1.1 Introduction:-

- Time domain approach (based on differential Equation) can be used to deal with linear and non-linear systems but transfer functions are limited to linear systems
- Beside controller design, time-domain approach offers a variety of methods to analyze dynamical behaviour of the plant
- Time-domain methods can solve control problems for MIMO (Multiple Input Multiple Output)
- With Time-domain, we can deal with control engg. problems with complex higher order plant. systems.

⇒ Laplace Transform - frequency domain

⇒ Differential equation in } - Time domain
matrix vector eqn & its derivative with time

⇒ Transfer function → that get from application of laplace transform → which describes the input-output behaviour of the plant

1.2 first Examples of Time-Domain System Models

1.2 ⇒ A typical approach to solve control problems using a model-based approach reads as follow.

STEP: 1 Modelling - Derive dynamic equations

STEP 2: Simplify Model equations - Eg. linearization, small angle assumptions....

STEP 3: Standard Representation - Eg: transfer function, state space equations....

STEP 4: Model Analysis - Eg: Stability, Controllability, observability.....

STEP 5: Controller Design - Eg: PID - Control, state vector feedback

STEP 6: Controller Validation - in simulation and on the real system

1.3 General form of Linear State-space Representation

① Linear SISO system:

- State equation $\dot{\underline{x}} = \underline{A} \cdot \underline{x} + \underline{b} \cdot u + (\underline{b}_2 z)$
- Output equation $y = \underline{c}^T \cdot \underline{x} + d \cdot u$
- Initial / starting condition for state vector $\underline{x}(t_0) = \underline{x}_0$

where $\underline{A}$ - state space / dynamic matrix $(n \times n)$; y - output signal (1×1)

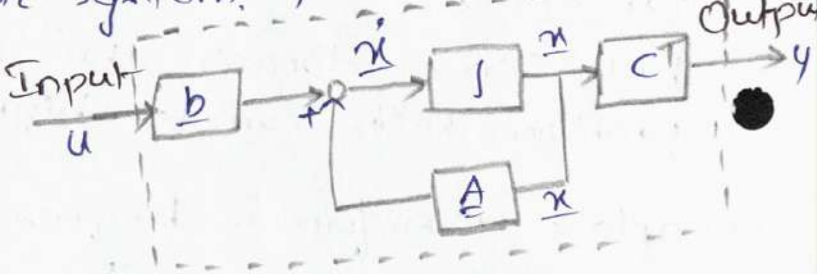
$\underline{b}$ - control vector $(n \times 1)$; $\underline{x}$ - state vector $(n \times 1)$

u - input signal (1×1) ; $\underline{c}^T$ - Output matrix vector $(1 \times n)$

d - feed-through vector factor (1×1)

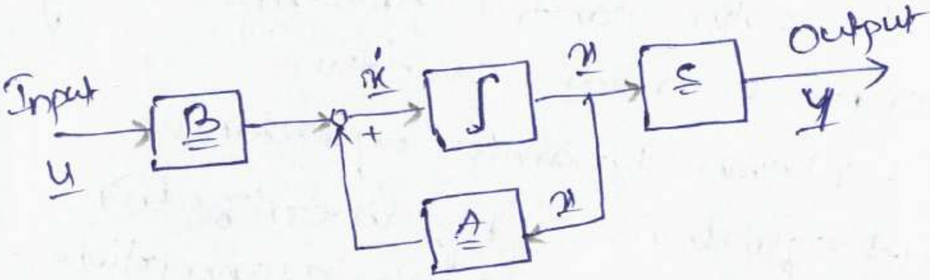
n - order of dynamic system, $\underline{x}_0$ - initial state

⇒ Block diagram of state-space representation of a (SISO) plant



② Linear MIMO system

- state equation $\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B} \cdot \underline{u}$ $(+ (\underline{B}_z \cdot \underline{z}))$ disturbance
- Output equation $\underline{y} = \underline{C} \cdot \underline{x} + \underline{D} \cdot \underline{u}$
- Initial condition of state vector $\underline{x}(t_0) = \underline{x}_0$



Here,

$\underline{x}$ - state vector $(n \times 1)$

$\underline{y}$ - output vector $(m \times 1)$

$\underline{u}$ - input vector $(r \times 1)$

$\underline{A}$ - Dynamic matrix $(n \times n)$

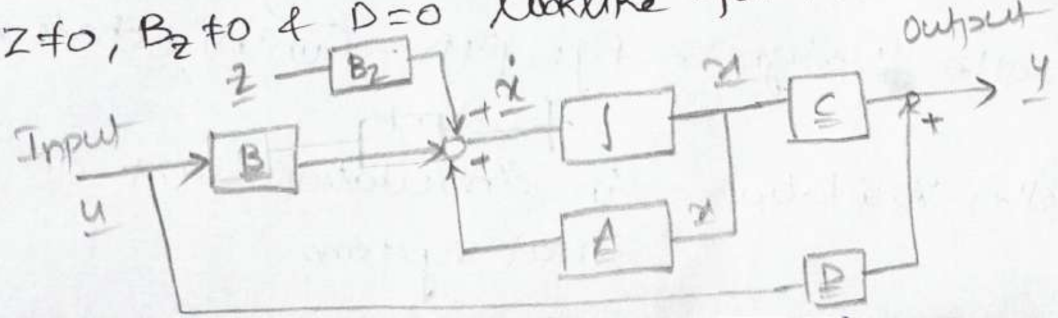
$\underline{B}$ - Control matrix $(n \times r)$

$\underline{C}$ - Output matrix $(m \times n)$

$\underline{D}$ - feedthrough matrix $(m \times r)$

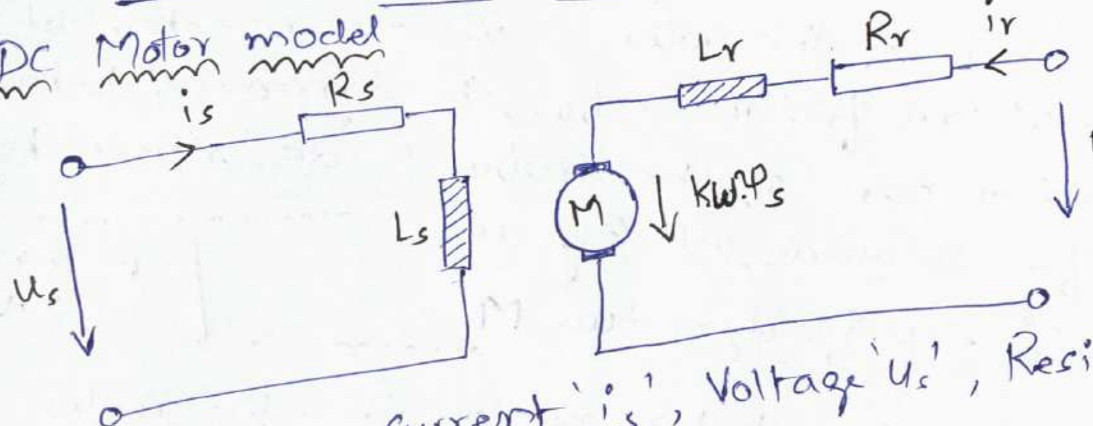
⇒ Block Diagram in general case for MIMO case looklike

$z \neq 0$, $B_z \neq 0$ & $D = 0$



1.2.1 L.N.2.1 Example of a non-linear plant model Lec-1

A DC Motor model



stator circuit: Current 'i_s', Voltage 'u_s', Resistance 'R_s', inductance 'L_s'

Rotor circuit: Current 'i_r', Voltage 'u_r', resistance 'R_r', inductance 'L_r', flux linkage ψ_s, induced voltage $u_{rw} = K \cdot w \cdot \psi_s$

Goal:- DE describing dynamical behaviour of the rotational speed (w - angular velocity) of rotor depending on 3 inputs (u_s, u_r, M_L)

The stator of motor comprises electromagnets (u_s, i_s) and generates electric flux linkage

$\psi_s = L_s \cdot i_s$

and induced voltage on stator is, $u_{sw} = L_s \cdot \frac{di_s}{dt}$

$$u_{sw} = \frac{d\psi_s}{dt}$$

Total voltage on stator,

$$u_s = u_{sw} + R_s \cdot i_s = \frac{d\psi_s}{dt} + R_s \cdot \frac{\psi_s}{L_s}$$

In the rotor winding, the flux linkage induces the voltage $u_{rw} = Kw\psi_s$ (K - m/c constant, w - ang velocity)

Total voltage on rotor is,

$$u_r = R_r \cdot i_r + L_r \cdot \frac{di_r}{dt} + K \cdot w \cdot \psi_s$$

$$\Leftrightarrow R_r \cdot i_r + L_r \cdot \frac{di_r}{dt} = u_r - K \cdot w \cdot \psi_s$$

The electric motor generates the resulting torque

$$M_{GMS} = K \cdot i_r \cdot \psi_s$$

with the external torque M_l acting on the rotor, the Inertia 'J' of the rotor & viscous mechanical friction '$d \cdot \omega$' (d - viscous friction coeff), the dynamical behaviour of mechanical part of the motor is described by the DE for angular velocity 'ω'

$$J \cdot \dot{\omega} = K \cdot i_r \cdot \Psi_s - d\omega - M_l \qquad \left[\because F = m \cdot a \atop = J \cdot \dot{\omega} \right]$$

so, the dynamical behaviour of the DC motor can be modelled by 3 coupled differential eqn's (DEs)

1\. ODE → flux dynamics : $\frac{d\Psi_s}{dt} + \frac{R_s}{L_s} \cdot \Psi_s = u_s$

2\. ODE → rotor current i_r : $L_r \cdot \frac{di_r}{dt} + R_r \cdot i_r = u_r - \underline{K \cdot \omega \cdot \Psi_s}$

3\. ODE → angular velocity 'ω' : $J \cdot \dot{\omega} + d \cdot \omega = K \cdot \underline{i_r \cdot} \Psi_s - M_l$

Above eqns contain products of time-varying quantities $(\omega * \Psi_s), (i_r * \Psi_s)$, so, those are set of non-linear ODEs & the related dynamical system is called non-linear system

⇒ Flux dynamics ($x_1 = \Psi_s, u_1 = u_s$): $\frac{dx_1}{dt} = \frac{-R_s}{L_s} \cdot x_1 + u_1$

⇒ Rotor current i_r ($x_2 = i_r, u_2 = u_r$) dynamics:

$$L_r \cdot \frac{dx_2}{dt} = -R_r \cdot x_2 - K \cdot \omega \cdot x_1 + u_r$$

⇒ Angular velocity dynamics [~~ω~~ $x_3 = \omega$, $u_3 = M_l$]

$$J \cdot \frac{dx_3}{dt} = -d\, x_3 + K \cdot x_2 \cdot x_1 - u_3$$

standard representation as a vector equation...

$$\text{state eqn} \rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -R_s/L_s \cdot x_1 + u_1 \\ (-R_r \cdot x_2 - K \cdot x_3 \cdot x_1 + u_2) \cdot L_r^{-1} \\ J^{-1} \cdot (-d \cdot x_3 + K \cdot x_2 \cdot x_1 - u_3) \end{bmatrix} \Leftrightarrow \dot{x} = \begin{bmatrix} f_1(x,u) \\ f_2(x,u) \\ f_3(x,u) \end{bmatrix} = \underline{f}(x,u)$$

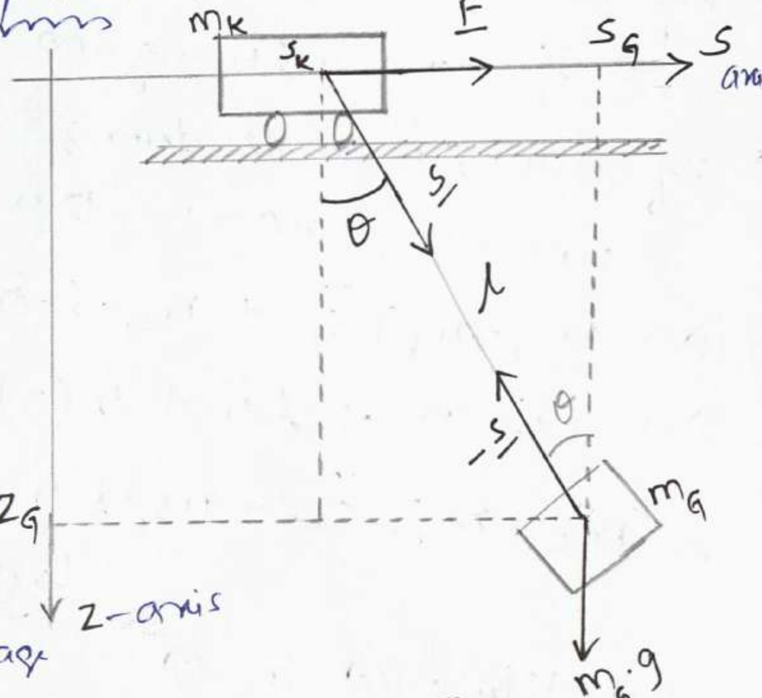
$$\text{output eqn} \rightarrow y_1 = \omega = x_3 \qquad \Rightarrow \underline{y} = \underline{g}(\underline{x}, \underline{u})$$

where $\underline{f}(\underline{x}, \underline{u})$ and $\underline{g}(x, u)$ are non-linear, vector valued function depen on $\underline{u}$ (varying with time) and system states ($\underline{x}$)

In general case : $\underline{\dot{x}}(t) = (\underline{f}(\underline{x}(t), \underline{u}(t)))$ & $\underline{y}(t) = g(\underline{x}(t), \underline{u}(t))$

1.2.2 Example of a Linear plant model

⇒ CONTAINER CARRIAGE SYSTEM



Control i/p (or) Variable : force $\underline{f}$

Variable to control : Position s_G

m_K - mass of carriage

m_G - " " container

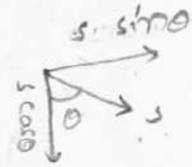
s_K - position of carriage

s_G - " " container

$\underline{f}$ - i/p force on ~~contain~~ carriage

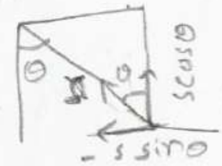
$\underline{s}$ - (Rope / cable) force on container

By applying Newton's law of body $\underline{K}$ (carriage) & G (container) with cable force 's' ;



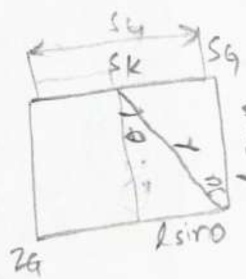
carriage : $m_K \cdot \ddot{s}_K = s \cdot \sin\theta + f \rightarrow (1)$

Z-axis : no vertical movement



Container : s-axis $\Rightarrow m_G \cdot \ddot{s}_G = -s \cdot \sin\theta \rightarrow (2)$

Z-axis $\Rightarrow m_G \cdot \ddot{Z}_G = m_G \cdot g - s \cdot \cos\theta \rightarrow (3)$



Geometric coupling of subsystem :

s-axis $S_G = S_K + l \cdot \sin\theta$ $\quad (s_G - s_K)\operatorname{cosec}\theta$

z-axis $Z_G = l\cos\theta$

$\dot{S}_G = \dot{S}_K + \underline{l\cos\theta \cdot \dot{\theta}}$ | $\dot{Z}_G = l(-\sin\theta)\cdot\dot{\theta}$

$\ddot{S}_G = \ddot{S}_K + l\dot{\theta}(-\sin\theta)\cdot\dot{\theta} + l\cos\theta\cdot\ddot{\theta}$

$= \ddot{S}_K + l\cdot\ddot{\theta}\cos\theta - l(\dot{\theta})^2 \sin\theta$

$\ddot{Z}_G = -l\sin\theta\cdot\ddot{\theta} - \cancel{s}\, l\dot{\theta}\cos\theta\cdot\dot{\theta} = -l\sin\theta\cdot\ddot{\theta} - l\dot{\theta}^2\cos\theta$

Add forces on 's' axis i.e. (1) + (2) ⇒

$m_G \cdot \ddot{s}_G + m_K \cdot \ddot{s}_K = -s \cdot \cancel{\sin\theta} + s \cdot \cancel{\sin\theta} + F$

$m_G(\ddot{s}_K + l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta) + m_K \cdot \ddot{s}_K = f$

$(m_K + m_G) \cdot \ddot{s}_K + m_G \cdot l \cdot \ddot{\theta} \cos\theta - m_G \cdot l \cdot \dot{\theta}^2 \sin\theta = F \rightarrow ①$

By solving z axis force,

$\rightarrow ②$

$\ddot{s}_K \cos\theta + l\ddot{\theta} + g\sin\theta = 0$

If $|\theta|$ & $|\dot{\theta}|$ small during operation, then

$\sin\theta \approx \theta$, $\cos\theta \approx 1$, $\dot{\theta}^2 \cdot \sin\theta \approx 0$

$① \Rightarrow (m_G + m_K)\ddot{s}_K + m_G \cdot l \cdot \ddot{\theta} \cdot (1) - 0 = F$

$② \Rightarrow \ddot{s}_K (1) + l\ddot{\theta} + \theta(g) = 0$

$\therefore (m_K + m_G)\ddot{s}_K + m_G \cdot l \cdot \ddot{\theta} = F$ & $\ddot{s}_K + l\ddot{\theta} + g\theta = 0$

$\hookrightarrow ③$ $\qquad l\ddot{\theta} = -\ddot{s}_K - g\theta$

$\hookrightarrow ④$

substitute ④ in ③

$(m_K + m_G)\ddot{s}_K + m_G(-\ddot{s}_K - g\theta) = F$

$m_K \cdot \ddot{s}_K + \cancel{m_G \ddot{s}_K} - \cancel{m_G \ddot{s}_K} - m_G g\theta = F$

$\ddot{s}_K = \frac{m_G}{m_K} \cdot g \cdot \theta + \frac{F}{m_K}$

$④ \Rightarrow \ddot{\theta} = \frac{-\ddot{s}_K}{l} - \frac{g\theta}{l} = \frac{1}{l}\left[-\frac{m_G}{m_K} \cdot g \cdot \theta\right] + \left(\frac{-F}{m_K \cdot l}\right) - \frac{g\theta}{l}$

$= \frac{-F}{m_K \cdot l} - \frac{g\theta}{l}\left[\frac{m_G}{m_K} + 1\right]$

$\ddot{\theta} = \frac{-g}{l}\left(\frac{m_G + m_K}{m_K}\right) \cdot \theta - \frac{F}{m_K \cdot l}$

$\therefore \ddot{s}_K = \frac{m_G}{m_K} \cdot g \cdot \theta + \frac{F}{m_K}$; $\ddot{\theta} = \frac{-g}{l}\left(\frac{m_G + m_K}{m_K}\right)\theta - \frac{F}{m_K \cdot l}$

New state variables

$x_1 = s_K$ (carriage position)	$x_3 = \theta$ (Rope angle)
$x_2 = \dot{s}_K$ (carriage velocity)	$x_4 = \dot{\theta}$ (Angular velocity)

$x_1 = s_K$

$\dot{x}_1 = \dot{s}_K = x_2$

$x_3 = \theta$

$\dot{x}_3 = \dot{\theta} = x_4$

$\dot{x}_4 = \ddot{\theta}$

$= -\frac{g}{l}\left(\frac{m_G + m_K}{m_K}\right)\cdot\theta - \frac{F}{l m_K}$

$\dot{x}_2 = \ddot{s}_K$

$= \frac{m_G}{m_K}\cdot g\cdot\theta + \frac{F}{m_K}$

$\dot{x}_2 = \frac{m_G}{m_K}\cdot g\cdot x_3 + \frac{F}{m_K}$

$\dot{x}_4 = \frac{-g}{l}\left(\frac{m_G + m_K}{m_K}\right)x_3 - \frac{F}{l m_K}$

Standard Representation

- $\dot{x}_1 = x_2$
  $\dot{x}_2 = \frac{m_G}{m_K}\cdot g\cdot x_3 + \frac{F}{m_K}$
  $\dot{x}_3 = x_4$
  $\dot{x}_4 = \frac{-g}{l}\left(\frac{m_G + m_K}{m_K}\right)x_3 - \frac{F}{l m_K}$

$$\underbrace{\underline{\dot{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{m_G}{m_K}\cdot g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-g}{l}\left(\frac{m_G+m_K}{m_K}\right) & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m_K} \\ 0 \\ -\frac{1}{l m_K} \end{bmatrix} F}_{\text{Dynamic state eqn.}}$$

- O/p eqn: $s_G = s_K + l\theta$
  $y = x_1 + l x_3$
  $y = (1 \;\; 0 \;\; l \;\; 0)\,\underline{x} + 0\cdot F$

The container carriage system has one scalar input 'u' (i.e motor force F at carriage) and one (just) scalar output y to be controlled (position of container s_G). Therefore this system is Single Input Single output (SISO) system

1.5 Linearization - from Non-linear to linear Equations

Suppose Non linear equations
$$\left.\begin{aligned} \dot{\underline{x}} &= \underline{f}(\underline{x},\underline{u}) \Rightarrow \text{state differential equations} \\ \underline{y} &= g(\underline{x},\underline{u}) \Rightarrow \text{output equation} \end{aligned}\right\}$$
, are non linear equations

① Calculation of Equilibrium Points (Equilibrium States)

In equilibrium state $\dot{\underline{x}}=0$ if $\underline{x} \to \underline{x}_\infty$, $\underline{u} \to \underline{u}_\infty$

$\therefore\ 0 = \underline{f}(\underline{x}_\infty, \underline{u}_\infty)$ (equilibrium equation for the states)

and $\underline{y}_\infty = (\underline{g}\ \underline{x}_\infty, \underline{u}_\infty)$, steady state output vector is calculated from the output equation of $\underline{x}_\infty, \underline{u}_\infty$ in equilibrium condition

② Linearization around Equilibrium

for the linearization, the difference between equilibrium and the actual value is required, which is denoted by Δ

$$\Delta \underline{x} = \underline{x} - \underline{x}_\infty$$
$$\Delta \underline{u} = \underline{u} - \underline{u}_\infty$$
$$\Delta \underline{y} = \underline{y} - \underline{y}_\infty$$

Linearization by Taylor series

$$\frac{d}{dt}(\Delta \underline{x}) = \Delta \dot{\underline{x}} \Rightarrow \dot{\underline{x}} = \underline{f}(\underline{x}_\infty + \Delta\underline{x}, \underline{u}_\infty + \Delta\underline{u}) \approx \underline{f}(\underline{x}_\infty, \underline{u}_\infty) + \left.\frac{\partial f}{\partial x}\right|_\infty \cdot \Delta x + \left.\frac{\partial f}{\partial u}\right|_\infty \cdot \Delta u$$

$\Rightarrow$ Jacobian representation

Example: Linearization of Non-linear state and Output equation

$\dot{x}_1 = (x_1+4)(x_2-5)$; $\dot{x}_2 = x_1^2 + 4x_2 + 2u$; $y = 2 \cdot x_1 \cdot x_2$

so, $\dot{x}_1 = f_1(x_1, x_2, u) = (x_1+4)(x_2-5)$

$\dot{x}_2 = f_2(x_1, x_2, u) = x_1^2 + 4x_2 + 2u$

$y = g(x_1, x_2, u) = 2 \cdot x_1 \cdot x_2$

$$\dot{\underline{x}} = \begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix} f_1(x_1,x_2,u) \\ f_2(x_1,x_2,u)\end{pmatrix} = \begin{bmatrix}(x_1+4)(x_2-5) \\ x_1^2+4x_2+2u\end{bmatrix}$$

$y = g(x_1, x_2, u) = 2 \cdot x_1 \cdot x_2$ for $\underline{x} = (x_1, x_2)^T$

i) Calculation of equilibrium state

$\dot{x}_1 = 0$ & $\dot{x}_2 = 0$ with $x \to x_\infty$, $u \to u_\infty$

$\cancel{x_{1,\infty}+4}\,(x_{1,\infty}+4)(x_{2,\infty}-5) = 0$ | $x_{1,\infty}^2 + 4x_{2,\infty} + 2u_\infty = 0$

$x_{1,\infty} = -4$; $x_{2,\infty} = 5$ | If $x_{1,\infty} = -4$

case(i) case(ii)

$$x_{2,\infty} = \frac{-16 - 2u_\infty}{4}$$
$$= -4 - 0.5u_\infty$$

As the equilibrium system has to real, the second case only exist for a steady state i/p

If $x_{2,\infty} = 5$

$$x_{1,\infty} = \sqrt{-20 - 2u_\infty}$$

$-20 - 2u_\infty \geq 0 \Rightarrow u_\infty \leq -10$

otherwise, the system has only one equilibrium point.

