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Problem Set 1

1. (a) In IEEE binary system, the single precision F (2, 24, L, U) for a fixed exponent, since the first digit is always 1 then, there are 2^{23} different numbers.
Similarly, in IEEE binary system, the double precision F (2, 53, L, U) for a fixed exponent there are 2^{52} different numbers.
Hence, there are $\frac{2^{52}}{2^{23}} - 1 = 2^{29} - 1$ double precision numbers between any two adjacent nonzero single precision numbers

   (b) In IEEE binary system, the double precision F (2, 53, L, U) carries 53 bits; hence, the largest p for double is, $p = 2^{53} - 1$
In IEEE binary system, the single precision F (2, 24, L, U) carries 24 bits; hence, the largest p for single precision is, $p = 2^{24} - 1$

2. (a) The first loop prints 5 values of a, and the second loop prints 4 values of b.
For the first loop we have:

   $0.1_2 = 0.0\,\overline{0011}$; hence, in IEEE representation for double,
   $0.1 = 1.1\ 0011...\ 0011\ 01_{52}0_{53} \times 2^{-4}$
   $0.2 = (0.1 + 0.1)_2 = 1.10\ 0110\ 0110...\ 0110\ 1_{52}0_{53} \times 2^{-3}$
   $0.3 = (0.1 + 0.1 + 0.1)_2 = 1.001\ 1001\ 1001...\ 1001\ 1010_{52}\ 0_{53} \times 2^{-2}$
   $0.4 = (0.1 + 0.1 + 0.1 + 0.1)_2 = 1.100\ 1100\ 1100...1100\ 1101_{52}\ 0_{53} \times 2^{-2}$
   $0.5 = (0.1 + 0.1 + 0.1 + 0.1 + 0.1)_2 = 1.0000\ 0000\ ...00_{52}0_{53} \times 2^{-1}$
   Hence, the first loop print out 5 numbers.

   For the second loop we have:

   $1.1_2 = 1.0\,\overline{0011}$; hence, in IEEE representation for double,
   $1.1 = 1.0\ 0011...\ 0011\ 01_{52}0_{53} \times 2^0$
   $1.2 = (1.1 + 0.1) = 1.0\ 0110\ 0110...\ 0110\ 10_{52}0_{53} \times 2^0$
   $1.3 = (1.1 + 0.1 + 0.1) = 1.0\ 1001\ 1001...\ 1001\ 11_{52}0_{53} \times 2^0$
   $1.4 = (1.1 + 0.1 + 0.1 + 0.1) = 1.0\ 1100\ 1100...1100\ 1101\ 00_{52}0_{53} \times 2^0$
   $1.5 = (1.1 + 0.1 + 0.1 + 0.1 + 0.1)_2 = 1.10000000...01_{52}0_{53} \times 2^0$
   Hence, we get 1.5_2's IEEE representation is bigger than 1.5; therefore, only 4 numbers are printed out.

   (b) I would recommend the second expression which is $\frac{x}{10.0}$, since from the previous question we knew that computer cannot represent 0.1 in a exact form in binary. However, it could represent 10.0 in exact form as $1010 = 1.01 \times 2^3$.

3. Python Program:

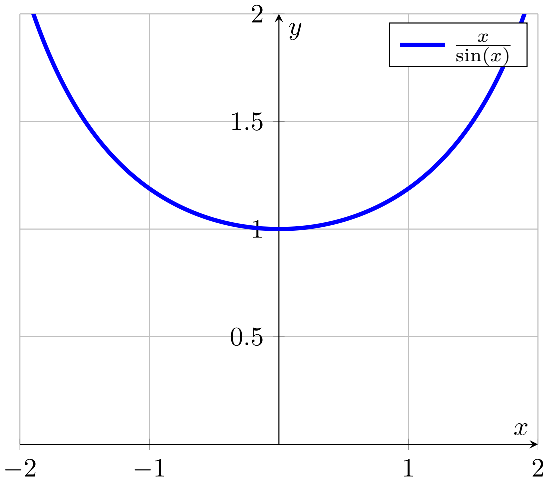
```
import math
def cos():
    for m in range(21):
```

```
x = 2 * pow(10, m)
print( "m =" + str (m) + ": " + "2 * m * pi = " + str(x * math.pi))
print (math.cos(x * math.pi))
```

