

Homework 6 Sample Solutions

1.

Theorem 1. *Satisfiability of existential equality formulas is NP-complete.*

Proof. To show the NP-hardness, we show a reduction from propositional satisfiability. Suppose we are given a propositional formula φ over the propositions p_1, \dots, p_k . We introduce individual variables $x_1, \dots, x_k, y_1, \dots, y_k$. We now obtain the formula ψ by replacing p_i in φ by the atomic formula $x_i \approx y_i$. It can then be seen that φ is satisfiable iff $(\exists x_1) \dots (\exists x_k)(\exists y_1) \dots (\exists y_k)(\psi)$ is satisfiable. Given an A, α that satisfies ψ , a satisfying truth assignment for φ is $\{p_i \mid \alpha(x_i) = \alpha(y_i)\}$. Given a truth assignment a that satisfies φ , we construct the structure over two elements $A = \{0, 1\}$ and variable assignment α where $\alpha(x_i) = 0$ and $\alpha(y_i) = \begin{cases} 0 & \text{when } p_i \in a, \\ 1 & \text{when } p_i \notin a. \end{cases}$

To show that satisfiability of existential equality formulas is in NP, we note that if the formula $\exists x_1 \dots \exists x_n \varphi$ is satisfiable, then it is satisfiable in a domain A with n elements, and we must guess the polynomially sized witness α which is an assignment to all variables in φ such that $A, \alpha \models \varphi$. To check the witness, we only need to compute $\varphi^A(a)$, which takes polynomial time.

Thus the satisfiability of equality formulas is NP-complete. \square

2.

Theorem 2. *φ is satisfiable iff $\exists * \varphi$ is satisfiable and φ is valid iff $\forall * \varphi$ is valid.*

Proof. φ is satisfiable iff there is some A, α where $A, \alpha \models \varphi$. Similarly, $A \models \exists * \varphi$, holds precisely when we can find domain elements $a_1 \dots a_n$ so that $A, [x_1 \mapsto a_1, \dots, x_n \mapsto a_n] \models \varphi$. If φ is satisfiable, then $\alpha(x_1) \dots \alpha(x_n)$ are precisely the domain elements that render $\exists * \varphi$ satisfiable. Similarly, if $\exists * \varphi$ is satisfiable, then $A, [x_1 \mapsto a_1, \dots, x_n \mapsto a_n] \models \varphi$ and φ is satisfiable.

Since φ valid iff $\neg \varphi$ not satisfiable and $\forall * \varphi$ valid iff $\neg \exists * \varphi$ not satisfiable, the above proof applies to the universal case also. \square

3. There are 2 possibilities: either Q is universal or Q is not universal. If Q is universal, then either P or Q are universal, and we are done. We have to consider the case when Q is not universal then.

Assume Q is not universal. Since Q is both *transitive* and *symmetric*, but there are at least two pairs (a, b) and (b, a) it does not contain, Q is composed of closed disjoint sets Q_i where each $Q_i = D_i^2$, and $D_i \cap D_j = \emptyset$ for $i \neq j$ (that is, if we think of Q as defining a graph, it has a number of connected components -each a clique- but disjoint from each other).

Now, since $P \cup Q$ is universal, P must contain at least all tuples not in Q . These include:

- All pairs (a, b) and (b, a) s.t. $a \in D_i, b \in D_j$ with $i \neq j$ (i.e. links between the connected components of Q)
- All other pairs (c, d) and (d, c) s.t. $c, d \in D - \cup_i D_i$ (the clique among elements of D not in Q)
- All pairs (e, f) and (f, e) s.t. $e \in Q_i, f \in D - \cup_i D_i$ (all connections to/from the components of Q and the rest of D)

Thus, under P , every element $a \in D$ has a "transitive path" to every other element $b \in D$ of the form $(a, x_1), (x_1, x_2), \dots, (x_{n-1}, x_n), (x_n, b)$ (i.e. if thinking of a graph, every element is reachable from every other element). But since P is *transitive*, then there is a tuple (a, b) for every pair $a, b \in D$. Thus P is universal.