(ii) Linearization by Taylor series Expansion :-

The choice of case (i) or (ii) is more or less arbitrary here and we just deal with case (ii) in the following

assume $u_\infty = -10.5$, then $x_{1,\infty} = 1$, $x_{2,\infty} = 5$

$$y_\infty = 2 \cdot x_{1,\infty} \cdot x_{2,\infty} = 2\cdot(1)(5) = 10$$

To get linearized state & output equation i.e. Standard state space representation for linear systems, the Jacobians

$\left.\frac{\partial f}{\partial \underline{x}}\right|_\infty$, $\left.\frac{\partial f}{\partial \underline{u}}\right|_\infty$ and $\left.\frac{\partial g}{\partial \underline{x}}\right|_\infty$, $\left.\frac{\partial g}{\partial \underline{u}}\right|_\infty$ have to be calculated.

$$\left.\frac{\partial f}{\partial \underline{x}}\right|_\infty = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\underline{x}_\infty, \underline{u}_\infty) & \frac{\partial f_1}{\partial x_2}(\underline{x}_\infty, \underline{u}_\infty) \\ \frac{\partial f_2}{\partial x_1}(\underline{x}_\infty, \underline{u}_\infty) & \frac{\partial f_2}{\partial x_2}(\underline{x}_\infty, \underline{u}_\infty) \end{bmatrix} = \begin{bmatrix} x_{2,\infty} - 5 & x_{1,\infty} + 4 \\ 2x_{1,\infty} & 4 \end{bmatrix} = \underline{A}$$

$$\left.\frac{\partial f}{\partial \underline{u}}\right|_\infty = \begin{bmatrix} \frac{\partial f_1}{\partial u}(u_\infty, x_\infty) \\ \frac{\partial f_2}{\partial u}(\underline{x}_\infty, \underline{u}_\infty) \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \underline{B}$$

$$\left.\frac{\partial g}{\partial \underline{x}}\right|_\infty = \begin{bmatrix} \frac{\partial g}{\partial x_1}(\underline{x}_\infty, \underline{u}_\infty) & \frac{\partial g}{\partial x_2}(\underline{x}_\infty, \underline{u}_\infty) \end{bmatrix} = \begin{pmatrix} 2x_{2,\infty} & 2x_{1,\infty} \end{pmatrix} = \underline{C}$$

$$\left.\frac{\partial g}{\partial \underline{u}}\right|_\infty = \frac{\partial g}{\partial \underline{u}}(\underline{x}_\infty, \underline{u}_\infty) = [0] = \underline{D}$$

∴ The original non-linear system can be described by the following linear state-space equation in the neighbour-hood of the linearization setpoint $x_{1,\infty}=1$

$x_{2,\infty}=5$ & $u_\infty=-10.5$

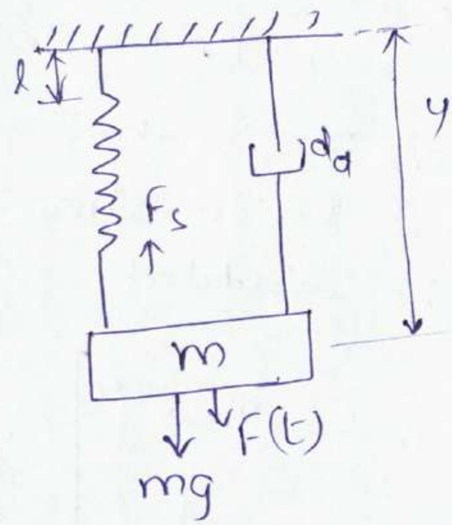
$$\underline{\underline{A}} = \begin{bmatrix} x_{2,\infty}-5 & x_{1,\infty}+4 \\ 2\cdot x_{1,\infty} & 4 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ 2 & 4 \end{bmatrix} ; \quad \underline{\underline{B}} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\underline{\underline{C}} = \begin{bmatrix} 2\cdot x_{2,\infty} & 2x_{1,\infty} \end{bmatrix} = \begin{bmatrix} 10 & 2 \end{bmatrix} ; \quad \underline{\underline{D}} = 0$$

$$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\cdot\underline{u} \Rightarrow \Delta\dot{\underline{x}} = \begin{bmatrix} 0 & 5 \\ 2 & 4 \end{bmatrix}\cdot\Delta\underline{x} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}\Delta u$$

$$\underline{y} = \underline{\underline{C}}\,\underline{x} + \underline{\underline{D}}\cdot u \Rightarrow \Delta y = \begin{bmatrix} 10 & 2 \end{bmatrix}\Delta\underline{x} + 0\cdot\Delta u$$

Example 2: Non-linear spring pendulum - Application Example



Here, y - measured position of the pendulum mass

m - mass of the pendulum

d_a - Co-efficient of linear damping proportional to velocity

l - Initial length of spring (w/o mass)

F_s - spring force

F_t - External force from an actuator used to control mass

$$m\ddot{y} = F(t) - F_s(y) + mg - d_a\cdot\dot{y}$$

Here we assume progressive spring defined by $F_s(y) = \underbrace{(y-l)^2}_{\Delta y}\cdot\alpha_s$

$$m\ddot{y} = F(t) - \alpha_s\cdot(y-l)^2 + mg - d_a\cdot\dot{y} \rightarrow (1)$$

$y(0) = y_0$ and $\dot{y}(0) = \dot{y}_0$

I) Non-linear state space form/model :-

New state variables $x_1 = y$, $x_2 = \dot{y}$ and $u = F(t)$

$\dot{x}_1 = \dot{y} = x_2$; $\dot{x}_2 = \ddot{y}$

$$\dot{x}_2 = \ddot{y} = \frac{1}{m}\left[F(t) - \alpha_s(y-l)^2 + mg - d_a\cdot\dot{y}\right]$$

$$\dot{x}_2 = \frac{1}{m}u(t) - \frac{\alpha_c}{m}(x_1 - l)^2 + g - \frac{d_d}{m} \cdot x_2$$

$$\Rightarrow \underline{\dot{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{\alpha_s}{m}(x_1 - l)^2 + g - \frac{d_d}{m}x_2 + \frac{1}{m}u(t) \end{bmatrix} = \underline{f}(\underline{x}, \underline{u})$$

$$y = x_1 = g(\underline{x}, \underline{u})$$ (I) non-linear

$$IC : \underline{x}_0 = \begin{bmatrix} x_{1,0} \\ x_{2,0} \end{bmatrix} = \begin{bmatrix} y(0) \\ \dot{y}(0) \end{bmatrix} = \begin{bmatrix} y_0 \\ \dot{y}_0 \end{bmatrix}$$

ii) Calculation of Equilibrium states: i.e

$$[\dot{x}] = 0 \Rightarrow x_{2,\infty} = 0$$

$$\Rightarrow -\frac{\alpha_s}{m}(x_{1,\infty} - l)^2 + g - \frac{d_d}{m}\overset{0}{x_{2,\infty}} + \frac{1}{m}u_\infty = 0$$

$$\Leftrightarrow -\frac{\alpha_s}{m}(x_{1,\infty} - l)^2 + g - 0 + \frac{1}{m}u_\infty = 0$$

$$\Leftrightarrow u_\infty = \left[\frac{\alpha_s}{m}(x_{1,\infty} - l)^2 - g\right] \cdot m$$

so, I/p force need to get the position $x_{1,\infty} = y_\infty$, we need a force

Eg: for $m = 2\,kg$, $\alpha_s = 250\, N/m^2$, $l = 0.1\,m$

$$u_\infty = (250(x_{1,\infty} - 0.1)^2 - (9.81)\,2$$

If $x_{1,\infty} = 0.25\,m \Rightarrow u_\infty = -14.0\,N$

so, from desired linearization point $x_{1,\infty}$, we can calculate necessary input u_∞

iii) Calculation of linearized model:

The linearized model describes the dynamical behaviour of $\Delta\underline{x} = \underline{x} - \underline{x}_\infty$, $\Delta y = y - y_\infty$ for an input $\Delta\underline{u} = \underline{u} - u_\infty$

This is sufficient later on to design a controller operating in the neighbourhood of the linearization point

Linearized state-space systems
$$\Delta\dot{\underline{x}} = \underline{A}\,\Delta\underline{x} + \underline{b} \cdot u$$
$$\Delta y = \underline{c}^T \cdot \Delta y + d\,u$$ (II) linear
$$IC : \Delta\underline{x}_0 = \underline{x}_0 - \underline{x}_\infty$$

$$\underline{A} = \left.\frac{\partial f}{\partial \underline{x}}\right|_\infty = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{\alpha_s}{m} \cdot 2(x_1 - l) & -\frac{d_d}{m} \end{bmatrix}$$

$$\underline{B} = \frac{\partial f}{\partial u}\Big|_{\infty} = \begin{bmatrix} \frac{\partial f_1}{\partial u}(x_\infty, u_\infty) \\ \frac{\partial f_2}{\partial u}(x_\infty, u_\infty) \end{bmatrix} = \begin{bmatrix} 0 \\ 1/m \end{bmatrix}$$

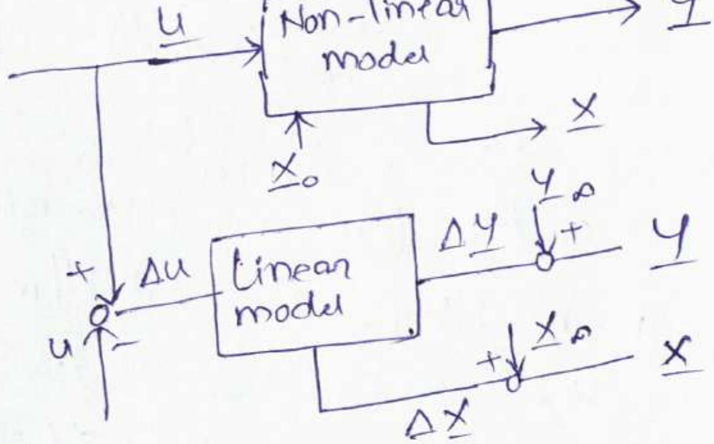
$$C = \frac{\partial g}{\partial x}\Big|_{\infty} = \begin{bmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$D = \frac{\partial g}{\partial u}\Big|_{\infty} = \frac{\partial g}{\partial u}(x_\infty, u_\infty) = [0]$$

IV) Non-linear and Linear Simulation:-

The linear system (II) describes the dynamical behaviour around the linearization point defined by $x_{1,\infty}, x_{2,\infty} = 0$ & u_∞. whereas the non-linear model describes the behaviour in absolute quantities

Both are related by



$\underline{x} = \Delta \underline{x} + \underline{x}_\infty$

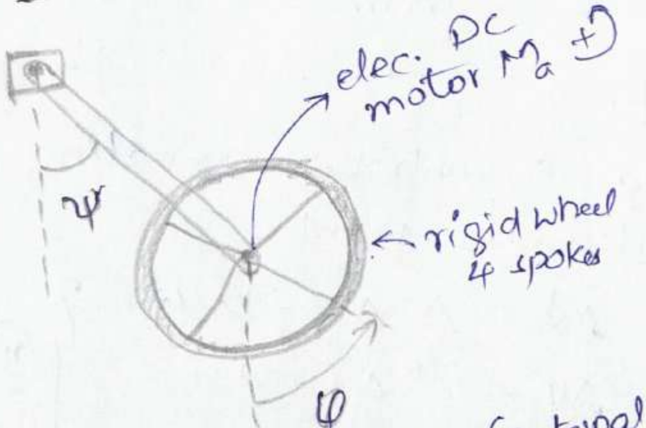
$\underline{y} = \Delta \underline{y} + \underline{y}_\infty$

$\underline{u} = \Delta \underline{u} + \underline{u}_\infty$

for same results close to $x_{1,\infty}, x_{2,\infty} = 0, u_\infty$ we have to convert the simulation values

$\Delta \underline{x}_0 = \underline{x}_0 - \underline{x}_\infty$

Example 3 : The Turning Wheel Pendulum - Application Example (2)



u: motor voltage (external)

Non-linear model equation

h_g - gear ratio

K - motor constant ($i \to M_a$)

R_a - resistance rotor

J - Inertia of the overall system including motor

m - overall mass

l - rod length

J_a - Inertia of rotor

$$\ddot{\psi} = \underbrace{\frac{-h_g}{R_a \cdot J}}_{a_1} \cdot u - \overbrace{\frac{m \cdot g\, l}{J}}^{a_2} \underline{\sin\psi} + \underbrace{\frac{h_g^2 k^2}{R_a \cdot J}}_{a_3} \dot{\varphi} - \underbrace{\frac{h_g \cdot k^2}{R_a \cdot J}}_{a_4} \dot{\psi} \to \text{Non linear}$$

$$\ddot{\varphi} = \underbrace{\frac{h_g K}{R_a' J_a}}_{b_1} \cdot u - \underbrace{\frac{h_g^2 \cdot K^2}{R_a' J_a}}_{b_2} \dot{\varphi} + \underbrace{\frac{h_g K^2}{R_a' J_a}}_{b_3} \cdot \dot{\psi}$$

above eqn's are 2 coupled ODE's - non linear

I) Non-linear standard Representation:

New state variables $x_1 = \psi$; $x_2 = \dot{\psi}$; $x_3 = \varphi$; $x_4 = \dot{\varphi}$

$\dot{x}_1 = \dot{\psi} = x_2$; | $\dot{x}_3 = \dot{\varphi} = x_4$

$\dot{x}_2 = \ddot{\psi}$ | $\dot{x}_4 = \ddot{\varphi}$

$$\dot{\underline{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ a_1 u + a_2 \sin x_1 + a_3 x_4 + a_4 x_2 \\ x_4 \\ b_1 u + b_2 x_4 + b_3 x_2 \end{bmatrix} = \underline{f}(\underline{u}, \underline{x})$$

Model output : $\underline{y} = \begin{bmatrix} \psi \\ \varphi \end{bmatrix} = \underline{g}(\underline{x}, \underline{u}) = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}$

II) Linearization process

Calculation of Equilibrium model

$$\dot{\underline{x}} \overset{!}{=} 0 \Rightarrow \underline{f}(\underline{x}_\infty, \underline{u}_\infty) = 0 \Rightarrow \begin{cases} x_2 = 0 \\ f_2(\underline{x}_\infty, \underline{u}_\infty) = 0 \\ x_4 = 0 \\ f_4(\underline{x}_\infty, \underline{u}_\infty) = 0 \end{cases}$$

i.e. $a_1 u + a_2 \sin x_1 + a_3 \cancel{x_4}^{0} + a_4 \cancel{x_2}^{0} = 0$ | $b_1 u + b_2 \cancel{x_4}^{0} + b_3 \cancel{x_2}^{0} = 0$

$a_1 u + a_2 \sin x_1 = 0$ | $b_1 u = 0 \Rightarrow u = 0$ for equilibrium

$0 + a_2 \sin x_1 = 0$

$a_2 \cdot x_{1,\infty} \overset{!}{=} 0 \Rightarrow$ 2 solutions in $\psi = x_1 \in (0, 2\pi]$

i.e. $x_{1,\infty_1} = 0$; $x_{1,\infty_2} = 2\pi$

$x_3 = \varphi$ can be chosen arbitarily?

So, Depending on our situation, $x_{1,\infty} = \psi$ can be chosen

Task 1: Inverted pendulum

- chose $\underline{x}_\infty = [\pi \;\; 0 \;\; 0 \;\; 0]^T$

Task 2 : maximize damping of pendulum in downward pos

$\rightsquigarrow$ choose $\underline{x}_\infty = [0, 0, 0, 0]^T$ & , $u_\infty = 0$

Calculation of linearized model : $\because x_{1,\infty} = \pi$

$$\underline{\underline{A}} = \left.\frac{\partial \underline{f}}{\partial \underline{x}}\right|_{\infty} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a_2 \cos x_{1,\infty} & a_4 & 0 & a_3 \\ 0 & 0 & 0 & 1 \\ 0 & b_3 & 0 & b_2 \end{bmatrix}_{\infty} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -a_2 & a_4 & 0 & a_3 \\ 0 & 0 & 0 & 1 \\ 0 & b_3 & 0 & b_2 \end{bmatrix}$$

$$\underline{B} = \left.\frac{\partial \underline{f}}{\partial \underline{u}}\right|_{\infty} = \begin{bmatrix} 0 \\ a_1 \\ 0 \\ b_1 \end{bmatrix} ; \quad \underline{\underline{C}} = \left.\frac{\partial \underline{g}}{\partial \underline{x}}\right|_{\infty} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\underline{\underline{d}} = \left.\frac{\partial \underline{g}}{\partial \underline{u}}\right|_{\infty} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The linearized model describes the dynamical behaviour in the neighbourhood of the equilibrium

$\Delta \underline{x} = \underline{x} - \underline{x}_\infty, \ \Delta \underline{y} = \underline{y} - \underline{y}_\infty, \ \Delta \underline{u} = \underline{u} - \underline{u}_\infty$

$$\left.\begin{aligned} \Delta \dot{\underline{x}} &= \underline{\underline{A}} \cdot \Delta \underline{x} + \underline{b} \cdot \Delta u \\ \Delta \underline{y} &= \underline{\underline{C}} \cdot \Delta \underline{x} + \underline{0} \cdot \Delta u \end{aligned}\right\} \rightarrow$$ It is the linearized model for analysis & controller design

To get an approximation of original variables we use
$$\underline{x} = \Delta \underline{x} + \underline{x}_\infty$$
$$\underline{u} = \Delta \underline{u} + \underline{u}_\infty$$
$$\underline{y} = \Delta \underline{y} + \underline{y}_\infty$$

1.6 Solution of the State-space Equation :-

Different methods for calculation of solution. But general goal (i) solution in the time domain to get $y(t)$ output vector signal & $x(t)$ - a state vector signal with $t \in [0, \infty)$

(ii) solution in the frequency domain seeks for corresponding laplace transformed signals $Y(s)$ & $X(s)$.

(i) Solution in Time Domain :

$$\left.\begin{aligned} \dot{\underline{x}}(t) &= \underline{\underline{A}}\, \underline{x} + \underline{\underline{B}} \cdot u(t) \rightarrow (1) \\ \underline{y}(t) &= \underline{\underline{C}}\, \underline{x}(t) + \underline{\underline{D}} \cdot u(t) \rightarrow (2) \end{aligned}\right\}$$ state space equations in time domain

for an initial value vector $\underline{x}_0(t_0) = \underline{x}(0) = \underline{x}_0$, so scalar DE is

$\dot{x} = a x + b \cdot u \qquad Ic ; x(t_0) = x_0$

The solution of this linear scalar DE is

$$x(t) = x_{hom}(t) + x_{inhom}(t) = e^{a(t-t_0)} \cdot x_0 + \int_{t_0}^{t} e^{a(t-\tau)} \cdot b \cdot u(\tau) d\tau$$

x_{hom} → homogeneous solution for initial condition x_0 of i/p $u(t)=0$

x_{inhom} → inhomogeneous solution for initial condition $x(t_0)=0$ & an arbitrary o/p $u(t)$

so, solution of eq ① is

$$\underline{x}(t) = \underline{x}_{hom}(t) + x_{inhom}(t)$$

$$= e^{\underline{A}(t-t_0)} \cdot \underline{x}_0 + \int_{t_0}^{t} e^{\underline{A}(t-\tau)} \cdot \underline{B} \cdot \underline{u}(\tau) \cdot d\tau$$

② ⇒ $\underline{y}(t) = \underline{C}\, x(t) + \underline{D}\, u(t)$

- $$y(t) = \underline{C}\left[e^{\underline{A}(t-t_0)} \cdot x_0 + \int_{t_0}^{t} e^{A(t-\tau)} \cdot \underline{B}\, \underline{u}(\tau)\, d\tau \right] + \underline{D} \cdot u(t)$$

(ii) Solution in frequency Domain :-

Initial condition $\underline{x}(0) = 0$

$\dot{\underline{x}}(t) = \underline{A}\, x(t) + \underline{B} \cdot u(t)$; $\underline{y}(t) = \underline{C}\, x(t) + \underline{D}\, u(t)$

Apply laplace transform for initial condition

$s \cdot X(s) = \underline{A}\, X(s) + \underline{B}\, \underline{U}(s)$; $Y(s) = \underline{C}\, X(s) + \underline{D}\, \underline{U}(s)$

$(s\underline{I} \cdot \underline{A})\, X(s) = \underline{B}\, \underline{U}(s)$ ∴ $\underline{\phi}(s) = (s \cdot \underline{I} - \underline{A})^{-1}$

- $$X(s) = \underbrace{(s \cdot \underline{I} - \underline{A})^{-1}}_{\text{fundamental matrix of state space system}} \cdot \underline{B} \cdot \underline{U}(s)$$

$\underline{Y}(s) = \underline{C}\, X(s) + \underline{D}\, \underline{U}(s)$

$\underline{Y}(s) = \underline{C}\, (s \cdot \underline{I} - \underline{A})^{-1} \cdot \underline{B} \cdot \underline{U}(s) + \underline{D} \cdot \underline{U}(s)$

$\underline{Y}(s) = \left(\underline{C} \cdot (s \cdot \underline{I} - \underline{A})^{-1} \cdot \underline{B} + \underline{D}\right) \cdot U(s)$

$$\text{Transfer function} = \frac{Y(s)}{U(s)} = C \cdot (s \cdot \underline{I} - \underline{A})^{-1} \cdot \underline{B} + \underline{D} \quad (\because \text{MIMO})$$

$$= \underline{c}^T \cdot (s \cdot \underline{I} - \underline{A})^{-1} \cdot \underline{b} + \underline{d} \quad (\because \text{SISO})$$

Note: for the calculation of transfer function of a system from a differential eqn. of higher order, initial values '0' $x_0 = 0$ were also assumed.

1.7 Coordinate Transformation of State-space System

Starting from the standard state-space representation in MIMO case.

$$\dot{\underline{x}} = \underline{A}\cdot\underline{x} + \underline{B}\cdot\underline{u} + \underline{B}_Z\cdot\underline{z} \rightarrow ①$$

$$\underline{y} = \underline{C}\,\underline{x} + \underline{D}\cdot\underline{u} \rightarrow ②$$

with $\underline{x} = n\times 1$; $\underline{u} = r\times 1$; $\underline{z} = p\times 1$; $\underline{y} = m\times 1$

and initial state $\underline{x}(t_0) = \underline{x}_0$

We can apply a coordinate transformation on the state vector $\underline{x}$. The transformation between the original state vector $\underline{x}$ and the "new/transformed" state vector $\tilde{\underline{x}}$ is defined by a constant transformation matrix $\underline{T}$ of dimension $n\times n$

$$\underline{x}(t) \underset{\underline{T}}{\overset{\underline{T}^{-1}}{\rightleftarrows}} \tilde{\underline{x}}(t)$$

$$\underline{x}(t) = \underline{T}\cdot\tilde{\underline{x}} \Leftrightarrow \tilde{\underline{x}} = \underline{T}^{-1}\cdot\underline{x}$$

$$\dot{\underline{x}}(t) = \underline{T}\,\dot{\tilde{\underline{x}}} \Leftrightarrow \dot{\tilde{\underline{x}}} = \underline{T}^{-1}\,\dot{\underline{x}}$$

$$① \Rightarrow \underline{T}\cdot\dot{\tilde{x}} = \underline{A}\cdot\underline{T}\cdot\tilde{x} + \underline{B}\cdot\underline{u} + \underline{B}_Z\cdot Z$$

$$\dot{\tilde{x}} = \underbrace{\underline{T}^{-1}\cdot\underline{A}\cdot\underline{T}}_{\tilde{A}}\,\tilde{x} + \underbrace{\underline{T}^{-1}\cdot\underline{B}}_{\tilde{B}}\cdot\underline{u} + \underbrace{\underline{T}^{-1}\cdot\underline{B}_Z}_{\tilde{B}_Z}\cdot Z$$

$$② \Rightarrow \underline{y} = \underbrace{\underline{C}\cdot\underline{T}}_{\tilde{C}}\cdot\tilde{\underline{x}} + \underbrace{\underline{D}}_{\tilde{D}}\cdot\underline{u}$$

so, new state space matrices, $\tilde{\underline{A}} = \underline{T}^{-1}\cdot\underline{A}\cdot\underline{T}$; $\tilde{\underline{B}} = \underline{T}^{-1}\cdot\underline{B}$, $\tilde{\underline{B}}_Z = \underline{T}^{-1}\cdot\underline{B}_Z$

$\tilde{\underline{C}} = \underline{C}\cdot\underline{T}$; $\tilde{\underline{D}} = \underline{D}$

Transformed state equation as follows:

$$\dot{\tilde{\underline{x}}} = \tilde{\underline{A}}\cdot\tilde{\underline{x}} + \tilde{\underline{B}}\cdot u + \tilde{\underline{B}}_Z\cdot z$$

$$\underline{y} = \tilde{\underline{C}}\cdot\tilde{\underline{x}} + \tilde{\underline{D}}\cdot u$$

and IC $\tilde{\underline{x}}(t_0) = \underline{T}^{-1}\cdot\underline{x}_0$

Note:-
⇒ As we are applying a so called equivalence transformation $\underline{T}^{-1}\cdot\underline{A}\cdot\underline{T}$ on the system matrix $\underline{A}$, "the eigenvalues of $\underline{A} = \tilde{\underline{A}}$"

⇒ For same input signal $u(t)$ & $z(t)$, we get same output $\underline{y}(t)$ for both, standard & transformed state space representation "Only the state vector over time is different."

⇒ Obviously, the state-space representation of a given system is NOT unique.

