Midterm Exam

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20754 DECISION THEORY AND HUMAN BEHAVIOR

1 Problem

To outline the main elements of rational choice theory for a Decision Maker (DM hereinafter) we start by defining $\boldsymbol{X}$, the all-inclusive *choice space*, the space of all conceivable alternatives, and $\mathcal{X}$, the collection of choice sets X which are subsets of $\boldsymbol{X}$.

The pair $(\boldsymbol{X}, \mathcal{X})$ forms a *decision framework*.

When choosing, the DM confronts a specific choice set X in $\mathcal{X}$, and has a specific preference over the alternatives available in X. Let the preorder $\succsim_X$ denote such contextualized preference. We define P as the *preference map* which associates each choice set X to a preference relation $\succsim_X$ defined on X, where $\succsim_X$ is a reflexive and transitive binary relation (i.e., a preorder). The preference map can be interpreted as the DM herself, who has preferences over contextualized alternatives.

We can now define a *decision environment* as the triple $(\boldsymbol{X}, \mathcal{X}, P)$.

Finally, given a choice set X, the DM confronts a *decision problem* given by the pair $(X, \succsim_X)$, that is, by a choice set and its associated preference relation. Within *the decision problem* a *rational* DM chooses, if it exists, an *optimal alternative* $\hat{x}$. That is, an alternative that is not strictly preferred by any other alternative in X. This definition could be formalized as:

$$\hat{x} \in X \colon \nexists x \in X, x \succ_X \hat{x}$$

Not every *decision problem* admits *optimal alternatives*, and so we define as $\mathcal{D}$ the collection of choice sets X for which *optimal alternatives* exist. Clearly, $\mathcal{D}$ is a subset of $\mathcal{X}$. On $\mathcal{D}$ we can define the *rational choice correspondence* which associates each set X in $\mathcal{D}$ to the *subset of optimal alternatives* $\sigma(X)$ in X.

Define the *rational choice correspondence* $\sigma \colon \mathcal{D} \rightrightarrows \boldsymbol{X}$ as:

$$\sigma(X) = \{\hat{x} \in X \colon \nexists x \in X, x \succ_X \hat{x}\}$$

A *rational* DM will thus choose an *optimal alternative* $\hat{x} \in \sigma(X)$.

Now note that, since $\succsim_X$ is a preorder, it lacks completeness and thus the set $\sigma(X)$ of optimal alternatives will contain indifferent alternatives (that is, alternatives x, y such that $x \succsim_X y$ and $y \succsim_X x$, and so $x \sim_X y$) as well as alternatives which cannot be compared with one another because of the lack of completeness of the binary relation $\succsim_X$. Any alternative x in $\sigma(X)$ is compatible with *rational choice*. When X is finite, abstracting from computational constraints, the DM can find optimal alternatives by sequential pairwise comparisons, a procedure formalized by the *ascent algorithm*.

Now given that $\succsim_X$ is a preorder, the induced binary relation $\sim_X$ is reflexive, symmetric and transitive and is thus an equivalence relation. This allows us to define *indifference curves* in $\sigma(X)$ for the preorder $\succsim_X$. Given any element $\hat{x} \in \sigma(X)$ the equivalence class of $\hat{x}$ defined as:

$$[\hat{x}]_X = \{\hat{y} \in \sigma(X) \colon \hat{x} \sim_X \hat{y}\}$$

denotes an *indifference curve*. Note that the collection of indifference curves $\{[\hat{x}]_X : \hat{x} \in \sigma(X)\}$ induces a partition of $\sigma(X)$. Since $\succsim_X$ is a preorder, each indifference curve $[\hat{x}]_X$ is represented by optimal alternatives that are pairwise incompatible.

Now, if the DM were to rank alternatives independently of which choice set she's facing, we could posit the existence of a *universal preference* $\succsim$ defined on the choice space $\boldsymbol{X}$ such that:

$$x' \succsim_X x \Longleftrightarrow x' \succsim x \ \forall x, x' \in X$$

Then it becomes clear how every preference $\succsim_X$ defined on a choice set X is merely a restriction of the universal preference $\succsim$ over X, and we can thus drop the subscript accordingly. Since $\succsim$ defines a preference map, the *decision environment* under a *universal preference* can be expressed with the triple $(\boldsymbol{X}, \mathcal{X}, \succsim)$. Note that since alternatives are ranked consistently irrespective of the choice set to which they belong, we abstract from *context effects*.

