

Duality in Linear Programming

Learning Goals.

- Introduce the Dual Linear Program.
- Widget Example and Graphical Solution.
- Basic Theory:
 - Mutual Bound Theorem.
 - Duality Theorem.

Readings: Read text section 11.6, and sections 1 and 2 of Tom Ferguson's notes (see course homepage).

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Standard Form for Linear Programs: Review

Consider a real-valued, unknown, n -vector $x = (x_1, x_2, \dots, x_n)^T$.

A linear programming problem in **standard form** (A, b, c) has the three components:

Objective Function: We wish to choose x to maximize:

$$c^T x = c_1 x_1 + c_2 x_2 + \dots + c_n x_n \quad \text{Linear function of } x$$

with x subject to the following constraints:

Problem Constraints: For an $m \times n$ matrix A , and an $m \times 1$ vector b :

$$Ax \leq b$$

Non-negativity Constraints:

$$x \geq 0$$

Linear inequality constraints on x

Notation: For two K -vectors x and y ,
 $x \leq y$ iff $x_k \leq y_k$ for each $k = 1, 2, \dots, K$.
 Other inequalities (\leq , etc.) defined similarly.

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Widget Factory Example: Revisited

Widget problem in Standard Form, constants (A, b, c).

Unknowns:

$x = (x_1, x_2)^T$ number (in thousands) of the two widget types.

Objective function (profit):

$$c^T x = c_1 x_1 + c_2 x_2 = x_1 + 2x_2, \text{ so } c^T = (c_1, c_2) = (1, 2).$$

Problem Constraints: $Ax \leq b$

$$\begin{aligned} x_1 + x_2 &\leq 4, \\ -x_1 + x_2 &\leq 1, \text{ so } A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ -3 & 10 \end{pmatrix}, b = \begin{pmatrix} 4 \\ 1 \\ 15 \end{pmatrix} \\ -3x_1 + 10x_2 &\leq 15. \end{aligned}$$

Non-negativity Constraints:

$$x \geq 0$$

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Widget Factory Example: Upper Bounds

Maximize profit: $c^T x$, where $c^T = (c_1, c_2) = (1, 2)$.

Subject to: $Ax \leq b$ and $x \geq 0$.

Notice, for any feasible x and any $y = (y_1, y_2, y_3)^T \geq 0$:

$$y^T A x = y^T \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ -3 & 10 \end{pmatrix} x \leq y^T b = y^T \begin{pmatrix} 4 \\ 1 \\ 15 \end{pmatrix}.$$

Could choose $y^T A \geq c^T$ and $y \geq 0$.

E.g., $y = (2, 0, 0)^T$ gives $y^T A = (2, 2) \geq c^T$. Therefore:

$$c^T x \leq y^T A x \leq y^T b = 2b_1 = 8, \text{ i.e. } \max \text{ profit } c^T x \leq 8.$$

Upper bound!

In general, for any feasible x and any y such that

$$\begin{aligned} y \geq 0 \text{ and } y^T A \geq c^T, \\ \text{we have the inequality:} \\ c^T x \leq y^T A x \leq y^T b. \end{aligned}$$

Feasible y

Dual LP

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Duality in Linear Programming

Defn. Consider the linear programming problem (in standard form):

$$\begin{aligned} & \text{maximize } c^T x \\ & \text{subject to } Ax \leq b \text{ and } x \geq 0, \end{aligned} \quad (1)$$

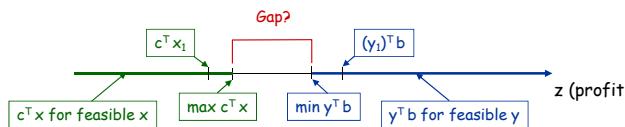
The dual of this LP problem is the LP minimization problem:

$$\begin{aligned} & \text{minimize } y^T b \\ & \text{subject to } y^T A \geq c^T \text{ and } y \geq 0. \end{aligned} \quad (2)$$

These two LP problems are said to be **duals** of each other.

Mutual Bound Theorem: If x is a feasible solution of LP (1) and y is a feasible solution of LP (2), then $c^T x \leq y^T A x \leq y^T b$.

Pf: See previous slide.



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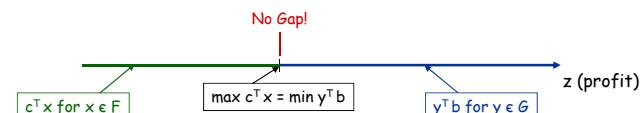
Duality Theorem of Linear Programming

LP Duality Theorem: Consider the linear programming problem:

$$\begin{aligned} & \text{maximize } c^T x \\ & \text{subject to } Ax \leq b \text{ and } x \geq 0. \end{aligned} \quad (1)$$

The feasible set F for (1) is not empty and $c^T x$ is bounded above for $x \in F$ if and only if the corresponding dual LP (2) (above) has a non-empty feasible set $G = \{y | y^T A \geq c^T \text{ and } y \geq 0\}$ and $y^T b$ is bounded below for $y \in G$.

Moreover, in this case, $\max \{c^T x | x \in F\} = \min \{y^T b | y \in G\}$.



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Vertices of Dual Linear Program

Consider the **Dual LP** problem:

$$\begin{aligned} & \text{minimize } y^T b \\ & \text{subject to } y^T A \geq c^T \text{ and } y \geq 0. \end{aligned} \quad (2)$$

We can rewrite the feasibility conditions (2) of the dual as

$$y^T D \equiv y^T \begin{pmatrix} A & I \end{pmatrix} \geq d^T \equiv \begin{pmatrix} c^T & 0^T \end{pmatrix}. \quad (3)$$

The dual LP is an LP, and vertices can be defined the same way as we did before.