1.9 Self-study Exercise

$$\underline{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} ; \underline{\underline{B}} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} ; \underline{\underline{C}} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} ;$$

$$a = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix} ; b = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$$

a) $\underline{\underline{A}} \cdot \underline{a} = \begin{bmatrix} 2-3+0 \\ -1+6-7 \\ 0-3+7 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 4 \end{bmatrix}$; $\underline{\underline{A}} \cdot \underline{b} = \begin{bmatrix} 4+0+0 \\ -2+0-4 \\ 0+0+4 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 4 \end{bmatrix}$

$a^T \cdot b = \begin{bmatrix} 1 & 3 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} = 2+0+28 = 30$; $b \cdot a^T = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 14 \\ 0 & 0 & 0 \\ 4 & 12 & 28 \end{bmatrix}$

$$\underline{\underline{A}} \cdot \underline{\underline{C}} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 1 \\ -3 & 6 & -3 \\ 1 & -3 & 2 \end{bmatrix}$$

$$\underline{\underline{A}}^2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -4 & 1 \\ -4 & 6 & -3 \\ 1 & -3 & 2 \end{bmatrix}$$

b) $|A| = 2(2-1) + 1(-1+0) = 2-1 = 1 \neq 0 \Rightarrow$ Rank $= 3$

$$|B| = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 2 & 8 & 26 \\ 0 & 3 & 15 & 63 \end{vmatrix} = 1 \cdot \det \begin{bmatrix} 1 & 3 & 7 \\ 2 & 8 & 26 \\ 3 & 15 & 63 \end{bmatrix} = 12 \neq 0, \text{ so Rank} = 4$$

$|C| = 1(2-1) + 1(-1+0) = 1-1 = 0$

If $|C| = 0 \Rightarrow$ Rank is either 1 or 2 $\Rightarrow \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} = 2-1 = 1 \neq 0$

so, Rank $= 2$

d) $A^{-1} = \frac{1}{|A|} \cdot \left[(-1)^{(i+j)} \text{cofactor } a_{ij} \right]^T = \frac{1}{1} \begin{bmatrix} 1 & +1 & 1 \\ +1 & 2 & +2 \\ 1 & +2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

e) $|s \cdot I - C| = 0 \Rightarrow \begin{vmatrix} s-1 & 1 & 0 \\ 1 & s-2 & 1 \\ 0 & 1 & s-1 \end{vmatrix} = 0$

$(s-1)\left[(s^2-3s+2) - 1\right] - 1(s-1) = 0 \Rightarrow (s-1)(s^2-3s+1) - s + 1 = 0$

$s^3 - 3s^2 + \cancel{s} - s^2 + 3s - \cancel{1} - \cancel{s} + \cancel{1} = 0 \Rightarrow s^3 - 4s^2 + 3s = 0$

$s(s^2-4s+3) = 0 \Rightarrow s(s-1)(s-3) = 0$

$s_1 = 0, \quad s_2 = 1 ; \quad s_3 = 3$

1) Eigenvectors for each eigen values of $\underline{C}$

$s=0 \quad \lambda = 0 \Rightarrow (\lambda \cdot \underline{\underline{I}} - \underline{\underline{C}})\vec{V} = \underline{0}$

$$\begin{pmatrix} s-1 & 1 & 0 \\ 1 & s-2 & 1 \\ 0 & -1 & s-1 \end{pmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

$(\lambda - A\cdot I) =$

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & -1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & -1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -4 \end{bmatrix}$$

$R_2 \to R_2 - R_1 \qquad R_3 \to 3R_3 - R_2$

$$\left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$(A - \lambda I)x = 0$

$$\begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & +1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_3 = 0$

$x_2 - x_3 = 0$

$x_2 =$

$x_3 = 1 \; ; \; x_2 = 1 \; ; \; x_1 = 1$

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\lambda = 0$

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + R_1$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$\lambda = 1$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] \Leftrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] \Leftrightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_2 - R_1 \qquad R_2 \to R_2 - R_1 \qquad R_3 \to R_3 + R_2$

$x_1 + (-2x_2) + x_3 = 0 \Rightarrow x_3 = -1$

$x_3 = 1 \; ; \; x_2 = 0 \; ;$

$$v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda = 3$

$\Rightarrow (\lambda \cdot I - A)(\vec{\underline{x}}) = \underline{0}$

$$\left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 1 & 1 & \Phi & 0 \\ 0 & -1 & 2 & 0 \end{array}\right] \Leftrightarrow \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & -1 & 2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} /4 \Leftrightarrow \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 2 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to 2R_2 - R_1 \qquad R_3 \to R_3 + R_2 \qquad R_1 \to R_1 - R_2$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}$$

Example Exercise 1: Q1.2

$\dot{x} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \cdot \dot{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \quad , y = (1 \quad 0)\, x$

$\underline{A} = \begin{bmatrix} 0 & +1 \\ -2 & -3 \end{bmatrix}$

$\underline{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\underline{C}^T = [1\ 0] \; ; \underline{D} = 0$

a) $\underline{\Phi}(s) = (s \cdot \underline{I} - \underline{A})^{-1} = \left[\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\right]^{-1}$

$= \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} = \frac{1}{s^2+3s+2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$

a) $\underline{\Phi}(s) = \begin{bmatrix} \frac{s+3}{(s+2)(s+1)} & \frac{1}{(s+2)(s+1)} \\ \frac{-2}{(s+2)(s+1)} & \frac{s}{(s+2)(s+1)} \end{bmatrix}$

b) $\underline{\Phi}(t) = \begin{bmatrix} \frac{s+2}{(s+2)(s+1)} + \frac{1}{(s+1)(s+2)} & \frac{1}{s+1} - \frac{1}{s+2} \\ -2\left(\frac{1}{s+1} - \frac{1}{s+2}\right) & \frac{2(s+1)}{(s+2)(s+1)} - \frac{s+2}{(s+2)(s+1)} \end{bmatrix}$

$= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & 2e^{-2t} - e^{-t} \end{bmatrix}$

c) $G(s) = \underline{c}^T (s \cdot \underline{I} - \underline{A})^{-1} \underline{b} + \underline{d}$

$\underline{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} ; \underline{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\underline{c}^T = [1\ 0] \; ; \underline{d} = 0$

$= [1\ 0] \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0$

$= \frac{1}{s^2+3s+2} \cdot [1\ 0] \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0$

$= \frac{1}{s^2+3s+2} \cdot [s+3 \quad 1] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{s^2+3s+2}$

$G(s) = \frac{1}{s^2+3s+2}$

d) $\dot{x}_1 = x_2 \quad ; \dot{x}_2 = -2x_1 - 3x_2 + u$

$y = x_1$

$\dot{y} = \dot{x}_1 = x_2 \;;$

$\ddot{y} = \dot{x}_2 = -2x_1 - 3x_2 + u$

$\ddot{y}(t) = -2y - 3\dot{y}(t) + u(t)$

Apply Laplace Transfn

$s^2 y(s) = -2y(s) - 3s\,y(s) + u(s)$

$(s^2+3s+2)\, y(s) = u(s)$

$G(s) = \frac{y(s)}{u(s)} = \frac{1}{(s+1)(s+2)}$

Given $u(s) = \frac{1}{s}$

$y(s) = \frac{1}{(s+1)(s+2)} \cdot \left(\frac{1}{s}\right)$

$$y(s) = \left(\frac{1}{s+2}\right)\left(\frac{1}{s+1}\right)\left(\frac{1}{s}\right) = \frac{(s+2)-(s+1)}{(s+2)(s+1)}\cdot\left(\frac{1}{s}\right)$$

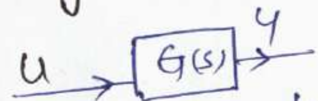
$$= \left(\frac{1}{s+1} - \frac{1}{s+2}\right)\frac{1}{s} = \frac{s+1-s}{s\cdot(s+1)} - \frac{2(s+2)-2s-3}{s\cdot(s+2)}$$

$$= \frac{1}{s} - \frac{1}{s+1} - \frac{2}{s} + \frac{2}{s+2} - \frac{3}{s\cdot(s+2)}$$

$$= 1 - e^{-t} - 2 + e^{-2t}$$

Chapter - 2 : State space system Analysis

2.1 Stability of LTI systems :-



SISO system

$$G(s) = \frac{\text{numerator}(s)}{\text{denominator}(s)}$$

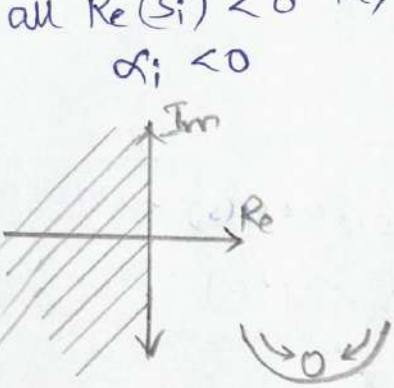
$$= \frac{p_K s^K + p_{K-1}\cdot s^{K-1} + \cdots + p_0}{q_l s^l + q_{l-1}\cdot s^{l-1} + \cdots + q_0}$$ i.e, transfer function G(s) is a function in s of two polynomials equation in num(s) and den(s).

⇒ for stability analysis of transfer function, the complex poles $s_i = \alpha_i + j\beta_i$ (which are roots of den(s)) play the key role

⇒ To get complex poles, solve the characterstic equation i.e (den(s)) of the system

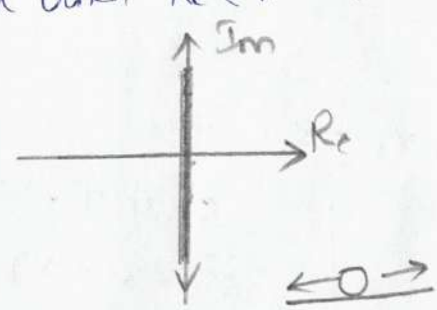
i.e, $q_l s^l + q_{l-1}\cdot s^{l-1} + \cdots + q_1 s^1 + q_0 = 0$

① Stable (Asymptotically)
all $Re(s_i) < 0$ i.e, $\alpha_i < 0$



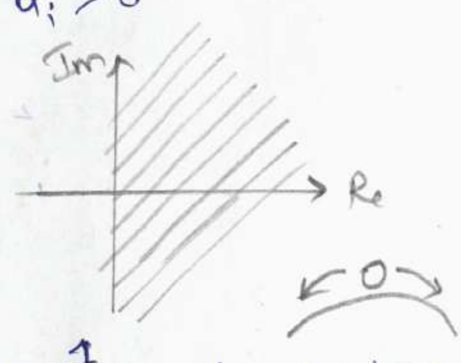
all poles must have negative real parts

② Stability limit
One $Re(s_i) = \alpha_i = 0$ and all other $Re(s_i) = \alpha_i < 0$



atleast one pole on the imaginary axis & other poles real parts on negative axis

③ Unstable
atleast one $Re(s_i) = \alpha_i > 0$



atleast 1 poles of system have positive real parts.

2.1.2 Stability of State-space Systems

A state space system should be "asymptotically stable" if the system tend back to their "equilibrium point"

stable

Definition of asymptotic stability:-

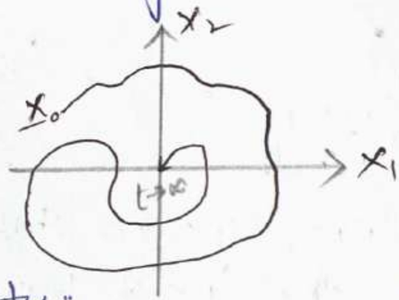
The state-space representation

$\underline{\dot{x}} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\cdot\underline{u}$, $\underline{x}(0) = \underline{x}_0$ is asymptotically stable

if for $u(t) = 0$, all state variable

$x_i(t)$ tends to zero over time

if $|x_i| \to 0$ for $t \to \infty$ [with $u(t)=0$]



- **Standard Criterion for asymptotic stability:-** system $\underline{\dot{x}} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\cdot\underline{u}$ is asymptotically stable if all eigenvalues of the dynamic matrix $\underline{\underline{A}}$ have negative real parts.

- Eigenvalues of the dynamic matrix $\underline{\underline{A}}$ are called the poles of the system.

Remainder:- Eigenvalues determined from $\boxed{\det(s\cdot\underline{\underline{I}} - \underline{\underline{A}}) = 0}$

** Stability check criterion

- Example: Consider following system [SISO]

$$\dot{\underline{x}}(t) = \begin{bmatrix} -1 & 2 \\ 0 & 1/2 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{x} + 2\cdot u$$

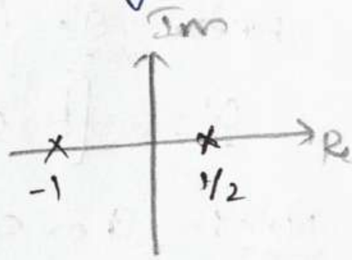
check whether the system is stable or not?

sol: $\underline{\underline{A}} = \begin{bmatrix} -1 & 2 \\ 0 & 1/2 \end{bmatrix}$

$\det(s\cdot\underline{\underline{I}} - \underline{\underline{A}}) = 0$, Criterion to check stability of system

$$\begin{vmatrix} s+1 & -2 \\ 0 & s-1/2 \end{vmatrix} = 0 \Rightarrow (s+1)(s-1/2) = 0$$

$$s = -1,\ s = 1/2$$



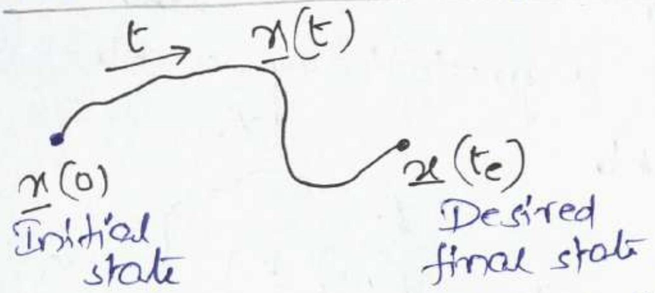
$Re(s_i) > 0$, so the system is unstable

Note: stability depends on Eigenvalues of dynamic matrix "$\underline{\underline{A}}$".

2.2 Controllability:-

Controllability & observability plays a key role for the controller design. for state space systems.

Definition:- The system is called controllable, if an input signal $\underline{u}(t)$ can be found which drives the system state from an arbitrary initial state $\underline{x}_0$ to a desired final state $\underline{x}(t_e)$.



⇒ Drive system state x_0 to $\underline{x}(t_{end})$ by the use of input $u(t)$?

- Simply, if the system is controllable, then it is possible to move from $\underline{x}_0$ to $\underline{x}_{tend}$ with input $u(t)$.

① Standard criterion for controllability of MIMO systems: ("KALMAN's criterion")

A system $(\underline{A}, \underline{B}, \underline{C}, \underline{D})$ in standard LTI state space representation is controllable, if $\boxed{\text{rank } \underline{Q}_s = n}$, where $\underline{Q}_s$ - Controllability matrix

$$\underline{Q}_s = [\underline{B} \mid \underline{A}\cdot\underline{B} \mid \underline{A}^2\cdot\underline{B} \mid \cdots\cdots \mid \underline{A}^{n-1}\cdot B]$$

② Standard criterion for controllability of SISO systems: (KALMAN criterion):

A system $(\underline{A}, \underline{b}, \underline{c}^T, d)$ in standard LTI state-space representation is controllable, if $\boxed{\det \underline{Q}_s \neq 0}$ i.e. $|\underline{Q}_s| \neq 0$

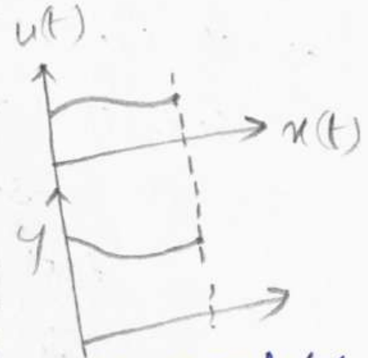
where $\underline{Q}_s$ is controllability matrix

$$\underline{Q}_s = [\underline{b} \mid \underline{A}\,\underline{b} \mid \underline{A}^2\underline{b} \mid \underline{A}_3\cdot\underline{b} \cdots\cdots \mid \underline{A}^{n-1}\cdot\underline{b}]$$

Note:- The controllability of a state-space system in standard representation is only depending on the dynamic matrix ($\underline{A}$) and the input matrix ($\underline{B}$ or $\underline{b}$)

2.3 Observability :- Observers are used to measure the feedback of all state variables i.e, observers observe the control signal $u(t)$ and the output signal $y(t)$ to estimate the current value of the state vector.

Definition :- A standard LTI state-space system is called observable, if the initial value of the state x_0 (or) current state value $\underline{x}(t)$ can be calculated based on the known input signal ~~y~~ $\underline{u}(t)$ and the measured output signal $\underline{y}(t)$ over the time span $(0, t_e]$.



① Standard criterion for observability of MIMO systems (KALMAN criterion) :-

A system in standard LTI state space representation is observable, if $\boxed{\text{rank } \underline{\underline{Q}}_B = n}$, where $\underline{\underline{Q}}_B$ is observability matrix

$$\underline{\underline{Q}}_B = \begin{bmatrix} \underline{\underline{C}} \\ \underline{\underline{C}} \cdot \underline{\underline{A}} \\ \underline{\underline{C}} \cdot \underline{\underline{A}}^2 \\ \vdots \\ \underline{\underline{C}} \cdot \underline{\underline{A}}^{n-1} \end{bmatrix}.$$

② Standard criterion for observability of SISO systems ("KALMAN criterion")

A system $(\underline{\underline{A}}, \underline{b}, \underline{c}^T, d)$ in standard LTI state space representation is observable, if $\boxed{\det \underline{\underline{Q}}_B \neq 0}$

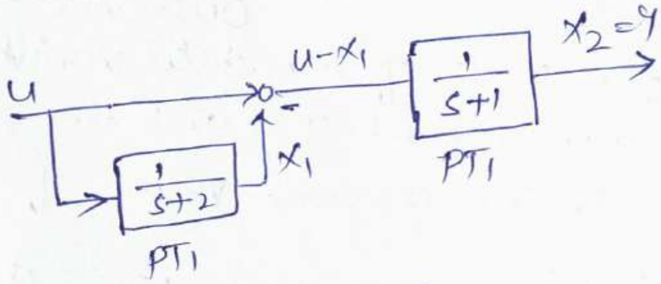
where $\underline{\underline{Q}}_B$ is observability matrix

$$\underline{\underline{Q}}_B = \begin{bmatrix} \underline{c}^T \\ \underline{c}^T \cdot \underline{\underline{A}} \\ \vdots \\ \underline{c}^T \cdot \underline{\underline{A}}^{n-1} \end{bmatrix}$$

Note: The observability of a state-space system in standard representation is only depending on the dynamic matrix $\underline{\underline{A}}$ & the output matrix ($\underline{\underline{C}}$ or $\underline{c}^T$).

2.4 A summarizing example :-

Block diagram of simple connection of PT1 systems



a) Calculation of state space systems:

output eqn: $y = x_2 = [0 \quad 1]\, \underline{x} + 0 \cdot \underline{d}u$

state equation: $X_2(s) = \frac{1}{s+1} \cdot (U(s) - X_1(s))$

$\Rightarrow X_2(s) \cdot (s+1) = U(s) - X_1(s)$

$s \cdot X_2(s) + X_2(s) = -X_1(s) + U(s)$

$s \cdot X_2(s) = -X_1(s) - X_2(s) + U(s)$

Apply Inverse laplace transform

$\dot{x}_2(t) \Rightarrow \dot{x}_2 = -x_1(t) - x_2(t) + u(t) \rightarrow ①$

$\Rightarrow X_1(s) = \frac{1}{s+2} \cdot U(s) \Rightarrow s \cdot X_1(s) = -2X_1(s) + U(s)$

Apply Laplace Inverse

$\dot{x}_1(t) = \dot{x}_1 = -2x_1(t) + u(t) \rightarrow ②$

so $\dot{x}_1 = -2x_1 + u$; $\dot{x}_2 = -x_1 - x_2 + u$

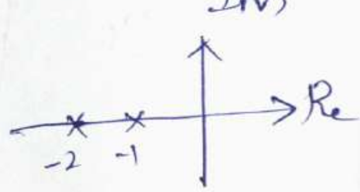
$$\underline{\dot{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 & 0 \\ -1 & -1 \end{bmatrix}}_{\underline{A}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\underline{b}} u$$

$$y = \underbrace{[0 \quad 1]}_{c^T} x + \underbrace{0}_{\underline{d}}\, \underline{d}u$$

$\therefore \underline{A} = \begin{pmatrix} -2 & 0 \\ -1 & -1 \end{pmatrix}$; $\underline{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $c^T = (0 \quad 1)$; $\underline{d} = 0$

b) Check for stability: Eigenvalues of $\underline{A}$

$\det(s \cdot \underline{I} - \underline{A}) = 0 \Rightarrow \det \begin{pmatrix} s+2 & 0 \\ 1 & s+1 \end{pmatrix} = 0$



$(s+1)(s+2) = 0 \Rightarrow s_1 = -1,\ s_2 = -2$

If $Re(s_i) < 0 \Rightarrow$ Then the system is stable.

c) Check for controllability : Condition ; $\det \underline{\underline{Q}}_s \neq 0$

$\cancel{\underline{\underline{Q}}_s \neq \underline{b}\ \underline{A}\,\underline{b}\ \underline{A}^2\underline{b} \ldots A^n}$ $\cancel{\left[\underline{b}\ \ \underline{b}\cdot\underline{A}\ \ \underline{b}\cdot\underline{A}^2 \ldots \underline{b}\cdot\underline{A}^{n-1}\right]}$

$\cancel{\underline{\underline{Q}}_s = \left[\underline{b}\ \ \ \underline{b}\cdot\underline{A}\right]}$ $(\because n = 2)$ $\cancel{\underline{b}\cdot\underline{A} = \begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}-2 & 0\\ -1 & -1\end{pmatrix} = -2}$

$$\underline{\underline{Q}}_s = \left[\underline{b}\ \ \underline{\underline{A}}\cdot\underline{b}\ \ \underline{\underline{A}}^2\cdot\underline{b} \ldots\ \underline{\underline{A}}^{n-1}\cdot\underline{b}\right]$$

$$\underline{\underline{A}}\cdot b = \begin{pmatrix}-2 & 0\\ -1 & -1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-2\\-2\end{pmatrix}$$

$$\underline{\underline{Q}}_s = \left[b\ \ \ \underline{\underline{A}}\cdot b\right] \quad (\because n=2)$$

CSbA

$$\det\left(\underline{\underline{Q}}_s\right) = \begin{vmatrix}1 & -2\\ 1 & -2\end{vmatrix} = -2+2 = 0$$

So, system is not controllable

OBCA

d) Check for observability   Condition : $\det(\underline{\underline{Q}}_B) \neq 0$

$$\underline{\underline{Q}}_B = \begin{bmatrix}\underline{c}^T\\ \underline{c}^T\cdot\underline{\underline{A}}\\ \vdots\\ \underline{c}^T\cdot\underline{\underline{A}}^{n-1}\end{bmatrix} \Rightarrow n=2, \text{ so } \underline{\underline{Q}}_B = \begin{bmatrix}\underline{c}^T\\ \underline{c}^T\cdot\underline{\underline{A}}\end{bmatrix}$$

$$\underline{c}^T\cdot\underline{\underline{A}} = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}-2 & 0\\ -1 & -1\end{bmatrix} = \begin{bmatrix}-1 & -1\end{bmatrix}$$

$$\det(\underline{\underline{Q}}_B) = \begin{vmatrix}0 & 1\\ -1 & -1\end{vmatrix} = 1 \neq 0$$

So system is observable

e) Calculation of Transfer function : $G(s) = \frac{Y(s)}{U(s)} = \underline{c}^T(s\cdot\underline{\underline{I}} - \underline{\underline{A}})^{-1}\cdot\underline{b} + d$

$$G(s) = \underline{c}^T\cdot(s\cdot\underline{\underline{I}} - \underline{\underline{A}})^{-1}\cdot\underline{b} + \underline{d}$$

$$= \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}s+2 & 0\\ 1 & s+1\end{bmatrix}^{-1}\cdot\begin{bmatrix}1\\1\end{bmatrix} + 0$$

$$= \begin{bmatrix}0 & 1\end{bmatrix}\cdot\frac{1}{(s+2)(s+1)}\begin{bmatrix}s+1 & 0\\ -1 & s+2\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}$$

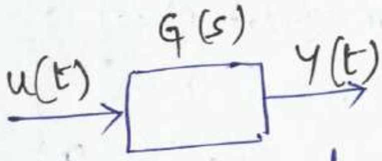
$$= \frac{1}{(s+2)(s+1)}\cdot\begin{bmatrix}-1 & s+2\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \frac{-1+s+2}{(s+2)(s+1)}$$

$$= \frac{1}{s+2}$$

Note : The transfer function $G(s)$ gives only one pole whereas the state space representation gives two eigenvalues of dynamic matrix & more detailed info. of internal dynamics.

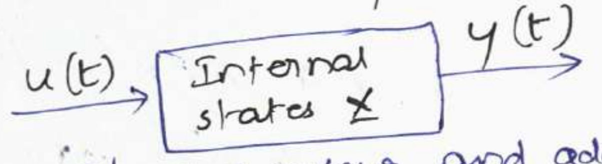
*** "State space representation provides more information about the control system than the transfer function."

POLES OF TRANSFER FUNCTION



only input/output behaviour of the plant

EIGEN VALUES:- State space

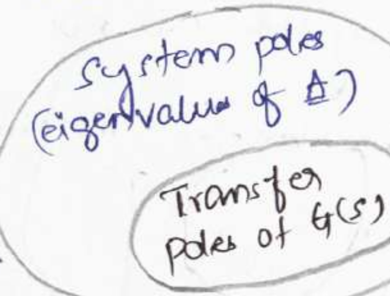


I/O behaviour and additional internal information about the plant

Relation between Transfer poles of G(s) and state system poles:

Transfer poles (or transmission poles i.e poles of TF)

System poles (eigenvalues of dynamic matrix of state space representation)



$\Leftrightarrow$ $\boxed{\text{Transfer poles} \subseteq \text{system poles}}$

Note:- If system is controllable and observable, then

Transfer poles $\simeq$ system poles

- If system is either not controllable or not observable, then Transfer poles of $G(s)$ < System Poles (eigenvalues of $\underline{A}$)

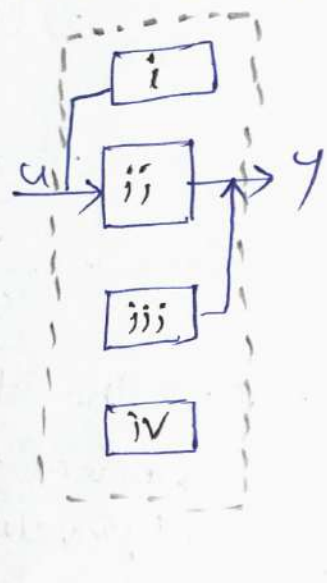
$\Rightarrow$ Since the transfer function $G(s)$ only contains information about I/O behaviour whereas the state space representation contains I/O behaviour and extra information about the internal dynamics of the system, we have to distinguish between 4 different types of dynamic matrix $\underline{A}$ eigenvalues.

i) Controllable, not observable

ii) Controllable & Observable (= Transfer poles)

iii) not controllable, observable

iv) not controllable, not observable



2.6 System Analysis using Canonical forms :-

What is Canonical forms?

As we know, state space representations have extra information apart from I/o behaviour; those extraordinary structures helps to analyze the system properties, to understand the dynamical behaviour of the system and/or to design state feedback controller and observers. These were called canonical forms (or normal forms) of state space system.

⇒ Normal forms of a state - space systems :-

1) Kalman Decomposition form (Kalman Normal form)
2) Controllability Normal form (Control Canonical form)
3) Observability Normal form (Observer Canonical form)
4) Diagonal form / Jordan Normal form (Modal Canonical form)

which can be calculated via a coordinate transformation of the state $\underline{\tilde{x}} = \underline{\underline{T}}^{-1} \cdot \underline{x}$

So, $(\underline{\underline{A}}, \underline{\underline{B}}, \underline{\underline{C}}, \underline{\underline{D}}) \rightarrow (\underline{\underline{T}}^{-1} \cdot \underline{\underline{A}} \cdot \underline{\underline{T}},\ \underline{\underline{T}}^{-1} \underline{\underline{B}},\ \underline{\underline{C}}\,\underline{\underline{T}},\ \underline{\underline{D}}) = (\underline{\underline{\tilde{A}}}, \underline{\underline{\tilde{B}}}, \underline{\underline{\tilde{C}}}, \underline{\underline{\tilde{D}}})$

2.6.1 Kalman Decomposition form / Kalman Normal form :-

Depends on Controllability (ctrb) and Observability (obsv), a system classified into 4 subsystems as discussed earlier,

a) Ctrb, not obsv ($\tilde{x}_1$).
b) ctrb and obsv ($\tilde{x}_2$)
c) not controllable, not obsv ($\tilde{x}_3$)
d) not ctrb, but ~~not~~ obsv ($\tilde{x}_4$)

The Kalman decomposition of state vector and the corresponding transformed state-space equation ($\tilde{x} = T^{-1} \cdot \underline{x}$) can be used.