Now, the universal preference allows us to specify an induced *indirect preference* $\succeq$ defined over the choice sets for which optimal alternatives exists, and so in $\mathcal{D}$, as:

$$X \succeq Y \Longleftrightarrow \forall y \in Y, \exists x \in X, x \succsim y$$

That is, we may (clearly, the completeness of $\succeq$ does not generally hold) rank choice sets according to the relation between their elements.

Now, recalling the rational choice correspondence $\sigma: \mathcal{D} \rightrightarrows \boldsymbol{X}$, an important proposition can be stated when $\succsim$ is a preorder. For each $X, Y \in \mathcal{D}$:

(i) $X \subseteq Y$ implies $X - \sigma(X) \subseteq Y - \sigma(Y)$ – that is, when a choice set is nested into another, nonoptimal alternatives in the included set remain nonoptimal in the including one. More simply: enlarging a choice set, nonoptimal alternatives remain such.

(ii) $X \subseteq Y$ implies $\sigma(Y) \cap X \subseteq \sigma(X)$ – that is, the intersection between optimal alternatives in the including set with the included one must be a subset of the set of optimal alternatives of the included one. More simply: shrinking a choice set, optimal alternatives remain such, provided they are still available.

Also note that we are disregarding utility theory in this general discussion since we have a universal preference relation $\succsim$ that is not complete, and thus does not admit a utility representation $u: \boldsymbol{X} \longrightarrow \mathbb{R}$.

Finally, if instead the decision maker rank alternatives differently according to the choice set under consideration, a universal preference is not admissible. To allow for context effects we denote the pair (x, X) as *contextualized alternative* and denote by $\mathcal{C}$ the collection of all such pairs. Now, preferences account for the set in which they are carried out and we assume that the DM can carry out comparisons across (as well as within) contexts defined by choice sets. Thus, the DM has a preference on $\mathcal{C}$ with form $(x, X) \succsim (y, Y)$ as well as a *contextualized universal preference* $\succsim$ over $\mathcal{C}$:

$$\forall X \in \mathcal{X}, \quad (x, X) \succsim (x', X) \Longleftrightarrow x \succsim_X x' \quad \forall x', x \in X,$$

with $\succsim_X = P(X)$

The *decision environment* thus becomes $(\boldsymbol{X}, \mathcal{X}, \succsim)$

Now, if we can parametrize choice sets by elements θ of a parameter set Θ via a *menu correspondence* $\varphi\colon \Theta \rightrightarrows \boldsymbol{X}$,we have that the collection of choice sets X can be expressed as:

$$\mathcal{X} = \{\varphi(\theta)\colon \theta \in \Theta\}$$

And, given a subset D of Θ of parameters inducing via φ a collection $\mathcal{D}$ of subsets of $\mathcal{X}$ with an optimal choice, we can write the collection of choice sets X for which *optimal alternatives* exist as:

$$\mathcal{D} = \{\varphi(\theta)\colon \theta \in D\}$$

Now we can see how the preference map P associates parameters to preorders on $\varphi(\theta)$ – that is, to preferences on the choice sets induced by the parameters. Then, P has the form: $\theta \mapsto \succsim_\theta$. And because of *extensionality*[1] we have that:

$$\varphi(\theta) = \varphi(\theta') \Longrightarrow P(\theta) = P(\theta')\ \forall \theta, \theta' \in \Theta$$

From this we can see how the *contextualized universal preference* $\succsim$ defined over $\mathcal{C}$ is a just a binary relation defined over pairs (x, θ) in the graph of the correspondence φ.

Finally, this induces a *parametric menu preference*, that is a preference over parameters defined as:

$$\theta \succeq_\Theta \theta' \Longleftrightarrow \forall y \in \varphi(\theta'), \exists x \in \varphi(\theta), (x, \theta) \succsim (y, \theta')$$

For all $\theta, \theta' \in \Theta$.