This theorem can be written as a F.O. sentence in the following way. Let:

$$\begin{aligned}
 \text{transitive}_Q &= (\forall x)(\forall y)(\forall z)(Q(x, y) \wedge Q(y, z) \rightarrow Q(x, z)) \\
 \text{transitive}_P &= (\forall x)(\forall y)(\forall z)(P(x, y) \wedge P(y, z) \rightarrow P(x, z)) \\
 \text{symmetric}_Q &= (\forall x)(\forall y)(Q(x, y) \rightarrow Q(y, x)) \\
 \text{universal}_Q &= (\forall x)(\forall y)Q(x, y) \\
 \text{universal}_P &= (\forall x)(\forall y)P(x, y) \\
 \text{universal}_{P \cup Q} &= (\forall x)(\forall y)(Q(x, y) \vee P(x, y))
 \end{aligned}$$

Then, the theorem can be written as:

$$(\text{transitive}_Q \wedge \text{transitive}_P \wedge \text{symmetric}_Q \wedge \text{universal}_{P \cup Q}) \rightarrow (\text{universal}_Q \vee \text{universal}_P)$$

4. Exercise 17(a) on p. 96.

17. (a) Consider a language with equality whose only parameter (aside from \forall) is a two-place predicate symbol P . Show that if $\mathcal{U} \equiv \mathcal{B}$, then \mathcal{U} is isomorphic to \mathcal{B} .

Since, U is finite, take $n = \text{size}(|U|)$. The predicate P is also finite. For $Q \subseteq |U|^2$, we can generate a formula which is true when P is an isomorphism of Q and false if not.

$$\varphi_Q = (\bigwedge_{i=1..n} \bigwedge_{j \in \{1..n\} - i} (\neg(x_i \approx x_j))) \wedge (\bigwedge_{\langle i,j \rangle \in Q} P(x_i, x_j) \wedge \bigwedge_{\langle i,j \rangle \notin Q} \neg P(x_i, x_j))$$

If $U, \alpha \models \varphi_Q$, Each x_i in α is assigned to a different member of $|U|$ by the first part of the formula. So, the formula is satisfiable iff there is a bijective mapping from $1..n$ to $|U|$, and α is the mapping. If $U \equiv B$, take the only φ_Q which is satisfiable on U , and thus on B . $U, \alpha \models \varphi_Q$, $B, \beta \models \varphi_Q$. And both α and β are bijective mappings. So $f = \alpha \circ \beta^{-1}$ is a bijection from $|U|$ to $|B|$ and $P^U(x_i, x_j)$ is true iff $P_B(f(x_i), f(x_j))$ and the same applies to f^{-1} . So U is isomorphic to B .

5. Let \mathcal{A} and \mathcal{B} be structures over the same vocabulary. A weak homomorphism from \mathcal{A} onto \mathcal{B} is a mapping (not necessarily one-to-one, but onto) $h : D^{\mathcal{A}} \rightarrow D^{\mathcal{B}}$ such that the following hold:

- For each k -ary predicate symbol P and a k -tuple $\langle a_1, \dots, a_k \rangle$ of elements of \mathcal{A} , we have that if $\langle a_1, \dots, a_k \rangle \in P^{\mathcal{A}}$ then $\langle h(a_1), \dots, h(a_k) \rangle \in P^{\mathcal{B}}$.
- For each k -ary function symbol f and a k -tuple $\langle a_1, \dots, a_k \rangle$ of elements of \mathcal{A} , we have that

$$h(f^{\mathcal{A}}(a_1, \dots, a_k)) = f^{\mathcal{B}}(h(a_1), \dots, h(a_k)).$$

Let h be a weak homomorphism from \mathcal{A} onto \mathcal{B} , and let φ be a positive formula, i.e., a formula built-up from atomic formulas using only the connectives \wedge and \vee and the quantifiers \forall and \exists (the connective \neg cannot be used). Let α be a variable assignment on \mathcal{A} . Show that if $\mathcal{A}, \alpha \models \varphi$ then $\mathcal{B}, h \circ \alpha \models \varphi$.