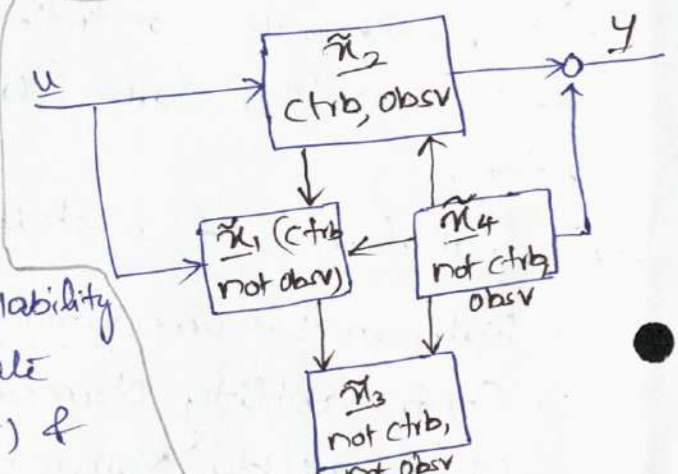
The Kalman decomposition of the state-space equation has the following structure:

$$\dot{\tilde{\underline{x}}} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} & \tilde{A}_{14} \\ 0 & \tilde{A}_{22} & 0 & \tilde{A}_{24} \\ 0 & 0 & \tilde{A}_{33} & \tilde{A}_{34} \\ 0 & 0 & 0 & \tilde{A}_{44} \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \end{bmatrix} + \begin{bmatrix} \tilde{B}_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix} \underline{u}$$

Interconnection of $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4$ of Kalman Decomposition

$$\underline{y} = [\underline{0} \;\; \tilde{\underline{c}}_2 \;\; \underline{0} \;\; \tilde{\underline{c}}_4] \cdot \tilde{\underline{x}} + \tilde{\underline{D}} \cdot \underline{u}$$

and $\tilde{\underline{x}}(t_0) = \underline{T}^{-1} \cdot \underline{x}_0$



2.6.2 Controllability Normal form:-

We focus on the case of controllability normal form for controllable state space systems with input $u(t)$ & arbitrary output equation

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B} \cdot \underline{u}$$
$$\underline{y} = \underline{c} \cdot \underline{x} + \underline{d} \cdot \underline{u}$$

The system shall be controllable with a characteristic polynomial given by

$$\det(\lambda \cdot \underline{I} - \underline{A}) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1 \cdot \lambda + a_0$$

By applying appropriate standard state vector transformation with transformation matrix $\underline{T}_s^{-1}$,

$\tilde{\underline{x}} = \underline{T}_s^{-1} \cdot \underline{x}$

$$\dot{\tilde{\underline{x}}} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \\ -a_0 & -a_1 & -a_2 & \cdots & & -a_{n-1} \end{bmatrix} \cdot \tilde{\underline{x}} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u + \tilde{\underline{B}}_2 \cdot \underline{z}$$

$$\underline{y} = \tilde{\underline{c}}\,\tilde{\underline{x}} + \tilde{\underline{D}} \cdot u = \underline{c} \cdot T_s \cdot \underline{x} + \underline{d} \cdot u$$

Controllability normal form state space representation

⇒ Transformation matrix ($\underline{T}_s^{-1}$):

Calculation of T_s^{-1} involves 2 steps

1) Determination of vector $\underline{t}^T$

2) Calculation of transform matrix

1) Determination of vector $\underline{t}^T$:

$t^T = [t_1 \ t_2 \ t_3 \ \dots \ t_n]$. In order to get $\underline{t}^T$ we have to solve 'n' scalar linear equations for unknown parameters $t_1, t_2 \dots t_n$.

i.e
1) $\underline{t}^T \cdot \underline{b} = 0$
2) $\underline{t}^T \cdot \underline{A} \cdot \underline{b} = 0$
3) $t^T \cdot \underline{A}^2 \cdot \underline{b} = 0$
$\vdots$
n) $\underline{t}^T \cdot \underline{A}^{n-1} \cdot \underline{b} = 1$

Eg: $n=2$, then $t^T = (t_1 \ t_2)$. so 2 linear eqns

$t^T \cdot \underline{b} = 0 \rightarrow (1)$

$t^T \cdot \underline{A} \cdot \underline{b} = 1 \rightarrow (2)$

on solving (1, 2) we get t_1, t_2

2) Calculation of the transformation matrix $\underline{T}_s^{-1}$:-

$$\underline{T}_s^{-1} = \begin{bmatrix} t^T \\ \underline{t}^T \cdot \underline{A} \\ \underline{t}^T \cdot \underline{A}^2 \\ \vdots \\ \underline{t}^T \cdot \underline{A}^{n-1} \end{bmatrix}$$

so,

$\therefore \dot{\tilde{\underline{x}}} = (\underline{T}_s^{-1} \cdot \underline{A} \cdot \underline{T}_s) \tilde{\underline{x}} + (\underline{T}_s^{-1} \cdot \underline{B}) \cdot \underline{u}$

$\tilde{y} = \underline{c} \cdot \underline{T} \ \tilde{\underline{x}} + d \cdot \underline{u}$

$\therefore \tilde{\underline{A}} = \underline{T}_s^{-1} \cdot \underline{A} \cdot \underline{T}_s$; $\tilde{\underline{B}} = \underline{T}_s^{-1} \cdot \underline{B}$

$\tilde{\underline{c}} = \underline{c} \cdot \underline{T}_s$; $\tilde{\underline{D}} = \underline{D}$

$\therefore$ The controllability normal form is beneficial for the derivation of state feedback controller as it reduces the calculation effort extremely.

Note: The characteristic polynomial of a matrix is invariant (i.e similar to) to similarity transformation $\tilde{\underline{A}} = \underline{T}_s^{-1} \cdot \underline{A} \cdot \underline{T}$

This means that characteristic polynomials of $\underline{A}$ and $\tilde{\underline{A}}$ are identical for every regular transformation matrix $\underline{T}$.

2.6.3 Observability Normal form

Here, we are only interested in the case of observable state space system with a scalar o/p $y(t)$ and input vector

$\dot{\underline{x}} = \underline{A} \underline{x} + \underline{B} u$ $\Rightarrow \dot{\underline{x}} = \underline{A} \ \underline{x} + \underline{B} \cdot \underline{u}$

$\underline{y} = \underline{c}^T \underline{x} + \underline{d}^T u$ $\quad \underline{y} = \underline{c}^T \cdot \underline{x} + \underline{d}^T \cdot \underline{u}$ } SISO

The system shall be observable with a characteristic polynomial given by

$$\det(\lambda \cdot \underline{I} - \underline{A}) = \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_1 \lambda + a_0$$

By applying appropriate vector transformation $\underline{\tilde{x}} = \underline{T}_B^{-1} \cdot \underline{x}$ with transformation matrix $\underline{T}_B^{-1}$, the observability normal form state-space representation is

$$\dot{\underline{\tilde{x}}} = \begin{bmatrix} 0 & 0 & 0 & \cdots & -a_0 \\ 1 & 0 & 0 & \cdots & -a_1 \\ 0 & 1 & 0 & \cdots & -a_2 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & -a_{n-1} \end{bmatrix} \cdot \underline{\tilde{x}} + \underline{\tilde{B}} \cdot \underline{u}$$

$$\underline{\tilde{y}} = [0 \quad 0 \quad 0 \cdots \quad 1]\, \underline{\tilde{x}} + \underline{\tilde{d}}^T \cdot \underline{u}$$

⇒ Transformation Matrix $\underline{T}_B^{-1}$

step 1) Determination of vector $\underline{s}$:

$\underline{s} = (s_1\ s_2 \cdots s_n)^T$ we have to solve 'n' scalar linear equation for n unknown parameter.

$s = (s_1 \quad s_2 \quad s_3 \cdots s_n)^T$

1) $\underline{c}^T \cdot \underline{A}\, \underline{s} = 0$

2) $\underline{c}^T \cdot \underline{A} \cdot \underline{s} = 0$

$\vdots$

n-1) $\underline{c}^T \cdot \underline{A}^{n-2} \cdot \underline{s} = 0$

n) $\underline{c}^T \cdot \underline{A}^{n-1} \cdot \underline{s} = 1$

step 2:

Calculation of transform matrix

$$\underline{T}_B^{-1} = [s \cdot \underline{A}s \cdot \quad \underline{A}^2 s \cdots \underline{A}^{n-1} \cdot s]^{-1}$$

$$\dot{\underline{\tilde{x}}} = (\underline{T}_B^{-1} \cdot \underline{A} \cdot \underline{T}_B) \underline{\tilde{x}} + (\underline{T}_B^{-1} \cdot \underline{B})\, \overline{u}$$

$$\overline{y} = (\underline{c}^T \underline{T}_B)\, \underline{\tilde{x}} + \underline{\tilde{d}}^T \cdot \underline{u}$$

is the state space representation in observability normal form

∴ Observability normal form is used to design state observers with reduced effort for the calculation of the observer gain matrix ($\underline{K}$)

2.6.4 Diagonal form / Jordan Normal form:-

step 1: Determine eigenvalues and eigen vector $v_1, v_2 \cdots v_n$

step 2: Transformation matrix $\underline{T}^* = [v_1 \ v_2 \cdots v_n]$

$$\begin{tikzcd} x(t) \arrow[r, bend left, "T^{-1}"] & \tilde{x}(t) \arrow[l, bend left, "T"] \end{tikzcd}$$

so, Diagonal form, $\dot{\underline{\tilde{x}}} = \underline{T}^{-1} \cdot \underline{A} \cdot \underline{T}\, \underline{\tilde{x}} + \underline{T}^{-1} \cdot \underline{B} \cdot \overline{u}$

$$y = \underline{c} \cdot \underline{T}\, \underline{\tilde{x}} + \underline{\tilde{d}}\, \underline{y}$$

2.8 Transmission Zeros & Decoupling Zeros in a Nutshell

We know that, the stability of a system depends on the system poles, but there are transmission zeros as well as input and output decoupling zeros exist, which play a key role for the dynamical behaviour of system. Hence, State-space systems have two main types of zeros.

1. Transmission Zeros
2. Decoupling Zeros

1. Transmission Zeros:-

In this type, the input of the system is sinusoidal with a special frequency where as the output is equal to zero at the same time.

Transmission zeros are no eigenvalues of dynamic matri

2. Decoupling zeros:-

A decoupling zero s_0 is always an eigenvalues of $\underline{A}$ (i.e nothing but a system pole).

3 different types of decoupling zero exists

a) Output-decoupling zero: if $\text{rank}\left(\begin{bmatrix} s_0 \cdot \underline{I} - \underline{A} \\ \underline{C} \end{bmatrix}\right) < n$,

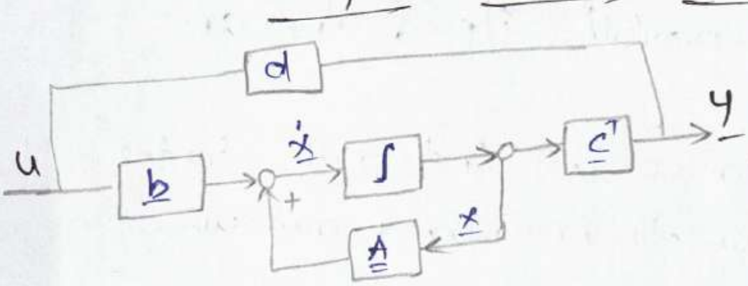
then pole ~~so~~ s_0 is an output decoupling zero.

b) Input-decoupling zero: If $\text{rank}\left(\begin{bmatrix} s_0\underline{I} - \underline{A} & -\underline{B} \end{bmatrix}\right) < n$,

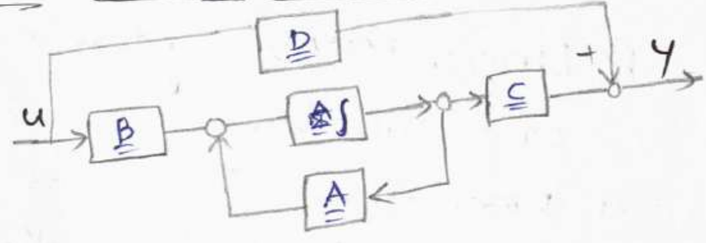
then pole s_0 is an input decoupling zero.

c) Input-output decoupling zero.

Chapter 3. State-Space Controller Design



Block diagram of standard SISO plant



Block diagram of standard MIMO plant

Pre-requisite for controller design and observer design:-

1) Plant must be controllable (for controller design) and observability (for observer design)
2) We deal only with plant models without direct feed-through i.e. $\underline{D} = 0$ or $\underline{d} = 0$

Goal of the controller Design:

i) Asymptotic stability of the closed control loop system
ii) Satisfying control performance of the close loop system for the nominal plant with regards to the application needs

- ⇒ Transition phase
  - ↳ Damping / overshoot
  - ↳ Reaction time / Bandwidth
- ⇒ Steady-state phase
  - ↳ Tracking of the desired values → $\underline{Y}_\infty = y(t \to \infty) = \underline{w}$
  - ↳ Disturbance reduction

When to use State-space Control?

State-space controller design is one of engineering techniques to solve feedback control problems. In general other engg. techniques such as PID controller, cascade control, fuzzy control, neural networks etc, can be used to solve feedback control problems.

For a linear plant, state space control is often good and simple choice in following situations:

a) Higher-dimensional SISO systems

b) Linear MIMO systems especially i/ps & o/ps are cross coupled

c) When there is a need to control internal states of the plant in addition to measured output.

3.1 Standard control loop structure with Pre-filter and State feedback :-

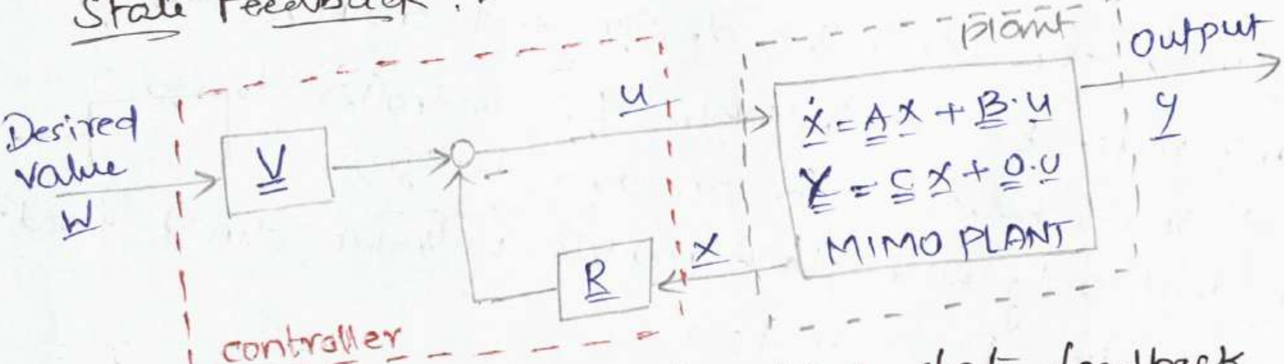


Fig: Block diagram of the standard state feedback controller for state-space plants

Here, $\underline{R}$ - Control matrix (or feedback matrix) - (r×n)

$\underline{V}$ - Pre-filter matrix - (r×m)

$\underline{w}$ - Vector of desired values (m×1)

- The controller behaviour is determined by the matrices which have to be suitabily designed for $\underline{R}$ & $\underline{V}$, from block diagram, the control task:

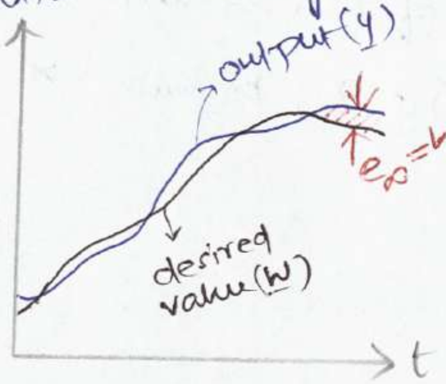
$$\boxed{u = -\underline{R}\cdot\underline{x} + \underline{V}\cdot\underline{w}} \Rightarrow \text{"Control law"}$$

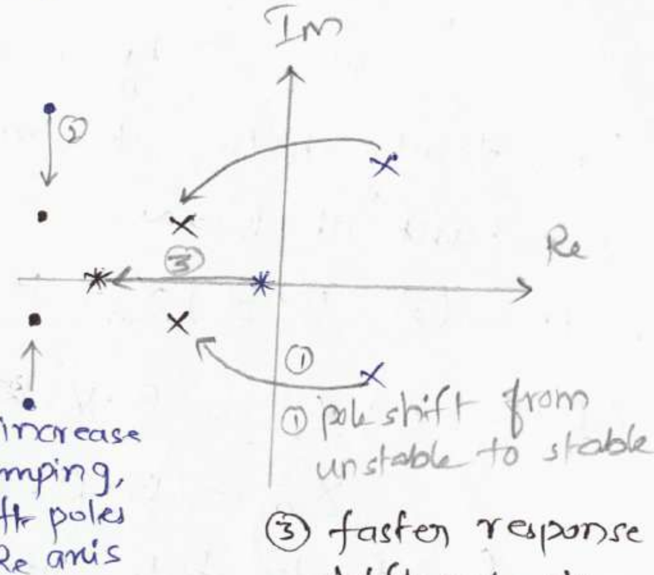
So the state feed back controller uses information of system state ($\underline{x}$). If we measure state ($\underline{x}$), we directly use in controller, but in practice not alway all state variables are measurable. Therefore observer can be used to estimate the current value of system state ($\underline{x}$) based on system equation, control signal and output signal.

How to design two controller matrices $\underline{V}$ and $\underline{R}$?

1) Use the pre-filter $\underline{V}$ to guarantee tracking behaviour $\underline{w} \rightarrow \underline{y}$

2) Use the feedback matrix $\underline{R}$ to shift poles of a plant to a desired closed-loop pole (dynamical properties of CL like stability, damping etc.,)





General Controller Design Process

step1:
Design $\underline{R}$ to meet dynamic requirements based on state-space model

step 2:
Based on $\underline{R}$ and state space model, calculate $\underline{V}$

Based on $\underline{R}$ and state space model, calculate $\underline{V}$ for Proportional State feedback :.

3.2 Pre-filter Design for Proportional State feedback

Pre-filter $\underline{V}$ is used to guarantee zero steady state offset of the closed control loop i.e (steady state behaviour).

$\underline{y}_\infty = \underline{y}(t \rightarrow \infty) = \underline{w}_\infty$ i.e $\underline{e}_\infty = 0$

from classical control ;
- control loop with proportional feedback,

$e_\infty = \underline{w}_\infty - \underline{y}_\infty \neq 0$ but with the help of Pre-filter $\underline{V}$, we can get $\underline{e}_\infty = \underline{w}_\infty - \underline{y}_\infty = 0$ (in ideal) $\simeq 0$ (in real)

* which is only possible if the no. of inputs r is atleast equal to /larger than the number of outputs m i.e $r = m$ (or) $r > m$

Computation for $r = m$:-

system; $\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\cdot\underline{u}$; Output - $\underline{y} = \underline{\underline{C}}\,\underline{x} + (\underbrace{\underline{\underline{D}}\cdot\underline{u}}_{0})$

Control law $\underline{u} = -\underline{\underline{R}}\cdot\underline{x} + \underline{\underline{V}}\cdot\underline{w}$

$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,(-\underline{\underline{R}}\cdot\underline{x} + \underline{\underline{V}}\cdot\underline{w})$

$\dot{\underline{x}} = \underbrace{(\underline{\underline{A}} - \underline{\underline{B}}\cdot\underline{\underline{R}})}_{\underline{\underline{A}}_{cl}}\,\underline{x} + \underline{\underline{B}}\,\underline{\underline{V}}\cdot\underline{w}$ and $\underline{y} = \underline{\underline{C}}\,\underline{x}$

(dynamic matric of closed loop stati sys.)

In steady state, $t \to \infty \Rightarrow \dot{\underline{x}}(t) = 0$, because $\underline{x}$ is constant at $t \to \infty$

$\therefore \underline{0} = (\underline{\underline{A}} - \underline{\underline{B}}\cdot\underline{\underline{R}})\,\underline{x}_\infty + \underline{\underline{B}}\cdot\underline{\underline{V}}\cdot\underline{w}_\infty$

$(\underline{\underline{B}}\cdot\underline{\underline{R}} - \underline{\underline{A}})\,\underline{x}_\infty = \underline{\underline{B}}\cdot\underline{\underline{V}}\cdot\underline{w}_\infty$

$\underline{x}_\infty = (\underline{\underline{B}}\cdot\underline{\underline{R}} - \underline{\underline{A}})^{-1}\cdot\underline{\underline{B}}\cdot\underline{\underline{V}}\cdot\underline{w}_\infty$

and for steady state output y_∞

$\underline{y}_\infty = \underline{\underline{C}}\cdot\underline{x}_\infty = \underline{\underline{C}}\,(\underline{\underline{B}}\cdot\underline{\underline{R}} - \underline{\underline{A}})^{-1}\cdot\underline{\underline{B}}\cdot\underline{\underline{V}}\cdot\underline{w}_\infty$

But, in steady state $y_\infty = w_\infty$,

$\therefore y_\infty = \underline{w}_\infty = \underline{\underline{C}}\cdot(\underline{\underline{B}}\cdot\underline{\underline{R}} - \underline{\underline{A}})^{-1}\cdot\underline{\underline{B}}\cdot\underline{\underline{V}}\cdot\underline{w}_\infty$

This equation is only fulfilled if, $\underline{\underline{C}}(\underline{\underline{B}}\cdot\underline{\underline{R}} - \underline{\underline{A}})^{-1}\cdot\underline{\underline{B}}\cdot\underline{\underline{V}} = \underline{\underline{I}}$

(where $\underline{\underline{I}}$ = unity matrix)

$\underline{\underline{C}}\cdot(\underline{\underline{B}}\cdot\underline{\underline{R}} - \underline{\underline{A}})^{-1}\cdot\underline{\underline{B}}\cdot\underline{\underline{V}} = \underline{\underline{I}}$

$\therefore$ Prefilter matrix, $\boxed{\underline{\underline{V}} = \left[C\,(\underline{\underline{B}}\cdot\underline{\underline{R}} - \underline{\underline{A}})^{-1}\cdot\underline{\underline{B}}\right]^{-1}}$

so, to get zero steady state error ($\underline{e}_\infty = 0$ i.e $y_\infty = \underline{w}_\infty$) for a perfect state plant model, use prefilter matrix

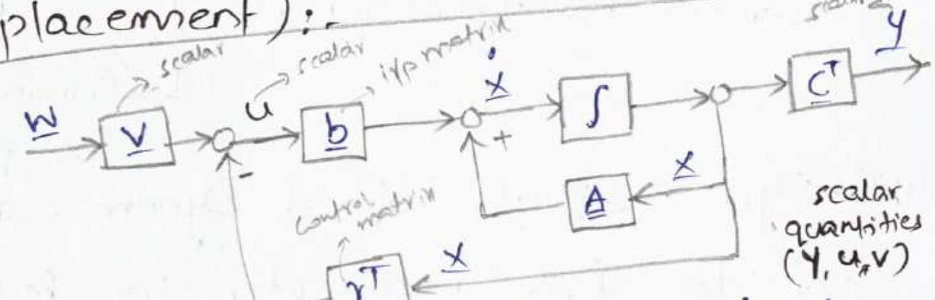
$\underline{\underline{V}} = \left(\underline{\underline{C}}\cdot(\underline{\underline{B}}\,\underline{\underline{R}} - \underline{\underline{A}})^{-1}\cdot\underline{\underline{B}}\right)^{-1}$.

* Note:- But, the prefilter matrix $\underline{\underline{V}}$ depends on the controller gain matrix "$\underline{\underline{R}}$"

Remarks:-

1) $\underline{\underline{R}}$ must be known before
2) Ideal case i.e, $\underline{e}_\infty = 0 \Rightarrow y_\infty = \underline{w}_\infty$
3) Real case $y_\infty \approx \underline{w}_\infty$

3.3 Computation of the feedback Matrix by Pole Placement for Proportional State feedback Control (SISO plant pole placement):-



Block Diagram of SISO plant control loop for proportional st. feedback

Plant :

system; $\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} + \underline{b}\cdot u \rightarrow ①$

output; $\underline{y} = \underline{c}^T\cdot\underline{x} \rightarrow ②$

Control law:

$\underline{u} = -\underline{r}^T\cdot\underline{x} + \underline{v}\cdot\underline{w} \rightarrow ③$

substitute controller equation ③ into plant dynamics,

$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{b}\cdot(-\underline{r}^T\cdot\underline{x} + \underline{v}\cdot\underline{w})$

- $\dot{\underline{x}} = (\underline{A} - \underline{b}\underline{r}^T\cdot \cancel{b})\,\underline{x} + \underline{b}\cdot\underline{v}\cdot\underline{w}$ for $\underline{\underline{A}}_{CL} = \underline{A} - \cancel{r^T}\cdot b\cdot r^T$

where, $\underline{r}^T = [r_1 \;\; r_2 \;\; \cdots \;\; r_n]$ i.e, feedback control vector contain 'n' gain factors r_i $(i = 1, 2, \cdots n)$ which can be chosen individually for controller design.

