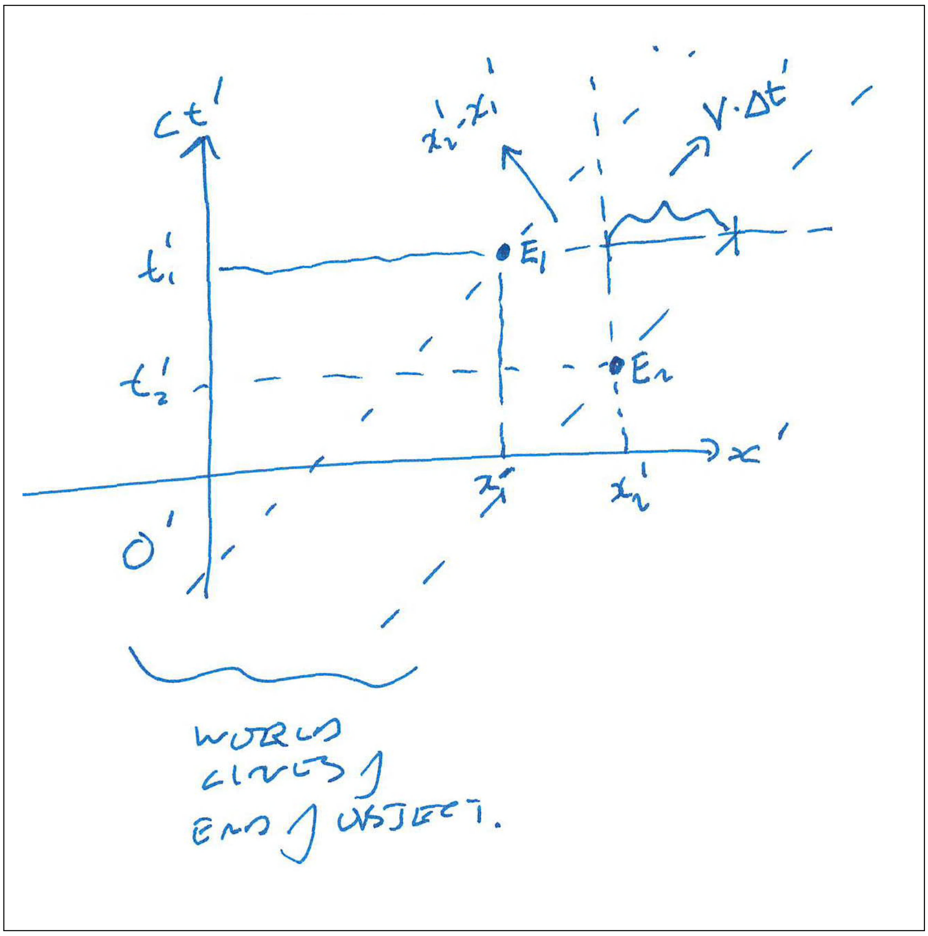


Figure 17.8: Measurement of a moving object.

the same time, not simultaneous, so $\Delta x'_{12}$ does not reflect the length of the object as measured by O'. Instead we can see that we need to add on a length associated with the time interval $\Delta t' = t'_2 - t'_1$.

$$t'_1 = \gamma\left(t_1 - \frac{V}{c^2}x_1\right) \tag{17.29}$$

$$t'_2 = \gamma\left(t_2 - \frac{V}{c^2}x_2\right) \tag{17.30}$$

$$\Delta t' = t'_2 - t'_1 \tag{17.31}$$

recall that in O the events were simultaneous, $t_1 = t_2$ hence

$$\Delta t' = -\gamma\left(\frac{V}{c^2}x_2 - \frac{V}{c^2}x_1\right) \tag{17.32}$$

$$= -\gamma\frac{V}{c^2}\Delta x_0. \tag{17.33}$$

Now we can add these two components up to give,

$$\Delta x' = \Delta x'_{12} - V \cdot \gamma\frac{V}{c^2}\Delta x_0 \tag{17.34}$$

$$= \gamma\Delta x_0 - \gamma\frac{V^2}{c^2}\Delta x_0 \tag{17.35}$$

$$= \gamma \Delta x_0 \underbrace{\left(1 - \frac{V^2}{c^2}\right)}_{\frac{1}{\gamma^2}} \tag{17.36}$$

$$\therefore \Delta x' = \frac{\Delta x_0}{\gamma} \tag{17.37}$$

just as before. Phew. The point of this exercise was to revisit these coordinate transforms and cement their use and their relationship to measurement. Also to show with without clear thought (and perhaps a diagram) it is easy to get led astray.

Section 18

Minkowski space [Non-examinable]

In the preceding section we used space-time diagrams to picture how an observer "here and now" views events in space and time. The questions we'll address here is, how can we plot, on the same digram, those same events relative to another observer in relative motion to the original observer? The answer, on a Minkowski diagram.

18.1 Minkowski diagram construction

We shall now construct a Minkowski diagram which is a single space-time diagram but with pairs of axis for both observer O and O'. This is quite advanced, but I feel you deserve it. Consider again the two IRF's yours O and O' that is moving in the $+ve\ x$ direction with speed V, figure 18.1.

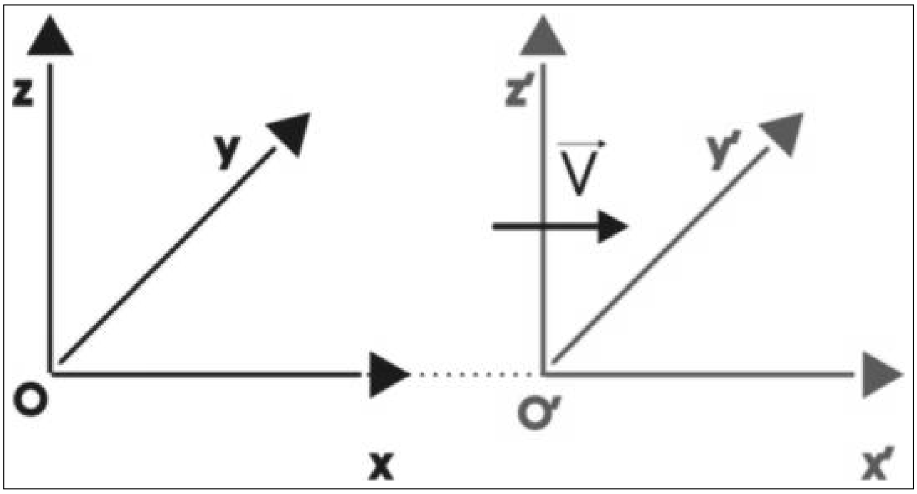


Figure 18.1: Our old friends O and O'.

18.2 Construction of O' time axis

Now O can make a number of measurement of the location of O' origin and will find (assuming the origins cross at $t = 0 = t'$),

$$x = Vt \tag{18.1a}$$

$$\therefore ct = \frac{c}{V}x \tag{18.1b}$$

$$ct = \frac{1}{\beta}x. \tag{18.1c}$$

Meanwhile O' will measure these same events to have $x' = 0$, so what we have found is the equation of the time axis of O' from Os IRF. We can plot this in figure 18.2. Notice as V gets bigger so this axis will cant over more, until at $V = c$ it will follow the light-front $ct = x$.

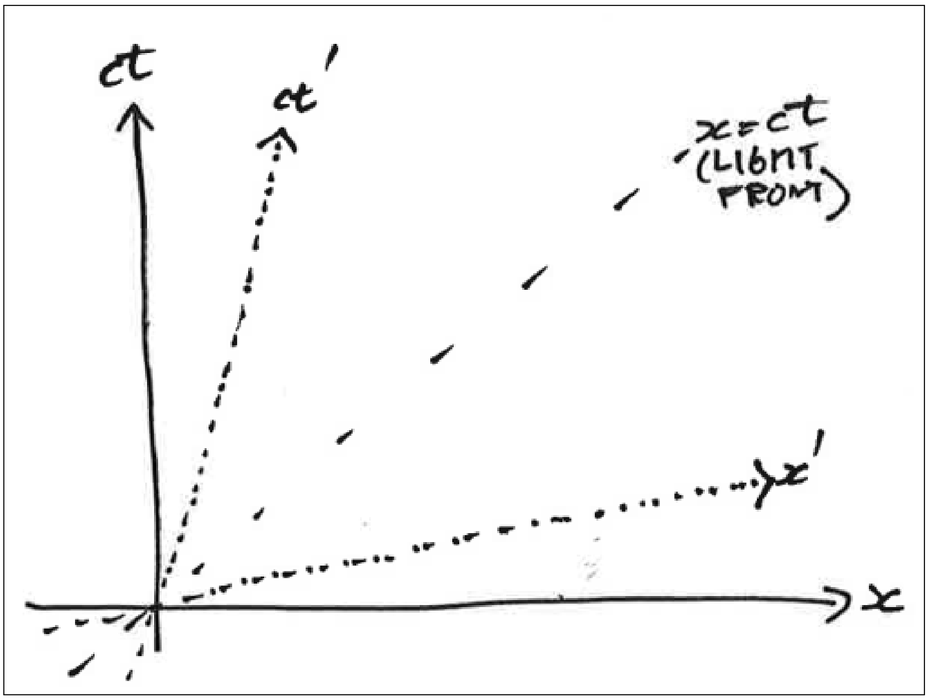


Figure 18.2: Constructing ct' and x axis in a Minkowski diagram.

