Rui Ji (1000340918)

1. PROOF:

$\Rightarrow$ Suppose $f(\vec{x})$ is a *n-ary* computable function. Then there exist some program $\{e\}$ computes f, by KNFT we get,

$$f(\vec{x}) = \{e\}(\vec{x}) = U(\mu y T_n(e, \vec{x}, y))$$

Let $graph(f) = \{(\vec{x}, y) \mid y = f(\vec{x})\}$, then we can get that,

$$(\vec{x}, y) = (x_1, x_2, ..., x_n, y) \in graph(f) \Leftrightarrow \exists z (T_n(e, \vec{x}, z) \wedge U(z) = y)$$

Hence, let $R(x_1, x_2, ..., x_n, y, z) = T_n(e, x_1, x_2, ..., x_n, y, z) \wedge U(z) = y)$. Clearly R is a recursive relation, then $graph(f)$ is r.e.

$\Leftarrow$ Before formalizing the argument for 'if' direction, let's first give an informal algorithm for computing f from an enumeration of the tuples in $graph(f)$

algorithm (for computing $f(\vec{x})$):

suppose $(a_1, a_2 ....)$ is a enumeration, where a_i is a $n+1$ tuple.
for $i = 1 ... \infty$
  if $a_i = (x_1, x_2, ..., x_n, y)$ where $(x_1, x_2, ..., x_n) = \vec{x}$
    OUTPUT y
  end if
end for

Now let's formalize the argument. Suppose $graph(f) = \{(\vec{x}, y) \mid y = f(\vec{x})\}$ is r.e., where f is a $n - ary$ function and we want to show that $f(\vec{x})$ is computable.

Since graph(f) is r.e., then there exist some $n + 2$-*ary* recursive relation $R(\vec{x}, z)$such that,

$$(x_1, x_2, ..., x_n, y) \in graph(f) \Leftrightarrow \exists z \, R(x_1, x_2, ..., x_n, y, z)$$

Then clearly $f(x_1, x_2, ..., x_n) = \mu y(\exists z \, R(x_1, x_2, ..., x_n, y, z))$, by Church-Turing Thesis, $f(x_1, x_2, ..., x_n)$ is computable.

□

2. CLAIM: A is neither recursive nor r.e., A^c is r.e but not recursive.

First we show that A is not r.e. Notice, it suffices to show that

$$K^c \leq_m A$$

Thus we want a total computable function $f(x)$ such that

$$x \in K^c \Leftrightarrow f(x) \in A$$

i.e we want

$$\{x\}_1(x) = \infty \Leftrightarrow range(\{f(x)\}_1) \subseteq ODD$$

We can define $f(x)$ implicitly using the S-m-n Theorem as follows:

$$\{f(x)\}_1(y) = \begin{cases} 0 \cdot \{x\}_1(x) & \text{if } y \neq 1 \\ 1 & \text{if } y = 1 \end{cases}$$

Thus if $\{x\}_1(x)$ is defined, then $range(\{f(x)\}_1) = \{0,1\} \not\subseteq ODD$.

But if $\{x\}_1(x)$ is undefined then $range(\{f(x)\}_1) = \{1\} \subseteq ODD$.

Hence, A is not r.e.

Now we we want to show that A^c is is r.e.

$$A^c = \{x | range(\{x\}_1) \not\subseteq ODD\}$$

Notice that $x \in A^c$ iff there is some input u and some v such that v codes a halting computation of program $\{x\}$ on input u, and $U(v)$ is not a odd. Using the T-predicate, we have,

$$x \in A^c \leftrightarrow \exists u\ \exists v\ [T(x,u,v) \wedge EVEN(U(v))]$$

$$EVEN(x) = \exists y \leq x\ \ 2y = x$$

using a pairing function to combine both existential quantifiers into one quantifier,

$$x \in A^c \leftrightarrow \exists z\ [T(x,K(z),L(z)) \wedge EVEN(U(L(z)))]$$

We get the form $x \in A^c \leftrightarrow \exists z\ R(x,z)$ where R is recursive. Hence A^c is r.e.

It follows that A is not recursive, and hence A^c is not recursive. It also follows that A is not r.e., because then A would be recursive.

□

3. SOLUTION:

Let $R(x,y)$ be a recursive relation, define $A = \{x | f(x)\ is\ defined\}$, where $f(x) = \mu y R(x,y)$.