This parametric analysis is of particular interest in Economics, where characteristics of the DM (the budget constraint, a production function, etc.) can then be to parametrize the induced preferences over the choice sets of a decision problem.

[1] That is, if $X = X'$, then $x \succsim_X y$ if and only if $x \succsim_{X'} y$ for all $x, y \in X$

Problem 2

Let $\mathbf{L}$ be the space of all lotteries defined over an all-inclusive prize (or consequences) space $\mathbf{C}$ and a preference relation $\succsim$ defined over such lotteries. Expected utility theory takes an axiomatic approach to show that $\succsim$ ranks lotteries according to their expected utility.

To introduce the main elements of expected utility theory is fundamental to first introduce a mathematical formalization of decisions under risk. Given $\mathbf{C}$, a lottery l is a function $l\colon \mathbf{C} \longrightarrow [0,1]$ which associates each sure consequence (or prize) c in $\mathbf{C}$ to a probability $l(c) \geq 0$, positive only for a finite number of c and such that $\sum_{c \in \mathbf{C}} l(c) = 1$. From this, we derive the graph of the lottery:

$$\text{Gr } l = \{(c, p) \in \mathbf{C} \times [0,1] \colon p = l(c)\}$$

therefore, a lottery is formally defined as

$$l = \{(c_1, p_1); \dots; (c_n, p_n)\} = \{(c, p) \in \text{Gr } l : p > 0\}$$

That is, a lottery is a way to summarize risky alternatives through a prospect in which consequences obtain with some probability. Note that the analysis of choice under risk encompasses that of choice under certainty, since any sure alternative c in $\mathbf{C}$ can be identified through the degenerate lottery $\{c, 1\}$. Therefore, $\mathbf{C} \subseteq \mathbf{L}$ and lotteries can be compared to sure prizes.

Note also that the space of all lotteries $\mathbf{L}$ defined over $\mathbf{C}$ is convex – that is, it contains every convex combination of its elements. This allows, when the DM only cares about final consequences, for a nice *reduction of compound lotteries* principle whereby lotteries where prizes are themselves lotteries can be reduced to the convex combination of their prizes.

Now we can define a *decision pre-framework under risk* as: $(\mathbf{A}, \mathcal{A}, \mathbf{L}, \rho)$. Where $\mathbf{A}$ is an all-inclusive action space, with the subset actions with certain consequences $\mathbf{A}_{\mathrm{c}} \subseteq \mathbf{A}$; $\mathcal{A}$ is a collection of action sets A in $\mathbf{A}$; and $\rho\colon \mathbf{A} \longrightarrow \mathbf{L}$ **is** a consequence function which assigns each action $a \in \mathbf{A}$ to its random consequences $\rho(a) \in \mathbf{L}$.

Now, abstracting from context effects, the DM has universal preference:

(i) $\succsim$ over actions – hence on $\mathbf{A}$
(ii) $\dot{\succsim}$ over consequences – hence on $\mathbf{C}$
(iii) $\ddot{\succsim}$ over lotteries – hence on $\mathbf{L}$

We also adopt a methodologically consequentialist approach whereby actions are instrumental to consequences, formally stated as follows.

S.1 RANDOM CONSEQUENTIALISM: for all actions $a, b \in \mathbf{A}$,

$$\rho(a) \ddot{\succsim} \rho(b) \Longrightarrow a \succsim b \qquad \text{and} \qquad \rho(a) \ddot{\succ} \rho(b) \Longrightarrow a \succ b$$

Now, when $\ddot{\succsim}$ is a weak order, S.1 holds if and only if:

$$a \succsim b \Longleftrightarrow \rho(a) \ddot{\succ} \rho(b)$$

Altogether, note that since $\mathbf{C} \subseteq \mathbf{L}$, then $\ddot{\succsim}$ subsumes $\dot{\succsim}$, and by random consequentialism $\ddot{\succsim}$ also subsumes $\succsim$. Then, when $\ddot{\succsim}$ admits a utility function $\ddot{u}\colon \mathbf{L} \longrightarrow \mathbb{R}$, $\succsim$ also does, with $u = \ddot{u} \circ \rho$.

