CSC438F/2404F **Solutions to Problem Set 1** **Fall, 2015**

Due: Friday, October 2, beginning of tutorial

NOTE: Each problem set counts 15% of your mark, and it is important to do your own work. You may consult with others concerning the general approach for solving problems on assignments, but you must write up all solutions entirely on your own. Copying assignments is a serious academic offence and will be dealt with accordingly.

1. Suppose that we allow $\oplus$ (exclusive OR) as a primitive connective in building propositional formulas. Give the appropriate left and right introduction rules for $\oplus$. The rules should be in the same style as the introduction rules for $\wedge, \vee, \neg$ (the formulas in the bottom sequent are constructed from formulas in the top sequent by sometimes adding $\oplus$). Make sure your rules satisfy the Sequent Soundness Principle (page 11 of the Notes) and the Inversion Principle (page 13).

   Solution:
   By definition, $\tau(P \oplus Q) = T$ iff exactly one of $\{\tau(P), \tau(Q)\}$ is T. Here are the rules:

$$\textbf{left}\ \frac{A, \Gamma \rightarrow \Delta, B \qquad B, \Gamma \rightarrow \Delta, A}{(A \oplus B), \Gamma \rightarrow \Delta} \qquad\qquad \textbf{right}\ \frac{A, B, \Gamma \rightarrow \Delta \qquad \Gamma \rightarrow \Delta, A, B}{\Gamma \rightarrow \Delta, (A \oplus B)}$$

2. Consider the algorithm on page 9 of the Notes which, given a set S of propositional clauses either outputs a satisfying assignment for S, or outputs a resolution refutation of S. Show that if every clause in S has at most two literals, then the procedure halts in time polynomial in the size of the input S.

   Solution:
   As explained in the introductory paragraph, each time Step 2 is is reached, no clause in S' is falsified by τ. Hence each time Step 4 is executed the resolvant R that is added to the set S' is new. Since the resolvant of two clauses with at most two literals each has at most two literals, it follows that there can be at most $O(n^2)$ clauses in S', where n is the number of input clauses. Between successive executions of Step 4, the stack of literals never decreases (and increases at least every second time Step 2 is executed). Hence there are at most $O(n)$ steps between successive additions of clauses to S', and so the algorithm halts in at most $O(n^3)$ steps.

   It remains to show that each step can be executed in polynomial time. This is easy to see, since there are $O(n)$ literals on the stack and $O(n^2)$ clauses in the set S'.

3. Do Excercise 14, page 17 of the Notes. (This involves using propositional formulas to express colourability of graphs.)

   Solution:
   (a) We are to give a propositional formula A_n using the variables $\{R_i, B_i, Y_i \mid 0 \leq i < n\}$ such that A_n is satisfiable iff the graph $G_n = (V_n, E_n)$ is 3-colourable.

We define $A_n = (C_n \wedge D_n)$, where C_n ensures that every vertex in V_n is assigned at least one of the colours R, B, Y, and D_n ensures that no two vertices connected by an edge have the same colour.

Let $C_n = \bigwedge_{i=0}^{n-1}(R_i \vee B_i \vee Y_i)$.

For each edge $e = (i, j)$ in G_n let

$$D'_{e,n} = (\neg R_i \vee \neg R_j) \wedge (\neg B_i \vee \neg B_j) \wedge (\neg Y_i \vee \neg Y_j)$$

Let $D_n = \bigwedge_{e \in E} D'_{e,n}$.

(b) Here G is a graph on the vertex set the natural numbers, and G_n is the induced subgraph of G on the vertex set $V_n = \{0, 1, \dots, n-1\}$.

Assume that G_n has a 3-colouring for all $n > 1$. Then it follows from (a) that A_n is satisfiable for all $n > 1$. Notice that any 3-colouring of G_n is also a 3-colouring of $G_2, \dots G_{n-1}$ and hence each set of formulas $\{A_2, \dots, A_n\}$ is satisfiable.

Let S be the infinite set of formulas $\{A_2, A_3, \dots\}$. Then every finite subset of S is satisfiable. By the propositional compactness theorem, it follows that S is satisfiable. But any truth assignment satisfying S provides a three colouring of G. Therefore G has a 3-colouring.

4. Exercise 9, page 25 in the Notes: Give a predicate calculus sentence A with vocabulary consisting of $=$ and a binary predicate symbol R such that for all positive integers n, A has a model whose universe has n elements if and only if n is even. Justify your answer. (*Hint*: think of R as a pairing relation.)

   Solution:
   Let $A = \forall x \exists y (x \neq y \wedge R(x, y) \wedge R(y, x) \wedge \forall z (z \neq y \supset \neg R(x, z)))$

   Suppose n is even, so $n = 2m$. We define a model $\mathcal{M}$ for A as follows.

   Let the universe $M = \{1, 2, \dots, n\}$. Define $R^{\mathcal{M}}(a, b)$ to hold iff $a \leq m$ and $b = a + m$ or $b \leq m$ and $a = b + m$. Then it is easy to check that $\mathcal{M}$ satisfies A

   Conversely, suppose that $\mathcal{M}$ is a structure with a finite universe M which satisfies A. Then M must be partitioned into a disjoint set of pairs $\{a, b\}$ such that $R^{\mathcal{M}}(a, b)$ and $R^{\mathcal{M}}(b, a)$. Therefore the cardinality of M must be even.

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Practice Exercises (Do not hand in):

- Exercise 6 page 12: Find PK proofs.
- Exercise 11 page 16: Prove the equivalence of the three formulations of the Propositional Compactness Theorem.
- Exercise 1, page 19 in the Notes: Prove the unique readability of terms.