Worth: 8% **Due:** By 5:59pm on Tuesday 31 November

Remember to write the *full name* and *student number* of *every group member* prominently on your submission.

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*Please read and understand the policy on Collaboration given on the Course Information Sheet. Then, to protect yourself, list on the front of your submission **every** source of information you used to complete this homework (other than your own lecture and tutorial notes). For example, indicate clearly the **name** of every student from another group with whom you had discussions, the **title and sections** of every textbook you consulted (including the course textbook), the **source** of every web document you used (including documents from the course webpage), etc.*

*For each question, please write up detailed answers carefully. Make sure that you use notation and terminology correctly, and that you explain and justify what you are doing. Marks **will** be deducted for incorrect or ambiguous use of notation and terminology, and for making incorrect, unjustified, ambiguous, or vague claims in your solutions.*

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1. Let $x_1, \ldots, x_n$ be n be a list of n people who have applied for a job at Google. They are interviewed in pairs: if x_i and x_j are interviewed together, one of them is chosen to be more qualified. You are given a list of outcomes of m interviews, each of the form "candidate x_i is more qualified than candidate x_j" Your task is to order the job candidates, with the best candidate first. Specifically, if x_i is more qualified than x_j (as the result of the outcome of some interview), then you must place candidate x_i ahead of x_j in the ordering.

   (a) How do you model this problem using a graph? Assume that all graphs in this problem set are implemented using adjacency list.

   (b) How do you determine whether it is possible at all to arrange all candidates in a line such that all constraints are satisfied? Explain your algorithm in clear English, and analyse its worst-case running time.

   (c) Assuming such arrangement is possible, how do you compute an actual arrangement? Explain your algorithm in clear English, and analyse its worst-case running time.

answer :

(a) We construct a directed graph $G(V,E)$ with the following rules,

- Construct a graph with n nodes, that is $G(V,E)$ where $V = \{x_1, \ldots, x_n\}$ *and* $E = \varnothing$. Each node stands for a candidate. Using adjacency list it would be create n empty list of size m and list $A[i]$ is for vertex x_i.
- For each of the m outcomes, we add a edge $\overrightarrow{x_i x_j}$ to E, if outcome says "candidate x_i is more qualified than candidate x_j". In other words, stores x_j in $A[i]$.

We attempt the solve the problem using topological sort such that the vertices are in an order that all edges are pointing to the right side. Then every candidate is more qualified than the candidates on its left.

(b) Arranging all candidates in a line such that all constraints are satisfied means such topological sort exist, which means the graph is acyclic.
First we construct the graph by the rules mentioned in (a). Then we do a depth first search, check for cycle. If the graph is acyclic then it is possible to arrange all candidates, otherwise

not possible.
The run time for construct the graph $G(V,E)$ would take $\mathcal{O}(m)$, where $|V| = n$ and $|E| = m$. DFS check for cycle using adjacency list would take $\mathcal{O}(n+m)$. Hence, the worst case runtime is $\mathcal{O}(n+m)$.

(c) We can compute an actual arrangement by topological sort such that the vertices are in an order that all edges are pointing to the right side. Since all edges are pointing to the right side which means every candidate is more qualified than the candidates on its left. Then the order generated by the topological sort would be a valid arrangement.
First we construct the graph $G(V,E)$ which takes $\mathcal{O}(m)$, then call DFS(G) to compute finishing times $f[x_i]$ foreach vertex x_i. And as each vertex is finished, insert it on to the front of a linked list return the linked list of vertices, this would take $\mathcal{O}(n+m)$. Hence, the worst case runtime is $\mathcal{O}(n+m)$.

2. Let $G = (V,E)$ be a weighted undirected connected graph that contains a cycle, and let e be the maximum-weight edge among all edges in the cycle. **Prove** that there exists a minimum spanning tree of G which does NOT include e.

Proof :

Suppose $v_1v_2...v_nv_1$ is the circle mentioned in the question, and v_i, v_j are the vertices adjacent to edge e, which with maximum-weight among all edges in the cycle. Let T be any minimum weight spanning tree of G .

If T does not contain edge e then we are done.

Now suppose T contain edge e, then $T - e$ has two connected component A, B, and each of them is a tree.

Since $v_1v_2...v_nv_1$ form a circle, then there must be some edge e' other than e with two ends point u, v are in A, B separately. Otherwise, $v_1v_2...v_nv_1$ could not form a circle.

Then connect A, B with edge e', then we get $T - e + e'$.

Since A, B are two connected component then $T - e + e'$ is a spanning tree of G. Also according to the question we know that weight of e' is no more than e; hence, $T - e + e'$ is also a minimum-weight spanning tree of G.

3. Consider a list of cities $c_1, c_2, \ldots, c_n$. Assume we have a relation R such that, for any i, j, $R(c_i, c_j)$ is 1 if cities c_i and c_j are in the same province, and 0 otherwise.

(a) If R is stored as a table, how much space does it require?