Finally, we move to expected utility theory by considering the following Axioms:

B.1 WEAK ORDER: $\ddot{\succsim}$ is complete and transitive

B.2 INDEPENDENCE: for all $l, l', l'' \in \mathbf{L}$ and $0 < p < 1$:

$$l \ddot{\succ} l' \Longrightarrow pl + (1-p)l'' \ddot{\succ} pl' + (1-p)l''$$

That is, a ranking between two lotteries is preserved when they are mixed with a common one.

B.3 ARCHIMEDEAN: for all $l, l', l'' \in \mathbf{L}$ with $l \ddot{\succ} l' \ddot{\succ} l''$, there exist $p, q \in (0{,}1)$ such that:

$$pl + (1-p)l'' \ddot{\succ} l' \ddot{\succ} ql + (1-q)l''$$

That is, no lottery is infinitely superior or inferior to the other.

Finally, the notorious von Neumann-Morgenstern (vN-M) representation theorem defines the expected utility representation of a preference over lotteries $\ddot{\succsim}$ which satisfies axioms B.1 to B3:

von Neumann-Morgenstern representation theorem: *let $\ddot{\succsim}$ be a preference on the set $\mathbf{L}$ of all lotteries defined on a prize space $\mathbf{C}$. The following are equivalent:*

(i) $\ddot{\succsim}$ *satisfies axioms B.1, B.2, and B3*

(ii) *there exists a function $\dot{u}\colon \mathbf{C} \longrightarrow \mathbb{R}$ such that the function $\ddot{u}\colon \mathbf{L} \longrightarrow \mathbb{R}$ given by:*

$$\ddot{u}(l) = \sum_{i=1}^{n} \dot{u}(c_i)p_i$$

represents $\ddot{\succsim}$

Moreover, the function $\dot{u}$ (and so $\ddot{u}$) is cardinal (i.e., unique up to positive affine transformations).

This means that, provided $\ddot{\succsim}$ satisfies axioms B.1, B.2, and B3, we can use $\ddot{u}$, the (cardinal) expected utility representation to rank lotteries and thus operationalize the analysis of choice under risk. That is, the *vN-M utility function* $\dot{u}$ on prizes admits a corresponding *Bernoulli utility function* $\ddot{u}$ on lotteries.

The theorem implies that two lotteries are ranked equally if and only if their expected utility are the same. In turn, the cardinality of $\dot{u}$ (and, consequently, of $\ddot{u}$) implies that only affine transformations $\dot{u}$ preserve the original ranking – monotone transformations can spoil the ordering. Thanks to the cardinality of $\dot{u}$, we can now also consider differences in utility, which previously had no interpretation.

Now, if we endow $\mathbf{C}$ with a partial order $\geq$ so that the pair $(\mathbf{C}, \geq)$ is an ordered space, and we add an axiom to relate this (objective) feature of the prize space to the (subjective) preference relation, expected utility theory allows to study the monotonicity of preferences over lotteries conveniently.

B.4 STRONG MONOTONICITY: for all consequences, $c, c' \in \mathbf{C}$,

$$c \gg c' \Longrightarrow c \dot{\succ} c' \quad \text{and} \quad c \geq c' \Longrightarrow c \dot{\succsim} c'$$

Now, a vN-M preference over lotteries $\ddot{\succsim}$ satisfies STRONG MONOTONICITY if and only if its vN-M utility function $\dot{u}\colon \mathbf{C} \longrightarrow \mathbb{R}$ is *strongly increasing*. When $\geq$ is also complete over $\mathbf{C}$, then when the axiom is satisfied the vN-M utility function must be *strictly increasing* (since then $\gg$

reduces to $>$). This is especially useful in Economics, where lotteries are often characterized by monetary or material prices, where there exists a complete order.

