1. Color code explained

Definition Block

Theorem Block

Proof Block

Proof that cannot be completed without more info so it is incomplete

2. Untyped Systems

2.1. Untyped Arithmetic Expressions

2.1.1. Language used

```
t ::=
      true
      false
      if t then t else t
      0
      succ t
      pred t
      iszero t
```

2.1.2. Inductive definition of the language

the smallest set T is defined as

1. $\{\text{true}, \text{false}, 0\} \subseteq T$
2. $t_1 \in T$, then $\{\text{succ } t_1, \text{pred } t_1, \text{iszero } t_1\} \subseteq T$
3. if $t_1, t_2, t_3 \in T$ then if t_1 then t_2 else $t_3 \in T$

2.1.3. Inference rules definition of the language

$$\frac{\text{true} \in T}{\text{succ } t_1 \in T \quad t_1 \in T} \qquad \frac{\text{false} \in T}{\text{pred } t_1 \in T \quad t_1 \in T} \qquad \frac{0 \in T}{\text{iszero } t_1 \in T \quad t_1 \in T}$$

$$\frac{\text{if } t_1 \text{ then } t_2 \text{ else } t_3}{t_1, t_2, t_3 \in T}$$

2.1.4. Concrete Procedural definition of the language

$$\begin{aligned} S_0 &= \emptyset \\ S_{i+1} &= \{\text{true}, \text{false}, 0\} \\ &\quad \cup \{\text{succ } t_1, \text{pred } t_1, \text{iszero } t_1 \mid t_1 \in S_i\} \\ &\quad \cup \{\text{if } t_1 \text{ then } t_2 \text{ else } t_3 \mid t_1, t_2, t_3 \in S_i\} \\ S &= \bigcup_i S_i \end{aligned}$$

2.1.4.1. Exercice

How many elements does S_3 have?

2.1.4.1.1. Answer

S_0 contains 0 elements (emptyset)
S_1 contains 3 elements (constants)
S_2 contains 39 elements (3 from constants, $3 * 3 = 9$ from univariable functions, $3^3 = 27$ from all the combinations of ifelse)
we can guess that $|S_{n+1}| = 3 + |S_n| \times 3 + |S_n|^3$ has
so S_3 contains 59439 elements

2.1.4.2. Exercice

Showing that forall i, $S_i \subseteq S_{i+1}$

2.1.4.2.1. Answer

from the previous exercice we have $\left|S_{n+1}\right| = 3 + \left|S_n\right| \times 3 + \left|S_n\right|^3$ since $\left|S_n\right|$ is always positve then it is trivial