Spectral DTQ Method, May 2017

We start with the SDE in $\mathbb{R}^N$:

$$dX_t = f(X_t)dt + gdW_t$$

Here $g$ is an $N \times N$ invertible matrix. Let

$$\phi(y) = y + f(y)h.$$

Assuming that $f$ is Lipschitz we see that $\phi$ is invertible for sufficiently small $h$; this follows from

$$D\phi(y) = I + Df(y)h.$$

Discretizing the SDE in time, we obtain

$$X_{n+1} = X_n + f(X_n)h + gh^{1/2}Z_{n+1}.$$

Here $Z_{n+1}$ is a sequence of independent multivariate Gaussians, each with mean vector 0 and covariance matrix equal to the identity matrix $I$. Hence $X_{n+1}$ given $X_n = y$ has multivariate Gaussian density with mean vector $\phi(y)$ and covariance matrix $hgg^T$. Now moving from sample paths to densities, we have

$$\widetilde{p}(x, t_{n+1}) = \int_{y \in \mathbb{R}^N} G(x - \phi(y); hgg^T)\widetilde{p}(y, t_n)\, dy.$$

Let

$$G(x - \phi(y); hgg^T) = \frac{1}{\sqrt{(2\pi h)^N |g|^2}} \exp\left(-\frac{1}{2h}(x - \phi(y))^T (gg^T)^{-1}(x - \phi(y))\right).$$

Now we let $z = \phi(y)$ so that $dz = D\phi(y)\, dy$. Then

$$\widetilde{p}(x, t_{n+1}) = \int_{z \in \mathbb{R}^N} G(x - z; hgg^T)\, \underbrace{\widetilde{p}(\phi^{-1}(z), t_n)\, [D\phi(\phi^{-1}(z))]^{-1}}_{\psi_n(z)}\, dz.$$

We now have a convolution!