University of Toronto
Faculty of Arts and Science
December 2012 Examinations

CSC 438H1F/2404H

Duration - 3 hours
No Aids Allowed

There are 8 questions worth a total of 90 marks.
Answer all questions on the question paper, using backs of pages for scratch work. Note that the last page of this exam gives the rules for LK and PK.

Check that your exam book has 11 pages (including this cover page).

PLEASE COMPLETE THIS SECTION:

Name ______________________________

(Please underline your family name.)

Student Number ______________________________

FOR USE IN MARKING:

1. ____________/ 6
2. ____________/ 20
3. ____________/10
4. ____________/24
5. ____________/6
6. ____________/8
7. ____________/10
8. ____________/6

Total:__________/90

[6]

1. Recall that $Con(PA)$ is a true formula of $\mathcal{L}_A$ asserting the consistency of **PA**. Prove that there is a consistent extension $\mathbf{PA}'$ of **PA** such that $\mathbf{PA}' \vdash \neg con(PA)$.

[20] 2. Let L be the set of numbers x such that x codes a Turing Machine program, and such that this program halts on an even number of inputs. (For example, if x codes a TM that halts only on the inputs 5,7,8, then $x \notin L$, or if the TM coded by x halts on infinitely many inputs, then $x \notin L$, but if the TM coded by x halts only on the inputs $100, 302$, then $x \in L$.) Is L r.e.? Is $\overline{L}$ r.e. Justify your answers (but do not use Rice's Theorem).

[10] 3. Give an LK proof of the sequent

$$\forall x(x + 0 = x) \to \forall x \forall y(x + (y + 0) = x + y)$$

You do not need to put in weakenings or exchanges. Indicate which LK equality axioms you use.

4. Let $\mathcal{L}_s$ (the vocabulary of successor) be the vocabulary $[0, s; =]$. Let $Th(s)$ (theory of successor) be the set of all sentences over this vocabulary which are logical consequences of the following infinite set Ψ_s of axioms:

P1) $\forall x(sx \neq 0)$
P2) $\forall x \forall y(sx = sy \supset x = y)$
Q) $\forall x(x = 0 \vee \exists y(x = sy))$ (every nonzero element has a predecessor)

S1) $\forall x(sx \neq x)$
S2) $\forall x(ssx \neq x)$
S3) $\forall x(sssx \neq x)$
.
.
.

[10] (a) Prove that for each $n \geq 1$ the axiom Sn is not a logical consequence of $\{$P1,P2,Q,S1,S2,...S$n-1\}$. (Do this by giving a model.)

[8] (b) Prove using (a) that $Th(s)$ is not finitely axiomatizable. That is, show that there is no finite set Γ of sentences in $Th(s)$ such that every sentence in $Th(s)$ is a logical consequence of Γ. (Note that the sentences in Γ are not necessarily among the original set Ψ_s of axioms.)

[6] (c) Use the fact that every sentence true in the standard model $\underline{\mathbb{N}}_s$ for the language $\mathcal{L}_s$ is in $Th(s)$ to show that $Th(s)$ is decidable.

[6] 5. Let $F(x, y)$ be a total computable function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$. For each $e \in \mathbb{N}$ let $g_e(x) = F(e, x)$ for all $x \in \mathbb{N}$. Show that there is a total unary computable function $f(x)$ such that f is not in the list $g_0, g_1, g_2, \ldots$.

6. Let Σ be a theory over the vocabulary $\mathcal{L}_A$. We say that a function $f : \mathbb{N} \rightarrow \mathbb{N}$ is *provably total* in Σ if there is an $\exists\Delta_0$ formula $A(x, y)$ which represents the relation $(y = f(x))$ (the graph of f) and further

$$\Sigma \vdash \forall x \exists! y A(x, y)$$

Here $\exists! y A(x, y)$ stands for the formula $\exists y(A(x, y) \wedge \forall z(A(x, z) \supset z = y))$, and it intuitively means there is a unique y satisfying $A(x, y)$.

[8] Show that if a function $f : \mathbb{N} \rightarrow \mathbb{N}$ is provably total in **TA** then f is computable. Show this by giving a high level algorithm for computing f. Refer to the $\exists\Delta_0$ formula $A(x, y)$ which represents the graph of f.

[10] 7. Prove that a theory is consistent if and only if it has a model.

[6] 8. State Godel's first and second incompleteness theorems.

The rules of PK are as follows.

- Structural rules: Exchange, weakening
- OR right: From $\Gamma \to \Delta, A, B$ derive $\Gamma \to \Delta, A \vee B$.
- OR left: From $A, \Gamma \to \Delta$ and $B, \Gamma \to \Delta$, derive $A \vee B, \Gamma \to \Delta$.
- AND right: From $\Gamma \to \Delta, A$ and $\Gamma \to \Delta, B$ derive $\Gamma \to \Delta, A \wedge B$.
- AND left: From $A, B, \Gamma \to \Delta$ derive $A \wedge B, \Gamma \to \Delta$.
- NEG right: From $A, \Gamma \to \Delta$ derive $\Gamma \to \Delta, \neg A$
- NEG left: From $\Gamma \to \Delta, A$ derive $\neg A, \Gamma \to \Delta$.
- CUT: From $A, \Gamma \to \Delta$ and $\Gamma \to \Delta, A$ derive $\Gamma \to \Delta$.

The two addition LK rules are as follows.

- $\forall$ left: From $A(t), \Gamma \to \Delta$ derive $\forall x A(x), \Gamma \to \Delta$.
- $\forall$ right: From $\Gamma \to \Delta, A(b)$ derive $\Gamma \to \Delta, \forall x A(x)$.
- $\exists$ left: From $A(b), \Gamma \to \Delta$ derive $\exists x A(x), \Gamma \to \Delta$.
- $\exists$ right: From $\Gamma \to \Delta, A(t)$ derive $\Gamma \to \Delta, \exists x A(x)$.

The free variable b must not occur in the conclusion in $\forall$ right and $\exists$ left, and t is a proper term (free variables only).