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Citation: Physics of Fluids (1958-1988) **10**, 658 (1967); doi: 10.1063/1.1762162
View online: http://dx.doi.org/10.1063/1.1762162
View Table of Contents: http://scitation.aip.org/content/aip/journal/pof1/10/3?ver=pdfcov
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Elastic Collisions in Simulating One-Dimensional Plasma Diodes on the Computer

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(Received 20 June 1966)

The inclusion of electron–neutral collisions in computer simulation calculations is demonstrated. Only one-dimensional fields are considered in diodes with plane–parallel symmetry. The method used is to change the velocities of particles in a random manner according to their probabilities of suffering a collision. The method is applied to the hard-sphere type of collisions between electrons and neutrals. Three examples are shown, and in two of them comparison between analytical and computer simulation results are possible. Reasonably good agreements are obtained. It is demonstrated that the inclusion of electron–neutral collisions into computer simulation procedures could be used for finding the damping effect of these collisions on the oscillatory behavior of unstable plasma diodes. With this method we can compute the large-signal, time-dependent characteristics of diodes at intermediate pressures in which the mean-free path is comparable to the diode distance.

1. INTRODUCTION

COMPUTER simulation techniques have been used for a large variety of plasma problems in the past. The digital computer became very useful in analyzing the nonlinear problems of two-stream instability,[1] or the kinetic behavior of a one-dimensional plasma[2,3] and collisionless randomization.[4] In other fields the large amplitude oscillations in a low-pressure thermionic converter was analyzed[5] and recently, two-dimensional plasmas were successfully simulated giving insight into the process of neutralization of ion beams by electrons[6] and the diffusion of a plasma across magnetic field lines.[7] In all these papers (and in many others that deal with the computer simulation of plasmas) the basic procedure of calculating the behavior of the given plasma system is the same. This basic procedure consists of calculating the trajectories of a large number of charged particles (charged sheets for the one-dimensional problems) whose motion is influenced by the fields that the charges themselves produce. The calculations are executed by a high-speed digital computer in small steps. Each step corresponds to a given time interval in the physical system. The fields are recalculated at each time step from the known positions of the charges at the given time, and it is hoped that when the time steps are small, the trajectories and fields of the simulated diode are similar to those that one might find in a physical experiment. Good agreement between simulated and experimental results have been demonstrated,[8] which gave the computer simulation techniques the importance of being the only available tool for analyzing large-signal time-varying plasma problems.

All computer simulation studies until now have analyzed low-pressure, or collisionless plasmas. In these plasmas particles follow trajectories that are determined by the fields alone once they have been injected into the space under study. It is true that randomness was introduced into problems that deal with thermionic emitters (see Refs. 5–8), but the randomness effected the injection velocities of the particles only, their trajectories were calculated strictly according to Newton's laws of motion. In this paper we describe the first attempt to introduce randomness into the trajectories of the particles in a computer simulated diode. The random effects could correspond to electron–neutral collisions in a physical experiment, and it is shown that realistic scattering cross sections could be incorporated into the calculations even though the present paper deals only with the simplest case: constant mean free path, hard-sphere collisions.

In the first section the general method of the calculations is described. In the next two sections two cases will be treated for which analytical results are known, and, finally, the effects of the electron–neutral collisions will be demonstrated on the unstable behavior of the low-pressure thermionic converter.

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[1] O. Buneman, Phys. Rev. **115**, 503 (1959).
[2] J. Dawson, Phys. Fluids **5**, 445 (1962).
[3] O. C. Eldridge and M. Feix, Phys. Fluids **5**, 1076 (1962).
[4] P. Burger, D. A. Dunn, and A. S. Halsted, Phys. Fluids **8**, 2263 (1965).
[5] P. Burger, J. Appl. Phys. **36**, 1938 (1965).
[6] O. Buneman, G. Kooyers, AIAA J. **1**, 2525 (1963).
[7] R. Hockney, Phys. Fluids **9**, 1826 (1966).
[8] W. H. Cutler amd P. Burger, J. Appl. Phys. **37**, 2867 (1966).

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It is well to keep in mind that we describe here our first attempt to introduce randomizing collisions into computer simulation techniques and therefore it is perhaps crude in many parts. These parts require later refinements. The results, however, show promise that problems of these types could be analyzed by computer simulation methods, and for time-varying problems this method is the only available one at the present which could give us some insight into the behavior of plasmas at moderate pressures.

2. INTRODUCTION OF ELECTRON-NEUTRAL ELASTIC COLLISIONS INTO COMPUTER SIMULATION CALCULATIONS OF A ONE-DIMENSIONAL DIODE

The method of calculating the trajectories of charged particles (i.e. charged sheets) in a one-dimensional collisionless diode have been described earlier (see the Appendix, Refs. 4 and 5). Time is advanced in equal steps Δt, and the distribution of charges are calculated at these times: $t = 0, \Delta t, 2\Delta t, \cdots$. The positions of the sheets are stored for two successive time steps t_{n-1}, t_n ($t_n = n\Delta t$). The sheets are advanced according to the difference equation,

$$x_i(t_{n+1}) = 2x_i(t_n) - x_i(t_{n-1}) + (q_i/m_i)\,\Delta t^2 E[x_i(t_n), t_n], \qquad (1)$$

where $x_i(t_n)$ is the position of the ith sheet at time t_n; q_i/m_i is the charge-to-mass ratio of this sheet; and $E(x, t_n)$ is the electric-field distribution in the diode calculated at time t_n from the known positions of the sheets at this time $x_i(t_n)$, $i = 1, 2, \cdots, N(t_n)$. [$N(t_n)$ is the total number of sheets in the diode at time t_n.] The quantity $[x_i(t_n) - x_i(t_{n-1})]/\Delta t$ is the velocity of the ith sheet somewhere between times t_n and t_{n-1}, if continuous variation is assumed for this velocity as the function of time.