Proof by induction on a formula φ :

- $\varphi = P(t_1, \dots, t_k)$

$$\text{Since } \mathcal{A}, \alpha \models \varphi \quad \text{then} \quad \langle \bar{\alpha}(t_1), \dots, \bar{\alpha}(t_k) \rangle \in P^{\mathcal{A}}$$

$$\text{From the weak homomorphism } h, \quad \text{then} \quad \langle h \circ \bar{\alpha}(t_1), \dots, h \circ \bar{\alpha}(t_k) \rangle \in P^{\mathcal{B}}$$

$$\text{and by definition} \quad \mathcal{B}, h \circ \alpha \models \varphi$$

- $\varphi = (\theta \wedge \psi)$

$$\text{Since } \mathcal{A}, \alpha \models \varphi \quad \text{then} \quad \mathcal{A}, \alpha \models \theta \text{ and } \mathcal{A}, \alpha \models \psi$$

$$\text{By I.H.} \quad \text{then} \quad \mathcal{B}, h \circ \alpha \models \theta \text{ and } \mathcal{B}, h \circ \alpha \models \psi$$

$$\text{and by definition} \quad \mathcal{B}, h \circ \alpha \models (\theta \wedge \psi) = \varphi$$

- $\varphi = (\theta \vee \psi)$

Since $A, \alpha \models \varphi$ then $A, \alpha \models \theta$ or $A, \alpha \models \psi$
 By I.H. then $B, h \circ \alpha \models \theta$ or $B, h \circ \alpha \models \psi$
 and by definition $B, h \circ \alpha \models (\theta \vee \psi) = \varphi$

- $\varphi = (\forall x)\theta$

Since $A, \alpha \models \varphi$ then for all $a \in D^A$ $A, \alpha[x \mapsto a] \models \theta$
 By I.H. then for all $a \in D^A$ $B, h(\alpha[x \mapsto a]) \models \theta$
 From the weak homomorphism $h \circ (\alpha[x \mapsto a]) = (h \circ \alpha)[x \mapsto h(a)]$
 Since h is ONTO D^B , this holds for all $b = h(a) \in D^B$
 so For all $b \in D^B$ $B, (h \circ \alpha)[x \mapsto b] \models \theta$
 so $B, h \circ \alpha \models \varphi$

- $\varphi = (\exists x)\theta$

Since $A, \alpha \models \varphi$ then there is some $a \in D^A$ s.t. $A, \alpha[x \mapsto a] \models \theta$
 By I.H. then there is some $a \in D^A$ s.t. $B, h(\alpha[x \mapsto a]) \models \theta$
 From the weak homomorphism $h \circ (\alpha[x \mapsto a]) = (h \circ \alpha)[x \mapsto h(a)]$
 Since h is ONTO D^B , there is some $b = h(a) \in D^B$
 for which this holds
 so There is some $b \in D^B$ s.t. $B, (h \circ \alpha)[x \mapsto b] \models \theta$
 so $B, h \circ \alpha \models \varphi$

6. Compactness:

- (a) Let Σ be a set of formulas and let α be a formula. Show that $\Sigma \models \alpha$ iff there is a finite subset Σ' of Σ such that $\Sigma' \models \alpha$.

Lemma. Let Δ be a set of formulas and φ be a formula. Then, $\Delta \models \varphi$ iff $\Delta \cup \{\neg\varphi\}$ is UNSAT.

Proof:

$\Delta \models \varphi$
 iff for all A, α , if $A, \alpha \models \Delta$ then $A, \alpha \models \varphi$
 iff for all A, α , $(A, \alpha \not\models \Delta \text{ or } A, \alpha \models \varphi)$
 iff for all A, α , $(A, \alpha \not\models \Delta \text{ or } A, \alpha \models \neg\varphi)$
 iff for all A, α , $(A, \alpha \not\models \Delta \cup \{\neg\varphi\})$
 iff $\Delta \cup \{\neg\varphi\}$ is UNSAT.

Now to the main result:

- $$\Sigma \models \alpha$$
- iff $\Sigma \cup \{\alpha\}$ is UNSAT
- iff there exists finite $\Sigma' \subseteq \Sigma$ such that $\Sigma' \cup \{\alpha\}$ is UNSAT
- iff there exists finite $\Sigma' \subseteq \Sigma$ such that $\Sigma' \models \alpha$.

- (b) Let $G = (V, E)$ be a (possibly infinite) graph. $G' = (V', E')$ is a finite subgraph of G if V' and E' are finite subsets of V and E , respectively.