∇ The dynamic properties (i.e stability) depends on the dynamic matrix $\underline{A}_{CL}$ $(= \underline{A} - \cancel{r^T}\cdot\underline{b}\underline{r}^T)$ of the closed control loop. so, we can modify $\underline{A}_{CL}$ by an appropriate choice of state feedback vector $\underline{r}^T$ to get desired closed loop behaviour

- As the pre-filter reduces to a scalar here, the equation is $v = \dfrac{1}{\underline{c}^T\cdot(\underline{b}\underline{r}^T - \underline{A})^{-1}\cdot\underline{b}}$

~~Hence, the choice of r^T, the~~ depend

By choice of r^T, the eigenvalues (i.e, poles) of $\underline{A}_{CL}$ closed loop dynamic matrix can be modified arbitrarily in s-plane for a controllable plant. This choice of choosing (gain factors or poles is called "Pole placement".

STEP1 : Specify / Locate closed loop desired poles

STEP2 : Calculate r^T feedback vector from those poles

STEP3 : Calculate pre-filter

Idea: If we understand the dynamics behaviour of REAL & ~~closed loop~~ conjugate pair Poles, then we can choose desired poles of CL

3.3.1 Location of Desired Poles of the Closed Loop System

General two types of poles → Single poles on Real axis
→ (Complex conjugate) pair of poles

(I) Design of Single Poles & Dynamics of 1st Order Systems:

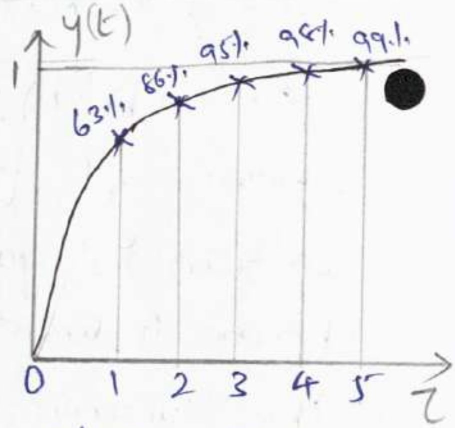
Consider first order system in state representation is

$$\dot{x} = -\sigma x + b \cdot u$$
$$y = x$$

Transfer function is $G(s) = \frac{Y(s)}{U(s)} = \frac{b}{s+\sigma}$

The eigenvalues of the dynamic matrix is "$-\sigma$" (ie, pole)

Sometimes the pole location is also denoted by time-constant 'τ' of a first order system $-\sigma = -\frac{1}{\tau}$ (by definition, $\tau = \frac{1}{\sigma}$)



step response y(t) Vs time constant (τ)

Method to get appropriate guess for one/more poles on the real axis

(i) choose the value for necessary rise time Δt to a certain percentage of step response. (Δt)

(ii) Take the necessary value λ for the multiple $\lambda \cdot \tau$ from figure for chosen percentage of step response (λ)

(iii) from $\Delta t = \lambda \cdot \tau = \lambda \cdot \left(\frac{1}{\sigma}\right)$. calculate σ ie $\sigma = \frac{\lambda}{\Delta t}$ from the results of (i) & (ii) (σ)

Then a good guess for desired pole will be, $\boxed{s_1 = -\sigma = -\frac{\lambda}{\Delta t}}$

Example: The desired pole location for a first order system is searched that reaches 95% of the final value in the step response after 0.1 sec.

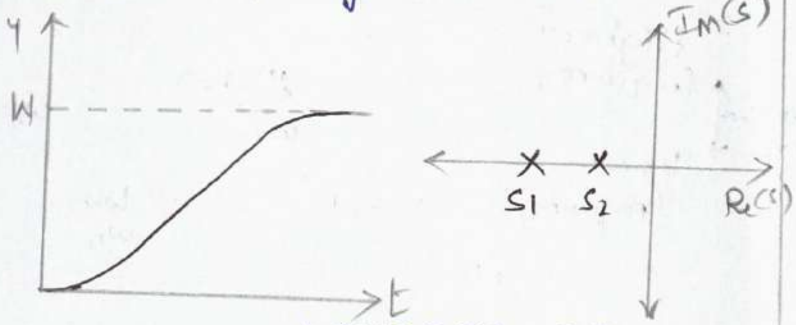
Sol. (i) $\Delta t = 0.1$ sec ; (ii) 95% step response $\Delta t = \lambda \cdot \tau = 3 \cdot \tau$ for 95% from figure

(iii) $\sigma = \frac{\lambda}{\Delta t} = \frac{3}{0.1} = 30$ 1/sec

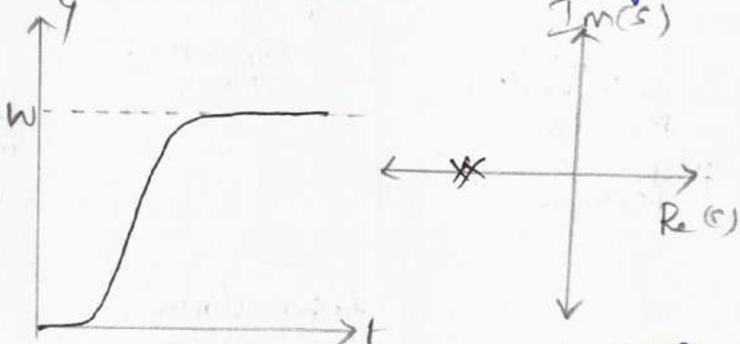
Hence desired pole $s_1 = -\sigma = -30$ 1/sec

(2) Design of Pairs of poles & Dynamic of 2^{nd} order System:-

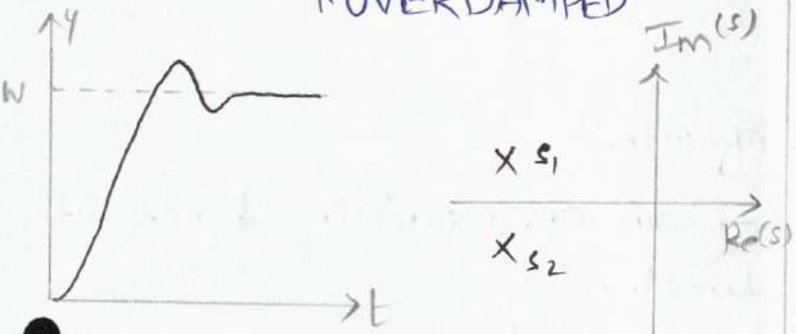
Overview of dynamic response to a step input for 2^{nd} Order systems:



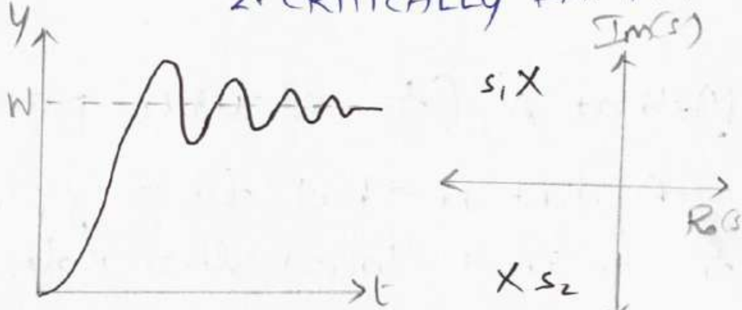
1. OVERDAMPED



2. CRITICALLY DAMPED

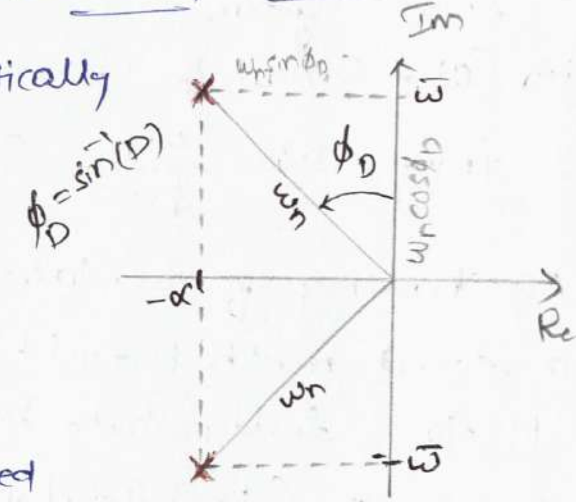


3. SMALL OVERSHOOT



4. STRONG OVERSHOOT WITH OSCILLATIONS

Transient response of 2^{nd} Order system to a step input:



A 2^{nd} order system with two asymptotically stable complex conjugate poles of the dynamic matrix is

$$S_{1,2} = -\alpha \pm j\bar{\omega}$$
$$= -D\omega_n \pm j\,\omega_n\sqrt{1-D^2}$$

Hence, the pole location $S_{1,2}$ is described by 2 different pairs of variables i.e, (α, ω) or (D, ω_n)

From fig; $\bar{\omega} = \omega_n \cos\phi_D$
$$= \omega_n\sqrt{1-\sin^2\phi_D}$$
$$\bar{\omega} = \omega_n\sqrt{1-D^2}$$

$$\alpha = \omega_n \sin\phi_D$$
$$\alpha = \omega \cdot D$$

where, $\bar{\omega}$ = oscillation frequency of the system

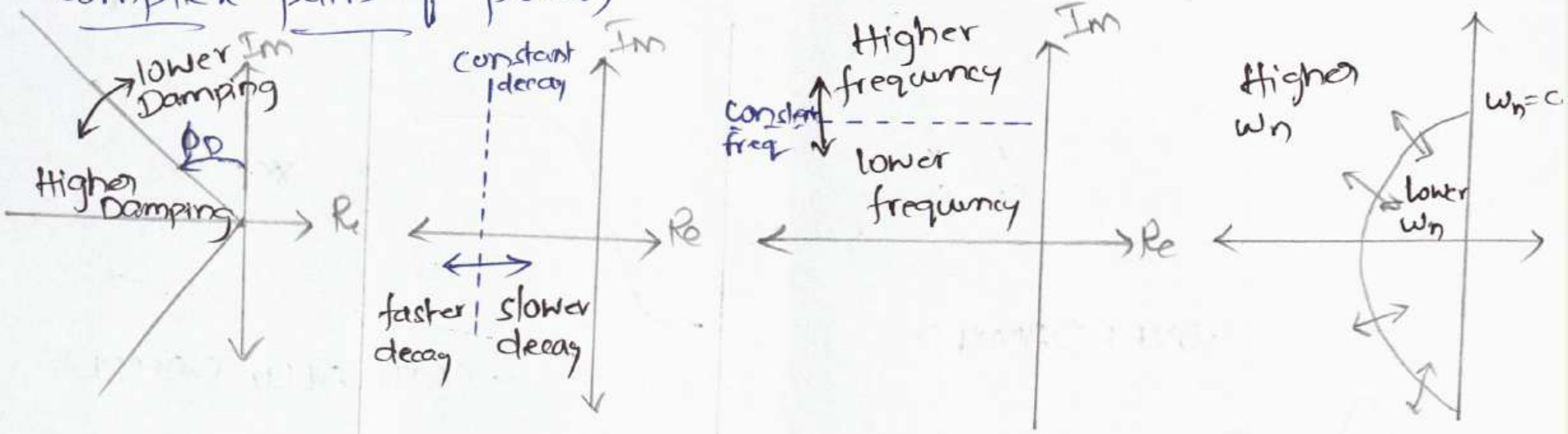
$-\alpha$ = Constant of exponential decay

D = damping ratio & ω_n = natural frequency

$\alpha = D\omega_n \geq 0$ and $\bar{\omega} = \omega_n\sqrt{1-D^2} \geq 0$

$G(s)$ of 2^{nd} order system is $G(s) = \dfrac{\omega_n^2}{s^2 + 2D\omega_n s + \omega_n^2}$

Behaviour of a system with pole placement (Conjugate complex pairs of poles)



Method 'A' (to calculate pairs of poles):

A first method for a guess of an appropriate choice of the desired closed loop pole location is

(i) Choose the desired damping ratio 'D' with respect to max. overshoot

(ii) choose the desired oscillation frequency $\bar{\omega}$ of the closed loop system

(iii) ~~Calc~~ Calculate the desired pole location $-\alpha + j\bar{\omega}$ from $\omega_n = \frac{\bar{\omega}}{\sqrt{1-D^2}}$ and $\alpha = D \cdot \omega_n$

As choosing of oscillation frequency is often complicated, a second method exist which is based on the settling time t_s to a steady state band of $\pm a$ around step input.

Method B (to calculate CL desired pairs of poles):

(i) Choose the desired damping ratio 'D' with respect to maximum overshoot

(ii) choose acceptable width a of the steady state band and describe the settling time t_s of steady state band

(iii) Calculate the desired pole location from $\alpha \pm j\bar{\omega}$

$$\alpha = \frac{-\ln(a)}{t_s} \quad \text{and} \quad \bar{\omega} = \omega_n \cdot \sqrt{1-D^2}$$

$$\bar{\omega} = \frac{\alpha}{D}\sqrt{1-D^2}$$

3.3.2 Calculation of state feedback Control Gain Vector (r^T/$\underline{R}$) for a given set of Desired Poles ("DIRECT METHOD")

Step-1 : Define desired poles of the Closed loop System

$$s_{d,1},\ s_{d,2},\ \dots\ s_{d,n-1},\ s_{d,n}$$

Step 2 : Calculation of desired characterstic polynomial of the closed loop

$$P_d(s) = (s - s_{d,1}) \cdot (s - s_{d,2}) \cdot \dots\dots (s - s_{d,n})$$

$$= \ell_{d,n} \cdot s^n + \ell_{d,n-1} \cdot s^{n-1} + \dots + \ell_{d,1} \cdot s + \ell_{d,0}$$

step 3: Calculation of characterstic equation of the closed control loop (depending on Controller gain r^T)

- $P_r(s) = \det(s \cdot \underline{\underline{I}} - \underline{\underline{A}}_{cl})$ $\qquad A_{cl} = (\underline{A} - b \cdot r^T)$

$$= \det(s \cdot \underline{\underline{I}} - \underline{\underline{A}} + \underline{b}\, \underline{r}^T) \qquad r^T = [r_1\ r_2\ \dots\ r_n]\ (1 \times n)$$

$$= a_{r,n} \cdot s^n + a_{r,n-1}\, s^{n-1} + \dots + a_{r,1}\, s^1 + a_{r,0}$$

step 4: Comparision of polynomial co-efficients of $P_d(s)$ & $P_r(s)$ to get the controller gain matrix (r^T)

i.e. s^0 : $\ell_{d,0} = a_{r,0}$

coeff. of s^1 : $\ell_{d,1} = a_{r,1}$ $\quad \dots\dots \quad s^{n-1}$: $\ell_{d,n-1} = a_{r,n-1}$,

- $s^n = \ell_{d,n} = a_{r,n}$

step 5: Solve linear equation (obtained from after comparing co-efficients) to get r^T

$$r^T = [r_1\ r_2\ r_3\ \dots\ r_n]$$

⇒ Calculation of feedback Control vector in Controllability Normal form :-

The standard state-space system is applying with state transformation $\tilde{\underline{x}} = \underline{\underline{T}}_s^{-1} x$ with matrix $\underline{\underline{T}}_s^{-1}$. Its controllibility normal form is

$$\dot{\underline{x}} = \underline{\underline{A}}\, \underline{x} + \underline{B} \cdot \underline{u} \quad \xrightarrow{T_s^{-1}} \quad \dot{\tilde{\underline{x}}} = (\underline{\underline{T}}_s^{-1} \cdot \underline{\underline{A}} \cdot \underline{\underline{T}}_s)\, \tilde{\underline{x}} + \underline{\underline{T}}_s^{-1} \cdot \underline{B}\, \underline{u}$$

$$y = \underline{C}\, \underline{x} \qquad\qquad \tilde{\underline{y}} = \underline{c}^T \cdot \underline{\underline{T}}_s\, \tilde{\underline{x}}$$

Then the dynamic matric $\underline{\tilde{A}}$ in controllability Normal form will be

$$\Rightarrow \underline{\tilde{A}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & 1 \\ -a_{r,0} & -a_{r,1} & -a_{r,2} & \cdots & -a_{r,n} \end{bmatrix}; \quad \underline{\tilde{x}} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_n \end{bmatrix}; \quad \underline{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

<u>step1</u>: calculation of the desired characterstic polynomial

$$P_d(s) = (s - s_{d,1})(s - s_{d,2}) \cdots (s - s_{d,n})$$
$$= s^n + P_{d,n-1} s^{n-1} + \cdots + P_{d,1} s^1 + P_{d,0}$$

<u>step2</u>: Compare coreffficients of $P_d(s)$ with ~~last~~ last row in controllability Normal form dynamic matrix

$$\tilde{A} - \tilde{b} \cdot r^T = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \cdots & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ -a_0 - r_1 & -a_1 - r_2 & \cdots & & & -a_{n-1} - r_n \end{bmatrix}$$

$$P_{d,0} = a_{r,0} = a_0 + r_1$$
$$P_{d,1} = a_{r,1} = a_1 + r_2$$
$$\vdots$$
$$P_{d,n-1} = a_{n-1} + r_n \qquad \therefore r^T = [r_1 \; r_2 \; \cdots \; r_n]$$

<u>Note</u> : The calculation of the feedback gain vector r^T for a system in controllability Normal form is extremely easy. Because of 2 reasons ;

1) avoid calculation of real charactestic polynomials of closed loop $\det(s \cdot \underline{I} - \underline{A} + \underline{r}^T \cdot \underline{b})$
2) No need to solve cross-coupled linear equation to get gain factors $(r_1, r_2, \cdots r_n)$

<u>Example</u>: $\dot{\underline{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \bar{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \bar{u}$ and $\bar{y} = [1 \; 0 \; 0] \underline{x}$

The above system representation is similar to controllability normal form where, $a_0 = 0$, $a_1 = 2$, $a_2 = 3$

<u>step1</u>: Desired poles charchterstic polynomial consider $s_{d,1} = -3$, $s_{d_{2,3}} = -1 \pm j$

$$P_d(s) = (s+3)(s - (-1+j))(s - (-1-j)) = s^3 + 5s^2 + 8s + 6$$

$P_d(s) = s^3 + 5s^2 + 8s + 6$

step 2: Compare coefeficients

$\Rightarrow a_0 + r_1 = p_{d,0} \Rightarrow r_1 = p_{d,0} - a_0 = 6 - 0 = 6$

$r_2 = p_{d,1} - a_1 = 8 - 2 = 6$

$r_3 = p_{d,2} - a_2 = 5 - 3 = 2$

$\therefore r^T = [r_1 \; r_2 \; r_3] = [6 \; 6 \; 2]$

3.3.5 Feedback Vector / Gain Matrix Calculation using Ackermann's formula (t-Vector Method)

first step: Definition of desired poles of closed loop systems

i.e, $s_{d,1}, s_{d,2}, \ldots s_{d,n}$

- step 2: Calculation of desired characteristic polynomials of the closed loop

$P_d(s) = (s - s_{d,1})(s - s_{d,2}) \ldots (s - s_{d,n})$

$P_d(s) = p_{d,n} \cdot s^n + p_{d,n-1} s^{n-1} + \ldots + p_{d,1} \cdot s^1 + p_{d,0}$

step 3: Determination of vector t^T

In order to get $t^T = (t_1 \; t_2 \ldots t_n)$, we have to solve n scalar linear equations i.e

$t^T \cdot b = 0$

$t^T \cdot A \cdot b = 0$

$t^T \cdot A^2 \cdot b = 0$

-

$t^T \cdot A^{n-1} \cdot b = 1$

step 4: Calculation of controller gain vector r^T

$r^T = p_{d,0} \cdot t^T \cdot I + p_{d,1} \cdot t^T \cdot A + \ldots + p_{d,n} \, t^T A^n$

$r^T = \sum_{i=0}^{n} p_{d,0} \cdot t^T \cdot A^i$

Example: $\dot{x} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$ and $y = [0 \; 1] x$

step 1: $s_{d,1} = s_{d,2} = -3$

step 2: $P_d(s) = (s - s_{d,1})(s - s_{d,2}) = (s+3)(s+3)$

$P_d(s) = s^2 + 6s + 9$

step 3: $\underline{t}^T$ vector $\Rightarrow \underline{t}^T = [t_1 \;\; t_2]$ $n=2$ states

linear eqn. $\underline{t}^T \cdot b = 0$

$$[t_1 \; t_2]\begin{bmatrix}1\\0\end{bmatrix} = 0$$

$$t_1 = 0$$

$\underline{t}^T \cdot \underline{A} \cdot \underline{b} = 1$

$$[t_1 \; t_2]\begin{bmatrix}1 & 0\\2 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = 1$$

$$t_1 + 2t_2 = 1$$

$$t_2 = 1/2$$

$$\underline{t}^T = [0 \;\; 1/2]$$

step 4 : $\gamma^T = p_{d,0} \cdot \underline{t}^T \cdot \underline{\underline{I}} + p_{d,1} \cdot \underline{t}^T \cdot \underline{A} + p_{d,2} \cdot \underline{t}^T \cdot \underline{A}^2$

$$= 9\,[0 \;\; 1/2]\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + 6\,[0 \;\; 1/2]\begin{bmatrix}1 & 0\\2 & -1\end{bmatrix} + 1\,[0 \;\; 1/2]\begin{bmatrix}1 & 0\\2 & -1\end{bmatrix}^2$$

$$\gamma^T = [6 \;\; 2]$$

3.4 feedback Matrix Design by Pole placement (MIMO)

① Dyadic Control (MIMO)

Idea: Reduce MIMO controller design task complexity to a controller design for a 1D scalar input systems

step1: a) Virtual scalar input ($\bar{u}$) is $\underline{u} = q \cdot \bar{u}$

b) New state equations, $\dot{\underline{x}} = \underline{A}\,\underline{x} + \underbrace{\underline{B} \cdot \underline{q}}_{\bar{b}} \cdot \bar{u}$;

$$y = \underline{C} \cdot \underline{x}$$

choose $\underline{q}$ (n×1) such that new state equation $\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{\bar{b}} \cdot \bar{u}$ is controllable for $\bar{b} = \underline{B} \cdot \underline{q}$

step2 : Pole placement for single input systems

Apply one of the SISO pole placement method to $\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{\bar{b}} \cdot \bar{u}$ for feedback control law $\bar{u} = -\underline{\gamma}^T \cdot \underline{x}$ to get virtual feedback gain vector γ^T

step3 : Calculate the state feedback matrix for MIMO sys.

The control law for the original MIMO system is $\underline{u} = -\underline{R} \cdot x$ and feedback gain matrix $\underline{R} = \underline{q} \cdot r^T$

$\underline{R} = q \cdot r^T$ (i.e, column vector $\underline{q}$ and row vector γ^T)

4. Dynamic State Observers

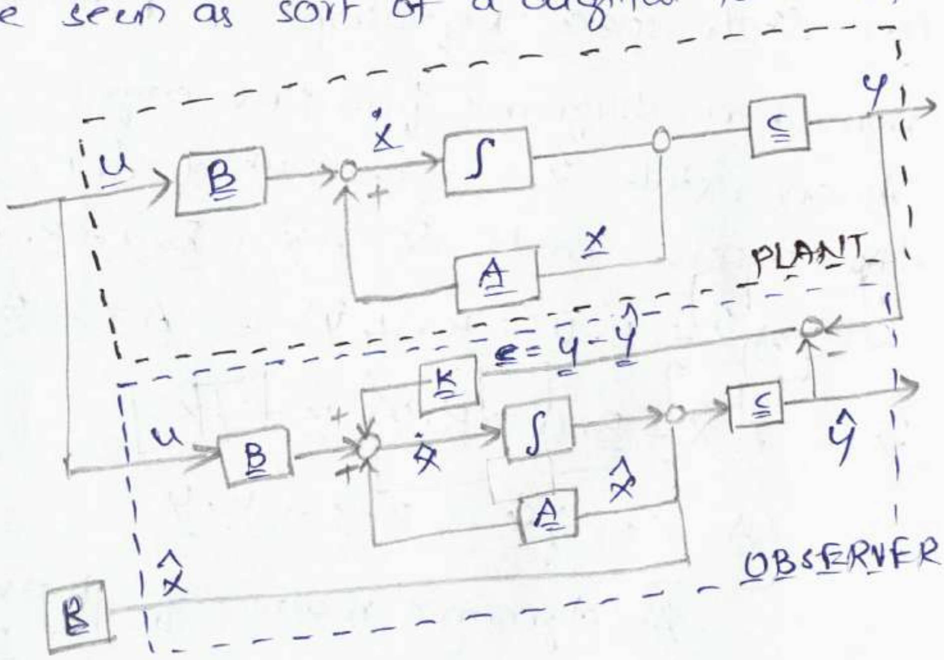
We know that state feedback controllers in state space controller design, are depends on feedback of the state vector $\underline{x}$ (i.e, $x_1, x_2, \dots x_n$). This state variables are measured by sensor of the control system and used this measurement to close the feedback control loop via feedback gain matrix $\underline{K}$.

In practice atleast some state variables x_i are not measurable, but can be calculated from other known state variables (eg. If x_1 is position, x_2 is velocity which can't be measured from sensor. But, if sensor gives x_1 value on derivation of x_1, we get x_2)

In most cases, a direct calculation of the missing variables x_i for state feedback control is not possible. These states have to be estimated by "dynamic state observer".

The observer can be seen as sort of a "digital twin" of the plant.

⇒ Block Diagram of MIMO plant with real state $\underline{x}$ and dynamic state observer($\hat{x}$)



- The figure shows that observer($\hat{x}$) as a digital twin of the plant
- Observer($\hat{x}$) tries to estimate x of plant from $\underline{y}$ & $\underline{u}$ (o/p & i/p signals). for a performant observer, the estimated state $\hat{x}$ is close to the real state of the plant

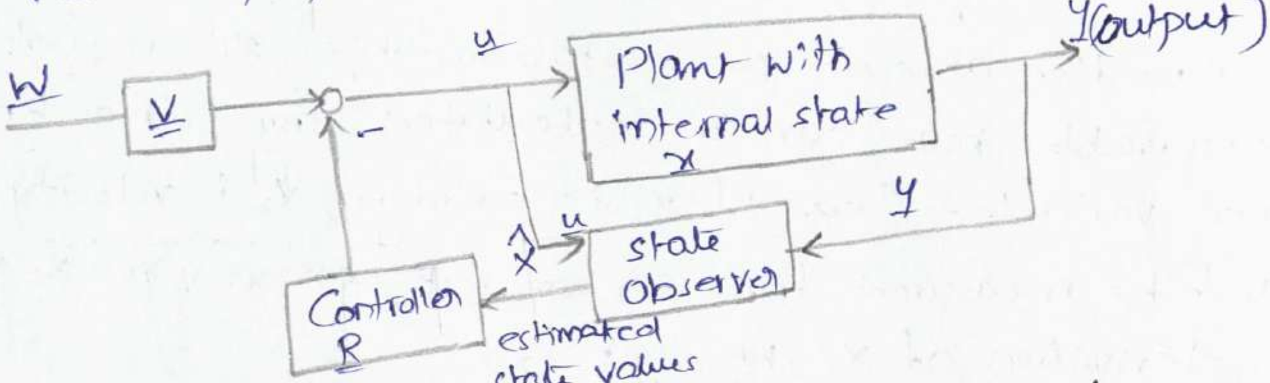
i.e $\hat{x} \simeq x$

so to get convergence of $\hat{\underline{x}}$ to $\underline{x}$, then $\hat{\underline{x}} \simeq \underline{x}$

for that we need,

- ⇒ Plant model ($\underline{\underline{A}}, \underline{\underline{B}}, \underline{\underline{C}}, \underline{\underline{D}}$)
- ⇒ Observer gain matrix ($\underline{\underline{K}}$)
- ⇒ Signal (both i/p & o/p) − $\underline{u}$ & $\underline{y}$

Once a performant observer is designed, the estimated state information can be used to close the feedback loop via the proportional feedback controller.