Every time step a given number of sheets are introduced at position $x = 0$. The velocities of the injected sheets are selected randomly according to a Maxwellian velocity distribution law of given temperature. Whenever the position of a sheet becomes less than zero or larger than d, the diode distance, it is taken out of the system of charges and is lost for successive time steps. Strict equilibrium is never reached because of fluctuations from time step to time step but a time-average equilibrium can be obtained when the time average rate of sheets lost is equal to the rate of their injection.

Now, we would like to extend the described calculations in order to include the effect of randomizing collisions between electrons and neutrals. The electric field remains one dimensional and its direction is in the principal direction x (see Fig. 1). No quantities are assumed to vary in the other two directions, therefore, the positions of charges are still recorded only in the principal direction $x(t_n)$. (We delete the index i for convenience and from now on the variables x, v without the index will always refer to one particular charge carrier.) The boundaries of our diode are still parallel infinite planes at positions $x = 0$, and $x = d$ with their planes perpendicular to the x direction (Fig. 1). But, in order to account for randomizing collisions, the velocity of each charge carrier has to be given in at least two directions—or what is equivalent, we have to know the total energy of the particle in addition to its velocity component in the x direction. The particle is still advanced according to the x velocity component only. The total energy of a particle influences the probability that the charge carrier will suffer a collision in a time interval Δt. Also the total energy of the particle is the quantity which is conserved in an elastic collision.

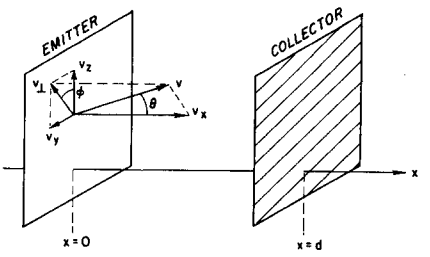

FIG. 1. The model of a one-dimensional plasma diode.

In our computer calculations we store an additional quantity to those described earlier $[x(t_n), v_x(t_{n-1})]$: the quantity $v_\perp^2(t_n)$, the square of the component of the particle's velocity lying in the plane perpendicular to the x direction (see Fig. 1). We set $v_x(t_n) = [x(t_n) - x(t_{n-1})]/\Delta t$ and assume that the charge carrier travels at this velocity during the time interval $t_{n-1} \leq t \leq t_n$. To calculate the effect of collisions we need only the magnitude of the total velocity of the particle, which is

$$v(t_n) = [v_x^2(t_n) + v_\perp^2(t_n)]^{\frac{1}{2}}.$$

We should expect that the error in the assumption that the particle travels with velocity $v(t_n)$ during the interval $t_{n-1} \leq t \leq t_n$ will decrease as we decrease Δt.

If we know the mean free path as the function of electron energy, i.e., total velocity, for the particular collision process we want to study, than we can calculate the probability of a collision for a

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given particle that travels with velocity $v(t_n)$ for a time interval Δt by the following expression:

$$P_c(t_n) = 1 - \exp\{-v(t_n)\,\Delta t/\lambda[v(t_n)]\}, \qquad (2)$$

where P_c is the probability that this particle suffers a collision during the time interval $t_{n-1} \leq t \leq t_n$ and $\lambda(v)$ is the mean free path given as the function of velocity. The probability given by Eq. (2) is evaluated for all the particles at every time step from the quantities $v^2(t_n)$, $v_x(t_n)$. After this calculation is made the decision has to be reached as to which particles should suffer collisions. Obviously, this decision should be made in a random manner. In the following examples (see Secs. 3, 4, 5) only hard-sphere collisions were considered with constant mean-free path, so λ was a constant. But Eq. (2) is valid for any known function of $\lambda(v)$ and therefore the calculations could be made for any known law of scattering process. Since the velocity of the particle is recalculated each time step, the effect of a variable cross section is incorporated in the calculations and the error will be small if the change in the velocity of the particles is small between time steps. This requires small time step again.

The decision as to which particle should suffer a collision was made in the following manner. The diode space was divided into 128 equal regions in the x direction. In each region the probability P_c were calculated for the particles and using the sequence in which they were recorded their probabilities were added. The particle whose probability made this sum larger than one suffered a collision, and the summing of probabilities was restarted with the sum decreased by one. The sums were saved from step to step and summing was always resumed where it was left off one time step earlier. The described procedure insures that in the average the number of particles that make collisions in a small region of the diode is equal to the expectation value calculated from their probabilities. Since particles travel from region to region fairly rapidly and are injected in a random manner, their sequence in storage could also be considered random.

Obviously, there are many different ways to devise procedures for the decision of a collision, and the effect of different methods on the final results should be explored in the future. The described method was well suited for computations and was particularly easy to apply. In the very early stages of this work we saved the history of each particle and calculated at each time step how long it has traveled from the instant of its last collision. For the first calculations, collisions were made whenever a particle traveled more than a mean free path. We found that the final results were not very seriously different when we changed to the procedure described above.