(b) Using a disjoint set ADT, write pseudo-code for an algorithm that puts each city in a set such that c_i and c_j are in the same set if and only if they are in the same province. (That is, you can assume that you have some implementation of the basic disjont set operations, MAKE-SET, FIND-SET, and UNION.)

(c) When the cities are stored in the disjoint set ADT, if you are given two cities c_i and c_j, how do you check if they are in the same province?

(d) If we use trees with the rank heuristic, what is the worst-case running time of the algorithm from (b) (Hint: the unions from your algorithm probably have a special form). Explain.

(e) If we use trees without the rank heuristic, what is the worst-case running time of the algorithm from (b). Explain. Are there more worst-case scenarios than in (d)?

Answer

(a) If R is stored as a table, then we have to store the relation between any two cities. Hence, it requires at least $\binom{n}{2} = \frac{n(n-1)}{2}$ brackets.

(b) CONSTRUCT:

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1  first we do a MAKE-SET for each city, and we get $c_1, c_2, \dots, c_n$, n sets
2  for $c_i$ in $c_1, c_2, \dots, c_n$:
3      for $c_j$ in $c_{i+1}, c_{i+2}, \dots, c_n$:
4          if $R(c_i, c_j)$ and FIND-SET($c_i$) $\neq$ FIND-SET($c_j$):
5              UNION($c_i$, $c_j$)
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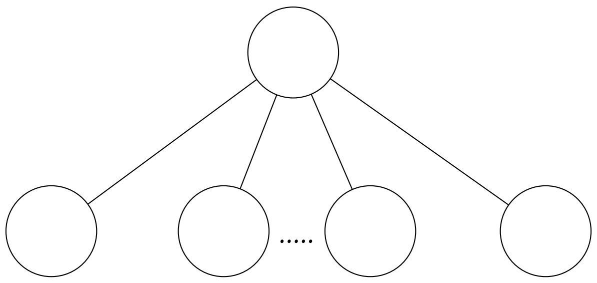
The function CONSTRUCT loops through every pair of (c_i, c_j). If c_i and c_j are in the same province but different sets, then UNION(c_i, c_j). Hence, every two cities which are in the same province are in the same set.

(c) Since R is clearly a equivalence relation, then to check if any two given sets c_i and c_j are stored in the same set, we call FIND-SET on c_i and c_j. If they return the same representative, then they are in the same set otherwise not.

(d) *Claim* : If we use trees with the rank heuristic, then every disjoint set we get from CONSTRUCT has height no more than 1.



It is sufficient to show after each iteration of the outer for loop every disjoint set has height no more than 1. We prove this by induction on the number of iterations of the outer for loop.

Base Case: $i = 0$, every disjoint set contains just one element; hence, has height 0.

Induction Hypothesis: After k iteration, every disjoint set has height no more than 1.

Induction Step: At $(k+1)_{th}$ iteration if CONSTRUCT calls UNION(c_i, c_j), by induction hypothesis, we know set contains c_i has height no more than 1.
Also, because FIND-SET(c_i) $\neq$ FIND-SET(c_j) then set c_j must has height 0. If set c_j does not have height 0, which means at some previous iteration (i.e m_{th} where $m < i$), $R(c_m, c_j)$, then clearly $R(c_m, c_i)$ as well. It contradicts with FIND-SET(c_i) $\neq$ FIND-SET(c_j), then set c_j must has height 0; therefore, UNION(c_i, c_j) generates a set of height no more than 1. Induction follows.

Now let's discuses the worst case runtime. It is clear that *line*#1, #2 always execute n times. Since CONSTRUCT checks all pair of (c_i, c_j), *line*#3, #4 always executes $\frac{n^2-n}{2}$ times. From the claim we know that unions from CONSTRUCT all have height no more than 1; hence, FIND-SET

and UNION take constant time. Then no matter in which cases $line\#1$ up to #4 always take the same time.
The worst case scenario occurs only when $line\#5$ executes the most which is $c_1, c_2, \ldots, c_n$ are all in the same province. Then $line\#5$ executes $n-1$ times takes $\Theta(n)$. The total runtime is still dominated by $line\#3, \#4$. Hence, the worst case run time is $\Theta(n^2)$ more precisely $n^2 + 3n$.

(e) If we use trees without the rank heuristic, the UNION takes constant time. Also the runtime for $line\#1, \#2, \#3$ is fixed for any cases. Hence, the worst-case runtime depends on the total runtime of $line\#3$,

$$\text{if } R(c_i, c_j) \text{ and FIND-SET}(c_i) \neq \text{FIND-SET}(c_j)$$

which in other words depends on the height of the tree.

Hence, worst-case scenario occurs only when all cities are in the same provence that is we will have a tree of height $n-1$.
We know that $line\#3$ will be executed $\frac{n^2-n}{2}$ times. After the first n times we will have a tree of height $n-1$ then the rest $\frac{n^2-3n}{2}$ times would take $(n-1)\frac{n^2-3n}{2}$. Therefore, worst-case runtime is $\Theta(n^3)$.
Like we already discussed, the worst-case scenario happens for d only when all cities are in the same province. Same for e; therefore, The worst-case scenario for both d and e are the same.