Block diagram of overall feedback system based on a state observer for the estimation of unmeasured internal states of a plant & a state feedback controller with $\underline{\underline{R}}$ & $\underline{\underline{V}}$

⇒ **4.2 State space Equations & Estimation Error Dynamics**

from block diagram (from prev. page)

Observer state $\hat{\underline{x}}$ is computed from $\underline{y}(t)$ & $\underline{u}(t)$ and the observer model $\dot{\hat{\underline{x}}} = \underline{\underline{A}}\hat{\underline{x}} + \underline{\underline{B}}\cdot\underline{u} + \underline{\underline{K}}\cdot\hat{\underline{e}}$

$$\dot{\hat{\underline{x}}} = \underline{\underline{A}}\hat{\underline{x}} + \underline{\underline{B}}\cdot\underline{u} + \underline{\underline{K}}(\underline{y}-\hat{\underline{y}}) = \underline{\underline{A}}\underline{x} + \underline{\underline{B}}\cdot\underline{u} + \underline{\underline{K}}\underline{y} - \underline{\underline{K}}\cdot\underbrace{\hat{\underline{y}}}_{\hat{y}=C\hat{x}}$$

$$\dot{\hat{\underline{x}}} = \underline{\underline{A}}\hat{\underline{x}} + \underline{\underline{B}}\cdot\underline{u} + \underline{\underline{K}}\cdot\underline{y} - \underline{\underline{K}}\cdot\underline{\underline{C}}\cdot\hat{\underline{x}}$$

$$= \underbrace{(\underline{\underline{A}} - \underline{\underline{K}}\cdot\underline{\underline{C}})}_{\hat{\underline{\underline{A}}}\text{ (dynamic matric of observer)}}\hat{\underline{x}} + \underline{\underline{B}}\cdot\underline{u} + \underline{\underline{K}}\cdot\underline{y}$$

state-space model of observer is $\dot{\hat{\underline{x}}} = \hat{\underline{\underline{A}}}\cdot\hat{\underline{x}} + \underline{\underline{B}}\cdot\underline{u} + \underline{\underline{K}}\cdot\underline{y}$

$\hat{y} = \underline{\underline{C}}\cdot\hat{\underline{x}}$

On the other hand, the estimation error $\underline{x}_e = \underline{x} - \hat{\underline{x}}$ for the real state variable $\underline{x}$ of the plant is given by

$$\dot{\underline{x}}_e = \dot{\underline{x}} - \dot{\hat{\underline{x}}} = \underline{\underline{A}}\underline{x} + \underline{\underline{B}}\cdot\underline{u} - (\underline{\underline{A}}\hat{\underline{x}} + \underline{\underline{B}}\cdot\underline{u} + \underline{\underline{K}}\cdot(\underline{y}-\hat{\underline{y}}))$$

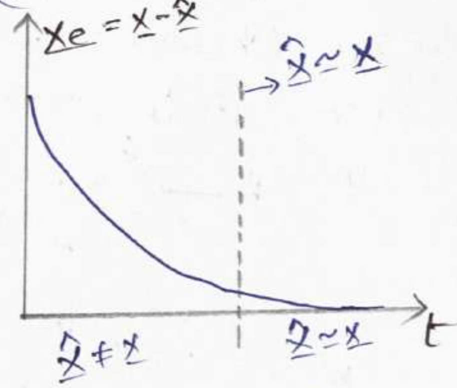
$$\dot{\underline{x}}_e = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\cdot\underline{u} - \underline{\underline{A}}\,\hat{\underline{x}} - \underline{\underline{B}}\cdot\underline{u} - \underline{\underline{K}}\,(\underline{\underline{C}}\,\underline{x} - \underline{\underline{C}}\cdot\hat{\underline{x}})$$

$$= \underline{\underline{A}}(\underline{x} - \hat{\underline{x}}) - \underline{\underline{K}}\cdot\underline{\underline{C}}\,(\underline{x} - \hat{\underline{x}})$$

$$\dot{\underline{x}}_e = (\underline{\underline{A}} - \underline{\underline{K}}\cdot\underline{\underline{C}})\,(\underline{x} - \hat{\underline{x}})$$

$$\dot{\underline{x}}_e = \hat{\underline{\underline{A}}}\cdot\underline{x}_e$$

Note ∇ : The dynamical behaviour of the estimation error is defined by the homogeneous (i.e no input) state space equation.



⇒ If this state space system describing the estimation error is asymptotically stable, the error $\underline{x}_e = \underline{x} - \hat{\underline{x}}$ will tend to zero over time and we get $\hat{\underline{x}} \simeq \underline{x}$.

As we aware that, the decay rate to zero is correlated to the pole location (i.e if poles are too far from origin on left side decay is faster and closes to origin, slow decay)

Hence, the convergence of the estimated state $\hat{\underline{x}}$ to the real state $\underline{x}$ of the plant is related to location of poles (eigenvalues) of the observer matrix $\hat{\underline{\underline{A}}}$ in the s-plane.

" Convergence will be faster for larger value of $\hat{\alpha}_k$ of eigenvalues : $\hat{s}_k = -\hat{\alpha}_k \pm j\hat{\omega}_k$ of $\hat{\underline{\underline{A}}}$.

Hence, $\hat{\underline{\underline{A}}} = \underline{\underline{A}} - \underline{\underline{K}}\cdot\underline{\underline{C}}$

so, the location of poles/eigenvalues of observer can be influenced by appropriate choice of observer gain matrix "$\underline{\underline{K}}$".

4.3 Observer Design:-

Idea: Here, we are comparing observer design to state feedback controller design.

In state feedback controller design, feedback gain matrix '$\underline{\underline{R}}$' has found with the help of desired pole of dynamic matrix $\underline{\underline{A}}_{cl} = \underline{\underline{A}} - \underline{\underline{B}}\cdot\underline{\underline{R}}$ of closed loop system, So, similarly in observer design, we are searching observer gain matrix $\underline{K}$ to place the pole of homogeneous state estimation error (i.e. $x_e = 0$)

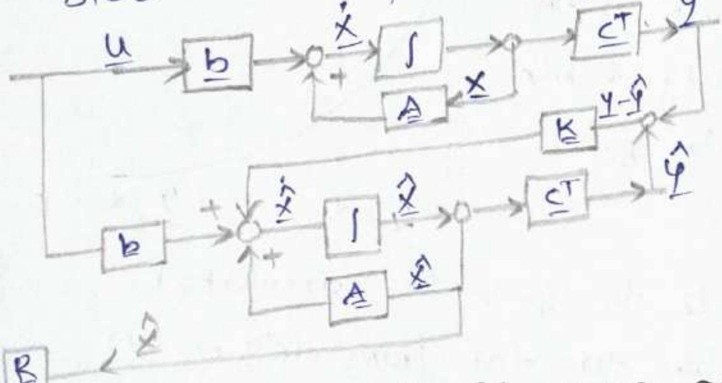
$$\underline{A}_{CL} = \underline{A} - \underline{B} \cdot \underline{R} \qquad \text{and} \qquad \hat{\underline{A}} = \underline{A} - \underline{K} \cdot \underline{C}$$

<u>Method</u>: How we get $\underline{R}$ in controller design, similarly we will get $\underline{K}$ in observer design.

4.3.2 Observer Design via "Direct Method" (SISO)

SISO state space system $\quad \dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{b} \cdot \underline{u}$

$\underline{y} = \underline{c}^T \cdot \underline{x}$



⇝ find $\underline{K}$ such that $\hat{\underline{A}} = \underline{A} - \underline{K} \cdot \underline{c}^T$ has desired eigenvalues

⇒ calculation of Observer gain vector in 5 steps.

<u>Step 1</u>: Define desired poles of the observer
i.e. $\hat{s}_{d,1}, \hat{s}_{d,2}, \hat{s}_{d,3} \cdots \hat{s}_{d,n}$

<u>Step 2</u>: Calculation of desired characteristic polynomial of the observer

$$\hat{P}_d(s) = (s - \hat{s}_{d,1})(s - \hat{s}_{d,2}) \cdots\cdots (s - \hat{s}_{d,n})$$

$$\hat{P}_d(s) = \hat{P}_{d,n} \cdot s^n + \hat{P}_{d,n-1} \cdot s^{n-1} + \cdots\cdots + \hat{P}_{d,1} \cdot s^1 + \hat{P}_{d,0}$$

<u>Step 3</u>: Calculation of real characteristic polynomial of observer depending on observer gain $\underline{K} = [K_1 \; K_2 \cdots K_n]^T$

$$\hat{P}_r(s) = \det(s \cdot \underline{\underline{I}} - \hat{\underline{\underline{A}}}) = \det(s \cdot \underline{\underline{I}} - (\underline{\underline{A}} - K \cdot c^T))$$

$$= \hat{a}_{r,n} \cdot s^n + \hat{a}_{r,n-1} \cdot s^{n-1} + \cdots\cdots + \hat{a}_{r,1}\, s + \hat{a}_{r,0}$$

Please note, the coefficients $\hat{a}_{r,k}$ of this polynomial are function of unknown 'k_i' values, $i = 1, 2, \cdots n$.

<u>Step 4</u>: Comparison of polynomial coefficients to get observer gain vector $\underline{K}$.

i.e. $\quad s^0 : \hat{P}_{d,0} = \hat{a}_{r,0}$

$s^1 : \hat{P}_{d,1} = \hat{a}_{r,1}$

$s^2: \hat{\ell}_{d,2} = \hat{a}_{r,2}$

$\vdots$

$s^n = \hat{\ell}_{d,n} = \hat{a}_{r,n}$

<u>step 5</u>: formulation of the observer gain vector

$$\underline{K} = \begin{bmatrix} K_1 \\ K_2 \\ \vdots \\ K_n \end{bmatrix}$$

$\underline{c}^T \to 1\times m$ | $b = (r \times 1)$

$\underline{K} \to m \times 1$ | $\underline{r} = (1 \times r)$

4.3.3 General Design Approach

The task of observer design is a question of pole placement like in the case of state feedback controller design. furthermore, all methods of controller design can be used to calculate necessary observer gain matrix $\underline{\underline{K}}$ (MIMO) or observer gain vector ($\underline{K}$)(SISO).

plant : $\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\cdot \underline{u}$

$\underline{y} = \underline{\underline{C}}\,\underline{x}$

Eigenvalues of $\hat{\underline{\underline{A}}}$ and eigenvalues of $(\hat{\underline{\underline{A}}})^T$ are same

$(\because \hat{\underline{\underline{A}}} = \underline{\underline{A}} - \underline{\underline{K}}\cdot\underline{\underline{C}})$ $\quad eig(\hat{\underline{\underline{A}}}) = eig(\hat{\underline{\underline{A}}})^T = eig(\underline{\underline{A}} - \underline{\underline{K}}\cdot\underline{\underline{C}})^T$

$= eig(\underline{\underline{A}}^T - \underline{\underline{C}}^T\cdot\underline{\underline{K}}^T)$

$\Rightarrow \underline{\underline{A}}^T - \underline{\underline{C}}^T\cdot K^T$ is similar to, $\underline{\underline{A}} - \underline{\underline{B}}\,\underline{\underline{R}}$ (the design of observer gain matrix $\underline{\underline{K}}$ can be replaced by design of controller for the so-called "dual system".

Controller

$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\cdot\underline{u}$

$\underline{y} = \underline{\underline{C}}\cdot\underline{x}$

$\underline{\underline{A}}_{cl} = \underline{\underline{A}} - \underline{\underline{B}}\cdot\underline{\underline{R}}$

$\underline{u} = -\underline{\underline{R}}\cdot\underline{x} + \underline{\underline{V}}\cdot\underline{w}$

$$\dot{\underline{x}}_D = \underline{\underline{A}}^T\cdot\underline{x}_D + \underline{\underline{C}}^T\cdot\underline{u}_D$$

$$\underline{y}_D = \underline{\underline{B}}^T\cdot\underline{x}_D$$

together with control feedback law

$$\underline{u}_D = -\underline{\underline{K}}^*\cdot\underline{x}_D$$

This controller design problem for the dual system is often called "dual control problem".

After getting $\underline{\underline{K}}^*$ with method from controller design, we get observer gain matrix by

$$\underline{\underline{K}} = (K^*)^T$$

Overview of this general procedure for pole placement :-

1) Define desired pole location for the observer
$\hat{s}_{d,1}, \hat{s}_{d,2} \ldots \ldots \hat{s}_{d,n}$

2) Define the dual system $\dot{\underline{x}}_D = \underline{\underline{A}}^T \cdot \underline{x}_D + \underline{\underline{C}}^T \cdot \underline{u}_D$, $\underline{y}_D = \underline{\underline{B}}^T \cdot \underline{x}_D$ to observer design task with feedback law $\underline{u}_D = -\underline{\underline{K}}^* \cdot \underline{x}_D$

③ Apply a pole placement method for SISO/MIMO systems for controller design to calculate $\underline{\underline{K}}^*$ for dual system

4) Observer gain matrix $\underline{\underline{K}} = (K^*)^T$

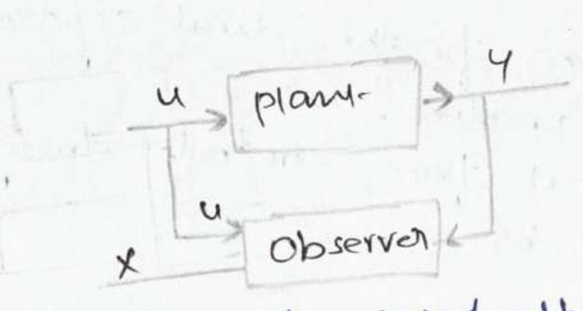
This procedure is also typically applied for the computation of the observer gain using some numerical tool like Matlab.

Choice of Desired Observer Poles :-

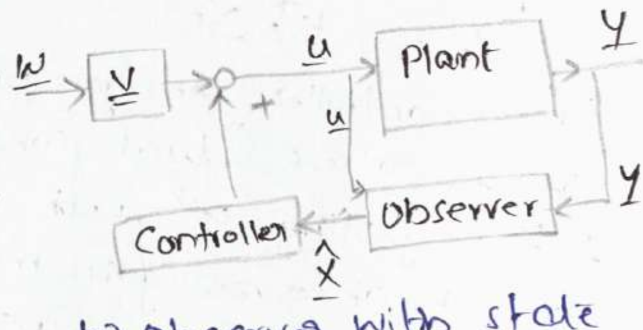
To acheive a performant state estimation, the observer dynamics has to be significantly faster than the dynamical behaviour of the state space system. Other-wise the state estimation of the observer is not able to follow the states of the real plant. To assure this, we have to design the pole location of the observer with regards to the pole location of the observed system.

a) Use of an observer to estimate the internal state of a plant without state feedback control

b) Use of an observer to estimate the internal state of a plant / system together with state-feedback controller based on the estimated state information ($\hat{x}$).



a) Observer w/o state feedback control



b) Observer with state feedback control

- In case (a), we have to place the eigenvalues of the dynamic matrix of the observer $\hat{\underline{A}} = \underline{\underline{A}} - \underline{\underline{K}} \cdot \underline{\underline{C}}$ on the left hand side of eigenvalues of the dynamic matrix of the plant by appropriate choice of $\underline{\underline{K}}$

- In case (b), we have to place the eigenvalues of the observer dynamic matrix $\hat{\underline{A}} = \underline{\underline{A}} - \underline{\underline{K}} \cdot \underline{\underline{C}}$ on the left hand side of the eigenvalues of closed-loop feedback system defined by matrix $\underline{\underline{A}}_{CL} = \underline{\underline{A}} - \underline{\underline{B}} \cdot \underline{\underline{R}}$ by an appropriate choice of $\underline{\underline{K}}$.

* A typical choice : $\boxed{\text{Real part of observer poles} = \text{2 to 6 times of real part of closed loop feedback system poles.}}$

□ $\hat{\underline{A}} = \underline{\underline{A}} - \underline{\underline{K}} \cdot \underline{\underline{C}}$ can be independently designed by the choice of $\underline{\underline{K}}$. (observer gain matrix)

□ $\underline{\underline{A}}_{CL} = \underline{\underline{A}} - \underline{\underline{B}} \cdot \underline{\underline{R}}$ also independently designed by the choice of $\underline{\underline{R}}$ (feedback control matrix)

This $\hat{\underline{A}}$ & $\underline{\underline{A}}_{CL}$ eigenvalues are defined seperately by the choice of $\underline{\underline{K}}$ & $\underline{\underline{R}}$. This is called "seperation principle".

⇒ Faster Convergence of the Estimation

To acheive a <u>faster convergence</u> of the estimation (i.e, $\hat{\underline{x}} \simeq \underline{x}$), the poles of observer have to be shifted <u>farer</u> to the <u>left in the s-plane</u>.

And to reduce the effect of <u>unavoidable, High frequency</u> measurement error on state estimation within the observer, the poles of the observer can be <u>shifted farer</u> to the <u>right in s-plane</u>.

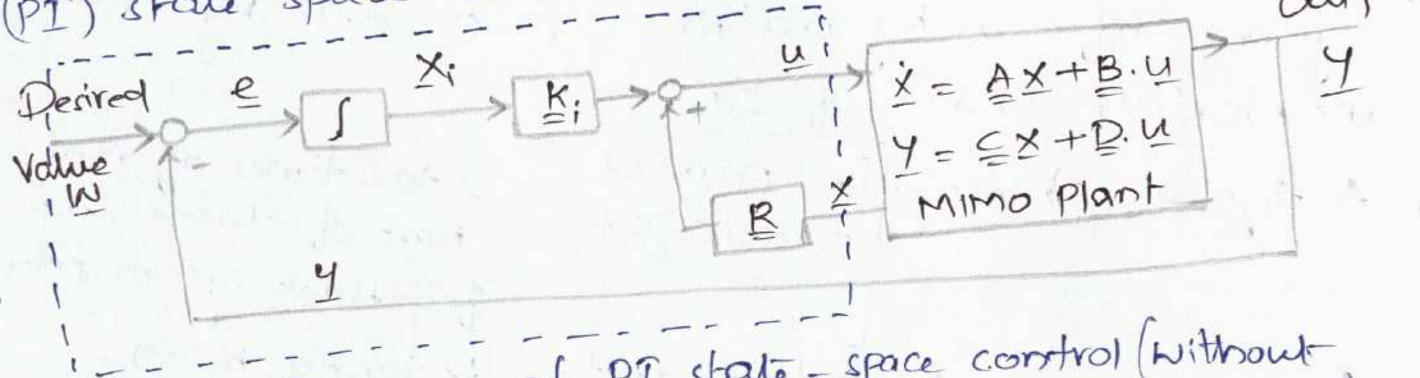
Overall Design Process for state-feedback Control with state Observer:-

first : Design the state feedback controller by pole placement, LQR design to get $\underline{\underline{R}}$

second : Design state observer 2-6 times faster than state feedback control loop (w/o observer). The knowledge of CL poles is a pre-requisite.

5. Proportional-Integral State space Control

We use pre-filter to get an acceptable tracking behaviour of the control loop and a static tracking error close to zero in Proportional state feedback control. But with pre-filter, small tracking error (eg. friction effects) can't be computed. To overcome this limitation, and to reduce static tracking error, Proportional-Integral (PI) state space control concept can be used.



Block Diagram of PI state-space control (without state observer)

⇒ General properties of PI state-space Control

As mentioned earlier, PI-state space control is an alternate approach to the pre-filter design and to get a small/no steady state tracking error [i.e $e(t\to\infty)=0$]

From figure; $x_i = \int e(t)\,dt \Leftrightarrow \dot{x}_i(t) = e(t)$

In steady state, $\dot{x}_i(t\to\infty)=0$ and $\dot{x}(t\to\infty)=0$

$\dot{x}_i(t\to\infty) = e(t\to\infty) = 0$, which means steady state error is zero.

$\Rightarrow \underline{W}(t\to\infty) - \underline{y}(t\to\infty) = \underline{0}$

$\Rightarrow \underline{W}(t\to\infty) = \underline{y}(t\to\infty)$

∴ By design of the control loop structure, the steady state tracking error $\underline{e}_\infty = \underline{W}_\infty - \underline{y}_\infty$ is always zero for PI-state space control.

→ Zero steady state is only achievable in general if $r=m$ (i.e no. of input = no. of outputs $\underline{u} = \underline{y}$)

→ PI-state space control can be applied to SISO as well as MIMO plants.

→ Proportional state feedback technique is limited to plant systems without feedthrough ($D=0$ / $d=0$), this PI state space control method can be used for system with a direct feedthrough

Controller Design and Derivation of the Extended Plant Model :-

Different approaches for the design of the two controller parameters $\underline{R}$ and $\underline{K}_i$

i) - Calculate $\underline{R}$ (by pole placement, LQR...) and then tune the matrix $\underline{K}_i$ in simulation or real experiment this method can be done in SISO case mainly.

ii) - Calculate $\underline{R}$ and $\underline{K}_i$ in one step (e.g: by pole placement, LQR...) for the extended plant model.

- No general rules for the first approach. but the design using an extended model of the plant is very systematic.

from block diagram, $\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B}\cdot\underline{u}$, $\underline{y} = \underline{C}\,\underline{x} + \underline{D}\cdot\underline{u}$

$\dot{\underline{x}}_i = \underline{e} = \underline{w} - \underline{y} = \underline{w} - \underline{C}\cdot\underline{x} - \underline{D}\cdot\underline{u}$

The dynamic state equation of the extended plant model is

$$\underbrace{\begin{bmatrix} \dot{\underline{x}} \\ \dot{\underline{x}}_i \end{bmatrix}}_{\dot{\underline{x}}_E} = \underbrace{\begin{bmatrix} \underline{A} & 0 \\ -\underline{C} & 0 \end{bmatrix}}_{\underline{A}_E} \underbrace{\begin{bmatrix} \underline{x} \\ \underline{x}_i \end{bmatrix}}_{\underline{x}_E} + \underbrace{\begin{bmatrix} \underline{B} \\ -\underline{D} \end{bmatrix}}_{\underline{B}_E}\cdot\underline{u} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\underline{w}$$

also, we get control law for the plant input

$$\begin{aligned} \underline{u} &= -\underline{R}\cdot\underline{x} + \underline{K}_i\cdot\underline{x}_i \\ &= \begin{bmatrix} -\underline{R} & \underline{K}_i \end{bmatrix}\begin{bmatrix} \underline{x} \\ \underline{x}_i \end{bmatrix} \\ &= -\underbrace{\begin{bmatrix} \underline{R} & -\underline{K}_i \end{bmatrix}}_{\underline{R}_E}\underline{x}_E \end{aligned}$$

Note:

- Extended model is similar to proportional state feedback so, we can use pole placemethods, LQR design method or any other controller method for proportional state feedbacks, to get controller gain matrix of Extended plant model $\underline{\underline{R}}_E$ of PI-state feedback control loop.