18.3 Construction of O' spatial axis

Consider all the events that could be measured at having $t' = 0$ these must lie in the plane of x-axis of O'. We can just take the lorentz transform of these to determine their position in O's IRF.

$$t' = 0 \tag{18.2a}$$

$$t' = \gamma\left(t - \frac{V}{c^2}x\right) \tag{18.2b}$$

$$\therefore ct = \beta x \tag{18.2c}$$

We can plot this in figure 18.2. Notice as V gets bigger so this axis will rotate up more, until at $V = c$ it will follow the light-front $ct = x$.

18.4 How to measure an event

Now let's set off a firework at A. What do O and O' measure? We simply use lines that are parallel to the corresponding axis as in figure 18.3.

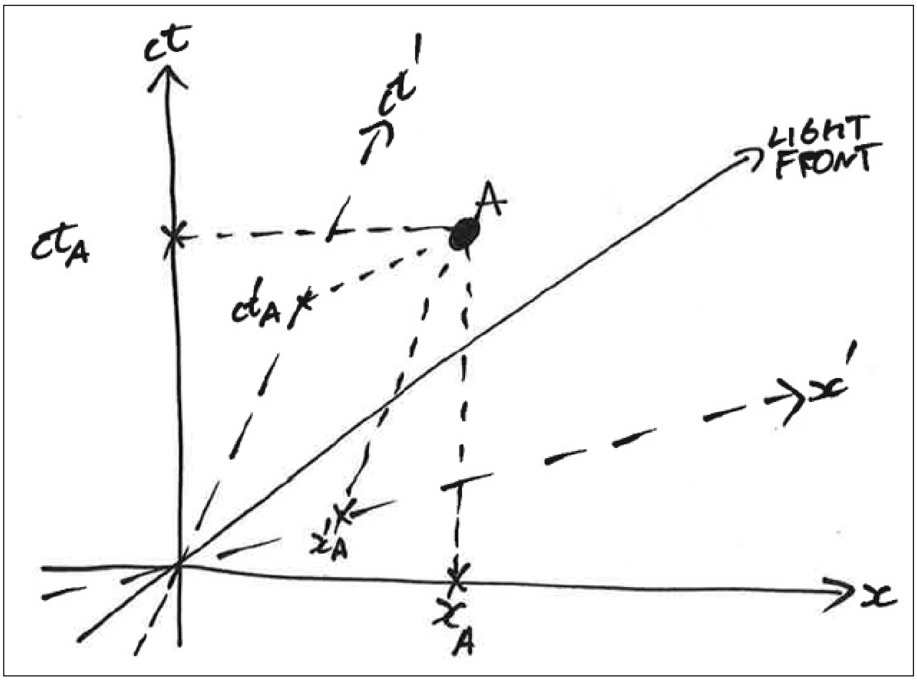


Figure 18.3: Measuring event A in a Minkowski diagram.

18.5 Relativity of simultaneity and position

The figure 18.4 shows two events measured to be simultaneous in reference frame O, but they are plainly not simultaneous in frame O'.

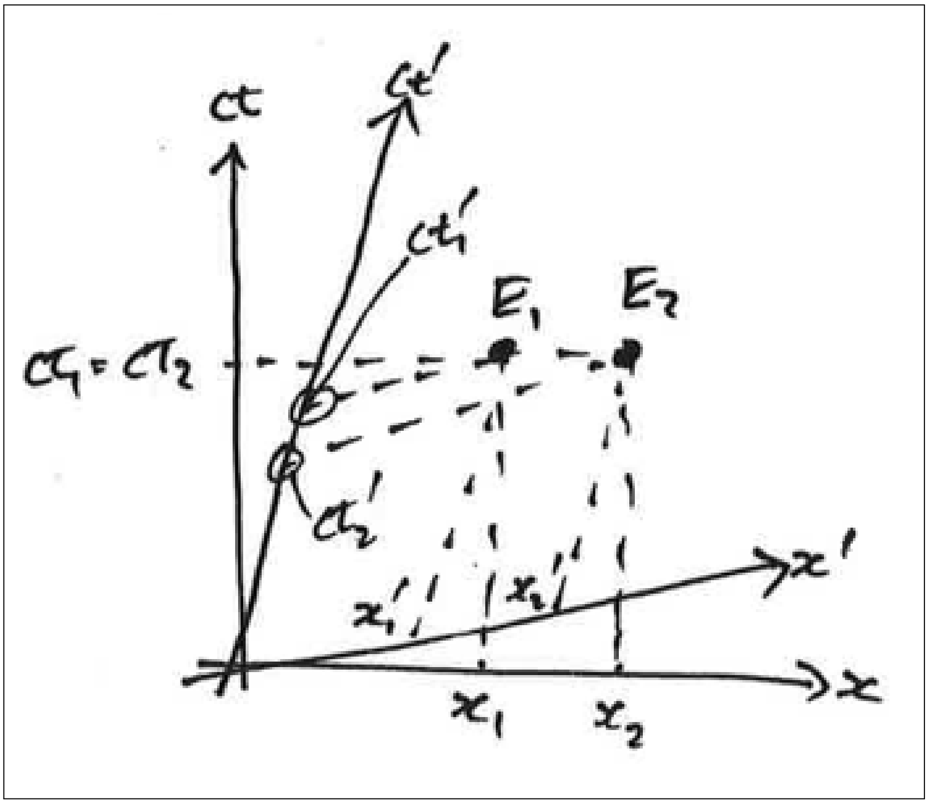


Figure 18.4: Not so simultaneous.

The figure 18.5 shows two events measured to occur at the same location in reference frame O, but they are plainly not at the same location in frame O'.

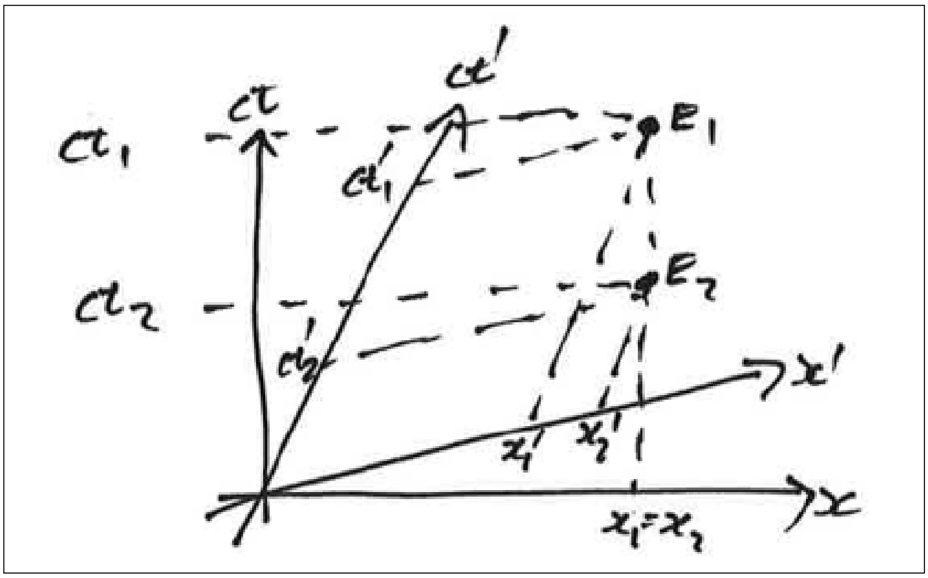


Figure 18.5: What position?

