UNIVERSITY OF TORONTO
Faculty of Arts and Science
APRIL 2014 EXAMINATIONS

CSC 438H1S/2404H1S

Duration - 3 hours
No Aids Allowed

There are 9 questions worth a total of 100 marks.
Answer all questions on the question paper, using backs of pages for scratch work.
Check that your exam book has 11 pages (including this cover page).

PLEASE COMPLETE THIS SECTION:

Name ______________________________

(Please underline your family name.)

Student Number ______________________________

FOR USE IN MARKING:

1. ____________/10
2. ____________/20
3. ____________/10
4. ____________/8
5. ____________/12
6. ____________/8
7. ____________/8
8. ____________/10
9. ____________/14

Total:__________/100

[10] 1. Let f and g be unary function symbols, and let A be the formula $\forall x fgx = x$ and let B be the formula $\forall x gfx = x$. Prove that $A \not\models B$.

[20] 2. Let PRIMES be the set of prime numbers. Define

$$A = \{x \mid dom(\{x\}_1) \subseteq \text{PRIMES}\}$$

Is A r.e.? Is A^c r.e. Justify your answers. (You may use Church's Thesis.)
DO NOT USE RICE'S THEOREM.

You may continue your solution on the next page.

(Continue your solution to Question 1 here).

[10] 3. Let f be a unary function (not necessarily total). Recall that $graph(f)$ is the relation $R_f(x, y) = (y = f(x))$. Prove that if $graph(f)$ is r.e. then f is recursive. DO NOT USE CHURCH'S THESIS. (Or use Church's thesis for part credit.)

[8] 4. Let $G(x, y)$ be a total computable function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$. For each $e \in \mathbb{N}$ let $f_e(x) = G(e, x)$ for all $x \in \mathbb{N}$. Show that there is a total unary computable function $h(x)$ such that h is not in the list $f_0, f_1, f_2, \ldots$.

[12] 5. Suppose that $A = range(f)$ for some computable unary function f. Give a primitive recursive relation $R(x, y)$ such that

$$A = \{x \mid \exists y R(x, y)\}$$

[8] 6. Let $\mathcal{L}$ be a first-order language with finitely many function and predicate symbols. Give an informal proof that the set of unsatisfiable $\mathcal{L}$-sentences is r.e., using Church's thesis together with the LK completeness theorem.

[8] 7. Recall that **RA** is a theory with 9 axioms P1, ... P9 over the language $\mathcal{L}_A$.
The **RA Representation Theorem** states that every r.e. relation is representable in **RA** by an $\exists\Delta_0$ formula. Use this theorem to prove that **RA** is undecidable.

[10] 8. Use the **RA Representation Theorem** (see previous question) to prove that every sound theory Σ with vocabulary $\mathcal{L}_A$ is undecidable. (Recall that Σ is *sound* if $\underline{\mathbb{N}}$ is a model of Σ.)

[14] 9. Let Σ be an axiomatizable theory over the vocabulary $\mathcal{L}_A$ of arithmetic such that every r.e. relation is representable in Σ by some $\exists\Delta_0$ formula. Show that there is a $\forall\Delta_0$ sentence (one of the form $\forall y B$, where B is bounded) such that $\Sigma \nvdash A$ and $\Sigma \nvdash \neg A$.

You may continue your solution on the next page.

(Continue your solution to Question 9 here.)