<u>Calculation of the Control Parameters</u> $\underline{\underline{R}}, \underline{\underline{K}}_i$ <u>in 2 steps</u>:-

<u>step 1</u>: Design of a proportional state feedback controller for the extended plant

$$\dot{\underline{x}}_E = \underline{\underline{A}}_E \underline{x}_E + \underline{\underline{B}}_E \cdot \underline{u}$$

eg: by pole placement, LQR... to get controller gain matrix $\underline{\underline{R}}_E$ of state feedback control law.

$$\underline{u} = -\underline{\underline{R}}_E \cdot \underline{x}_E$$

<u>step 2</u>: Extraction of the two controller parameters $\underline{\underline{R}}$ and $\underline{\underline{K}}_i$ from the first n and last m columns of the gain matrix $\underline{\underline{R}}_E$.

$$\underline{\underline{R}}_E = [\underline{\underline{R}} \quad -\underline{\underline{K}}_i]$$

<u>Note</u>: To design a proportional state feedback controller for the extended system with state eqn. $\dot{\underline{x}}_E = \underline{\underline{A}}_E \underline{x}_E + \underline{\underline{B}}_E \cdot \underline{u}$, "<u>controllability of the extended system is sufficient</u>"

ACS

① Time domain - linear & nonlinear sys - MIMO - analyze dynamical behaviour - Higher order plant - ODE

Frequency domain - transfer function - Laplace transform - only IO behaviour; Non-linear plant model:

DC motor :- $\psi_s = L_s i_s$, $u_{sw} = \frac{d\psi_s}{dt}$; $\frac{d\psi_s}{dt} + \frac{R_s}{L_s}\psi_s = u_s$; $L_r \frac{di_r}{dt} + R_r i_r = u_r - k\cdot\omega\cdot\psi_s$; $J\cdot\dot\omega + d\cdot\omega = k i_r \psi_s - M_L$; $y = x_3 = (0, 0, 1)$

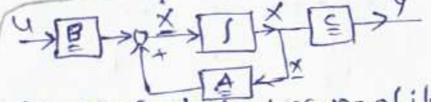
Linear plant model: container carriage system: $m_K \ddot{s}_K = s \cdot \sin\theta + F$; $m_G \ddot{s}_G = -s \sin\theta$; $m_G \cdot \ddot{z}_G = m_G g - s\sin\theta$; $s_G = s_K + l\sin\theta$, $z_G = l\cos\theta$; $\ddot{s}_K = \frac{m_G}{m_K}\cdot g\cdot\theta + \frac{F}{m_K}$; $\ddot\theta = \frac{-g}{l}\left(\frac{m_G + m_K}{m_K}\right)\theta - \frac{F}{m_K \cdot l}$; $x_1 = s_K$ (carriage pos)

$x_2 = \dot{s}_K$ (carriage vel); $x_3 = \theta$ (Rope angle); $x_4 = \dot\theta$ (angular velocity);

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{m_G}{m_K}\cdot g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-g}{l}\left(\frac{m_G+m_K}{m_K}\right) & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1/m_K \\ 0 \\ -1/m_K\cdot l \end{bmatrix} \underline{u} \quad (F = u) \; ; \; y = s_G = s_K + l\theta \quad \underline{y} = [1 \; 0 \; l \; 0]\,\underline{x} + 0\cdot F$$

$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B}\cdot\underline{u}$, $\underline{y} = \underline{C}\underline{x} + \underline{D}\cdot\underline{u}$ } MIMO ; $\dot{\underline{x}} = \underline{A}\underline{x} + \underline{b}u$, $y = \underline{c}^T\cdot\underline{x} + d\cdot u$ } SISO

State space representation:



$\underline{x}(n\times1)$; $\underline{y}(m\times1)$; $\underline{A}(n\times n)$; $\underline{B}(n\times r)$; $\underline{C}(m\times n)$; $\underline{u}(r\times1)$ — SISO: $x(n\times1)$, y 1×1; $\underline{A}(n\times n)$, $\underline{b}(n\times1)$; $\underline{c}^T(1\times n)$

n = no. of states; r = no. of i/p; m = no. of o/p

$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B}\cdot\underline{u}\;(+\underline{B}_z\cdot\underline{z})$ → disturbance

Linearization - from non linear to linear:-

$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}) = \begin{bmatrix} f_1(x_1, x_2 \dots x_n, u_1, u_2 \dots u_n) \\ \vdots \\ f_n(x_1, x_2 \dots x_n, u_1, u_2 \dots u_n) \end{bmatrix}$ Non-linear state equation

$\underline{y} = \underline{g}(\underline{x}, \underline{u}) = \begin{bmatrix} g_1(x_1, x_2 \dots x_n, u_1, u_2 \dots u_n) \\ \vdots \\ g_n(x_1, x_2 \dots x_n, u_1, u_2 \dots u_n) \end{bmatrix}$ non-linear o/p equation

(i) At equilibrium, $\dot{x} = 0$ if $x \to x_\infty$, $u \to u_\infty$

$0 = \underline{f}(\underline{x}_\infty, u_\infty)$, $\underline{y}_\infty = \underline{g}(x_\infty, u_\infty)$

(ii) Calculation of linearized model

$\Delta x = \underline{x} - \underline{x}_\infty$, $\Delta \underline{y} = \underline{y} - \underline{y}_\infty$ for $\Delta\underline{u} = \underline{u} - \underline{u}_\infty$

$\underline{A} = \frac{\partial \underline{f}}{\partial \underline{x}}\Big|_\infty$; $\underline{B} = \frac{\partial \underline{f}}{\partial \underline{u}}\Big|_\infty$; $\underline{C} = \frac{\partial \underline{g}}{\partial \underline{x}}\Big|_\infty$, $\underline{D} = \frac{\partial \underline{g}}{\partial \underline{u}}\Big|_\infty$

(iii) Non-linear & linear simulation:

$\underline{x} = \Delta\underline{x} + \underline{x}_\infty$, $\underline{y} = \Delta\underline{y} + \underline{y}_\infty$, $\underline{u} = \underline{u}_\infty + \Delta\underline{u}$

Eg. Turning wheel pendulum: 2 non linear ODE are

$\ddot\psi = \frac{-h_g k}{R_a\cdot J}\cdot u - \frac{m g l}{J}\sin\psi + \frac{h_g^2 k^2}{R_a\cdot J}\dot\phi - \frac{h_g k^2}{R_a\cdot J}\dot\psi$

$\ddot\phi = \frac{h_g k}{R_a\cdot J_a}\cdot u - \frac{h_g^2 k^2}{R_a\cdot J_a}\dot\phi + \frac{h_g k^2}{R_a\cdot J_a}\dot\psi$; $x_1 = \psi$, $x_2 = \dot\psi$, $x_3 = \phi$, $x_4 = \dot\phi$

Solution of state-space Equation:

Frequency domain

$X(s) = \underbrace{(s\cdot\underline{I} - \underline{A})^{-1}}_{\phi(s)}\cdot B\cdot U(s)$

fundamental matrix, $\phi(s) = (s\cdot\underline{I} - \underline{A})^{-1}$

Transfer function:

$G(s) = \frac{Y(s)}{U(s)} = \underline{C}\cdot(s\cdot\underline{I} - \underline{A})^{-1} * \underline{B} + \underline{D}$ (MIMO)

$G(s) = \underline{c}^T(s\cdot\underline{I} - \underline{A})^{-1}\cdot\underline{b} + \underline{d}$ (SISO)

Time domain:

$x(t) = x_{hom}(t) + x_{inhom}(t)$ — I.C, $u(t)$

$x(t) = e^{\underline{A}(t - t_0)} + \int_{t_0}^{t} e^{\underline{A}(t-\tau)}\cdot\underline{B}u(\tau)\,d\tau$

$y(t) = \underline{C}\cdot x(t) + \underline{D}\cdot u(t)$

$y(t) = \underline{C}\left[e^{\underline{A}(t-t_0)} + \int_{t_0}^{t} e^{\underline{A}(t-\tau)}\cdot\underline{B}\cdot u(\tau)\,d\tau\right] + \underline{D}\cdot u$

$x(t) = T\,\tilde{x}(t)$

Coordinate tranformation of state space: $\dot{x}(t) = \underline{T}\cdot\dot{\tilde{x}}(t)$

$\tilde{\underline{A}} = \underline{T}^{-1}\cdot\underline{A}\cdot\underline{T}$; $\tilde{\underline{B}} = \underline{T}^{-1}\underline{B}$; $\tilde{\underline{C}} = \underline{C}\cdot\underline{T}$; $\tilde{\underline{D}} = \underline{D}$

Connection of Different space sys: $\dot{x}_1 = A_1 x_1 + B_1 u_1$, $\dot{x}_2 = A_2 x_2 + B_2 u_2$, $y_1 = C_1 x_1 + D_1 u_1$, $y_2 = C_2 x_2 + D_2 u_2$

(i) parallel: $y = y_1 + y_2$

$\underline{A}_N = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$; $\underline{B}_N = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$; $\underline{C}_N = [C_1 \; C_2]$; $\underline{D}_N = D_1 + D_2$

(ii) series: $u_2 = y_1$, $y = y_2$

$\dot{\underline{x}} = \begin{bmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{bmatrix}\underline{x} + \begin{bmatrix} B_1 \\ B_2 D_1 \end{bmatrix}\cdot\underline{u}$; $\underline{y} = [D_2 C_1 \;\; C_2]\,\underline{x} + D_2 D_1\cdot\underline{u}$

(iii) Feedback: $u_1 = u - y_2$, $u_2 = y_1$, $y = y_1$ & $D_1 = 0$

$\underline{A}_N = \begin{bmatrix} A_1 - B_1 D_2 C_1 & -B_1 C_2 \\ B_2 C_1 & A_2 \end{bmatrix}$; $\underline{B}_N = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$; $\underline{C}_N = [C_1 \; 0]$; $\underline{D}_N = 0$

Stability: ② if $|x_i| \to 0$ for $t \to \infty$, with $u(t) = 0$

Eigenvalues of $\underline{A}$ matrix ⇒ $\det(s\cdot\underline{I} - \underline{A}) = 0$

Asymptotically stable: all $Re(s_i) = \alpha_i < 0$;

stability limit: one $Re(s_i) = \alpha_i = 0$, all other $Re(s_i)$ $Re(s_i) = \alpha_i < 0$; unstable: atleast one $Re(s_i) = \alpha_i > 0$

$eig(\underline{A}) \to$ poles ‖ TF = $\frac{zeros}{poles}$

Controllability: with $u(t)$, $x_0 \to x(t_e)$

KALMAN criterion: MIMO - rank$(\underline{Q}_S) = n$, SISO - det$(\underline{Q}_S) \neq 0$

controllibility matrix,

$\underline{Q}_S = [\underline{B} \;\; \underline{A}\cdot\underline{B} \;\; \underline{A}^2\underline{B} \cdots \underline{A}^{n-1}\underline{B}]$

$\underline{Q}_S = [\underline{b} \;\; \underline{A}\cdot\underline{b} \;\; \underline{A}^2\underline{b} \cdots \underline{A}^{n-1}\underline{b}]$

Observability: Calculate $x(t)$, from $u(t)$, $y(t)$

KALMAN Criterion:- MIMO - rank$(\underline{Q}_B) = n$, SISO - det$(\underline{Q}_B) \neq 0$

$\underline{Q}_B = \begin{bmatrix} \underline{C} \\ \underline{C}\cdot\underline{A} \\ \vdots \\ \underline{C}\cdot\underline{A}^{n-1} \end{bmatrix} \Rightarrow \begin{bmatrix} \underline{c}^T \\ \underline{c}^T\underline{A} \\ \underline{c}^T\underline{A}^2 \\ \underline{c}^T\cdot\underline{A}^{n-1} \end{bmatrix}$

Observability Normal form:

(i) Determination of s vector $\underline{s}$:

$\underline{s} = (s_1 \; s_2 \dots s_n)^T$ and

$\underline{c}^T\cdot\underline{s} = 0$, $\underline{c}^T\cdot\underline{A}\cdot\underline{s} = 0$, $\underline{c}^T\cdot\underline{A}^2\cdot\underline{s} = 0 \cdots \underline{c}^T\cdot\underline{A}^{n-1}\cdot\underline{s} = 1$ - After solving linear eqn.s, we can get vector $\underline{s}$

(ii) Cal. of Transformation matrix

$\underline{T}_B^{-1} = [\underline{s} \;\; \underline{A}\cdot\underline{s} \;\; \underline{A}^2\cdot s \cdots \underline{A}^{n-1}\cdot s]$

$\tilde{\underline{A}} = \underline{T}_B^{-1}\cdot\underline{A}\cdot\underline{T}_B$, $\tilde{\underline{B}} = \underline{T}_B^{-1}\cdot\underline{B}$, $\tilde{\underline{C}} = \underline{C}\cdot\underline{T}_B$

Diagonal / Jordan Normal Form:-

(i) Determine eigenvalues and eigenvectors

eigenvector $v_1, v_2, \dots v_n$

(ii) Transformation matrix

$\underline{T} = [v_1 \; v_2 \cdots v_n]$

$\tilde{\underline{A}} = \underline{T}^{-1}\underline{A}\,\underline{T}$, $\tilde{\underline{B}} = \underline{T}^{-1}\cdot\underline{B}$

$\tilde{\underline{C}} = \underline{C}\cdot\underline{T}$, $\tilde{\underline{D}} = \underline{D}$

Transfer Function } - only IO behaviour of plant

state space model I/O behaviour and info. about internal dynamics of plant.

Controllability Normal form (CNF):

(i) Determination of vector $\underline{t}^T$:

$\underline{t}^T = (t_1 \; t_2 \cdots t_n) \Rightarrow \underline{t}^T\cdot\underline{b} = 0$, $\underline{t}^T\underline{A}\cdot\underline{b} = 0$, $\cdots \underline{t}^T\underline{A}^{n-1}\cdot\underline{b} = 1$

solve above linear eqns to get vector $\underline{t}^T$.

(ii) Calculation of Transformation matrix

$\underline{T}_S^{-1} = \begin{bmatrix} \underline{t}^T \\ \underline{t}^T\underline{A} \\ \underline{t}^T\underline{A}^{n-1} \end{bmatrix}$; $\tilde{\underline{A}} = \underline{T}_S^{-1}\cdot\underline{A}\cdot\underline{T}$, $\tilde{\underline{B}} = \underline{T}_S^{-1}\cdot\underline{B}$, $\tilde{C} = C\cdot T$

Transfer poles ⊂ system poles, If system is ctrb, o'bsv then Transfer poles ≃ system poles

$e_\infty = w_\infty - y_\infty = 0$, only possible, if $r = m$.

In LQR, $x^T\cdot Q_x\, x \to$ reduces deviation of CL state $x(t)$ from equilibrium point and

$u^T\cdot Q_u\cdot u \to$ reduces necessary control energy.

→ shifting poles to farer left will need more control energy from actuators / higher values in u!

~~Time domain~~ $\boxed{3}$

control law, $u = -\underline{R}\cdot\underline{x} + \underline{V}\cdot\underline{w}$

$\underline{R}$ - control/feedback matrix $(r\times n)$

$\underline{V}$ - pre-filter matrix $(r\times m)$

$\underline{w}$ - desired value vector $(m\times 1)$

Controller Design Process:

(i) Design $\underline{R}$

(ii) Calculate $\underline{V}$ based on $\underline{R}$

Dynamic matrix of closed loop:

$\underline{A}_{cl} = \underline{A} - \underline{B}\cdot\underline{R}$ (MIMO), $\underline{A}_{cl} = \underline{A} - \underline{b}\cdot\underline{r}^T$

$r^T = (r_1 \; r_2 \cdots r_n)$ r^T-row vector

$\underline{V} = [c^T(b\cdot r^T - A)^{-1}\cdot b]^{-1}$ or $(\underline{C}(\underline{B}\cdot\underline{R} - \underline{A})^{-1}\cdot\underline{B})^{-1}$

Calculation of feedback matrix (r^T):

(i) define desired poles $s_{d,1}, s_{d,2} \cdots s_{d,n}$

(ii) Cal. of desired characteristic polynomial

$P_d(s) = (s - s_{d,1})(s - s_{d,2})\cdots(s - s_{d,n})$

$= \rho_{d,n}\cdot s^n + \rho_{d,n-1} s^{n-1} + \cdots \rho_{d,1} s + \rho_{d,0}$

(iii) Real char. polynomial of closed loop

$P_r(s) = \det(s\cdot\underline{I} - \underline{A}_{cl}) = \det(s\cdot\underline{I} - \underline{A} + \underline{b}\cdot r^T)$

$= a_{r,n}\cdot s^n + a_{r,n-1}\cdot s^{n-1} + \cdots + a_{r,1}\cdot s + a_{r,0}$

(iv) Compare polynomial co-efficients to get $r_1, r_2, \cdots r_n$

(v) Combine & get $r^T = [r_1 \; r_2 \cdots r_n]$

DIRECT METHOD

Locate desired single poles on Real axis: $s_d = -\sigma = -\frac{\lambda}{\Delta t} = \frac{-1}{\tau}$

λ - multiple of τ in step response for % rise

Δt - rising time. for 95% $\Rightarrow \lambda = 3$, 98% $\Rightarrow \lambda = 4$; 99% $\Rightarrow \lambda = 5$, 86% $\Rightarrow \lambda = 2$

Design complex conjugate pair of poles: $s_{1,2} = -\alpha \pm j\bar{\omega}$

$s_{1,2} = -D\cdot\omega_n \pm j\omega_n\sqrt{1-D^2}$

$\alpha = D\cdot\omega_n$; $\bar{\omega} = \omega_n\cdot\sqrt{1-D^2}$

D = Damping ratio, ω_n = natural frequency

α = constant exp. decay

$\bar{\omega}$ = oscillation frequency

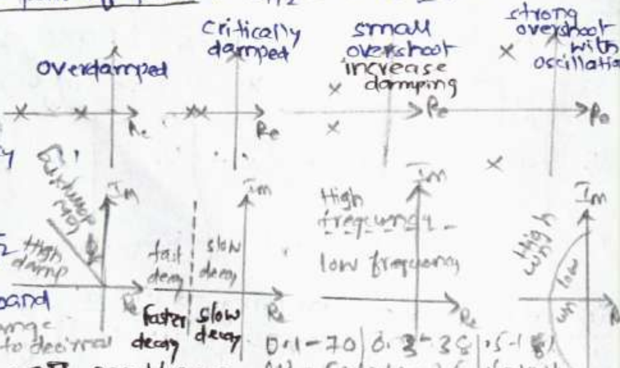
$\alpha = \frac{-\ln(a)}{t_s}$; $\bar{\omega} = \frac{\alpha}{D}\sqrt{1-D^2}$

a = width of steady state band ($\hookrightarrow$ change % to decimal)

t_s = settling time

$\phi_0 = \sin^{-1}(D)$

1 Hz = 2π rad/sec



If system representation in CNF i.e. $\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} x + \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} u$

Controllability Normal form,

2 steps are similar to direct method. $-a_0, -a_1, -a_2$

(iii) $r_1 = \rho_{d,0} - a_0$, $r_2 = \rho_{d,1} - a_1, \cdots r_n = \rho_{d,n-1} - a_{n-1}$ (+ve value)

(iv) $r^T = [r_1 \; r_2 \cdots r_n]$

Ackermann / t-vector method

(i),(ii) are similar

(iii) vector $t^T = [t_1 \; t_2 \cdots t_n]$

with $t^T\cdot b = 0$, $t^T\cdot A\cdot b = 0 \cdots$

$\cdots t^T\cdot A^{n-1}\cdot b = 1$

calculate vector t^T

(iv) $r^T = \rho_{d,0}\cdot t^T\cdot I + \rho_{d,1}\cdot t^T\cdot A + \cdots \rho_{d,n}\cdot t^T A^n$

Dyadic control (MIMO):

(i) Choice of $\underline{q}$: $\underline{u} = \underline{q}\cdot\bar{u}$

$\dot{\underline{x}} = \underline{A}\underline{x} + \underline{B}\cdot\underline{q}\cdot\bar{u} \Rightarrow \dot{\underline{x}} = \underline{A}\underline{x} + \bar{b}\cdot\bar{u}$

with $\bar{b} = \underline{B}\cdot\underline{q}$, $\underline{q} = r\times 1$

(ii) pole placement for SISO

$\bar{u} = -r^T x$ & cal. r^T

(iii) $\underline{R}$ matrix for MIMO

$\underline{R} = \underline{q}\cdot r^T$ $\quad q\,(r\times 1)$, $r^T(1\times n)$ so, dyadic

$\boxed{4}$ $\underline{K}$ - observer gain matrix $(n\times 1)$

$\dot{\hat{x}} = \underline{A}\hat{\underline{x}} + \underline{B}\cdot\underline{u} + \underline{K}\cdot e = \underline{A}\hat{x} + \underline{B}\cdot\underline{u} + \underline{K}\cdot(y - \hat{y})$

$= (\underline{A} - \underline{K}\,\underline{C})\hat{x} + \underline{B}\cdot\underline{u} + \underline{K}\cdot\underline{y}$

$\hat{y} = \underline{C}\hat{x}$

$\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} = (\underline{A} - \underline{K}\cdot\underline{C})(x - \hat{x}) = \underbrace{(\underline{A} - \underline{K}\cdot\underline{C})}_{\hat{A}}\, x_e$

$\rightarrow$ Calculation of $(\underline{K})$: (SISO)

(i) define desired poles $\hat{s}_{d,1}, \hat{s}_{d,2}, \cdots \hat{s}_{d,n}$

(ii) Cal. desired characteristic polynomial

$\hat{P}_d(s) = (s - \hat{s}_{d,1})(s - \hat{s}_{d,2})\cdots(s - \hat{s}_{d,3})$

(iii) Cal. Real char. polynomial

$P_r(s) = \det(s\cdot\underline{I} - \hat{A}) = \det(s\cdot\underline{I} - \underline{A} + \underline{k}\cdot\underline{c}^T)$

(iv) compare polynomial co-efficients & get $K_1, K_2, \cdots K_n$ K - column vector

(v) combine, $K = [K_1 \; K_2 \; K_3 \cdots K_n]^T$

General Design Approach

(i) def. desired poles $\hat{s}_{d,1}, \hat{s}_{d,2} \cdots \hat{s}_{d,n}$

(ii) def. dual system

$\dot{x}_D = \underline{A}^T x_D + \underline{C}^T u_D$,

$y_D = \underline{B}^T x_D$ with

$u_D = -\underline{K}^{*}\cdot x_D$

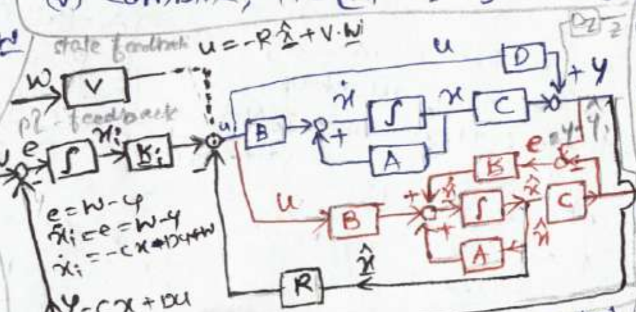
(iii) Cal. K^* from controller technique

(iv) Cal K (observer) by $\underline{K} = (K^*)^T$

$\boxed{5}$ PI - st. space control - to get small/no steady state tracking error (i.e. $e(t\to\infty) = 0$)

$$\underbrace{\begin{bmatrix} \dot{\underline{x}} \\ \dot{x}_i \end{bmatrix}}_{\dot{x}_E} = \underbrace{\begin{bmatrix} \underline{A} & 0 \\ -\underline{C} & 0 \end{bmatrix}}_{A_E} \underbrace{\begin{bmatrix} \underline{x} \\ x_i \end{bmatrix}}_{x_E} + \underbrace{\begin{bmatrix} \underline{B} \\ -\underline{D} \end{bmatrix}}_{B_E} \cdot\underline{u} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\underline{w}$$

$u = -[\underline{R} \;\; -K_i]\, x_E \Rightarrow R_E = [\underline{R} \;\; -K_i]$



Properties of resultant control CL sys & Benefits of LQR:

1. Closed control loop is asymptotically stable

2. The closed control loop has guaranteed robustness properties w.r.t. regards to model/ parameter errors of our state space plant.

a) The gain margin of CL system is atleast 50% i.e, CL still asymptotically stable for unmodelled gains in CL

b) Closed controlloop has guaranteed phase margin Eg: In SISO sys, Phase margin of CL is atleast 60°.

K_i - integral control gain, $\underline{R}$ - feedback control matrix $(r\times n)$ $(r\times m)$

$L^{-1}(1/s) = 1$; $L^{-1}(\frac{1}{s^2}) = t$ $\underline{K}$ - observer gain matrix $(n\times m)$

$L^{-1}(\frac{1}{s-a}) = e^{at}$ rank$\begin{bmatrix} s_0\cdot I - A \\ c \end{bmatrix} < n \Rightarrow$ o/p decoupling zero

$[s\cdot I - A ; -B] < n \Rightarrow$ i/p decoupling zero

unit step response: $\lim_{t\to\infty} y(t) = G(0)$; dc gain means, $s = 0$ in transfer function

- If poles contain imaginary parts, oscillations in o/p
- If poles (real part) is far left on s-plane faster decay rate.
- 6 faster decay compared to -2

inv(M); eig(M); det(M); rank(M); $s = tf('s')$ $Gs = (2*s+1)/(s\^2+3*s+1)$ $Gs = tf([2,1]/[1,3,1])$	M = parallel(M_1, M_2) M = series(M_1, M_2) M = feedback(M_1, M_2) M = feedback$(M_1, M_2, +1)$ '+1' for +ve feedback	basic sys. analysis: step(sys) bode(sys) pzmap(sys) dcgain(sys) — State space rep. $Gss = ss(A,B,C,D)$ sys_TF = tf(Gss)	basic system analysis: ② ctrb(Gss) obsv(Gss) rank(obsv(Gss)) rank(ctrb(Gss)) $Q_S = [b, A*b, A*A*b \ldots]$ $(1\times C)$ $Q_B = [c; c*A; c*A*A \ldots]$ $(r\times 1)$

③ R = place(A, B, poles) $\Rightarrow$ feedback matrix

V = inv(C * inv(B*R - A) * B) $\Rightarrow$ prefilter

R = lqr(A, B, Q_x, Q_u) (or) lqr(sys, Q_x, Q_u)

④ observer general design approach

KT = place(A', C', poles)

K = (KT)' $\Rightarrow$ Transpose

⑤ PI : R_E = place(AE, BE, Poles)

$R = (r\times n)$, $K_i = r\times m$

ACS SHORT NOTES

Classical controller Design	Modern controller Design
→ SISO plant sys only	→ SISO & MIMO plant sys.
→ Transfer function in frequency domain	→ often internally cross coupled problems
→ Controller design by IO behaviour of TF	→ system of ODE in time domain
	→ Controller design by manipulating internal dynamics of cl system & dynamic state estimation.

Rank of matrix is no. of its linearly independent rows/col.