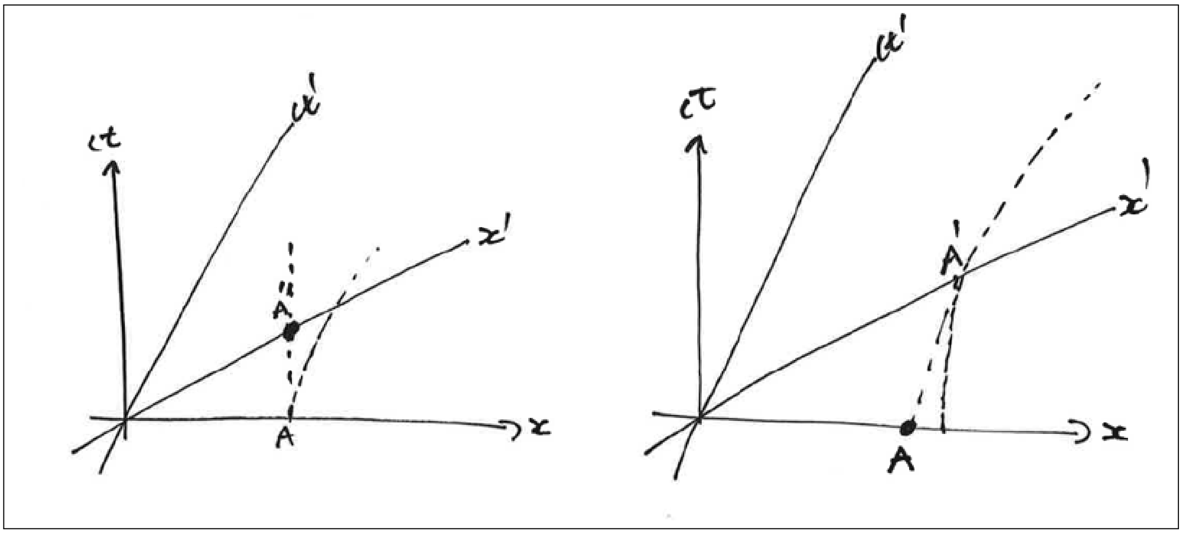


Figure 18.6: Lorentz contraction:LHS O measuring O', RHS O' measuring O

18.6 Minkowski calibration and Lorentz contraction

To calibrate the ct' and x' axis of our Minkowski diagram we simply plot constant space-time hyperbola as in figure 17.4. So lets say O' measures the end of a stick to be at A' , figure 18.6, then O will measure the position of this event at A. From O we can draw a constant space-time hyperbola and see that $A' < A$. As we know, moving rods shrink due to Lorentz contraction.

Conversely, O measures his stick to be at A, O' measures this event to occur at A', from O' was can draw a constant space-time hyperbola and see that from O''s perspective $A < A'$. And again, moving rods shrink due Lorentz contraction and is symmetric.

Using similar arguments as above you should now be able to show that time dilation holds in Minkowski diagrams.

Section 19

Problems for part C

P12: Balmer lines [synthesis, medium]

The H_α Balmer series transition for a hydrogen atom causes the emission of light at 656.2nm as measured in the inertial reference frame of the atom at rest. If the hydrogen atom moves in a circular path around an observer fixed at the centre of the circle, at what speed must the atom be moving so the observer measures the emitted light to have a wavelength of 680.6 nm? [3]

P13: Doppler [evaluation, hard]

A spaceship travelling towards Earth with speed $|v_s|$ has a long rod sticking out at right angles to its direction of travel. When a light at the spaceship end of the rod flashes, the light pulse which travels along the rod is reflected back to the spaceship by a mirror at the end of the rod. The returning light pulse activates a very fast acting mechanism which makes the light flash again. Let the frequency of flashes, as determined from the spaceship by f_s.

(a) If the time dilation were the only effect to take into account, what would be the period of the flash as measured by Earth clocks?

(b) Light from the flash also travels towards Earth. This period of flashing will be less than the period found above since the spaceship moves towards the Earth between flashes and the second flash had less far to travel than the first one. How far towards the Earth will an Earth observer say the light has travelled in the time between flashes?

(c) How far towards the Earth will an Earth observer say the ship has travelled between flashes?

(d) What is the distance between successive flashes for an Earth observer?

(e) What is the frequency of flashes as measured by an Earth observer? [Relativistic Doppler effect for an approaching source.]

P14: Expanding universe [analysis, easy]

Most stars consist of principally of hydrogen and obtain their energy from nuclear fusion of hydrogen to form helium. Thus the starlight from different stars exhibits certain common properties, specifically, light at characteristic wavelengths corresponding to transitions between particular energy levels in the hydrogen atom, here we'll examine the H_α line at 656 nm. Using sensitive spectrometers, this line can be identified in light from distant galaxies, but it is observed to be shifted in wavelength (and therefore also in frequency as the frequency f is related to the wavelength λ by $f = \frac{c}{\lambda}$). Calculate the velocities relative to the earth of the following galaxies, indicating whether they are receding or approaching:

(a) In light from the Virgo cluster of galaxies, the wavelength is observed to increase by 2.63 nm

(b) From Corona the wavelength is increased by 50 nm.

(c) From Hydra the wavelength is increased by 150 nm.

P15: Events and space-time [analysis, medium, past exam]

(a) Define all the terms in the Lorentz x-coordinate transform. Make use of diagrams if you wish. [3]

(b) A reference frame O' travels at speed $V = +0.7c$ along the x-direction relative to reference frame O. As the origins of these two frames pass each other $t = t' = 0$.

(i) An event $E1$ is measured from frame O with (x, y, z, t) coordinates of

$$E1(89\ \mathrm{m}, 0, 10\ \mathrm{m}, 2 \times 10^{-6}\ \mathrm{s}).$$

What are the coordinates $E1'$ for this event as measured in frame O'? [2]

(ii) A second event $E2$ is measured by O to have coordinates

$$E2(839\ \mathrm{m}, 0, 10\ \mathrm{m}, t),$$

and is light-like separated from $E1$. What is the time coordinate t of $E2$ as measured by O? [2]

P16: Chicken [knowledge, easy]

A chicken crosses the road. Is the space-time separation between the beginning and the end of its journey time-like, space-like or light-like? [1]

P17: Barn paradox? [evaluation, hard]

A pole 10 m long lies on the ground next to a barn that is 8 m long. An athlete picks up the pole, carries it far away and then runs with it towards the barn at speed $0.8c$. The athlete's friend remains at rest, standing by the open door of the barn.

(a) How long does the friend measure the pole to be as it approaches the barn? [2]

(b) Immediately after the pole is entirely inside the barn, the fried shuts the door. How long after the door is shut does it take for the front of the pole to strike the back of the barn as measured by the friend? [2]

(c) In the reference of the athlete what is the length of the pole and the barn? [2]

(d) How do you reconcile the closing of the barn door with the experience of the athlete? [4]

P18: Maxwell's waves [application, hard]

Show that the electromagnetic wave equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0$$

is invariant under Lorentz transform. [60]

Tactics, first change variables to x' etc. $\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial \phi}{\partial y'}\frac{\partial y'}{\partial x} + \frac{\partial \phi}{\partial z'}\frac{\partial z'}{\partial x} + \frac{\partial \phi}{\partial t'}\frac{\partial t'}{\partial x}$ using the Lorentz coordinate transforms. Keep calm and accurately differentiate. Also $\phi = \phi(x, y, z, t)$, so take care with partials and changing variable. You should (will!) find,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial x'^2} + \frac{\partial^2 \phi}{\partial y'^2} + \frac{\partial^2 \phi}{\partial z'^2} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t'^2}$$

This is a crazy hard question and really it's a bit beyond what I can expect of you, but could be fun for those that find these things fun.

Section 20

Ideas and Projects for part C

Some extra ideas to play with. These are for you to talk and discuss with your friends and perhaps tutor. They are not going to have solutions, they are for you to play with and see if you can make sense of relativity. They have purposely been written as sort of mini-projects that if you tackled a few you would learn a lot along the way. This year, 2021/2022, is the first year I've included them. So the rest of the course is identical to the past year's course, these are just interesting extras. Please feel free to use moodle or the google-doc to discuss with other students.