Once we have decided that a particle will suffer a collision we have the problem of determining the two components of its velocity after the collision has taken place v_x^*, $(v_\perp^*)^2$ (where the star indicates quantities after the collision). The first assumption we make is that there is no energy loss due to the collision. This assumption means that the neutrals have infinite mass and zero velocity in the laboratory frame. Consequently the equation

$$(v_x^*)^2 + (v_\perp^*)^2 = v_x^2 + v_\perp^2 \qquad (3)$$

must hold, and we can choose only one velocity component freely. The selection of this velocity component (we selected v_x^* in our calculations) depends upon the angle-dependence of the collision process under study and in general it is a function of v_x and $v_\perp$. We again simplified our calculations by assuming hard-sphere collisions. This process has no preferred direction, therefore the distribution of velocities v_x^* is uniform in the interval $-v \leq v_x^* \leq v$. We have generated a sequence of random numbers in the computer whose distribution was uniform in the unit interval and used numbers from this sequence (R) to determine the x component of the velocities of particles that suffered collisions. The equation $v_x^* = \pm Rv$ was used with the sign alternating from collision to collision.

After v_x^* has been chosen for the colliding particle, Eq. (3) is used to determine $(v_\perp^*)^2$ which is then stored and used for this particle at successive time steps until the next collision takes place.

The following examples were calculated with the described method of introducing collisions into the computer simulation procedure. For each particle the quantities $x(t_n)$, $v_x(t_n)$, $[v_\perp(t_n)]^2$ were stored. At each time step the total velocity $v = [v_x^2 + v_\perp^2]^{\frac{1}{2}}$ was calculated and the probability of a collision $P_c = 1 - \exp(-v\Delta t/\lambda)$ was determined for each particle. The velocities of those particles which suffered collisions were changed according to the equations: $v_x^* = \pm Rv$, where $0 < R < 1$ and R's are taken sequentially from a sequence of random numbers uniformly distributed in the unit interval. After v_x^* has been found the new perpendicular velocity components were calculated: $(v_\perp^*)^2 = v^2 - (v_x^*)^2$. All the particles then were advanced according to the equations

$$\begin{aligned} v_x(t_{n+1}) &= v_x(t_n) + (q/m)\,\Delta t E(x, t_n), \\ x(t_{n+1}) &= x(t_n) + v_x(t_{n+1})\,\Delta t. \end{aligned} \qquad (4)$$

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Equation (4) is equivalent to Eq. (1) with the definition of velocity

$$v_x(t_n)\,\Delta t = x(t_n) - x(t_{n-1}).$$

The only remaining problem is to choose the initial velocities of particles which are injected at the emitter plane. We assume throughout this paper that particles are emitted thermionically. For a thermionic emitter the differential number of electrons that pass the emitter plane in a given time interval and have velocities within the ranges from v_x to $v_x + dv_x$, from $v_\perp$ to $v_\perp + dv_\perp$, and from φ to $\varphi + d\varphi$ is given by the expression

$$n_0\pi(m/2\pi kT)^{5/2}v_x v_\perp \exp(-mv^2/2kT)\,dv_x\,dv_\perp\,d\varphi\,\Delta t, \quad (5)$$

where $v^2 = v_x^2 + v_\perp^2$, n_0 is the number density of electrons inside the emitter, m is the electron mass, T is the emitter temperature and k is Boltzmann's constant. We can integrate Eq. (5) over the whole range of φ's $(0 - 2\pi)$ since our variables do not depend on φ. After integration we arrive at the result that the number of particles with velocities in the ranges v_x to $v_x + dx$, and $v_\perp$ to $v_\perp + dv_\perp$ is proportional to

$$v_x \exp(-mv_x^2/2kT)v_\perp \exp(-mv_\perp^2/2kT)\,dv_x\,dv_\perp,$$

and hence symmetrical in the variables v_x and $v_\perp$. We have shown earlier (Ref. 5) that we can simulate a thermionic emitter by choosing initial velocities for the x component of the velocities according to the distribution law

$$v_x(\text{injected}) = (2kT/m)^{\frac{1}{2}}(-\ln R)^{\frac{1}{2}}, \quad (6)$$

where R is a random variable uniformly distributed in the interval $0 < R < 1$. The distribution in v_x(injected) is the same as in earlier works and because of the symmetry of Eq. (5) in v_x and $v_\perp$ we have to chose the same procedure for $v_\perp$ of the injected particles as was used for v_x, i.e.,

$$v_\perp(\text{injected}) = (2kT/m)^{\frac{1}{2}}(-\ln R)^{\frac{1}{2}}$$

or (7)

$$[v_\perp(\text{injected})]^2 = -(2kT/m)\ln R.$$

The random variable in Eq. (7) must be uncorrelated to the random variable in Eq. (6). The generation of a sequence of random numbers in a digital computer that are uncorrelated and distributed uniformly in the interval $0 < R < 1$ has been investigated in great detail.[9] We have used one of the standard procedures from Ref. 9 to generate a sequence of random numbers in the computer and used successive members of this sequence for the determination of injection velocities of particles according to Eqs. (6) and (7).

At every time step the electric field $E(x, t)$ is determined by a difference equation form of Poisson's equation $(\partial E/\partial x = \rho/\epsilon_0)$ with the appropriate boundary condition $\int_0^d E\,dx = -V_d$, where V_d is the diode potential. The space–charge distribution ρ is given by the known positions of the charged particles in the diode. This phase of the calculations is equivalent to those used for collisionless diodes described in an earlier paper.[5]

We selected the current as the function of time for our output quantity. At every time step the number of particles that leave the diode at the collector were recorded along with the change of the electric field. The current through the diode was determined then adding the convection current carried by the particles to the displacement current caused by the changing electric field at the collector.