Expected utility theory is best modelled over probabilistic settings where lotteries are modelled as simple probability measures on $\mathbf{C}$. Denote by $\Delta_0(\mathbf{C})$ the set of all simple probability measures on $\mathbf{C}$, which now replaces $\mathbf{L}$. Now $p(c)$ is the probability with which prize c obtains for a lottery p. Since p is a finite probability measure, it is finitely additive and thus has finite support. With this, we can define the *expected utility* as the mean value of $\dot{u}$ under p, defined by:

$$\mathbb{E}_p\dot{u} = \sum_{c\in \text{supp } p} \dot{u}(c)p(c)$$

The convex combination of lotteries in $\Delta_0(\mathbf{C})$ maintains the interpretation in terms of compound lotteries previously states, and letting $\ddot{\succsim}$ be a preference on the lottery space $\Delta_0(\mathbf{C})$, we can restate axioms B.1 to B.3 as:

C.1 WEAK ORDER: $\ddot{\succsim}$ on $\Delta_0(\mathbf{C})$ is complete and transitive

C.2 INDEPENDENCE: for all $p, p', p'' \in \Delta_0(\mathbf{C})$ and $0 < \alpha < 1$:

$$p \ddot{\succ} p' \Longrightarrow \alpha p + (1-\alpha)p'' \ddot{\succ} \alpha p' + (1-\alpha)p''$$

C.3ARCHIMEDEAN: for all $p, p', p'' \in \Delta_0(\mathbf{C})$ with $p \ddot{\succ} p' \ddot{\succ} p''$, there exist $\alpha, \beta \in (0,1)$ such that:

$$\alpha p + (1-\alpha)p'' \ddot{\succ} p' \ddot{\succ} \beta p + (1-\beta)p''$$

And we can thus finally restate the vN-M representation theorem, *mutatis mutandis,* in mean value representation form:

vN-M representation theorem *let* $\ddot{\succsim}$ *be a preference on* $\Delta_0(\mathbf{C})$*. The following are equivalent:*

(i) $\ddot{\succsim}$ *satisfies axioms C.1, C.2, and C3*

(ii) there exists a function $\dot{u}\colon \mathbf{C} \longrightarrow \mathbb{R}$ *such that the function* $\ddot{u}\colon \Delta_0(\mathbf{C}) \longrightarrow \mathbb{R}$ *given by:*

$$\ddot{u}(p) = \mathbb{E}_p\dot{u}$$

represents $\ddot{\succsim}$

Moreover, the function $\dot{u}$ *(and so* $\ddot{u}$*) is cardinal.*

And we close by pointing out that, in this probabilistic framework, for all $p, q \in \Delta_0(\mathbf{C})$, we have:

$$p \ddot{\succsim} q \Longleftrightarrow \mathbb{E}_p\dot{u} \geq \mathbb{E}_q\dot{u}$$

That is, preferences over risky outcomes are represented by the (familiar) comparison of the cardinal expected utility of consequences.

Problem 3

Proposition *An Archimedean weak order $\succsim$ on $\mathbf{L}$ satisfies* B.2 *if and only if, for all $l, l', l'' \in \mathbf{L}$ and all $0 < p < 1$,*

$$l \succsim l' \iff pl + (1-p)l'' \succsim pl' + (1-p)l'' \tag{1}$$

Proof

Let $\succsim$ be an Archimedean weak order on $\mathbf{L}$ – i.e., $\succsim$ satisfies axioms B.1 and B.3:

B.1 WEAK ORDER: $\succsim$ is complete and transitive

B.3 ARCHIMEDEAN: for all $l, l', l'' \in \mathbf{L}$ with $l \succ l' \succ l''$, there exist $p, q \in (0,1)$ such that:

$$pl + (1-p)l'' \succ l' \succ ql + (1-q)l''$$

"Only if" part of the proof $\implies$

Assume that $\succsim$ satisfies B.1, B.2 and B.3. Then, by the von Neumann-Morgenstern representation theory, there exists a function u: $\mathbf{C} \longrightarrow \mathbb{R}$ such that the function $\ddot{u}$: $\mathbf{L} \longrightarrow \mathbb{R}$ given by

$$\ddot{u}(l) = \sum_{i=1}^{n} u(c_i)p_i$$

represents $\succsim$ and is cardinal.