State space representation

- state equation : $\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B}\cdot\underline{u} + \underline{B_z}\cdot z$
  output eqn : $\underline{y} = \underline{C}\,\underline{x} + \underline{D}\cdot\underline{u}$
  IC : $\underline{x}(t_0) = \underline{x}_0$

$$\underset{(n\times1)}{\dot{\underline{x}}} = \underset{(n\times n)}{\underline{A}}\,\underline{x} + \underset{(n\times r)}{\underline{b}}\cdot\underset{(r\times1)}{\underline{u}}$$

$$\underline{y} = \underset{(1\times m)}{\underline{c}^T}\cdot\underline{x} + \underset{(1\times m)}{\underline{d}}\cdot u$$

n - no. of states ; m - no. of outputs ; r - no. of inputs

2 different types of plant/plant models:

1) Non linear behaviour of plant → non-linear model eqns.
2) Linearized behaviour of plant → linear model eqn.s

Non-linear → linearization

- $\dot{\underline{x}} = \underline{f}(\underline{x},\underline{u}) = \begin{bmatrix} f_1(x_1, x_2 \cdots x_n, u_1 \cdots u_r) \\ \vdots \\ f_n(x_1, x_2, \cdots x_n, u_1 \cdots u_r) \end{bmatrix}$ & $y = \underline{g}(\underline{x},\underline{u}) = \begin{bmatrix} g_1(x_1, x_2 \cdots x_n, u_1, u_r) \\ \vdots \\ g_m\ x_1 \cdots x_n, u_1 \cdots u_r \end{bmatrix}$

1. calculate equilibrium points for linearizing f & g
   i.e $\dot{\underline{x}}_\infty = 0 = \underline{f}(\underline{x}_\infty, \underline{u}_\infty)$.
2. Calculate Jacobian matrix of f & g to get state space rep.

$$\Delta\dot{\underline{x}} = \frac{\partial f}{\partial \underline{x}}\bigg|_\infty \cdot \Delta\underline{x} + \frac{\partial f}{\partial \underline{u}}\bigg|_\infty \Delta\underline{u} \quad ; \quad \Delta\underline{y} = \frac{\partial g}{\partial \underline{x}}\bigg|_\infty \Delta\underline{x} + \frac{\partial g}{\partial \underline{u}}\bigg|_\infty \cdot \Delta\underline{u}$$

with $\Delta\underline{x} = \underline{x} - \underline{x}_\infty$ and $\Delta\underline{u} = \underline{u} - \underline{u}_\infty$

$$\underline{A} = \frac{\partial \underline{f}}{\partial \underline{x}}\bigg|_\infty(\Delta x) = \begin{bmatrix} \frac{\partial}{\partial x_1} f_1(\underline{x}_\infty, \underline{u}_\infty) \cdots \frac{\partial}{\partial x_n} f_1(\underline{x}_\infty, \underline{u}_\infty) \\ \vdots \\ \frac{\partial}{\partial x_1} f_n(\underline{x}_\infty, \underline{u}_\infty) \cdots \frac{\partial f_n}{\partial x_n}(\underline{x}_\infty, \underline{u}_\infty) \end{bmatrix}$$

$$B = \frac{\partial f}{\partial u}\bigg|_\infty = \begin{bmatrix} \frac{\partial}{\partial u_1} f_1(\underline{x}_\infty, \underline{u}_\infty) \cdots \frac{\partial}{\partial u_n} f_1(\underline{x}_\infty, \underline{u}_\infty) \\ \vdots \\ \frac{\partial}{\partial u_1} f_n(\underline{x}_\infty, \underline{u}_\infty) \cdots \frac{\partial}{\partial u_n} f_n(\underline{x}_\infty, \underline{u}_\infty) \end{bmatrix} ; \quad \underline{C} = \frac{\partial g}{\partial \underline{x}}\bigg|_{x_\infty}$$

Solution of state space Equations

in Time Domain	in frequency Domain
- To get better understanding of influence of initial states & i/p $\underline{u}$ on state $\underline{x}$ and o/p $\underline{y}$	- To get the relation to classical control and transfer function
$\underline{x}(t) = \underline{x}_{hom}(t) + \underline{x}_{inhom}(t)$	$\underline{X}(s) = (s \cdot \underline{\underline{I}} - \underline{\underline{A}})^{-1} \cdot \underline{\underline{B}} \cdot \underline{U}(s)$
$= e^{\underline{\underline{A}}(t-t_0)} + \int_{t_0}^{t} e^{\underline{\underline{A}}(t-\tau)} \cdot \underline{\underline{B}} u(\tau) d\tau$	$\underline{Y}(s) = (\underline{\underline{C}} \cdot (s \cdot \underline{\underline{I}} - \underline{\underline{A}})^{-1} \cdot \underline{\underline{B}} + \underline{\underline{D}}) \underline{U}(s)$
$y(t) = \underline{\underline{C}} \left\{ e^{\underline{\underline{A}}(t-t_0)} + \int_{t_0}^{t} e^{\underline{\underline{A}}(t-\tau)} \underline{\underline{B}} \cdot u(\tau) d\tau \right\}$	$G(s) = \frac{Y(s)}{U(s)} = C \cdot (s \cdot \underline{\underline{I}} - \underline{\underline{A}})^{-1} \cdot \underline{\underline{B}} + \underline{\underline{D}}$
	fundamental matrix in laplace domain $\underline{\underline{\Phi}}(s) = (s \cdot \underline{\underline{I}} - \underline{\underline{A}})^{-1}$

$\underline{\underline{\Phi}}(s) = (s \cdot \underline{\underline{I}} - \underline{\underline{A}})^{-1} \xleftrightarrow{\text{laplace}} e^{At}$

fundamental matrix (freq) $\xleftrightarrow{\text{Laplace}}$ Matrix exponential (time)

$\underline{x}(t) = \underline{\underline{T}} \cdot \tilde{\underline{x}}(t)$

$\tilde{\underline{x}}(t) = \underline{\underline{T}}^{-1} \cdot \underline{x}(t)$

⇒ Co-ordinate Transformation of state space

$\dot{\tilde{\underline{x}}} = \tilde{\underline{\underline{A}}} \tilde{\underline{x}} + \tilde{\underline{\underline{B}}} \underline{u}$; $\underline{y} = \tilde{\underline{\underline{C}}} \cdot \tilde{\underline{x}} + \tilde{\underline{\underline{D}}} \cdot \underline{u}$

$\tilde{\underline{\underline{A}}} = \underline{\underline{T}}^{-1} \cdot \underline{\underline{A}} \cdot \underline{\underline{T}}$; $\tilde{\underline{\underline{B}}} = \underline{\underline{T}}^{-1} \cdot \underline{\underline{B}}$; $\tilde{\underline{\underline{C}}} = \underline{\underline{C}} \cdot \underline{\underline{T}}$; $\tilde{\underline{\underline{D}}} = \underline{\underline{D}}$

* Eigenvalues of $\underline{\underline{A}}$ and $\tilde{\underline{\underline{A}}}$ $(= \underline{\underline{T}}^{-1} \underline{\underline{A}} \cdot T)$ are equal

* We get same output for same input in $\underline{x}$ and $\tilde{\underline{x}}$ (std. & transform) only the state vector over time is different.

Connection of different state space systems

1) parallel.

$\underline{\underline{A}}_N = \begin{bmatrix} \underline{\underline{A}}_1 & 0 \\ 0 & \underline{\underline{A}}_2 \end{bmatrix}$

$\underline{\underline{B}}_N = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$; $\underline{\underline{C}}_N = [\underline{\underline{C}}_1 \; \underline{\underline{C}}_2]$; $\underline{\underline{D}}_N = (\underline{\underline{D}}_1 + \underline{\underline{D}}_2)$

$y = y_1 + y_2$

2) series

$u_2 = y_1$; $y = y_2$; $\underline{\underline{A}}_s = \begin{bmatrix} A & 0 \\ \underline{\underline{B}}_2 C_1 & \underline{\underline{A}}_2 \end{bmatrix}$; $\underline{\underline{B}}_s = \begin{bmatrix} B_1 \\ \underline{\underline{B}}_2 \underline{\underline{D}}_1 \end{bmatrix}$

$\underline{\underline{C}}_s = [\underline{\underline{D}}_2 \underline{\underline{C}}_1 \;\; \underline{\underline{C}}_2]$

$\underline{\underline{D}}_s = [\underline{\underline{D}}_1 \cdot \underline{\underline{D}}_2]$

3) Feedback $u_1 = u - y_2$; $u_2 = y_1$; $D_1 = 0$

$\underline{\underline{A}}_f = \begin{bmatrix} \underline{\underline{A}}_1 - \underline{\underline{B}}_1 \underline{\underline{D}}_2 \underline{\underline{C}}_1 & -\underline{\underline{B}}_1 \underline{\underline{C}}_2 \\ \underline{\underline{B}}_2 \cdot \underline{\underline{C}}_1 & \underline{\underline{A}}_2 \end{bmatrix}$

$\underline{\underline{B}}_f = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$

$\underline{\underline{C}}_f = [\underline{\underline{C}}_1 \;\; \underline{\underline{0}}]$,

$\underline{\underline{D}}_f = 0$ $[\because D_1 = 0]$

- Poles of Transfer function = Roots of denomination of $G(s)$.
- stability is connected to poles (i.e. stability of sys (or TF) depends on poles of the system.
  - $\text{all } Re(s_i) = \alpha_i < 0$
    - ↳ Asymptotically stable all $Re(s_i) < 0$
    - ↳ stability limit : one $Re(s_i) = 0$ all other $Re(s_i) < 0$
    - ↳ Unstable : atleast one $Re(s_i) = \alpha_i > 0$
- If all eigenvalues of the dynamic matrix $\underline{\underline{A}}$ have negative real part i.e $Re(s_i) = \alpha_i < 0$, then system is stable.
- Eigenvalues of $\underline{\underline{A}}$ also called systems poles.

Def: If for $u(t) = 0$, all state variables x_i tends to zero over time

$x_i(t \to \infty) = 0$ with $u(t) = 0$

Controllability: can we find each desired final state & each final time, with an input $u(t)$ to get there? (tracking the systems)

KALMAN criterion:- controllability matrix $(\underline{Q}_s) = \cancel{B} \cdot \cancel{B \cdot A} \ldots \cancel{B}$

MIMO: $\text{rank}(\underline{Q}_s) = n$ $Q_s = [\underline{B} \cdot \underline{A}\underline{B} \; \underline{A}^2 B \ldots \underline{A}^{n-1} \cdot B]$

SISO: $\det(\underline{Q}_s) \neq 0$

$\hookrightarrow (\underline{Q}_s) = [\underline{b} \; \underline{A} \cdot b \; \underline{A}^2 \cdot b \ldots A^{n-1} \cdot b]$

- If systems have no connections internally/externally (i/p's), then system is not controllable

Observability: state space system is observable if we can calculate initial state (x_0) from output (y) & i/p ($\underline{u}$) for a time span $[0, t_e]$

KALMAN Criterion

MIMO: $\text{rank}(\underline{Q}_B) = n$

SISO: $\det(\underline{Q}_B) \neq 0$

$\underline{Q}_B$ = observability matrix = $\begin{bmatrix} \underline{C} \\ \underline{C} \cdot \underline{A} \\ \underline{C} \cdot \underline{A}^2 \\ \vdots \\ \underline{C} \cdot \underline{A}^{n-1} \end{bmatrix}$

$\underline{Q}_B = \begin{bmatrix} \underline{C}^T \\ \underline{C}^T \underline{A} \\ \vdots \\ \underline{C}^T \cdot \underline{A}^{n-1} \end{bmatrix}$

- observability depends on internal dynamic ($\underline{A}$) and sensors ($\underline{C}$).
- Controllability depends on internal dynamics ($\underline{A}$) and actuators ($\underline{B}$).
- TF describes only the I/O behaviour of the plant
- state space model contains IO behaviour and internal dynamics of plant
- If transfer poles < system poles ⇒ system is either not ctrb / not obsv
- If transfer poles = system poles → system is ctrb & obsv.

KALMAN Decomposition:

$$\dot{\tilde{x}} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} & \tilde{A}_{14} \\ \underline{0} & \tilde{A}_{22} & 0 & \tilde{A}_{24} \\ \underline{0} & \underline{0} & \tilde{A}_{33} & \tilde{A}_{34} \\ \underline{0} & \underline{0} & \underline{0} & \tilde{A}_{44} \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \end{bmatrix} + \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \\ \underline{0} \\ \underline{0} \end{bmatrix} \underline{u}$$

$\tilde{x}_1$ → ctrb, not obsv

$\tilde{x}_2$ → ctrb, obsv

$\tilde{x}_3$ → not ctrb, not obsv

$\tilde{x}_4$ → not ctrb, obsv

$x_1 = T\,\tilde{x}(t) \Rightarrow \tilde{x} = T^{-1} \cdot x$

Controllability Normal form (sys. shall be ctrb)

1. Vector: $t^T = (t_1 \; t_2 \ldots t_n)$

$\left.\begin{array}{l} t^T \cdot b = 0 \\ t^T \cdot \underline{A} \cdot \underline{b} = 0 \\ \vdots \\ t^T \cdot \underline{A}^{n-1} \cdot b = 1 \end{array}\right\}$ solve linear eqn. to get $t^T = (t_1 \; t_2 \ldots t_n)$

2. transformation matrix

$\underline{T}_s^{-1} = \begin{pmatrix} t^T \\ t^T \cdot \underline{A} \\ \vdots \\ t^T \cdot \underline{A}^{n-1} \end{pmatrix}$

$\tilde{A} = \underline{T}_s^{-1} \cdot \underline{A} \cdot T_s$

$\tilde{B} = \underline{T}_s^{-1} \cdot B$

$\tilde{C} = \underline{C} \cdot \underline{T}_s$

$\tilde{D} = \underline{D}$

state-space representation: $\dot{\tilde{x}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \tilde{x} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \cdot u$

"t-vector Method"

Observability normal form

1. vector $\underline{s} = (s_1 \; s_2 \ldots s_n)^T$

$\left.\begin{array}{l} c^T \cdot \underline{s} = 0 \\ c^T \cdot \underline{A} \cdot \underline{s} = 0 \\ \vdots \\ c^T \cdot A^{n-1} \cdot s = 1 \end{array}\right\}$ solve n linear eqn to get $s = (s_1 \; s_2 \ldots s_n)^T$

2. transformation matrix

$T_B^{-1} = [s \; \underline{A}s \cdot \underline{A}^2 s \ldots \underline{A}^{n-1} s]^{-1}$

$T_B = [s \; \underline{A} \cdot s \; \underline{A}^2 s \ldots \underline{A}^{n-1} s]$

s-vector Method

state space representation of observability normal form is

$$\dot{\tilde{\underline{x}}} = \begin{bmatrix} 0 & 0 & \cdots & -a_0 \\ 0 & 1 & \cdots & -a_1 \\ 0 & 0 & 1 \cdots & -a_2 \\ \vdots & & & \\ 0 & 0 & 0 \cdots & -a_{n-1} \end{bmatrix} \tilde{\underline{x}} + \underline{\tilde{B}} \cdot u \quad ; \quad y = [0 \;\; 0 \;\cdots\; 1] \underline{\tilde{x}} + \underline{\tilde{d}}^T \cdot u$$

Jordan / Diagonal form:

1. eigenvalues & eigenvector of $\underline{A}$
2. $\underline{T} = [v_1 \; v_2 \cdots v_n] \Rightarrow \underline{T}^{-1} = (v_1 \; v_2 \cdots v_n)^{-1}$
3. $\tilde{\underline{A}} = \underline{T}^{-1} \cdot \underline{A} \cdot \underline{T}$; $\tilde{\underline{B}} = \underline{T}^{-1} \underline{B}$; $\tilde{C} = \underline{C} \cdot \underline{T}$

$\Rightarrow \underline{v}_k \cdot e^{\lambda_k \cdot t} = \underline{v}_k \cdot e^{-(\alpha_k + j\omega_k)t}$ is called (eigen) mode of the system

Mode = direction $\underline{v}_K$, decay rate α_K, eigenfrequency ω_K rad/sec

Transmission zeros: They are No eigenvalues of $\underline{A}$ (dynamic matrix)
I/p of system is sinusoidal but o/p is zero.
Decoupling zeros (s_0) is always an eigenvalue of $\underline{A}$

for controller design, plant must be ctrb., and obsr.

Controller Design Goals
- Asymptotically stability of CL
- satisfying control performance

Transition phase
- → Damping / overshoot
- → Reaction time / Bandwidth / settling time

steady state phase
- → Tracking of desired value
- → Disturbance reduction

When to use state space controller
- Higher dimensional SISO sys.
- Linear MIMO sys with cross coupled and need to control internal states of the plant.

control law: $u = -\underline{R} \cdot \underline{x} + \underline{V} \cdot w$

$\Rightarrow$ Pre-filter ($\underline{V}$): designed to guarantee tracking behaviour $w \to y$

$\Rightarrow$ controller / feedback matrix ($\underline{R}$) is used to shift open loop plant poles to desired closed loop poles

$\Rightarrow$ Dynamic matrix CL: $\underline{A}_{CL} = \underline{A} - \underline{B}\,\underline{R}$
$\underline{B}_{CL} = \underline{B} \cdot \underline{V}$

$\Rightarrow$ char. eqn of CL to get poles is $\det(s \cdot \underline{I} - \underline{A}_{CL}) = 0$

Two step controller Design:
1. Design $\underline{R}$ to match dynamic req. based on plant model.
2. Design $\underline{V}$ based on $\underline{R}$ & plant model

In steady state, error $e = 0 \Rightarrow \underline{y}_\infty = \underline{w}_\infty$ which is possible if no of i/ps = no. of o/ps

$$\underline{V} = \left[\underline{C} (\underline{B} \cdot \underline{R} - \underline{A})^{-1} \cdot \underline{B} \right]^{-1} = \left(\underline{c}^T \cdot (\underline{b} \cdot \underline{r}^T - \underline{A})^{-1} \cdot \underline{b} \right)^{-1}$$

$$s_d = -\sigma = -\frac{\gamma}{\Delta t} = -\frac{\text{multiple of } \tau}{\text{rise time}}$$

$$s_{1,2} = -D\omega_n \pm j\,\omega_n \sqrt{1-D^2} = -\alpha \pm j\bar{\omega}$$

$\therefore \alpha = D\omega_n$; $\bar{\omega} = \omega_n \cdot \sqrt{1-D^2}$; $\bar{\omega}$ - oscillation freq.
D - Damping ratio, $\bar{\omega}$ - oscillation, ω_n - natural freq
α - exp. decay of oscillation

$$\alpha = -\frac{\ln(a)}{t_s}$$

$a \to$ steady state band
t_s - desired settling time

$\Rightarrow$ Calculation of $\underline{r}^T / \underline{R}$ (DIRECT Method)

i) Define desired poles of CL $s_{d,1}, s_{d,2}, \cdots s_{d,n}$

ii) desired characterstic polynomial

$$P_d(s) = (s - s_{d,1})(s - s_{d,2}) \cdots (s - s_{d,n})$$
$$= P_{d,n} s^n + P_{d,n-1} s^{n-1} + \cdots P_{d,1} s + P_{d,0}$$

iii) Real characterstic polynomial

$$P_r(s) = a_{r,n} s^n + a_{r,n-1} s^{n-1} + \cdots a_{r,1} s + a_{r,0}$$

iv) compare $P_d(s)$ & $P_r(s)$ co-efficient & get $r_1, r_2 \cdots r_n$

v) Combine to $\underline{r}^T = [r_1 \; r_2 \cdots r_n]$

Cal. of $\underline{r}^T$ in Controllability Normal form

(i) & (ii) same as direct method.

$r_1 = P_{d,0} - a_0$
$r_2 = P_{d,1} - a_1$
$\vdots$
$r_n = P_{d,n-1} - a_{n-1}$

$\therefore \underline{r}^T = [r_1 \; r_2 \cdots r_n]$

$\Rightarrow$ Very simple to cal. $\underline{r}^T$ in ctrb NF i.e, No step 3 & step 4

Ackermann's (t-vector) method to calc ($\underline{r}^T$):

i) Define desired poles of CL i.e.
$s_{d,1}, s_{d,2}, \dots s_{d,n}$

ii) desired char. polynomials
$$P_d(s) = (s - s_{d,1})(s - s_{d,2}) \dots (s - s_{d,n})$$
$$= p_{d,n} \cdot s^n + p_{d,n-1} s^{n-1} \dots + p_{d,1} \cdot s + p_{d,0}$$

iii) Determine t-vector, $t^T = [t_1 \ t_2 \dots t_n]$
$t^T \cdot \underline{b} = 0$; $t^T \cdot \underline{A} \cdot \underline{b} = 0$; $t^T \cdot \underline{A}^2 \cdot \underline{b} = 0$
$\dots$; $t^T \cdot \underline{A}^{n-2} \cdot \underline{b} = 0$; $t^T \cdot \underline{A}^{n-1} \cdot \underline{b} = 1$
solve linear eqn.s to get t^T

iv) $r^T = p_{d,0} \cdot t^T \cdot \underline{\underline{I}} + p_{d,1} \cdot t^T \cdot \underline{\underline{A}} + \dots p_{d,n} \cdot t^T \cdot A^n$

Cal. $\underline{R}$ (MIMO) - Dyadic control

i) Virtual scalar i/p, $\bar{u} \Rightarrow \underline{u} = \underline{q} \cdot \bar{u}$
Virtual system $\dot{\underline{x}} = \underline{A}\underline{x} + \bar{b} \cdot \bar{u}$ with $\bar{b} = \underline{B} \cdot \underline{q}$

(ii) Now its SISO sys. use previous methods to get r^T with control law $\bar{u} = -r^T \cdot x$

(iii) overall control law $\underline{u} = -\underline{q} \cdot r^T \cdot x$
i.e. $\underline{R} = \underline{q} r^T$

$\underline{V} = (\underline{C}(\underline{B} \cdot \underline{R} - \underline{A})^{-1} \cdot \underline{B})^{-1}$

Prefilter ($\underline{V}$) assures a good tracking behaviour. To get a very small/no steady state error, PI state space control approach is good

Extended plant model

$$\underline{\underline{A}}_E = \begin{bmatrix} \underline{A} & \underline{0} \\ -\underline{C} & \underline{0} \end{bmatrix} ; \ \underline{B}_E = \begin{bmatrix} \underline{B} \\ -\underline{D} \end{bmatrix} ;$$

control law $\underline{u} = -\underline{R}\underline{x} + \underline{K}_i \cdot x_i$
$$R_E = [\underline{R} \ \ -K_i]$$

Cal. of $\underline{R}$ & $\underline{K_i}$

i) Design to proportional state feedback controller for Extended plant model ($\underline{R}_E$)
i.e. cal $\underline{R}_E$ with $\underline{u} = -\underline{R}_E \cdot \underline{x}_E$

ii) split $\underline{R}_E$ into $\underline{R}$ & $\underline{K}_i$ by
$$R_E = [\underline{R}_i \ \ -\underline{K}_i]$$

PI state space control loop:
- Use of prefilter (V) assures a good tracking behaviour for accurate plant models.
- PI state space control is an alternative approach to get very small/no steady state error
- Combination with state observer is possible
- for SISO & MIMO plants.

For state feedback control,
we need, state information ($\underline{x}$)
(or) an estimate ($\hat{\underline{x}}$) of plant state

3 typical possibilities to know $\hat{x}$ / $\underline{x}$:-
① full state $\underline{x}$ is measured
② Calculate $\underline{x}$ from o/p y
③ dynamic state observer for estimates $\hat{\underline{x}}$

Observer as "digital twin" of the plant

→ for observer, we need
- plant models $\underline{A}, \underline{B}, \underline{C}, \underline{D}$
- observer gain matrix $\underline{K}$
- signals $u(t), y(t)$

$$\dot{\hat{\underline{x}}} = \underline{A}\hat{\underline{x}} + \underline{B}\underline{u} + K(\underline{y} - \hat{\underline{y}})$$
$$= \underbrace{(\underline{A} - \underline{K} \cdot \underline{C})}_{\hat{\underline{A}}} \cdot \hat{\underline{x}} + [\underline{B} \ \ \underline{K}] \begin{bmatrix} \underline{u} \\ \underline{y} \end{bmatrix}$$

$\hat{\underline{A}}$ defines dynamic behaviour of the observer
e.g.: stability, convergence rate ....

$x_e = \underline{x} - \hat{\underline{x}} \Rightarrow \dot{x}_e = \dot{\underline{x}} - \dot{\hat{\underline{x}}}$

$$\dot{x}_e = \underbrace{(\underline{A} - \underline{K} \cdot \underline{C})}_{\hat{\underline{A}}} \cdot x_e$$

→ x_e defines error dynamics $\underline{x} - \hat{\underline{x}}$ i.e. stable, convergence, fast
→ x_e is homogeneous state space system i.e. i/p is zero
$\therefore \dot{x}_e = \hat{\underline{A}} \cdot x_e$

No i/p to system, Then system is Homogeneous

Controller	Observer
$\underline{A}_{CL} = \underline{A} - \underline{B} \cdot \underline{R}$	$\hat{\underline{A}} = \underline{A} - \underline{K} \cdot \underline{C}$
$\underline{R}$ to be calc for desired poles	$\underline{K}$ to be calc for desired poles

Direct Method for SISO to calculate observer gain matrix ($\underline{K}$)

i) set of desired poles
$\hat{s}_{d,1}, \hat{s}_{d,2} \dots \hat{s}_{d,3}$

ii) Desired characteristic polynomial
$$\hat{P}_d(s) = (s-\hat{s}_{d,1})(s-\hat{s}_{d,2})\dots(s-\hat{s}_{d,n})$$
$$= \hat{P}_{d,n} \cdot s^n + \hat{P}_{d,n-1} \cdot s^{n-1} + \dots + \hat{P}_{d,1} s + \hat{P}_{d,0}$$

(iii) Real characteristic polynomial
$$\hat{P}_r(s) = \det(s \cdot I - \hat{\underline{A}}) = \det(s \cdot \underline{I} - \underline{A} + K C^T)$$
$$= \hat{a}_{r,n} \cdot s^n + a_{r,n-1} \cdot s^{n-1} + \dots + a_{r,1} \cdot s + a_0$$

(iv) Compare $\hat{P}_d(s)$ & $\hat{P}_r(s)$ coefficients & get $K_1, K_2 \dots K_n$

(v) Combine to get $\underline{K} = \begin{bmatrix} K_1 \\ K_2 \\ \vdots \\ K_n \end{bmatrix}$

General Approach for observer

(i) Define desired observer behaviour or for pole placement by OBSV poles
$\hat{s}_{d,1}, \hat{s}_{d,2} \dots \hat{s}_{d,n}$

(ii) Define dual problems
$\dot{\underline{x}}_D = \underline{A}^T \cdot \underline{x}_D + \underline{c}^T \cdot \underline{u}_D$, $\underline{y}_D = \underline{B}^T \cdot \underline{x}_D$
with dual control law, $u = -K^* \underline{x}_D$

(iii) solve dual control problem for desired dynamical behaviour from s(i) i.e. based on static feed back controll method, calculate K^*.

(iv) Get observer gain vector/matrix from $\underline{K} = (K^*)^T$

Observer dynamics ($\hat{\underline{A}}$)
i.e. - stability (convergence) of OBSV
- convergence rate (fast/slow) of OBSV
$\hat{\underline{A}}$, defined by observer poles $= eig(\underline{A} - \underline{K} \cdot \underline{C})$

Observer dynamics/poles are faster than dynamics/poles of the observed system
i.e. $eig(A - \underline{\underline{K}}\,\underline{\underline{C}})$ is faster $eig(\underline{A})$
OBSV system