We have used the described method of computer simulation technique for three examples that will be discussed in the following sections.

3. TRANSFER OF THERMIONICALLY EMITTED PARTICLES ACROSS A MEDIUM WITH ISOTROPIC SCATTERING PROPERTIES

Our first aim was to test our method of simulating the process of electron–neutral collisions without the effect of the electric field on the motions of the particles. We considered the problem of a plane parallel region (Fig. 1) with particles emitted thermionically at the plane $x = 0$. While traveling across the medium $0 \le x \le d$ the particles suffer elastic collisions. The mean free path of the particles is λ (independent of velocity) and the scattering process is isotropic. This problem is equivalent to one of the basic problems of radiative transfer and a detailed analysis of it can be found in Chandrasekhar's book.[10] The equation of transfer is derived from Boltzmann's equation. The distribution function of particles does not depend on the angle φ or time. The equation of transfer for the distribution of particles $f(x, v, \theta)$ is given by

$$v\mu\frac{\partial f(x, v, \mu)}{\partial x} = \frac{v}{2\lambda}\int_{-1}^{+1} f(x, v, \mu)\,d\mu - \frac{v}{\lambda}f(x, v, \mu), \quad (8)$$

where $\mu = \cos\theta$. The boundary conditions are

$$f(0, v, \mu) = n_0(m/2\pi kT)^{\frac{3}{2}}\exp(-mv^2/2kT), \quad \text{for } 0 \le \mu \le 1;$$
$$f(d, v, \mu) = 0, \quad \text{for } -1 \le \mu \le 0. \quad (9)$$

[9] T. E. Hull and A. R. Dobell, SIAM Rev. 4, 230 (1962).

[10] S. Chandrasekhar, *Radiative Transfer* (Dover Publications, Inc., New York, 1960), Chaps. VIII, IX, pp. 183–232.

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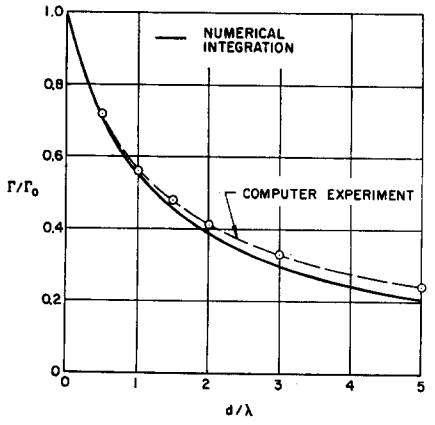

FIG. 2. Comparison between analytical and computer simulated results for the problem of Milne. The points of the computer simulation calculations are the average values of 600 points after the system reached equilibrium.

If we substitute for the distribution function

$$f(x, v, \mu) = n_0(m/2\pi kT)^{\frac{3}{2}} \exp(-mv^2/2kT)\psi(\xi, \mu),$$

where we normalized distance to the mean free path λ, so $\xi = x/\lambda$ then the velocity dependence in Eq. (9) is eliminated and the equation for the function $\psi(\xi, \mu)$ is given by

$$\mu \frac{\partial \psi(\xi, \mu)}{\partial \xi} = \frac{1}{2}\int_{-1}^{1} \psi(\xi, \mu)\, d\mu - \psi(\xi, \mu), \qquad (10)$$

with boundary conditions

$$\begin{aligned} \psi(0, \mu) &= 1, \quad \text{for} \quad 0 \leq \mu \leq 1, \\ \psi(d/\lambda, \mu) &= 0, \quad \text{for} \quad -1 \leq \mu \leq 0, \end{aligned} \qquad (11)$$

which is the equation of Milne for a finite interval $0 \leq \xi \leq d/\lambda$. The normalized flux transmitted across the region $\xi = d/\lambda$ wide is given by the expression

$$\frac{\Gamma}{\Gamma_0} = \int_0^1 \mu\psi(d/\lambda, \mu)\, d\mu = 1 - \int_{-1}^{0} \mu\psi(0, \mu)\, d\mu, \qquad (12)$$

where Γ_0 is the total incident flux at the emitter plane $x = 0$. The normalized transmitted flux as the function of (d/λ) is shown in Fig. 2 as a solid curve which was calculated from the solution of Eqs. (10) and (11) numerically. Our method of calculating Γ/Γ_0 as the function of d/λ is given in the Appendix.

We have conducted computer simulation calculations for the same problem with d/λ values ranging from 0.5 to 5. The transmitted flux as a function of time is shown in Fig. 3 for the value of $d/\lambda = 1$ and for the planes $\xi = 0$, $\xi = d/\lambda$. Time is normalized to the average transit time of the particles in the absence of collisions $\bar{\tau} = d/(2kT/m)^{\frac{1}{2}}$. The normalized particle flux at the emitter is given by the injected number of particles per time step minus the number of particles that return to the emitter in one time step divided by the injected number. The flux at the collector is given as the number of particles that reach the collector in one time step divided by the injected number of particles per time step. Both values of flux are calculated at every time step. There were 200 time steps in one normalized unit of time and the total number of particles present in the diode was around 5000 when the system reached equilibrium. The average values of the flux versus time functions for different values of d/λ are shown in Fig. 2. The average was taken for 600 time steps after the diode has reached equilibrium in all the computed cases. Reasonable agreement was obtained between theoretical and computer simulated results as shown in Fig. 2. The error at higher values of d/λ indicated that the time step was not small enough for these calculations. That the error decreases with decreasing time step is demonstrated in the next section where the space-charge effect of electrons as well as the effect of an applied potential difference across the diode on the transfer of electrons will be investigated. The results in Fig. 2 demonstrate that at moderate values of d/λ the random selection of collisions give the right integrated effect on the transfer of particles across an isotropically scattering medium. At higher values of d/λ the effect of collisions in the computer simulated diode seem to be smaller than they should be according to exact calculations. It is not clear