Therefore, $l \succsim l' \iff \ddot{u}(l) \geq \ddot{u}(l')$

Now, since $\ddot{u}$: $\mathbf{L} \longrightarrow \mathbb{R}$ is cardinal, it is invariant up to positive affine transformations. That is, for any $\alpha > 0$ and $\beta \in \mathbb{R}$, it holds that:

$$l \succsim l' \iff \ddot{\mathrm{u}}(l) \geq \ddot{\mathrm{u}}(l') \iff \alpha\ddot{\mathrm{u}}(l) + \beta \geq \alpha\ddot{\mathrm{u}}(l') + \beta$$

Now, since $p > 0$, and for all $l'' \in \mathbf{L}$, $\ddot{\mathrm{u}}(l'') \in \mathbb{R}$, let $\alpha = p$ and $\beta = (1-p)\ddot{\mathrm{u}}(l'')$.
Then it holds that:

$$l \succsim l' \iff \ddot{\mathrm{u}}(l) \geq \ddot{\mathrm{u}}(l') \iff p\ddot{\mathrm{u}}(l) + (1-p)\ddot{\mathrm{u}}(l'') \geq p\ddot{\mathrm{u}}(l') + (1-p)\ddot{\mathrm{u}}(l'')$$

since, letting u represent the affine positive transformation $\mathrm{u} = pu + (1-p)\ddot{u}(l'')$, we have that:

$$\sum_{i=1}^{n} \mathrm{u}(c_i)p_i = \sum_{i=1}^{n} [p(c_i) + (1-p)\ddot{u}(l'')]p_i = (1-p)\ddot{u}(l'') + p\sum_{i=1}^{n} u(c_i)p_i \geq$$

$$\geq (1-p)\ddot{u}(l'') + p\sum_{i=1}^{n} u(c'_i)p'_i = \sum_{i=1}^{n} [p(c'_i) + (1-p)\ddot{u}(l'')]p'_i = \sum_{i=1}^{n} \mathrm{u}(c'_i)p'_i$$

Which shows that the original ranking is preserved. Now, since $\ddot{u}$ represents $\succsim$ then this implies:

$$p\ddot{u}(l) + (1-p)\ddot{u}(l'') \geq p\ddot{u}(l') + (1-p)\ddot{u}(l'') \iff pl + (1-p)l'' \succsim pl'pl + (1-p)l''$$

Which finally shows that (1) holds since, putting everything together:

$$l \succsim l' \iff pl + (1-p)l'' \succsim pl'pl + (1-p)l''$$

"If" part of the proof $\Longleftarrow$

For proof by contrapositive, assume $\neg$B. 2, we want to show that it implies $\neg(1)$. Consider $\neg$B. 2:

$$\neg(l \succ l' \Longrightarrow pl + (1-p)l'' \succ pl' + (1-p)l'') \Longleftrightarrow l \succ l' \ \wedge \neg(pl + (1-p)l'' \succ pl' + (1-p)l'')$$

Then it follows that (1) does not hold, since $\neg(pl + (1-p)l'' \succ pl' + (1-p)l'')$ implies either:

(i) $pl' + (1-p)l'' \succ pl + (1-p)l''$, in which case (1) is negated

(ii) $pl' + (1-p)l'' \sim pl + (1-p)l''$, which also contradicts (1) whenever $l \succ l'$ since it violate the affinity induced on $\succsim$ when (1) is satisfied.

Therefore, we conclude that:

$$\neg(l \succ l' \Longrightarrow pl + (1-p)l'' \succ pl' + (1-p)l'') \implies \neg(l \succsim l' \Longleftrightarrow pl + (1-p)l'' \succsim pl' + (1-p)l'')$$

That is, $\neg$B. 2 implies $\neg(1)$, and therefore we conclude that B. 2 $\Longleftarrow$ (1). ■

Problem 4

Let $\boldsymbol{X}$ be the all-inclusive choice space of all possible restaurant meals in town. A restaurant in this context is a choice set X consisting of the collection of meals available in the restaurant, i.e., its menu. Then, $\mathcal{X}$ is the collection of all restaurants (or, more properly, of their menus).