- Read: Relativity : the Special and General Theory by Albert Einstein, Part I, chapter XII.
- Draw space-time diagrams of all of Section 8.
- Draw a spacetime diagram of the muon decay of Section 11 from both IRFs.
- Draw a space-time diagram of a ship emitting a pulsed signal as it moves away form you or approaches you. Do this for both Galilean relativity and Einsteinian relativity. Can you see you've drawn the Doppler effect?

Part D: Kinematics

Section 21

Lorentz velocity transforms

In this section we will apply the Lorentz transforms to the velocity as measured in an IRF. We'll see that parallel and transverse velocities are both affected by relativistic effects.

21.1 Recap: Galilean velocity transforms

Recall the Galilean velocity transforms of section 2.4 for two frames in relative motion to each other along the x axis, as figure 21.1, and for a measured velocity with only x component, then

$$v_x{}' = v_x - V \tag{21.1a}$$
$$v_y{}' = v_y \tag{21.1b}$$
$$v_z{}' = v_z \tag{21.1c}$$
$$\vec{v}\,' = \vec{v} - \vec{V} \tag{21.1d}$$
$$\vec{v} = \vec{v}\,' + \vec{V} \tag{21.1e}$$

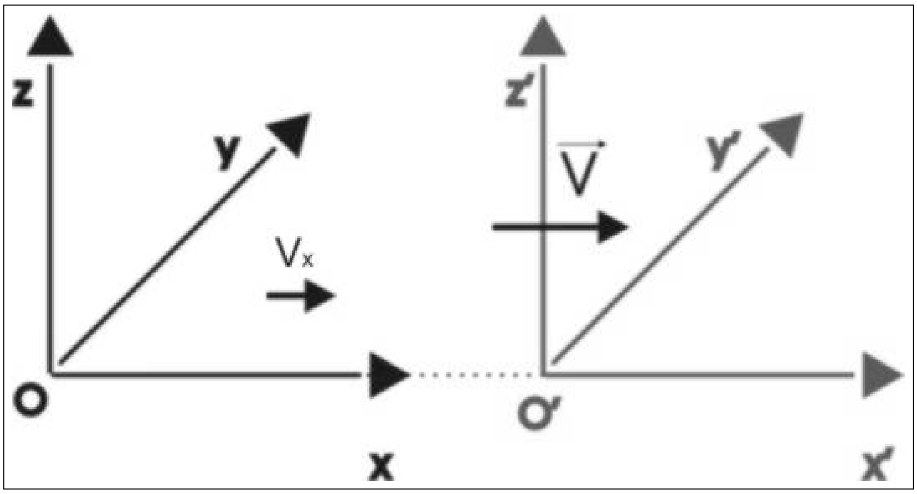


Figure 21.1: Two inertial reference frames in relative motion with the velocity of some object V_x as measured by O.

21.2 Relativistic x velocity transform

We want to compute this simple derivative

$$v_x{}' = \frac{dx'}{dt'} \tag{21.2a}$$

but now we know that

$$\begin{aligned}
x' &= \gamma\left(x - Vt\right) && (21.2b)\\
y' &= y && (21.2c)\\
z' &= z && (21.2d)\\
t' &= \gamma\left(t - \frac{V}{c^2}x\right) && (21.2e)
\end{aligned}$$

so we have to consider the differential of $x' = f(x,t)$ and $t' = f(x,t)$. For any function $f(x,t)$ we can write its differential as

$$df = \left(\frac{\partial f}{\partial x}\right)dx + \left(\frac{\partial f}{\partial t}\right)dt \tag{21.3}$$

therefore we can write

$$\begin{aligned}
dx' &= \left(\frac{\partial x'}{\partial x}\right)dx + \left(\frac{\partial x'}{\partial t}\right)dt && (21.4a)\\
&= (\gamma)\,dx + (-\gamma V)\,dt && (21.4b)\\
&= \gamma\left(dx - V dt\right) && (21.4c)\\
dy' &= \left(\frac{\partial y'}{\partial y}\right)dy && (21.4d)\\
&= dy && (21.4e)\\
dz' &= \left(\frac{\partial z'}{\partial z}\right)dz && (21.4f)\\
&= dz && (21.4g)\\
dt' &= \left(\frac{\partial t'}{\partial x}\right)dx + \left(\frac{\partial t'}{\partial t}\right)dt && (21.4h)\\
&= \gamma\left(dt - \frac{V}{c^2}dx\right). && (21.4i)
\end{aligned}$$

Finally we can find what we wanted from the start

$$\begin{aligned}
v_x{}' &= \frac{dx'}{dt'} && (21.4j)\\
&= \frac{\gamma\left(dx - V dt\right)}{\gamma\left(dt - \frac{V}{c^2}dx\right)} && (21.4k)\\
&= \frac{dt\left(\frac{dx}{dt} - V\right)}{dt\left(1 - \frac{V}{c^2}\frac{dx}{dt}\right)} && (21.4l)
\end{aligned}$$

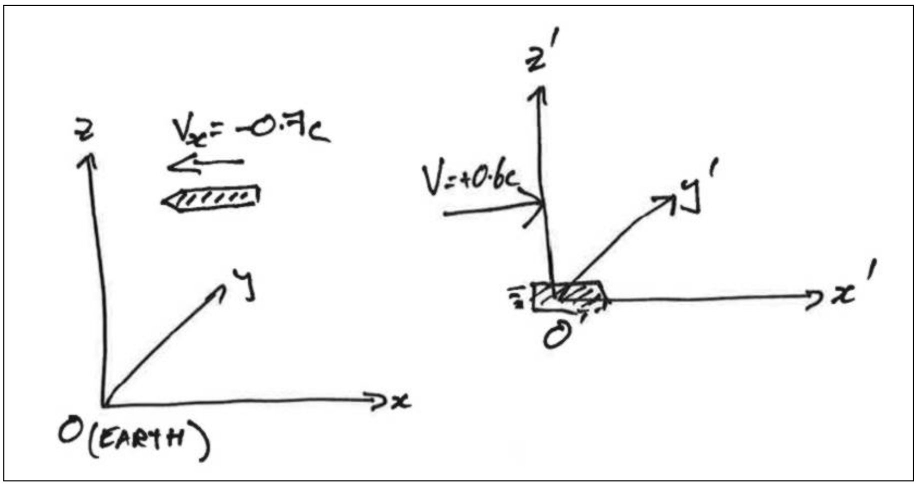


Figure 21.2: Two inertial reference frames in relative motion with the velocity of an object V_x as measured by O.

with the final result for measurement in O to O'

$$\boxed{{v_x}' = \frac{(v_x - V)}{\left(1 - \frac{V}{c^2}v_x\right)}} \tag{21.5}$$

and after some more algebra for O' to O

$$\boxed{v_x = \frac{({v_x}' + V)}{\left(1 + \frac{V}{c^2}{v_x}'\right)}} \tag{21.6}$$

21.3 Example

Consider the situation of figure 21.2. We have one rocket (O') zooming to $+x$ with speed 0.6c and missile zooming to $-x$ with speed $v_x = 0.7c$. What is the speed of the missile as measured by the rocket O'? Galileo would have us believe that

$$
\begin{aligned}
{v_x}' &= v_x - V && (21.7a)\\
\therefore &= -0.7c - 0.6c && (21.7b)\\
&= -1.3c && (21.7c)
\end{aligned}
$$

which is faster than the speed of light! Lorentz on the other hand gives us

$$
\begin{aligned}
{v_x}' &= \frac{(v_x - V)}{\left(1 - \frac{V}{c^2}v_x\right)} && (21.8a)\\
&= \frac{(-0.7c - 0.6c)}{\left(1 - \frac{0.6c \times -0.7c}{c^2}\right)} && (21.8b)\\
&= -0.915c. && (21.8c)
\end{aligned}
$$

The main problem you'll face using the Lorentz velocity transforms is getting confused as to which velocity is relative to which observer. But with a bit of practice you can overcome this hurdle.

21.4 3 frames

In the example above we considered O to be the Earth, O' to be a rocket and v_x to be the velocity of a missile as measured from Earth. What we really have is three reference frames, that of the Earth, rocket and missile. Using these Lorentz velocity transforms you should, with some clear thinking, be able to compute any velocity from any of the thee reference frames.