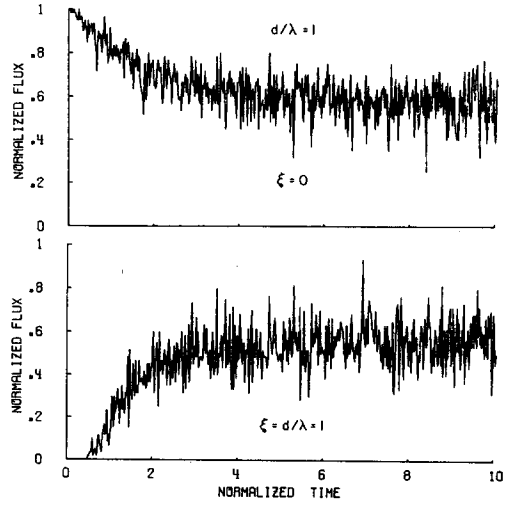

FIG. 3. Flux versus time in the computer simulated diode for the positions $\xi = 0$, and $\xi = d/\lambda$. Parameters are: $d/\lambda = 1$, Δt (time step) $= 0.02[d/(2kT/m)^{\frac{1}{2}}]$, average number of particles = 5000.

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at the present where this error comes from, and it shows up again in the next section.

4. CURRENT VERSUS VOLTAGE CURVE IN AN ELECTRON DIODE WITH ELECTRON–NEUTRAL (ELASTIC) COLLISIONS

The problem of particle transfer across a finite slab of scattering medium analyzed in the last section is now modified by making the particles electrons and by considering a potential difference V_d across the collector and emitter planes. The electric field in the diode naturally effects the transfer of electrons. Results calculated in the last section showed that the randomly selected collisions did add the right effect to the transfer properties of particles in an isotropically scattering medium. Earlier computer calculations for collisionless plasma diodes[11] demonstrated that the described step-by-step method of calculating particle trajectories gave the same voltage–current characteristics as that of static solutions whenever the static solutions were stable under time-varying conditions. It is now expected that the computer simulation calculations with electron–neutral collisions as described in Sec. 2 should give the static voltage–current characteristics of an electron diode in which the ratio d/λ has moderate values. There is no reason to expect that the static solution in the electron diode is unstable.

The calculation of the static solutions for the electron diode with electron–neutral collisions is not a trivial problem. Some results have been calculated by Sockol[12] for monotonically increasing potential functions. We will compare our computer simulated results to the static solutions of Sockol. The analytical results were obtained by solving an integral equation derived from Boltzmann's equation of the same form as Eq. (8). In the case of electrons there are two additional terms added on the left-hand side of Eq. (8) which express the effect of the electric field (negative gradient of the potential) on the distribution function of the electrons. The integral equation form of the Boltzmann equation is solved self-consistently with Poisson's equation $d^2V/dx^2 = -\rho(x)/\epsilon_0$ where $\rho(x)$, the charge density function of the electrons is given by an integral of the distribution function over all velocities.

The analytical results are shown in Figs. 4 and 5 for the ratio of diode distance to electron Debye length $d/\lambda_{DB} = 10$ and the ratio of diode distance to mean free path $d/\lambda = 1$ and 2, respectively.

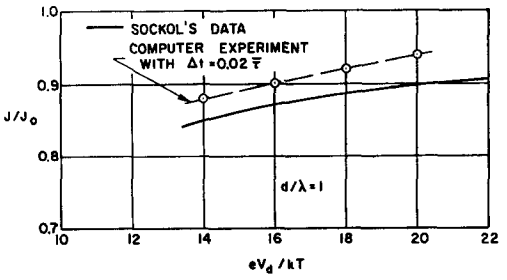

FIG. 4. Comparison between analytical and computer simulated results for the electron diode with electron–neutral collisions. Diode parameters: $d/\lambda_{DB} = 10$, $d/\lambda = 1$. Computational parameters: $\Delta t = 0.02\tau$, average number of particles: 5000.

There is a minimum voltage for all of the analytical voltage versus current curves of Sockol because the smallest diode potential for which solutions can be obtained arises from solutions with the field becoming zero at the emitter. At lower than this minimum diode potential the potential becomes negative at the front of the emitter and the potential distribution is not monotonic any more; hence results could not be obtained by the method used in Ref. 12.

The voltage–current curves of Figs. 4 and 5 are given in normalized form where the normalized diode potential η_d is expressed by $\eta_d = eV_d/kT$ and current is normalized to the electron saturation current of the emitter. Points obtained from computer simulation calculations are shown for comparison. The manner by which these results were obtained is shown in Fig. 6 where the normalized current versus normalized time curve of one of our "computer experiments" is demonstrated. Calculations started with parameters $d/\lambda_{DB} = 10$, $d/\lambda = 1$, $\eta_d = 20$

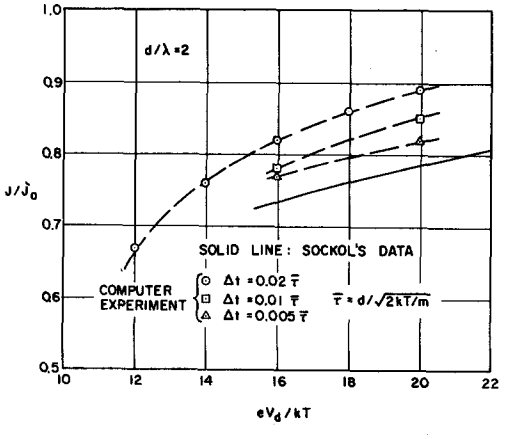

FIG. 5. Comparison between analytical and computer simulated results for the electron diode with collisions. Parameters: $d/\lambda_{DB} = 10$, $d/\lambda = 2$. The effect of decreasing the time step on the disagreement between analytic and computer results is also shown.