Then we have,

$$x \in A \Leftrightarrow f(x)\ is\ defined \Leftrightarrow \exists y\ R(x,y)$$

Clearly f is computable. Suppose some program $\{e\}$ computes f, then using KNFT we get,

$$f(x) = \{e\}_1(x) = U(\mu z T(e,x,z))$$

Then clearly $f(x)\ is\ defined \Leftrightarrow \exists z\ T(e,x,z)$

let $S(x,z) = T(e,x,z)$, clearly $S(x,z)$ is primative recursive and $\exists y\ R(x,y) \Leftrightarrow \exists z\ T(e,x,z)$.

□

4. PROOF:

Now consider the case $\exists\ \leq$, say A is $\exists x \leq t\ B(x)$, and this is in **TA**. Since this is a sentence, and by definition of $\exists x \leq t$, we know x cannot occur in t, it follows that t is a closed term. Thus by Lemma A, **RA** can prove $t = s_n$ for some n.

Now we do a induction on n

Base Case: n=0, then we have $\exists x \leq 0\ B(x)$. By axiom $P7$ we have $x = 0$, then $\exists x \leq 0\ B(x)$ is in **TA** means $B(0)$ is true. So by the induction hypothesis $B(0)$ is in $\boldsymbol{RA}_{\leq}$.

Induction Hypothesis: for some k, $\exists x \leq s_k\ B(x)$ is in TA then is also in $\boldsymbol{RA}_{\leq}$.

Inductive Step: $\exists x \leq s(s_k)\ B(x)$ is in **TA** . By $P8$ we know that $x \leq s(s_k) \supset (x \leq s_k \vee x = s(s_k))$. Hence we have,

$$\exists x \leq s(s_k)\ B(x) \Leftrightarrow \exists x \leq s_k\ B(x) \vee B(s(s_k))$$

Then $\exists x \leq s(s_k)\ B(x)$ is in **TA** means $\exists x \leq s_k\ B(x)$ is in **TA** or $B(S(s_k))$ is in **TA**. By induction hypothesis, we know $\exists x \leq s(s_k)\ B(x)$ is in $\boldsymbol{RA}_{\leq}$.

Therefore, if $\exists x \leq t\ B(x)$ is in **TA** then it is also in $\boldsymbol{RA}_{\leq}$.

□

5. PROOF: In order to show that $\forall x \forall y\ \ x+y = y+x$, we first show $\forall x\ \ 0+x = x$ and $\forall x \forall y\ \ sx+y = s(x+y)$, then using them to prove $\forall x \forall y\ \ x+y = y+x$.

First let's show $\forall x\ \ 0+x = x$, we call this property $A1$.

$$f(x) = 0 + x = x$$

We use the induction axiom $Ind(f(x))$.
Basis: x = 0

$$0 + 0 = 0 \qquad P3$$

Inductive Step: $x \leftarrow sx$

$$\begin{aligned} 0 + sx &= s(0 + x) && P4 \\ &= sx && Inductive\ Hypothesis \end{aligned}$$

Thus by $Ind(f(x))$, it follows that

$$\mathbf{PA} \vdash \forall x\ f(x)$$

Then let's show $\forall x \forall y\ \ sx + y = s(x + y)$ and we call this property $A2$.

$$g(y) = sx + y = s(x + y)$$

We use the induction axiom $Ind(g(y))$.
Basis: y = 0

$$\begin{aligned} sx + 0 &= sx && P3 \\ &= s(x + 0) && P3 \end{aligned}$$

Inductive Step: $y \leftarrow sy$

$$\begin{aligned} sx + sy &= s(sx + y) && P4 \\ &= s(s(x + y)) && Inductive\ Hypothesis \\ &= s(x + sy)) && P4 \end{aligned}$$

Thus by $Ind(g(y))$, it follows that

$$\mathbf{PA} \vdash \forall x\ \forall y\ g(y)$$

Finally let's show $\forall x \forall y \;\; x + y = y + x$

$$A(y) = x + y = y + x$$

We use the induction axiom $Ind(A(y))$.
Basis: y = 0

$$\begin{aligned} x + 0 &= x && P3 \\ &= 0 + x && A1 \end{aligned}$$

Inductive Step: $y \leftarrow sy$

$$\begin{aligned} x + sy &= s(x + y) && P4 \\ &= s(y + x) && Inductive\ Hypothesis \\ &= sy + x && A2 \end{aligned}$$

Thus by $Ind(A(y))$, it follows that

$$\mathbf{PA} \vdash \forall x \; \forall y \; A(y)$$

$\square$