Use compactness to prove that G is 3-colorable iff every finite subgraph of G is 3-colorable.

Suppose $G = (V, E)$ is 3-colorable. Then it has a 3-coloring $c : V \rightarrow \{1, 2, 3\}$. If $G' = (V', E')$ is a subgraph of G , then $c|_{V'}$ is a 3-coloring of G' . So every subgraph of G is also 3-colorable, and in particular, every finite subgraph is 3-colorable.

Now suppose every finite subgraph of $G = (V, E)$ is 3-colorable. We use compactness to show that G is also 3-colorable.

Our vocabulary will consist of the following: three unary predicate symbols, C_1, C_2, C_3 , and a constant symbol c_v for every $v \in V$. So if G is infinite, then our vocabulary will also be infinite. We define the following sentences:

$$\theta = (\forall x)[(C_1(x) \wedge \neg C_2(x) \wedge \neg C_3(x)) \vee (\neg C_1(x) \wedge C_2(x) \wedge \neg C_3(x)) \vee (\neg C_1(x) \wedge \neg C_2(x) \wedge C_3(x))]$$

$$\varphi_{(u,v)} = \bigvee_{i \neq j} C_i(c_u) \wedge C_j(c_v)$$

and define the set of sentences Σ as follows:

$$\Sigma = \{\theta\} \cup \{\varphi_{(u,v)} : (u,v) \in E\}$$

Let $V' = \{v_1, v_2, \dots, v_n\}$ be a finite subset of V and let $E' = E \cap (V' \times V')$. Then $G' = (V', E')$ is finite subgraph of G and so G' is 3-colorable. Let $c' : V' \rightarrow \{1, 2, 3\}$ be a 3-coloring of G' . Consider the structure $A = (V', C_1^A, C_2^A, C_3^A, c_{v_1}^A, c_{v_2}^A, \dots, c_{v_n}^A)$, where $C_i^A = \{v \in V' : c'(v) = i\}$ and $c_{v_i}^A = v_i$. Then $\Sigma_{V'} = \{\theta\} \cup \{\varphi_{(u,v)} : (u,v) \in E'\}$ is satisfied by A . Since every finite subset of Σ must be contained in $\Sigma_{V'}$ for some finite subset V' of V , therefore Σ is finitely satisfiable. Therefore, by compactness, Σ is satisfiable.

Let B be a model of Σ . Define the function $f : V \rightarrow \{1, 2, 3\}$ as follows:

$$f(v) = \begin{cases} 1, & c_v^B \in C_1^B \\ 2, & c_v^B \in C_2^B \\ 3, & c_v^B \in C_3^B \end{cases}$$

Since B satisfies θ , each c_v^B lies in exactly one of the C_i^B . So f is well defined as a function. Now consider an arbitrary edge $(u, v) \in E$ of graph G . Since B satisfies $\varphi_{(u,v)}$, we have $f(u) \neq f(v)$. Therefore f is a 3-coloring of G .

- (c) Recall that a set is recursive if there is an algorithm that decides membership in the set. A set is recursively enumerable if there is a procedure that enumerates all elements of the set in some order, perhaps with repetitions.

Let Σ be an exhaustive, recursively enumerable set of formulas. Show that the set $\{\alpha : \Sigma \models \alpha\}$ is recursive (i.e., describe an algorithm for membership in this set).

Let $con(\Sigma) = \{\alpha : \Sigma \models \alpha\}$ be the set of consequences of Σ . We have to show that $con(\Sigma)$ is decidable.

If Σ is unsatisfiable, then it will logically imply every formula. Then $con(\Sigma)$ is simply the set of all formulas which is decidable (the algorithm will only have to check that the input is a valid formula).

If Σ is satisfiable, then for any formula φ , $con(\Sigma)$ cannot contain both φ and $\neg\varphi$. But since Σ is exhaustive and $con(\Sigma)$ contains Σ , therefore $con(\Sigma)$ contains φ or $\neg\varphi$. Thus, $con(\Sigma)$ contains exactly one of φ or $\neg\varphi$, for all formulas φ . Then the following algorithm decides $con(\Sigma)$:

On input φ , begin enumerating all formulas in Σ . If φ is produced then accept. If $\neg\varphi$ is produced then reject.

