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1. Proof:

   Because the *Restriction* for this rule, the variable $b$ cannot occur in $\Gamma$ or $\Delta$. Hence, it suffices to verify that

   $$\forall x(\bigwedge \Gamma, A(x) \supset \bigvee \Delta) \models (\bigwedge \Gamma, \exists A(x)) \supset \bigvee \Delta$$

   to see that this logic consequence holds, suppose $\mathcal{M}$ satisfies the left hand side, then $\mathcal{M}$ satisfies $\forall$ A, hence, $\mathcal{M}$ satisfies A which means, $\mathcal{M}$ satisfies that right hand side.

2. We need the following LK Equality Axioms:

   EL1: $\rightarrow a = a$

   EL4: $a = a,\ b + 0 = b \rightarrow a + (b + 0) = a + b$

   Here is the LK proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{EL1}{\rightarrow a = a} \qquad \dfrac{EL4}{a = a,\ b + 0 = b \rightarrow a + (b + 0) = a + b}}{b + 0 = b \rightarrow a + (b + 0) = a + b}\ \mathit{cut\ rule}}{\forall x(x + 0 = x) \rightarrow a + (b + 0) = a + b)}\ \forall\ \mathit{left}}{\forall x(x + 0 = x) \rightarrow \forall y\ a + (y + 0) = a + y}\ \forall\ \mathit{right}}{\forall x(x + 0 = x) \rightarrow \forall x \forall y\ x + (y + 0) = x + y}\ \forall\ \mathit{right}$$

3. Proof:

   Using Compactness, we know that if $A$ is a logic consequence of $\Gamma$ then $A$ is a logic consequence of some finite subset $\Gamma$. Hence, it is surfficent to show that every finite subset of $\Gamma$ has a finite model.

   Suppose $S' \subset S$ is finite. Then there is some natural number $n$ such that $S'$ has no constant $c_i$ with $i > n$.

   Let $M$ be a model consist of a cycle of $n$ nodes. Let the constants $c_1, c_2, ..., c_n$ to be nodes in the cycle and $P^M(c_i, c_j)$ means there is a edge between nodes $c_i$ and $c_j$. Clearly, $M$ is a finite model for $S'$.

   Hence, every finite subset of $\Gamma$ has a finite model which means $A$ has a finite model.

4. (a)

let $P_1, P_2, ...P_n$ be the predicate symbols in A. and we know they all unary symbols.

Suppose A is satisfiable, then let $\mathcal{M}$ be a model for A over universe M, and let $\sigma$ be the associated object assignment, then we have $\mathcal{M} \models A[\sigma]$.

Now define a equivalence relation $\sim$ on M:

$$\forall a, b \in M \ \ (a \sim b \iff P_i^M[a] = P_i^M[b]) \ i \leq i \leq n$$

Clearly, $\sim$ is a equivalence relation.

Then every equivalence class $[x]$ is a n-tuple $(P_1^M(a), P_2^M(a), ..., P_n^M(a))$. Since $P_i^M$ is a boolean predicate then there are t most $2^n$ such n-tuples; hence, there are at most $2^n$ equivalence classes in M.

Now define a structure $\mathcal{M}'$ with a universe M' consisting all classes in M.

$$M' = \{[x] \mid x \in M\}$$

$$P_i^{M'}([x]) = P_i^M(x)$$

and the associated object assignment $\sigma'$

$$\sigma'([x]) = [\sigma(x)]$$

Then based on *Lemma* 2: $\mathcal{M}' \models A[\sigma']$ iff $\mathcal{M} \models A[\sigma]$, on page 45 we know that $\mathcal{M}' \models A[\sigma']$.

(b)

From (a) we know that for any model $\mathcal{M}$ associated object assignment $\sigma$ for A, we can choose an equivalent model with a finite universe which has at most $2^n$ elements. Hence, A is valid iff it is satisfied by every structure with at most $2^n$ elements in its universe.

Hence, we processing formula by replacing universal quantified sub-formulas with a conjunction of the sub-formulas over $2^n$ distinct elements and replacing existentially-quantified sub-formulas with a disjunction of the sub-formulas over $2^n$ distinct elements.

We know that there are $2^n$ possible truth assignments for all $n$ predicates in the formula. Then we can just substitute all possible truth assignments and the the formula is valid iff , for all substitutions, it to true.

Since the number of the all possible truth assignments is bounded; hence, the algorithm will always halts.