21.5 Relativistic y and z velocity transform

Because velocity is the time derivative of position and time is relative, so the y and z velocities also have relativistic properties. For the y and z we follow the same reasoning as above and we know that

$$y' = f(y) \tag{21.9a}$$
$$\therefore dy' = dy \tag{21.9b}$$
$$z' = f(z) \tag{21.9c}$$
$$\therefore dz' = dz \tag{21.9d}$$
$$t' = f(x,t) \tag{21.9e}$$
$$dt' = \gamma\left(dt - \frac{V}{c^2}dx\right) \tag{21.9f}$$

and so finally we have for the transform O to O'

$$v_y{}' = \frac{dy'}{dt'} \tag{21.10a}$$
$$= \frac{v_y}{\gamma\left(1 - \frac{V}{c^2}v_x\right)} \tag{21.10b}$$

and

$$v_z{}' = \frac{dz'}{dt'} \tag{21.10c}$$
$$= \frac{v_z}{\gamma\left(1 - \frac{V}{c^2}v_x\right)} \tag{21.10d}$$

and for the transform O' to O

$$v_y = \frac{v'_y}{\gamma\left(1 + \frac{V}{c^2}v'_x\right)} \tag{21.11a}$$
$$v_z = \frac{v'_z}{\gamma\left(1 + \frac{V}{c^2}v'_x\right)} \tag{21.11b}$$

Section 22

Relativistic Momentum

In this section we cover the implications of special relativity on dynamics (e.g. acceleration, energy, momentum etc.). However, in order to remain within the boundaries of special relativity it is important to recall that we are dealing only with IRFs. Consequently what we must consider here are only inertial observers who can measure accelerating bodies.

The aim is to find out whether the concepts of mass and momentum as we know them, from the Newtonian relativity, are altered or not by special relativity. In order to examine this let's see what happens when an object changes its velocity.

22.1 Do we need a new description of momentum?

Galilean/Newtonian classical physics tells us that momentum is

$$\vec{p} = m\vec{v} \tag{22.1}$$

and that the addition of velocities (see section 2.4) is

$$\vec{v}\,' = \vec{v} - \vec{V}. \tag{22.2}$$

The figure 22.1 shows before and after snapshots of a collision of two particles that have mass m_1 and m_2, only travel in the x direction and have speeds u_1 and u_2 before collision and speeds v_1 and v_2 after. Classically we write for the conservation of momentum

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_1 v_2. \tag{22.3}$$

Now from the point of view of another observer O' travelling at speed V (as ever) we use the Galilean transforms to get $u_1 = u_1' - V$ and so on which gives

$$m_1 u_1' + m_2 u_2' - (m_1 + m_2)\,V = m_1 v_1' + m_2 v_2' - (m_1 + m_2)\,V \tag{22.4a}$$

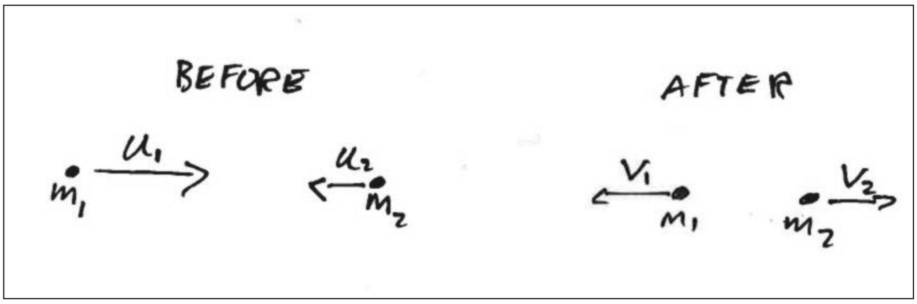


Figure 22.1: 1D classical scattering of two masses.

$$\therefore m_1 u_1' + m_2 u_2' = m_1 v_1' + m_2 v_2'. \tag{22.4b}$$

But now we know now to properly relate the two velocities we should write

$$u_1 = \frac{(u_1' + V)}{\left(1 + \frac{V}{c^2} u_1'\right)} \tag{22.5}$$

in which case equation (22.4b) recast with this relativistic view of velocities doesn't balance, momentum $p = m\vec{v}$ is not conserved! We can either abandon momentum, or try and find a new definition of momentum that does result in conservation.

22.2 The thesis

We want our new momentum to

- be parallel to the particle velocity
- be equal to the classical result $m\vec{v}$ in the limit of low speed

our thesis is that momentum is

$$\vec{p} = f(v)\, m\vec{v}. \tag{22.6}$$

where the mass m (the rest mass) is independent of the motion of the particle and $f(v)$ is a dimensionless function which ≈ 1 at low speed.

22.3 The thought experiment

This is going to take a bit of time but is nice to work through so you see there is no magic and no slight of hand, it's just physics. The figure 22.2 shows the process we will explore.

22.3.1 Experimental initial conditions

There are, as ever, two inertial reference frames O and O' moving relative to each other with speed V along the x axis, and each has an identical particle.

- O throws their particle A directly along their y-axis with speed as measured by themselves of u_0.
- O sees B travelling with velocity $v_x = V$ and $v_y = -u$.
- O' throws their particle B directly along their y'-axis with speed as measured by themselves of $-u_0$.
- O' sees A travelling with velocity $v_x' = -V$ and $v_y' = u$.
- $u_0 \ll v$ and $v \sim c$

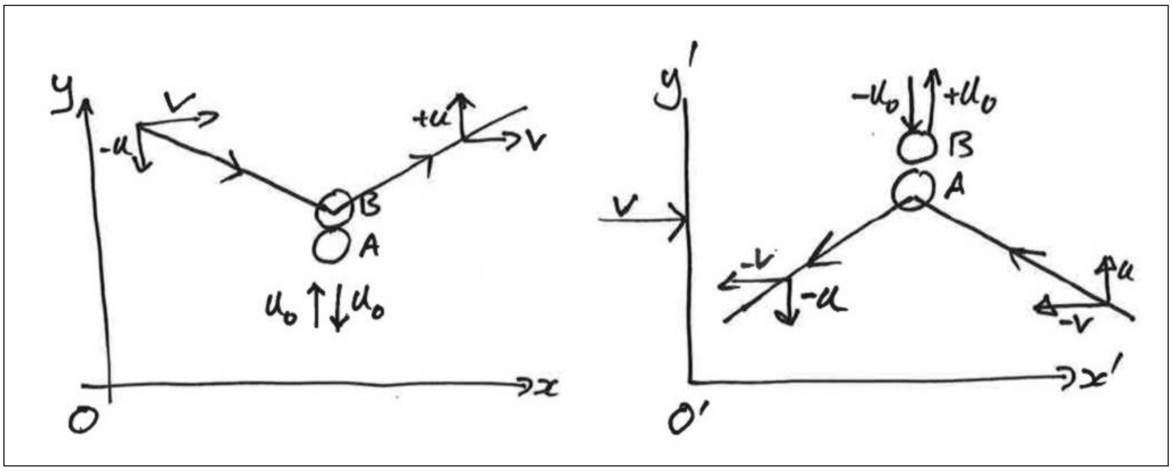


Figure 22.2: Two views of the same scattering event. (a) from O and (b) from O'.

22.3.2 The elastic collision

The experiments are so good that the particles collide along their centres of mass and so, as figure 22.2 depicts:

- O sees their particle A bounce back to themselves directly along their y-axis with speed $-u_0$.
- O sees no change in particle B's x velocity.
- O sees particle B's y-velocity change from $-u$ to $+u$.

and

- O' sees their particle B bounce back to themselves directly along their y'-axis with speed u_0.
- O' sees no change in particle A's x' velocity.
- O' sees particle A's y' velocity change from $+u$ to $-u$.

Since there is no change in the speeds of the particles this is an elastic collision.

22.3.3 Analysis

22.3.3.1 Observer O

Observer O measures the y-component of velocity of A

$$+u_0 \to -u_0 \tag{22.7a}$$

and they measure the y-component of B

$$-u \to +u. \tag{22.7b}$$

And we can relate these two together through

$$v_y = \frac{v'_y}{\gamma\left(1 + \frac{V}{c^2} v'_x\right)} \tag{22.7c}$$

the Lorentz y-velocity transform (equation (21.11a)), where in frame O' we know that

$$v'_{x_B} = 0 \tag{22.7d}$$

and that

$$v'_{y_B} = u_0 \tag{22.7e}$$
$$v_{y_B} = u \tag{22.7f}$$

therefore

$$u = \frac{v'_{y_B}}{\gamma\left(1 + \frac{V}{c^2} v'_{B_x}\right)} \tag{22.7g}$$

hence

$$v_{y_B} = \frac{u_0}{\gamma} \tag{22.7h}$$

finally

$$v_{y_B} = u = \frac{u_0}{\gamma} \tag{22.7i}$$

(22.7j)

22.3.3.2 Observer O'

The roles of A and B are interchangeable, just rotate the figure by 180 and we see an identical scattering process.