Let $\succsim_X$ be the DM's preference over the alternatives in X, that is, over meals in each restaurant. Since the DM ranks restaurants according to their best meal, she is able to compare meals across choice sets, and hence has a *universal preference over meals* $\succsim$ defined over the choice space $\boldsymbol{X}$ such that, for all $X \in \mathcal{X}$, it holds that

$$x' \succsim_X x \Longleftrightarrow x' \succsim x \qquad \forall x, x' \in X$$

Now consider the *rational choice correspondence* $\sigma : \mathcal{D} \rightrightarrows \boldsymbol{X}$ defined as:

$$\sigma(X) = \{\hat{x} \in X \colon \nexists x \in X, x \succ_X \hat{x}\}$$

Now, clearly $\emptyset \neq \sigma(X) \subseteq X$.

Let $X, Y \in \mathcal{D} \subseteq \mathcal{X}$ be restaurants for which optimal alternatives exist, and so $\sigma(X) \neq \emptyset$.

We can define the *indirect (menu) preference* $\succeq$ *over restaurants* in $\mathcal{D}$ induced by the *universal preference over meals* $\succsim$ as:

$$X \succeq Y \Longleftrightarrow \forall y \in Y, \exists x \in X, x \succsim y$$

Now, assuming that $\succsim$ is a weak order (and so complete), it follows (from Kreps 1979) that for any $X \in \mathcal{D}$, $X \sim \sigma(X)$. Therefore, if follows that for each $X, Y \in \mathcal{D}$:

$$X \succeq Y \Longleftrightarrow \sigma(X) \succeq \sigma(Y)$$

Which means that choice sets – restaurants – are ranked according to their optimal alternatives – best meals. Now, exploiting completeness, we can define the optimal restaurant $\hat{X} \in \mathcal{D}$ as:

$$\hat{X} \text{ s.t. } \hat{X} \succeq X \quad \forall X \in \mathcal{D}$$

Finally, this means that $\hat{X}$ is optimal if and only if:

$$\sigma\big(\hat{X}\big) \succeq \sigma(X) \quad \forall X \in \mathcal{D}$$

Then, $\hat{X}$ such that $\sigma\big(\hat{X}\big) \succeq \sigma(X)$ for all $X \in \mathcal{D}$ formalizes the claim "The best restaurant in town is the one that serves the best meal in town".

Problem 5

Proposition 2 *if there exists an antisymmetric and monotone binary relation* $\succsim$ *on* $\mathbb{R}^n$ *that admits a continuous utility function* u*:*$X \longrightarrow \mathbb{R}$*, then* $n = 1$*.*

Proof (sketch)

Since $\succsim$ is antisymmetric, then $x \succsim y$ and $y \succsim x$ imply $x = y$ for all $x, y \in \mathbb{R}^n$. Then, it holds that, for every $x \neq y$, either $x \succ y$, $y \succ x$ or they cannot be compared, i.e., $x \bowtie y$.

Then, by monotonicity of $\succsim$ on $\mathbb{R}^n$ it holds that for all $x, y \in X$, if $x \geq y$ then $x \succsim y$.

But then, on $\mathbb{R}^n$, for $n > 1$ this implies the existence of bundles $x \neq y$ with $x \geq y$ and $y \geq x$, and thus such that $x \bowtie y$ – i.e., they are incomparable since the indifference relation is not allowed between different bundles as it would violate the antisymmetry of $\succsim$.

It follows that, for $n > 1$, the preference relation $\succsim$ is not complete, and therefore it cannot admit a utility representation. For $n = 1$ on the other hand, completeness is preserved and $\succsim$ may allow a utility representation.

In other words, for $n > 1$, the order cannot be complete since monotonicity is not a strong enough property to ensure that different vectors are always ranked differently, and thus, since antisymmetry does not allow for an indifference relationship, the order would not be complete and thus would not admit a utility representation.

Alternatively, we can see how the cardinality of the collection of indifference curves $X \backslash \sim$ for the preference antisymmetric and monotone preference relation $\succsim$ on $\mathbb{R}^n$ has the power of n, and so a cardinality larger than $\mathbb{R}^1$ for $n > 1$:

$$|X \backslash \sim| > |\mathbb{R}^1|$$

Therefore, there cannot be a *utility function* u*:*$X \longrightarrow \mathbb{R}$ if $n > 1$ since the set of indifference curves would be infinitely uncountable for an antisymmetric preference relation $\succsim$. ■