Singular value decomposition

→ Matrix factorization technique
product of matrices : U, E and V^T $\boxed{A = UEV^T}$

→ U and V are orthogonal matrices
[E is diagonal matrix with non-negative elements
└→ orthogonal means AA^T.

SVD applications : 1. Data analysis
2. Signal processing
3. Image compression
4. ML

→ smaller singular values.
→ Ability to reduce dimensionality of dataset.

Process :

1. Anat A with m×n
2. Compute A^TA. & its eigen vectors, eigen values
   Eigen vectors form col of V,
   Eigen values are square of values in Σ
3. Arrange eigen values in Decreasing

   If symmetric or square matrix U and V are same

SVD of A is the triplet (U, Σ, V^T)

various methods,
- Jacobi method
- power iteration
- QR decomposition

$$A = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 6 \end{bmatrix}$$

Raw, column are same, so symmetric
so U and V are same

A be $m \times n$ matrix

Orthogonal matrices U, V & Diagonal matrix of Σ $\quad A = U\Sigma V^T$

V are column vectors of $A^T A$

$$A^T A = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 6 \end{bmatrix} \begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 6 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 12 & 12 & -20 \\ 12 & 12 & -20 \\ -20 & -20 & 44 \end{bmatrix} = 4 \underbrace{\begin{bmatrix} 3 & 3 & -5 \\ 3 & 3 & -5 \\ -5 & -5 & 11 \end{bmatrix}}_{B} \qquad AA^T = 4B$$

Eigen Values of B :

CE: $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = 3+3+11 = 17$ (sum of diagonal)

$$S_2 = \begin{vmatrix} 3 & -5 \\ -5 & 11 \end{vmatrix} + \begin{vmatrix} 3 & -5 \\ -5 & 11 \end{vmatrix} + \begin{vmatrix} 3 & 3 \\ 3 & 3 \end{vmatrix} = (33-25) + (33-25) + (9-9) = 16$$

$$\begin{array}{l} \\ S_3 = \\ \det(A) \end{array} \begin{array}{ccc} + & - & (+) \end{array}$$

$$S_3 = \begin{vmatrix} 3 & 3 & -5 \\ 3 & 3 & -5 \\ -5 & -5 & 11 \end{vmatrix} = 3\begin{vmatrix} 3 & -5 \\ -5 & 11 \end{vmatrix} - 3\begin{vmatrix} 3 & -5 \\ -5 & 11 \end{vmatrix} - 5\begin{vmatrix} 3 & 3 \\ -5 & -5 \end{vmatrix} = 0$$

eqn : $\lambda^3 - 17\lambda^2 + 16\lambda = 0$

$\lambda(\lambda^2 - 17\lambda + 16) = 0$

$\lambda = 0; \ (\lambda^2 - \lambda - 16\lambda + 16) = 0$

$\lambda = 0; \ (\lambda - 1)(\lambda - 16) = 0 \qquad \lambda = 1, 16$

Eigen values of B are 0, 1, 16

Eigen values of $A^T A = 4B$

$= 4(0, 1, 16)$

$\boxed{= 0, 4, 64}$

$\lambda_1 = 64, \ \lambda_2 = 4, \ \lambda_3 = 0$

Eigen vectors: $(A^T A - \lambda I)X = 0$

$$\begin{bmatrix} 12-\lambda & 12 & -20 \\ 12 & 12-\lambda & -20 \\ -20 & -20 & 44-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda_1 = 64$:

$$\begin{bmatrix} 12-64 & 12 & -20 \\ 12 & 12-64 & -20 \\ -20 & -20 & 44-64 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -52 & 12 & -20 \\ 12 & -52 & -20 \\ -20 & -20 & -20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-52x_1 + 12x_2 - 20x_3 = 0 \div 4 \Rightarrow -13x_1 + 3x_2 - 5x_3 = 0$$
$$12x_1 - 52x_2 - 20x_3 = 0 \div 4 \Rightarrow 3x_1 - 13x_2 - 5x_3 = 0$$

cross multiply:

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 3 & -5 & -13 & 3 \\ -13 & -5 & 3 & -13 \end{array}$$

$$\frac{x_1}{-15-65} = \frac{x_2}{-15-65} = \frac{x_3}{169-9}$$

$$\frac{x_1}{-80} = \frac{x_2}{-80} = \frac{x_3}{160}$$

$$(\div 80) \Rightarrow \frac{x}{-1} = \frac{x}{-1} = \frac{x_3}{2}$$

$$X_1 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

Norm of X_1

$$|X_1| = \sqrt{(-1)^2 + (-1)^2 + 2^2}$$
$$= \sqrt{1+1+4}$$
$$= \sqrt{6}$$

Normalized eigen vector $= V_1 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{6}} \end{bmatrix}$

$\lambda = 4$:
$$\begin{bmatrix} 12-4 & 12 & -20 \\ 12 & 12-4 & -20 \\ -20 & -20 & 44-4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 12 & -20 \\ 12 & 8 & -20 \\ -20 & -20 & 40 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$8x_1 + 12x_2 - 20x_3 = 0 \Rightarrow 2x_1 + 3x_2 - 5x_3 = 0$

$12x_1 + 8x_2 - 20x_3 = 0 \Rightarrow 3x_1 + 2x_2 - 5x_3 = 0$

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 3 & -5 & 2 & 3 \\ 2 & -5 & 3 & 2 \end{array}$$

$$\frac{x_1}{-15+10} = \frac{x_2}{-15+10} = \frac{x_3}{4-9}$$

$$\frac{x_1}{-5} = \frac{x_2}{-5} = \frac{x_3}{-5}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Norm of X_1

$|X_2| = \sqrt{1^2+1^2+1^2}$

$= \sqrt{1+1+1} = \sqrt{3}$

Normalized eigen vector

$$v_2 = \begin{bmatrix} 1/\sqrt{3} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

$\underline{\underline{\lambda = 0}}$:

$$\begin{bmatrix} 12 & 12 & -20 \\ 12 & 12 & -20 \\ -20 & -20 & 44 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$12x_1 + 12x_2 - 20x_3 \Rightarrow 3x_1 + 3x_2 - 5x_3 = 0$

$-20x_1 - 20x_2 + 44x_3 \Rightarrow -5x_1 - 5x_2 + 11x_3 = 0$

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 3 & -5 & 3 & 3 \\ -5 & 11 & -5 & -5 \end{array}$$

$$\frac{x_1}{33-25} = \frac{x_2}{25-33} = \frac{x_3}{-15+15}$$

$$\frac{x_1}{8} = \frac{x_2}{-8} = \frac{x_3}{0}$$

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{0}$$

$X_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

Norm $(X_3) = \sqrt{1^2 + (-1)^2 + 0}$

$= \sqrt{1+1} = \sqrt{2}$

Normalized vector

$V_3 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$

A symmetric $\boxed{U = V}$

$$U = \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 \\ 0 & 0 & \sqrt{\lambda_3} \end{bmatrix} = \begin{bmatrix} \sqrt{64} & 0 & 0 \\ 0 & \sqrt{4} & 0 \\ 0 & 0 & \sqrt{0} \end{bmatrix}$$

square root of eigen values of AA^T

$$= \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$A = U\Sigma V'$

$$= \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} \\ \frac{-2}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \end{bmatrix} \begin{bmatrix} 8 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{bmatrix}$$