22.3.4 Conservation of momentum

For both observers the speed of each particle is unchanged by the collision. O sees

- A has speed u_0
- B has speed $W = \sqrt{u^2 + V^2}$

We had proposed that

$$\vec{p} = f(v)\, m\vec{v}. \tag{22.8a}$$

so in the y-direction

$$p_y^{\text{Before}} = f(u_0)\, mu_0 - f(W)\, mu \tag{22.8b}$$
$$p_y^{\text{After}} = -f(u_0)\, mu_0 + f(W)\, mu \tag{22.8c}$$

$$\therefore \frac{f(W)}{f(u_0)} = \frac{u_0}{u} \tag{22.8d}$$

and in the limit $u_0 \to 0$ we said that $f \to 1$ therefore

$$f(W) = \frac{u_0}{u} \tag{22.8e}$$

$$= \frac{u_0}{\left(\frac{u_0}{\gamma}\right)} \tag{22.8f}$$

$$f(W) = \gamma \tag{22.8g}$$

and so we get the form for relativistic momentum

$$\vec{p} = \gamma m \vec{v} \tag{22.8h}$$

22.4 An ultimate speed

Consider a constant force acting on a particle of mass m classically we have

$$F = \frac{dp}{dt} \tag{22.9a}$$

$$= m\frac{dv}{dt} \tag{22.9b}$$

$$\therefore Fdt = mdv \tag{22.9c}$$

$$\int Fdt = \int mdv \tag{22.9d}$$

$$\therefore Ft = mv \tag{22.9e}$$

$$\therefore v = \frac{Ft}{m} \tag{22.9f}$$

as our force acts our particle just gets faster and faster. But now we know that relativistically we have

$$F = \frac{dp}{dt} \tag{22.10a}$$

$$= \frac{d}{dt}(\gamma m v) \tag{22.10b}$$

$$\therefore Fdt = d(\gamma m v) \tag{22.10c}$$

$$\int Fdt = \int d(\gamma m v) \tag{22.10d}$$

$$\therefore Ft = \gamma m v \tag{22.10e}$$

$$\therefore v = \left(\frac{Ft}{m}\right)\frac{1}{\sqrt{1+\frac{F^2t^2}{m^2c^2}}}. \tag{22.10f}$$

At short time this is the same as the classical result, but as $t \to \infty$ we have $v \to c$, see figure 22.3. Yet the force is acting all this time and over some distance, so where is all the work going? It also implies that the force would have to work for an infinite amount to time, doing an infinite amount of work, to get our particle up to c. Therefore no body with a mass can reach c: we have an ultimate speed limit in the universe.

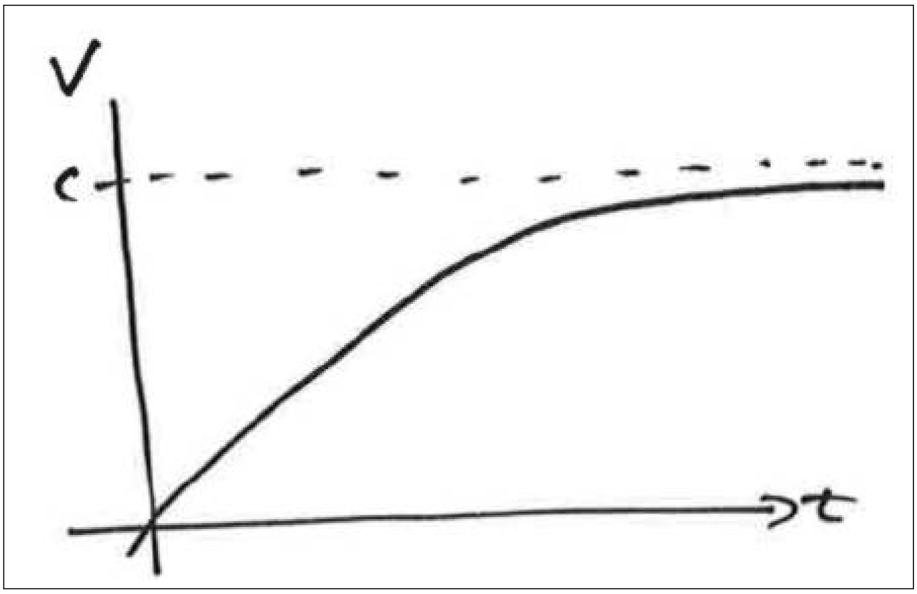


Figure 22.3: Plot of speed v versus time for a constant force acting on a body with relativistic momentum.

Section 23

Relativistic Energy

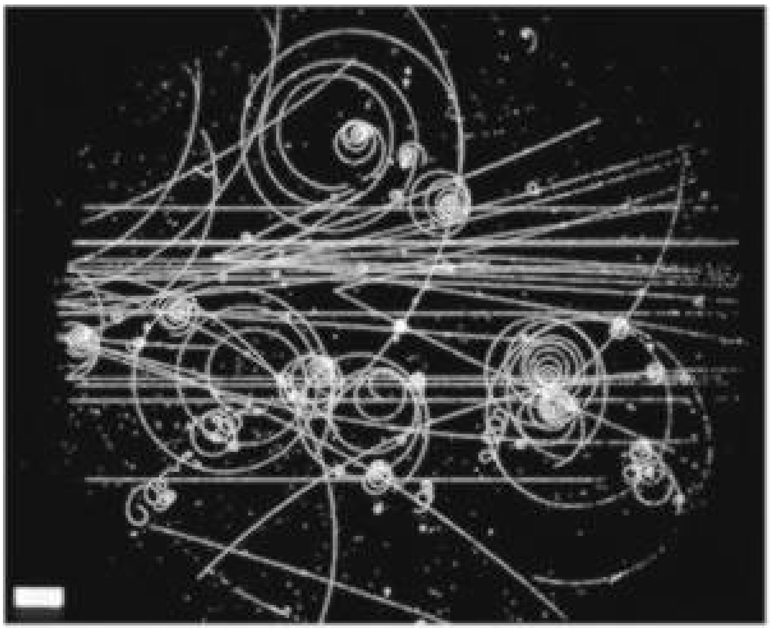


Figure 23.1: Smashing matter. Particle physics in a bubble chamber [8].

In this section we will (partly) derive the most famous equation of physics $E = mc^2$. We'll consider the total energy and the rest energy of a system, see how it all applied to light, and apply our new knowledge of relativistic momentum and mass to various scattering and nuclear processes. We'll see how to produce matter out of energy, figure 23.1.

23.1 Kinetic energy

23.1.1 Classical kinetic energy

We all know that classically the kinetic energy of a body is

$$K = \frac{1}{2}mv^2. \tag{23.1a}$$

But rather than start here we state that the kinetic energy is equal to the work done on a body to accelerate it from rest to v

$$K = \int F dx \tag{23.1b}$$

$$F = \frac{dp}{dt} \tag{23.1c}$$

$$= ma \tag{23.1d}$$

$$= m\frac{dv}{dt} \tag{23.1e}$$

$$K = \int \underbrace{m\frac{dv}{dt}}_{F} dx \tag{23.1f}$$

$$= \int m\frac{dv}{dx} \underbrace{\frac{dx}{dt}}_{v} dx \tag{23.1g}$$

$$= \int_0^v mvdv \tag{23.1h}$$

$$= \frac{1}{2}mv^2. \tag{23.1i}$$

Nice but dull.

23.1.2 Relativistic kinetic energy

But now we know that this classical view of momentum $p = mv$ is wrong. So we should have relativistically

$$K = \int F dx \tag{23.2a}$$

$$F = ma \tag{23.2b}$$

$$= \frac{dp}{dt} \tag{23.2c}$$

$$= m\frac{d(\gamma v)}{dt} \tag{23.2d}$$

$$K = \int m\frac{d(\gamma v)}{dt} dx \tag{23.2e}$$

Which after much algebra and integrating gives the relativistic expression for the kinetic energy of a body as

$$K = (\gamma - 1)\, mc^2. \tag{23.2f}$$

23.1.3 Recovery of classical kinetic energy

At low speed (as we've previously shown)

$$\gamma \approx 1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 \tag{23.3a}$$

$$\therefore K(v \ll c) \approx \left(1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 - 1\right) mc^2 \tag{23.3b}$$

$$\approx \frac{1}{2}mv^2 \tag{23.3c}$$

And so we recover the classical view of kinetic energy at low speed - as we should!