[11] P. Burger, Ph.D. thesis, Stanford University (1964).
[12] P. M. Sockol, in Report on the Thermionic Converter Specialist Conference, Cleveland (1964), p. 170.

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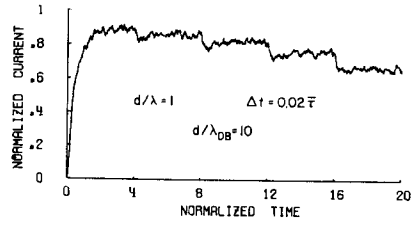

FIG. 6. Sample of computer results for the electron diode of $d/\lambda_{DB} = 10$, $d/\lambda = 1$. The diode potential $\eta_d = 20$ for normalized time $0 \leq t/\tau \leq 4$ and is decreased by 2 at time $t/\tau = 4$. The potential is decreased to $\eta_d = 16, 14$ at times $t/\tau = 8, 12$.

and after a time interval of four average electron transit time [$4\bar{\tau} = 4d/(2kT/m)^{\frac{1}{2}}$] the diode potential was decreased to the value $\eta_d = 18$. The diode potential was later decreased to successive values 16, 14 in four average transit time intervals. It was expected that the diode reaches equilibrium in four average transit time, and, as shown in Fig. 6, this was indeed the case. For diode potentials $\eta_d < 15$ the potential is not monotonic in the diode but this fact causes no difficulty for the computer simulation calculations. In fact, we should expect better agreement between dc calculations (if there were any available) and computer simulated results for lower values of diode potentials. It is apparent that the random collisions in the computer simulated diode are not as effective as should be because the currents are always larger in the simulated diode than those predicted by exact calculations. When the fields help the effects of collisions (decelerating fields), we should expect the error to be small. In situations when collisions work against the fields (accelerating fields), the fields transfer a larger current across the diode than exact calculations indicate.

The effect of the size of time step on the error in our results is shown in Fig. 5. Diode parameters $d/\lambda_{DB} = 10$, and $d/\lambda = 2$ were used and several computer runs were made that resulted in characteristics similar to those shown in Fig. 6. Different time steps were used for different runs and results are shown for three values of the time step: $\Delta t = 0.02\bar{\tau}$, $0.01\bar{\tau}$, and $0.005\bar{\tau}$, where $\bar{\tau}$ is defined as before. These results in Fig. 6 indicate that the error between the computer simulation results and the theoretically predicted values decreases when Δt decreases—but the convergence is not as fast as we have hoped. It is possible that a different scheme of deciding about collisions in the computer simulation calculations could give better or faster converging results. With the decribed procedure one should be able to determine a point on the voltage versus current curve for moderate ratios of d/λ within 5% of error. Using 5000 particles, one point was determined in about 5 min of IBM 7090 computer time. There is no restriction on the shape of the potential distribution function and this method of calculating static characteristics of plasma diodes with collisions could be considered useful when a complicated shape of the potential function does not allow exact analysis. In the case of complicated potential functions a Monte Carlo type of attack should show more promise and work has been progressing in this field.[13] The problems for which no other methods are available than computer simulation techniques, however, are the time-varying problems. An example of a time-varying situation is demonstrated in the next section.

5. THE EFFECT OF ELECTRON–NEUTRAL COLLISIONS ON OSCILLATORY BEHAVIOR OF THE LOW-PRESSURE CESIUM DIODE

The model of the low-pressure cesium diode defers from the model discussed in the preceding section by the inclusion of thermionically emitted ions at the emitter plane. The parameter

$$\alpha = (J_{si}/J_{se})(M/m)^{\frac{1}{2}} \tag{13}$$

is important for this diode where J_{si} and J_{se} are the ion and electron saturation currents, respectively, and M/m is the ratio of their masses.

A large signal instability was shown to be present in the collisionless diode under ion-rich conditions ($\alpha > 1$).[14] It was found recently that the diode is unstable if $\alpha > 0.401$ and the collector potential is larger than a critical value. This critical diode potential would draw saturation electron current if static solutions were possible with the field becoming zero at the emitter.[15]

We have conducted computer simulation runs that included the thermionic injection of ions in addition to the injection of electrons and the described procedure of electron–neutral collisions. In computer simulation calculations it is impossible to use realistic ion-to-electron mass ratios because it would be too costly to calculate the behavior of the system until it obtains equilibrium. Equilibrium can be reached only after a few average ion transit times. The average ion transit time is $\bar{\tau}(M/m)^{\frac{1}{2}}$, where $\bar{\tau} = d/(2kT/m)^{\frac{1}{2}}$, the average electron transit time. The best one can do is to calculate the behavior of the diode for increasing mass ratios and note when a further increase in the mass ratio has no

[13] C. K. Goldstein (private communication).
[14] P. Burger, J. Appl. Phys. 35, 3048 (1964).
[15] P. Burger, J. Appl. Phys. (to be published).