Let $t = \{t_1, t_2, \dots, t_m\}$ be a selection of m columns of (3), $1 \leq t_i \leq m+n$. Define $E(t)$ to be the $m \times m$ matrix formed from the t -columns of D , and $e^T(t)$ the $(1 \times m)$ -vector formed from the same columns of d^T .

A point $v \in \mathbb{R}^m$ is a vertex of the feasible set (3) iff there exists an t such that $E(t)$ is nonsingular, $v^T = e^T(t)[E(t)]^{-1}$, and v satisfies (3).

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Vertices of LP and Dual LP

Define the $m+n$ dimensional binary valued indicator vector $\delta(s)$ where $\delta_j = 1$ if $j \in s$, and $\delta_j = 0$ otherwise. Define $\delta(t)$ similarly.

$$\begin{aligned} \delta(s) &= (b_1, b_2, \dots, b_m, b_{m+1}, \dots, b_{m+n}), \\ \delta(t) &= (a_1, \dots, a_n, a_{n+1}, a_{n+2}, \dots, a_{n+m}). \end{aligned}$$

Vertex of LP: If the j^{th} coefficient of $\delta(s)$ is one (i.e., $[\delta(s)]_j = 1$) then the j^{th} row below is an equality for vertex x :

$$Px \equiv \begin{pmatrix} A \\ -I \end{pmatrix} x \leq p \equiv \begin{pmatrix} b \\ 0 \end{pmatrix}.$$

Vertex of Dual LP: If the i^{th} coefficient of $\delta(t)$ is one (i.e., $[\delta(t)]_i = 1$) then the i^{th} column below is an equality for vertex y :

$$y^T D \equiv y^T \begin{pmatrix} A & I \end{pmatrix} \geq d^T \equiv \begin{pmatrix} c^T & 0^T \end{pmatrix}.$$

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Complementary Slackness

Complementary Slackness: Given feasible solutions x and y of the LP and the dual LP, respectively. Then x and y are optimal iff

$$\sum_{j=1}^n A_{i,j}x_j < b_i \text{ implies } y_i = 0,$$

and

$$\sum_{i=1}^m y_i A_{i,j} > c_j \text{ implies } x_j = 0.$$

Pf: Follows from $c^T x = y^T A x = y^T b$ as a necessary and sufficient condition for the optimality of the feasible solutions x and y .

Suggests choosing of the sets s and t (defining the vertex x of the LP and the vertex y of the dual LP) such that the bit vectors satisfy:

$$[\delta(s)]_i = \text{not } [\delta(t)]_{i+n}, \\ [\delta(t)]_j = \text{not } [\delta(s)]_{j+m}.$$

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Proposing Vertices for the Dual LP

Spatially, complementary slackness suggests:

$$\delta(s) = (\alpha_1, \alpha_2, \dots, \alpha_m, \quad \alpha_{m+1}, \dots, \alpha_{m+n}), \\ \delta(t) = (\beta_1, \dots, \beta_n, \quad \beta_{n+1}, \beta_{n+2}, \dots, \beta_{n+m}).$$

Where $\beta_{i+n} = \text{bitFlip}(\alpha_i)$ for $i = 1, 2, \dots, m$.
And $\beta_j = \text{bitFlip}(\alpha_{j+m})$ for $j = 1, 2, \dots, n$.

Since $\text{sum}(\delta(s)) = n$, $\text{length}(\delta(\cdot)) = n+m$, it follows $\text{sum}(\delta(t)) = m$.

Given a vertex x of the LP, defined by s , we can use the rule above to try to construct t and the corresponding vertex of the dual LP. We can use the pair to check for optimality. See the following example.

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Graphing the Widget Factory Example: Cont.

Example: $x = (x_1, x_2)^T$. Linear Program specified by (A, b, c) .

Objective Function: $c^T x$,
 $c = (1, 2)^T$

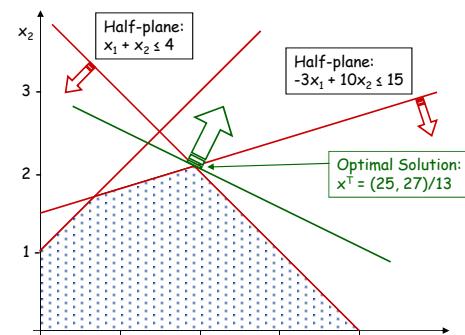
Problem Constraints:
 $Ax \leq b$,

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ -3 & 10 \end{pmatrix},$$

$$b = \begin{pmatrix} 4 \\ 1 \\ 15 \end{pmatrix}.$$

Non-negativity:
 $x \geq 0$.

Optimal Vertex:
 $s = \{1, 3\}$



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Widget Factory Example: Optimal Dual Solution

E.G. (Cont.): This vertex of the LP was obtained using $s = \{1, 3\}$. Generate corresponding column selection t (possibly a feasible vertex for dual LP):

$$\delta(s) = (1, 0, 1, 0, 0)$$

$$\delta(t) = (1, 1, 0, 1, 0)$$

So $t = \{1, 2, 4\}$ and we select columns 1, 2, and 4 from (3) below.

$$y^T (A^T I) \geq y^T (c^T 0^T). \quad (3)$$

$$y^T \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ -3 & 10 & 0 \end{pmatrix} = (1 \ 2 \ 0). \quad A^T = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ -3 & 10 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 1 \\ 15 \end{pmatrix}, \\ c^T = (1 \ 2).$$

Soln: $y^T = (16, 0, 1)/13$.

Check: $c^T x = (1, 2)(25, 27)^T/13 = 79/13 = y^T b = (16, 0, 1)/13(4, 1, 15)^T$

Conclude: y is a feasible vertex of the dual LP, and x, y are optimal.

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