23.1.4 High speed kinetic energy

What about at high speed? The $\boxed{\text{figure 23.2}}$ shows how as $v \to c$ so the kinetic energy $\to \infty$. Thus we cannot ever accelerate a body to the speed of light as it would take an infinite amount of energy. The speed of light is a speed limit.

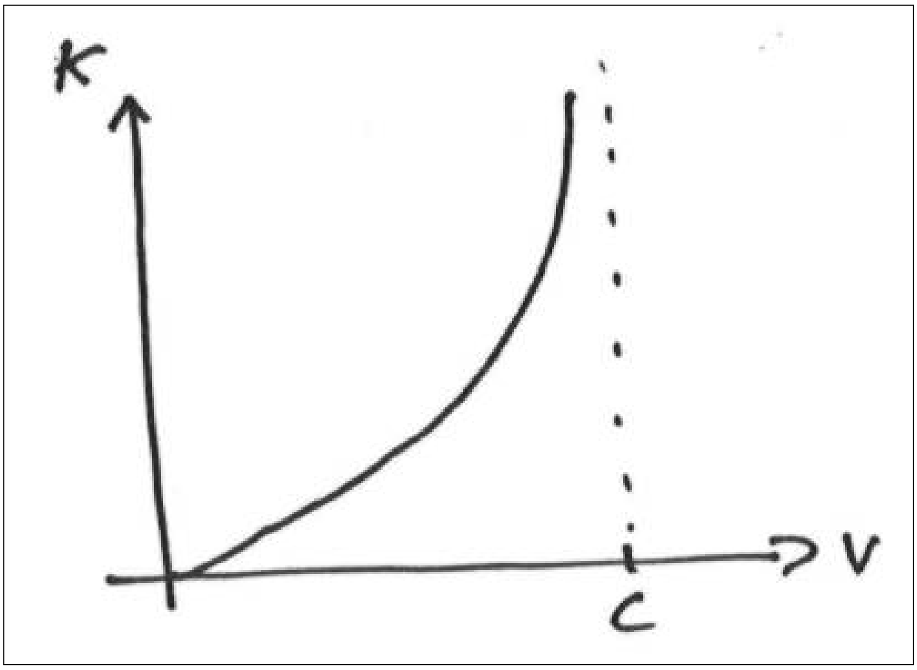


Figure 23.2: Relativistic kinetic energy and speed.

23.2 Total relativistic energy

If we take our result for the relativistic kinetic energy and rearrange we get

$$\gamma mc^2 = K + mc^2 \tag{23.4a}$$

so all the components of this equation have the dimensions of energy and this represents the total energy of a body, kinetic energy and the rest mass energy (i.e, potential energy).

$$\gamma mc^2 = \text{Total energy} \tag{23.4b}$$

$$K = \text{Kinetic energy} \tag{23.4c}$$

$$mc^2 = \text{Rest mass energy} \tag{23.4d}$$

hence for a particle stationary in our I.R.F.

$$E = mc^2. \tag{23.4e}$$

23.3 Mass-Energy equivalence

In section 16 relativity linked two entities (space and time) that we assumed, with our low speed common sense, were unconnected. Now we see that mass is just another form of energy. How much energy?

The rest mass energy of a proton is

$$E_{\text{rest}} = mc^2 \tag{23.5a}$$
$$= 1.67^{-27} \times 3 \times 10^8 \times 3 \times 10^8 \tag{23.5b}$$
$$= 1.5 \times 10^{-10} J \tag{23.5c}$$
$$\approx 1GeV \tag{23.5d}$$

23.4 More problems with photons

Photons have no rest mass and travel at the speed of light. We therefore have $\gamma = \infty$ and so neither the momentum, nor total energy as we've defined them are well defined functions. Instead we play with our equations

$$\frac{cp}{E} = \frac{c\gamma mv}{\gamma mc^2} \tag{23.6a}$$
$$= \frac{v}{c} \tag{23.6b}$$

This relation allows us to compute any particle's velocity without bothering with computing γ. And for photons

$$E = cp \tag{23.6c}$$

figure 23.3 depicts this energy-momentum relationship.

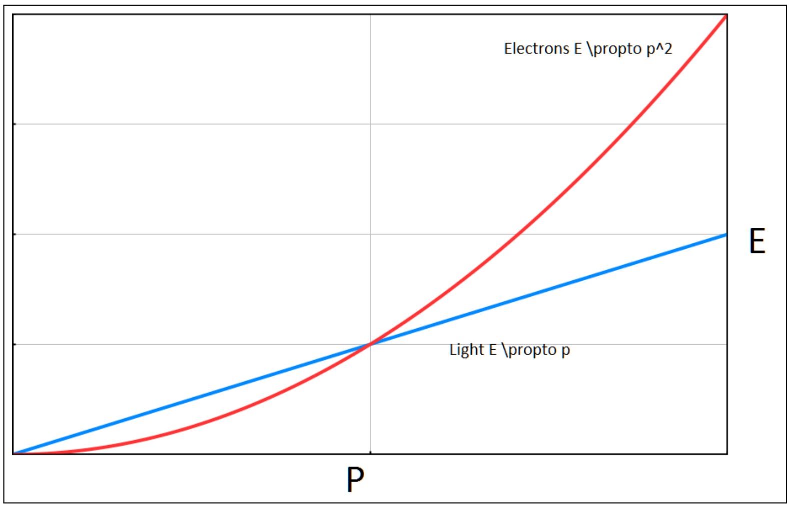


Figure 23.3: Momentum and energy scaling for photons and electrons

Photons have momentum, but no mass. In the 2D material graphene, the electrons energy-momentum relation is also liner, and hence the electrons obey the physics of mass-less, relativistic particles - odd.

23.5 Lorentz invariant energy

We've now seen that the total energy of a system is γmc^2 hence

$$E^2_{\text{total}} = \gamma^2 m^2 c^4 \tag{23.7a}$$

using the Calum method

$$E^2_{\text{total}} = \frac{m^2c^4}{1-\frac{v^2}{c^2}} \tag{23.7b}$$

e-write as

$$\begin{aligned} m^2c^4 &= E^2_{\text{total}}\left(1-\frac{v^2}{c^2}\right) && (23.7c)\\ &= E^2_{\text{total}} - \frac{E^2_{\text{total}}v^2}{c^2} && (23.7d)\\ &= E^2_{\text{total}} - \frac{\gamma^2 m^2 c^4 v^2}{c^2} && (23.7e)\end{aligned}$$

identifying

$$p = \gamma m v \tag{23.7f}$$

hence finally

$$m^2c^4 = E^2_{\text{total}} - p^2c^2 \tag{23.7g}$$

which has a left-hand-side that is the same in all inertial reference frames. Equation (23.7g) is therefore invariant under Lorentz transformation. Energy is conserved in relativity!

23.6 Example collision: evidence for relativity

Consider the process depicted in figure 23.4. A π meson (pion) travelling with kinetic energy 1 GeV and rest mass of 135 MeVc^{-2}, decays into two photons of equal energy. What is their angle α?