k	m	$2 \times m \times \pi$	$cos(2 \times m \times \pi)$
0	10^0	$6.2831\ 85307\ 17959 \times 10^0$	$1.0000\ 00000\ 00000 \times 10^0$
1	10^1	$6.2831\ 85307\ 17959 \times 10^1$	$1.0000\ 00000\ 00000 \times 10^0$
2	10^2	$6.2831\ 85307\ 17959 \times 10^2$	$1.0000\ 00000\ 00000 \times 10^0$
3	10^3	$6.2831\ 85307\ 17959 \times 10^3$	$1.0000\ 00000\ 00000 \times 10^0$
4	10^4	$6.2831\ 85307\ 17959 \times 10^4$	$1.0000\ 00000\ 00000 \times 10^0$
5	10^5	$6.2831\ 85307\ 17959 \times 10^5$	$1.0000\ 00000\ 00000 \times 10^0$
6	10^6	$6.2831\ 85307\ 17959 \times 10^6$	$1.0000\ 00000\ 00000 \times 10^0$
7	10^7	$6.2831\ 85307\ 17959 \times 10^7$	$1.0000\ 00000\ 00000 \times 10^0$
8	10^8	$6.2831\ 85307\ 17959 \times 10^8$	$9.9999\ 99999\ 99999 \times 10^{-1}$
9	10^9	$6.2831\ 85307\ 17959 \times 10^9$	$9.9999\ 99999\ 99999 \times 10^{-1}$
10	10^0	$6.2831\ 85307\ 17959 \times 10^0$	$9.9999\ 99999\ 89970 \times 10^{-1}$
11	10^{11}	$6.2831\ 85307\ 17959 \times 10^{11}$	$9.9999\ 99995\ 64035 \times 10^{-1}$
12	10^{12}	$6.2831\ 85307\ 17959 \times 10^{12}$	$9.9999\ 98545\ 10184 \times 10^{-1}$
13	10^{13}	$6.2831\ 85307\ 17959 \times 10^{13}$	$9.9998\ 54510\ 53279 \times 10^{-1}$
14	10^{14}	$6.2831\ 85307\ 17959 \times 10^{14}$	$9.9974\ 25356\ 19873 \times 10^{-1}$
15	10^{15}	$6.2831\ 85307\ 17959 \times 10^{15}$	$8.8841\ 05663\ 23832 \times 10^{-1}$
16	10^{16}	$6.2831\ 85307\ 17959 \times 10^{16}$	$7.1843\ 05743\ 37184 \times 10^{-1}$
17	10^{17}	$6.2831\ 85307\ 17959 \times 10^{17}$	$-4.3810\ 51599\ 26831 \times 10^{-1}$
18	10^{18}	$6.2831\ 85307\ 17959 \times 10^{18}$	$1.7656\ 16183\ 04251 \times 10^{-1}$
19	10^{19}	$6.2831\ 85307\ 17959 \times 10^{19}$	$-1.1403\ 69783\ 90490 \times 10^{-1}$
20	10^{20}	$6.2831\ 85307\ 17959 \times 10^{20}$	$6.8941\ 61562\ 99807 \times 10^{-1}$

We know that π is irrational suppose $math.pi = \pi + \epsilon$, let $f(m) = cos(2m(\pi + \epsilon))$
Then, $\mathcal{K}(f(m)) = \frac{m \times cos'(2m(\pi+\epsilon))}{cos(2m(\pi+\epsilon))}$
Then, $\mathcal{K}(f(m)) = 2m(\pi + \epsilon) \times \frac{sin(2m\pi+2m\epsilon)}{cos(2m\pi+2m\epsilon)} = 2m(\pi + \epsilon) \times \frac{sin(2m\pi)cos(2m\epsilon)+sin(2m\epsilon)cos(2m\pi)}{cos(2m\pi)cos(2m\epsilon)-sin(2m\pi)sin(2m\epsilon)}$ Then, $\mathcal{K}(f(m)) = 2m(\pi + \epsilon) \times \frac{sin(2m\epsilon)cos(2m\pi)}{cos(2m\pi)cos(2m\epsilon)} = 2m(\pi + \epsilon) \times tan(2m\epsilon)$
Then, as m increasing, $\mathcal{K}(f(m))$ increases as well; therefore, the question is ill-condition for big m.

4. (a) $\mathcal{K}(f(x)) = \frac{xf'(x)}{f(x)} = \frac{x(\frac{1-cos(x)}{sin(x)})'}{\frac{1-cos(x)}{sin(x)}} = \frac{x\frac{1-cos(x)}{sin^2(x)}}{\frac{1-cos(x)}{sin(x)}} = \frac{x}{sin(x)}$.



We can see that $1 \leq \frac{x}{\sin(x)} \leq \frac{\pi}{2}$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$; hence, $f(x)$ is well condition. However, we notice that $\lim_{x\to 0} cos(x) = 1$ causes catastrophic cancellation for the numerator, which means the algorithm is unstable.

(b) Alternative algorithm: $f(x) = \frac{1-cos(x)}{sin(x)} \times \frac{1+cos(x)}{1+cos(x)} = \frac{sin(x)}{1+cos(x)}$.
Clearly this avoid catastrophic cancellation, when x is close to 1.
Python Program:

```
import math
def q4():
    left = - math.pi / 2
    right = math.pi / 2
    if (x == 0):
        return 0
    elif (x == 0):
        numerator = math.sin(x)
        denominator = math.cos(x) + 1
      return numerator / denominator
    else:
      return " not define"
```