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effect on the final results. The results presented here were calculated with $M/m = 16$. Trial runs with $M/m = 64$ showed no essential differences from the results presented here and it was concluded that the mass ratio $M/m = 16$ was large enough for our purposes. The effect of the electron–neutral collisions is shown in Fig. 7 for a diode $d/\lambda_{DB} = 50$ with $\alpha = 0.5$ and $\eta_d = 10$. Calculations start with an empty diode into which electrons and ions are injected at the emitter planes starting at time $t = 0$. Ions do not suffer collisions, but electrons collide according to the procedure described before. The resulting current versus time functions are shown in Fig. 7 where time is normalized to the average electron transit time and current to the electron saturation current. As shown, oscillations are effectively damped by electron–neutral collisions as the mean free path becomes comparable to d. We have found from similar runs that the oscillations are damped when d/λ becomes of the order of two regardless of the ratio of d/λ_{DB}. It was shown earlier[5] that the oscillations in the collisionless diode were due to the rapid rearrangement of the electron charge distribution in a time interval during which the ion distribution remains comparatively stationary. The rearrangement of the electron distribution is an electron transit time effect because there is a large change in the distribution of electrons when they leave the diode space. Collisions slow down the mobility of the electrons and the average time in which an electron can leave the diode space is decreased rapidly when the electron–neutral mean free path becomes comparable to the diode distance. Even though this space-charge instability has a characteristic frequency of the order of one over the average transit time of the ions it is driven by the large mobility of the electrons. When the mobility of the electrons decreases the diode is stabilized.

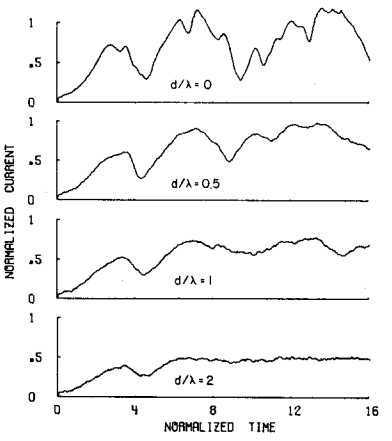

FIG. 7. The effect of electron–neutral collisions on the oscillatory behavior of the low-pressure cesium diode. Time is normalized to the average electron transit time $d/(2kT/m)^{\frac{1}{2}}$.
Parameters:
$d/\lambda_{DB} = 50$,
$\alpha = 0.5$,
$eV_d/kT = 10$.

6. CONCLUSIONS

We have demonstrated the possibility of the use of computer simulation techniques for analyzing the effects of randomizing collisions in one-dimensional plasma diodes. The results obtained with 5–7000 particles showed little dependence on the number of particles used—a necessary condition for a successful "computer experiment." We have tried one possible way of simulating random collisions in the diode which gave good agreement for the Milne problem between the results of computer simulation techniques and of analytic calculations. The limited analytical data that were available for the electron-diode case was also compared to computer simulated results showing good agreement. It was found that the relaxation effect of collisions in the simulated diode was smaller than it was predicted by exact calculations but this error decreased with decreasing time steps. Finally, the use of computer simulation techniques for calculating the damping effect of electron–neutral collisions on the oscillatory behavior of a plasma diode was demonstrated. These calculations show that it is possible to include the effect of randomizing collisions by a "brute-force" method into the computer simulation technique. The method can not be called mathematically "elegant," but in the case of large amplitude oscillations in a plasma system this is the only method at present which could give us some insight into the relevant mechanisms of these problems.

ACKNOWLEDGEMENTS

The author wishes to thank Professor O. Buneman, for helpful discussions on the subject, and Dr. Heinrich Derfler, for suggestions and for reading and correcting the manuscript.

This research was supported by the National Aeronautics and Space Administration Grant (Project 0254) NsG-299-63 at Stanford University

APPENDIX

The solution of the integro-differential equation

$$\mu \frac{\partial \psi(\xi, \mu)}{\partial \xi} = \frac{1}{2} \int_{-1}^{1} d\mu \psi(\xi, \mu) - \psi(\xi, \mu) \qquad \text{(A1)}$$

for the finite slab $0 < \xi < d/\lambda$ with boundary conditions

$$\begin{aligned} \psi(0, \mu) &= 1, \quad \text{for} \quad 0 < \mu < 1, \\ \psi(d/\lambda, \mu) &= 0, \quad \text{for} \quad -1 < \mu < 0, \end{aligned} \qquad \text{(A2)}$$

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is one of the basic problems of radiative transfer and has been analyzed extensively in the past.[9] However, we have not been able to find numerical results for the total transmitted flux as the function of slab thickness d/λ for the range $0 \leq d/\lambda \leq 5$; therefore, we have integrated Eq. (A1) numerically using a simple iterative procedure. Let us define $\psi(\xi, \mu)$ separately for $\mu > 0$ and $\mu < 0$ by the following expression:

$$\psi(\xi, \mu) = \begin{cases} \psi^+(\xi, \mu), & \text{for } \mu > 0, \\ \psi^-(\xi, -\mu), & \text{for } \mu < 0, \end{cases} \tag{A3}$$

and divide the region $0 < \mu < 1$ into K equal intervals. The midpoints of these intervals are

$$\mu_k = (2k - 1)/2K \qquad k = 1, 2, \cdots, K. \tag{A4}$$

Equation (A1) can be approximated by the following set of coupled differential equations:

$$\frac{d\psi^{\pm}(\xi, \mu_k)}{\partial \xi} = \pm \frac{1}{\mu_k} \left(\frac{1}{2K} \sum_{k=1}^{K} \psi^+(\xi, \mu_k) + \frac{1}{2K} \sum_{k=1}^{K} \psi^-(\xi, \mu_k) - \psi^{\pm}(\xi, \mu_k) \right) \tag{A5}$$

for $k = 1, 2, 3, \cdots, K$. We start the iteration by assuming that $\psi^-(\xi, \mu_k) = \psi_0^-(\xi, \mu_k) = 0$ for all μ_k and ξ and integrate $\psi^+(\xi, \mu_k)$ numerically [see Eq. (A5)] from $\xi = 0$ to $\xi = d/\lambda$ starting with the values $\psi^+(0, \mu_k) = 1$. The resulting function $\psi_1^+(\xi, \mu_k)$ then satisfies the following integral equation:

$$\psi_1^+(\xi, \mu_k) = 1 + \int_0^{\xi} \frac{1}{\mu_k} \cdot \left(\frac{1}{2K} \sum \psi_1^+(\xi, \mu_k) - \psi_1^+(\xi, \mu_k) \right) d\xi. \tag{A6}$$

After this step we integrate from $\xi = d/\lambda$ to $\xi = 0$ the function $\psi^-(\xi, \mu_k)$ [Eq. (A5)] using the boundary condition $\psi^-(d/\lambda, \mu_k) = 0$ (which is actually the initial value of ψ^-) and the known function $\psi_1^+(\xi, \mu_k)$. The resulting function $\psi_1^-(\xi, \mu_k)$ satisfies the following integral equation:

$$\psi_1^-(\xi, \mu_k) = -\int_{d/\lambda}^{\xi} \frac{1}{\mu_k} \left(\frac{1}{2K} \sum_{k=1}^{N} [\psi_1^+(\xi, \mu_k) + \psi_1^-(\xi, \mu_k)] - \psi_1^-(\xi, \mu_k) \right) d\xi \tag{A7}$$

for $k = 1, 2, 3, \cdots, K$. Having calculated the function $\psi_1^-(\xi, \mu_k)$, we can proceed with a forward integration again using the values of $\psi_1^-(\xi, \mu_k)$ and calculate $\psi_2^+(\xi, \mu_k)$ for $0 \leq \xi \leq d/\lambda$. After arriving at $\xi = d/\lambda$, we integrate backward and calculate $\psi_2^-(\xi, \mu_k)$. The iterative sequence satisfies the successive integral equations

$$\psi_{n+1}^+(\xi, \mu_k) = 1 + \int_0^{\xi} \frac{1}{\mu_k} \left(\frac{1}{2K} \sum_{k=1}^{K} [\psi_{n+1}^+(\xi, \mu_k) + \psi_n^-(\xi, \mu_k)] - \psi_{n+1}^+(\xi, \mu_k) \right) d\xi,$$

$$\psi_{n+1}^-(\xi, \mu_k) = -\int_{d/\lambda}^{\xi} \frac{1}{\mu_k} \left(\frac{1}{2K} \sum_{k=1}^{K} [\psi_{n+1}^+(\xi, \mu_k) + \psi_{n+1}^-(\xi, \mu_k)] - \psi_{n+1}^-(\xi, \mu_k) \right) d\xi. \tag{A8}$$

We are interested here only in the normalized flux, which is for the nth iteration,

$$\frac{\Gamma_n(\xi = 0)}{\Gamma_0} = 1 - \frac{2}{K} \sum_{k=1}^{K} \mu_k \psi_n^-(0, \mu_k),$$

$$\frac{\Gamma_n(\xi = d/\lambda)}{\Gamma_0} = \frac{2}{K} \sum_{k=1}^{K} \mu_k \psi_n^+(d/\lambda, \mu_k). \tag{A9}$$

Theoretically $\Gamma(\xi = 0)$ should be equal to $\Gamma(\xi = d/\lambda)$. The difference between the two computed values of the flux is an indication of the accuracy of our results. We have used fourth-order Runge–Kutta method for integration and stopped the calculations when the relative difference between iteration steps was less than 0.001, or

$$\frac{\Gamma_n(\xi = 0) - \Gamma_{n-1}(\xi = 0)}{\Gamma_n(\xi = 0)} < 0.001 \tag{A10}$$

was satisfied. The numerical results are shown in Table I.

TABLE I. Numerical results.

d/λ	Number of iteration	$\Gamma(\xi = 0)$	$\Gamma(\xi = d/\lambda)$	Average Γ
0.3	3	0.7936	0.7935	0.7935
0.5	4	0.7042	0.7041	0.7041
0.7	5	0.6343	0.6342	0.6343
1.0	6	0.5533	0.5531	0.5532
1.2	6	0.5103	0.5099	0.5101
1.5	7	0.4571	0.4567	0.4569
1.7	8	0.4275	0.4271	0.4273
2.0	9	0.3898	0.3892	0.3895
2.5	11	0.3398	0.3392	0.3395
3.0	13	0.3015	0.3008	0.3012
3.5	15	0.2709	0.2700	0.2704
4.0	18	0.2459	0.2451	0.2455
4.5	20	0.2252	0.2241	0.2247
5.0	23	0.2079	0.2068	0.2073

It is worthwhile to mention that the average flux $\frac{1}{2}[\Gamma_n(\xi = 0) + \Gamma_n(\xi = d/\lambda)]$ converged much more rapidly than is indicated by Table I, where convergence was reached only when Eq. (A10) was satisfied. We have used the value $K = 20$ throughout our computations.

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