7. A class \mathcal{C} of structures is EC if $\mathcal{C} = models(\psi)$ for a first-order sentence ψ and is EC_Δ if $\mathcal{C} = models(\Psi)$ for a set Ψ of first-order sentences. Let $3C$ be the class of 3-colorable graphs. Show that $3C$ is EC_Δ . (Hint: Recall that we showed that a graph is 3-colorable iff all its finite subgraphs are 3-colorable.)

$G \in 3C$ iff all finite subgraphs of G are in $3C$. Let G_k be a k -sized finite subgraph of G . G is in $3C$ iff $G_1 \in 3C, G_2 \in 3C$, etc. Now, given a coloration c for G , $G_{n-1} \cup \{v\}$ is 3-colorable if $\neg\exists x, y, z (E(v, x) \wedge E(v, y) \wedge E(v, z) \wedge c(x) \neq c(y) \wedge c(y) \neq c(z) \wedge c(z) \neq c(x))$. The predicate $3colorable_k(G)$ is TRUE if all k -size subgraphs G_k of G are 3-colorable.

- $3colorable_1(G)$ - trivially true
- $3colorable_n(G) = 3colorable_{n-1}(G) \wedge \forall v (\neg\exists x, y, z (E(v, x) \wedge E(v, y) \wedge E(v, z) \wedge \neg(c(x) = c(y)) \wedge \neg(c(y) = c(z)) \wedge \neg(c(z) = c(x))))$

So by 6.(b), $3C = \text{models}(3\text{colorable}_1, 3\text{colorable}_2 \dots \text{etc.})$ or $3C = \text{models}(\Psi)$. Therefore, $3C$ is EC_Δ .

8. *The Infinite Ramsey Theorem says that if $G = (V, E)$ has infinitely many nodes, then there is an infinite subset $V' \subseteq V$ such that either V' is a clique (i.e., every two nodes in V' are connected by an edge) or it is independent (i.e., no two nodes in V' are connected by an edge).*

Use the Compactness Theorem and the Infinite Ramsey Theorem to prove the Finite Ramsey Theorem: For every $n \geq 1$ there is an integer R_n such that if $G = (V, E)$ has at least R_n nodes then there is a set $V' \subseteq V$ with n nodes such that either V' is a clique or it is independent. (Hint: Assume that the Finite Ramsey Theorem fails and use compactness to show that the Infinite Ramsey Theorem fails.)

Suppose that the Finite Ramsey Theorem fails. That is, there is a number $n \geq 1$ such that for every number $m > n$ there is a graph $G_m = (V_m, E)$ with at least m nodes such that every subset $V' \subseteq V_m$ with at least n nodes is neither a clique nor an independent set (i.e., there are two nodes in V' that are connected by an edge and there are two nodes not connected by an edge).

We can express the fact that a graph does not have a subset with n nodes that is either a clique or an independent set by the sentence ψ defined as:

$$(\forall y_1) \dots (\forall y_n) [(\bigwedge_{1 \leq i < j \leq n} y_i \not\approx y_j) \rightarrow \bigvee_{1 \leq i < j \leq n} E(x_i, x_j) \wedge \bigvee_{1 \leq i < j \leq n} \neg E(x_i, x_j)].$$

We can express the fact that a graph has at least m nodes by the sentence θ_m defined as:

$$(\exists x_1) \dots (\exists x_m) \bigwedge_{1 \leq i < j \leq m} x_i \not\approx x_j.$$

Clearly $G_m \models (\psi \wedge \theta_{m'})$ if $m \geq m'$.

Consider the set T of formulas $\{\psi\} \cup \{\theta_m : m > n\}$. We claim that T is finitely satisfiable. Indeed, consider a finite subset $T' \subseteq T$. Let m be maximal such that $\theta_m \in T'$. Then $G_m \models T'$. By compactness, T is satisfiable. Let $G \models T$. Since $G \models \theta_m$ for all $m \geq n$, it follows that G is infinite. Since $G \models \psi$, it follows that G does not have a subset with n nodes that is either a clique or an independent set. This contradicts the Infinite Ramsey Theorem.