Rest mass

Note,

$$E = mc^2 \tag{23.8}$$

$$m = \frac{E}{c^2} \tag{23.9}$$

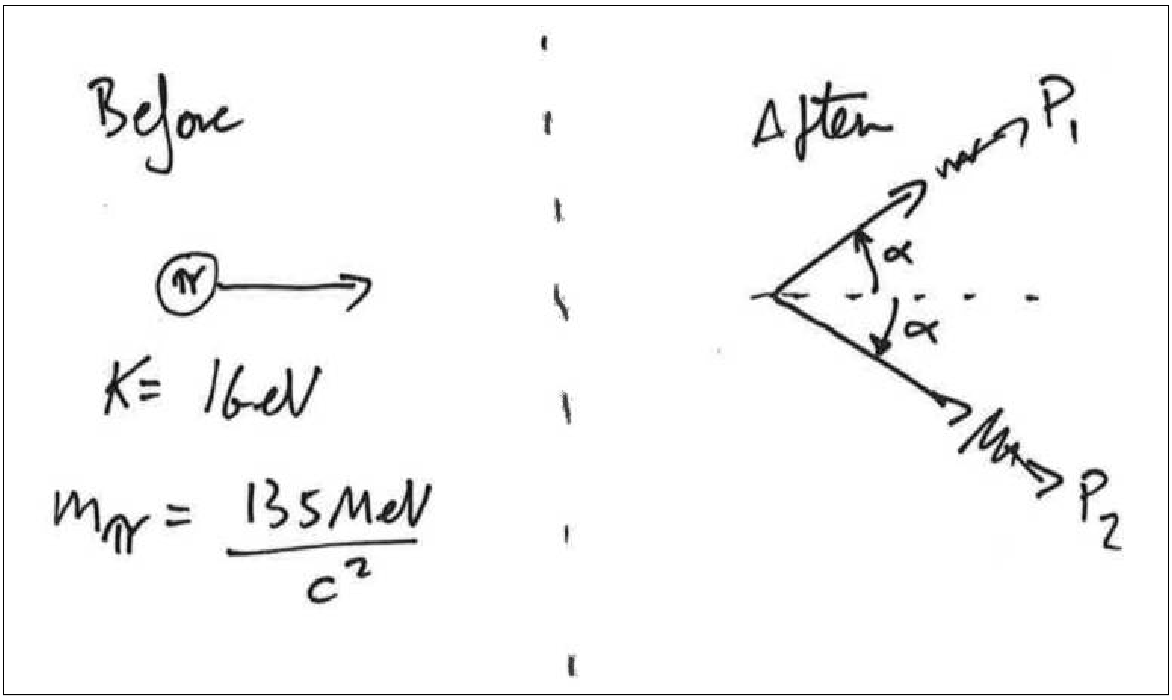


Figure 23.4: A π meson decays into two photons.

Conserve energy

The total energy of the system is conserved (we don't change reference frame during the collision) so

$$E_{\text{tot}} = m_\pi c^2 + K \tag{23.10a}$$

$$\therefore E_{\text{Photon}} = \frac{E_{\text{tot}}}{2} \tag{23.10b}$$

$$\text{and } p_{\text{Photon}} = \frac{E_{\text{tot}}}{2c} \tag{23.10c}$$

$$\therefore |p_{\text{Photon}}| = \frac{m_\pi c^2 + K}{2c} \tag{23.10d}$$

and so the photon momentum in the x-direction is just

$$p_{x \text{ Photon}} = |p_{\text{Photon}}| \cos\alpha \tag{23.10e}$$

$$= \frac{m_\pi c^2 + K}{2c} \cos\alpha \tag{23.10f}$$

Thus the x-momentum before collision must equal twice this photon momentum (as there are two photons) and from equation (23.7g) we have

$$p_x = \frac{\sqrt{E^2 - m_\pi^2 c^4}}{c} \tag{23.10g}$$

$$= 2\frac{m_\pi c^2 + K}{2c} \cos\alpha \tag{23.10h}$$

$$\therefore \cos\alpha = \frac{\sqrt{(m_\pi c^2 + K)^2 - m_\pi^2 c^4}}{m_\pi c^2 + K} \tag{23.10i}$$

$$\alpha = 6.8^\circ \tag{23.10j}$$

In 1962 this purely relativistic result was experimentally confirmed.

23.7 Antimatter

Figure 23.5: Paul Dirac (1902-1984). Born and educated in Bristol. Dirac shared the 1933 Nobel Prize for physics with Erwin Schrödinger "for the discovery of new productive forms of atomic theory." Was also awarded the Royal Medal in 1939 and both the Copley Medal and the Max Planck medal in 1952. He was elected a Fellow of the Royal Society in 1930, an Honorary Fellow of the American Physical Society in 1948, and an Honorary Fellow of the Institute of Physics, London in 1971. Dirac became a member of the Order of Merit in 1973. Dirac declined a knighthood as he did not want to be addressed by his first name. [9]

One more thing, consider the energy from equation (23.7g)

$$E_{\text{total}} = \pm\sqrt{p^2c^2 + m^2c^4} \tag{23.11}$$

Which in one square-root (kind of) gives us particles and antiparticles. British physicist Paul Dirac figure 23.5 combined quantum mechanics and relativity and gave us both the spin of an electron and antimatter (the $-ve$ square-root). You can find out more about this in PH40084 Advanced Quantum Mechanics.

Section 24

Problems for part D

P19: Speed of light is constant? [application, medium]

Using the Lorentz transformations for velocity between two inertial frames in moving relative to each other at a speed V is the x-direction, demonstrate that the speed of light is measured to be c in both frames. [3]

P20: Rocket velocities [comprehension, easy, past exam]

(a) Explain, with the aid of a diagram, all the terms in the Lorentz velocity transform equation [2]

$$v_x{}' = \frac{(v_x - V)}{\left(1 - \frac{V}{c^2} v_x\right)}$$

(b) Two rockets, A and B, zoom away from the Earth in opposite directions. As measured from Earth A has a velocity $+0.8c$ and B has a velocity of $-0.6c$. What is the velocity of A as measured by B? [2]

P21: Speeding protons [application, easy]

How may protons that have speed 10 ms^{-1} would be required to have the same total momentum as a single proton moving at speed 0.999c? [2]

P22: Speeding electrons (again) [application, easy]

For an electron travelling at $0.8c$

(a) What is its total energy? [3]

(b) What is its momentum? [2]

P23: UK energy [comprehension, easy]

The UK uses 1.5×10^{18} J per year of energy.

(a) What mass of protons would I have to destroy to run the UK per year? [2]

(b) What mass of feathers would I have to destroy to run the UK? [1]

(c) 1 kg of coal produces ∼30 MJ of energy. What mass of coal do I have to burn to run the UK? [2]

(d) If you burnt this coal in a sealed container, so no material could enter of leave, just the heat can leave, you will find the container is lighter than when you started. By how much? [2]

P24: LHC smasher [application, easy]

The LHC can smash together two protons at an energy of 7 TeV (7×10^{12}eV) per proton. Out of such a collision, how may stationary protons could one make? [2]

P25: Nuclear chemistry [application, medium, past exam]

The rest mass in kg of an electron, proton, neutron and deuteron (an atomic nucleus consisting of a proton and a neutron) are

$$m_e = 9.11 \times 10^{-31}, \qquad m_p = 1.6726^{-27}, \qquad m_n = 1.6749^{-27} \text{ and } \qquad m_d = 3.3436^{-27}.$$

How much energy is liberated upon the formation of a deuteron from the fusion of a fee proton and a free neutron (assume all particles are at rest). Give your answer in eV. [2]

P26: LHC in a spin [evaluation, medium, past exam]

The Large Hadron Collider (LHC) is designed to produce head on collisions between two protons, one proton circulating clockwise around the ring and other anti-clockwise. During the testing phase of the LHC, each proton has a kinetic energy of $\sim 1 \times 10^9$eV. The rest-mass energy of a proton is $\sim 1 \times 10^9$eV.

(a) From the point of view of the scientists, show that the speed of one of the protons is $\frac{\sqrt{3}}{2}$c and therefore that $\gamma = 2$. [2]

(b) From the point of view of one of the protons, show that the other proton travels at speed $\frac{4\sqrt{3}}{7}$c and that the associated $\gamma = 7$. [3]

(c) The quantity

$$m^2c^4 = E^2_{\text{total}} - p^2c^2$$

is Lorentz invariant, that is, it has the same value in all inertial reference frames. Show that this is the case by computing the right hand side of this equation for a single proton from:

(i) the reference frame of the proton itself. [2]

(ii) from the reference frame of the LHC scientists. [2]

P27: Back to normal speed [comprehension, medium]

The total relativistic energy of particle is given by

$$\gamma mc^2 = K + mc^2.$$

Where all the symbols are their usual meaning. Show that at low speeds, the kinetic energy K reduces to its classical form. [2]

Section 25

Ideas and Projects for part D

Some extra ideas to play with. These are for you to talk and discuss with your friends and perhaps tutor. They are not going to have solutions, they are for you to play with and see if you can make sense of relativity. They have purposely been written as sort of mini-projects that if you tackled a few you would learn a lot along the way. This year, 2021/2022, is the first year I've included them. So the rest of the course is identical to the past year's course, these are just interesting extras. Please feel free to use moodle or the google-doc to discuss with other students.

- Read: Relativity: the Special and General Theory by Albert Einstein, Part I, chapter XIII to end of Part I.
- Compare the relationships between energy, momentum with space and time. What symmetry do you see? Can you plot energy-momenta equivalents to space-time diagrams for some of the following problems?

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