Due: Friday, October 23, beginning of tutorial

NOTE: Each problem set counts 15% of your mark, and it is important to do your own work. You may consult with others concerning the general approach for solving problems on assignments, but you must write up all solutions entirely on your own. Copying assignments is a serious academic offence and will be dealt with accordingly.

1. Prove the soundness of the LK rule $\exists$ Left. That is, prove that the universal closure of the meaning of the bottom sequent is a logical consequence of the universal closure of the meaning of the top sequent.

   Show where in your proof you are using the restriction that the free variable b must not occur in the conclusion.

   (See pages 28-29 of the Notes. Note that the proof on page 29 for the $\forall$-right case is confusing, because the left and right side of $\models$ should be replaced by the universal closures of those formulas, so neither side has any free variables, so there is no need to discuss the object assignment σ.)

2. Give an LK proof of the sequent

$$\forall x(x + 0 = x) \to \forall x \forall y(x + (y + 0) = x + y)$$

   You do not need to put in weakenings or exchanges. Indicate which LK equality axioms you use.

3. Let $\mathcal{L}$ be a language consisting of a countably infinite set $\{c_1, c_2, \dots\}$ of constant symbols and the binary predicate symbol P, and also $=$. Let Γ be the set of sentences

$$\Gamma = \{c_i \neq c_j \mid i, j \in \mathbb{N} \text{ and } i < j\}$$

   Let A be an $\mathcal{L}$ sentence such that $\Gamma \models A$. Prove that A has a finite model.

4. Let us call a formula A of the predicate calculus *simple* if all predicate symbols in A are unary, and A has no function symbols (and no constants).

   a) Show that if A is a satisfiable simple formula with n predicate symbols, then A is satisfied by some structure whose universe has at most 2^n elements. (Hint: Start with a model for A with universe M, and define a certain equivalence relation on M. Show that there are at most 2^n equivalence classes. Now show how to define a model for A whose universe is the set M' of these equivalence classes. It may help to look at the proof of Lemma 2 on page 45 of the Notes.)

   b) Show that the set of valid simple formulas is decidable. (That is, give an algorithm which, given a simple formula A, determines whether A is valid. Your algorithm should halt on all inputs.)