

Expectation Maximization (EM) with Bridge Sampling

The governing equation for the problem in \mathbb{R}^d is:

$$dX(t) = f(X(t))dt + \Gamma dW_t \quad (1)$$

with Γ equal to a constant diagonal matrix and W_t denoting Brownian motion in \mathbb{R}^d . Consider an additive model for $f(x)$:

$$f(x) = \sum_{k=1}^M \beta_k \phi_k(x) \quad (2)$$

Here $\{\phi_i\}$ is some family of functions we prescribe, e.g., tensor products of orthogonal polynomials. Each $\phi_i : \mathbb{R}^d \rightarrow \mathbb{R}^d$ should be fairly easy to compute.

We assume that $\Gamma = \text{diag } \gamma$ and that there exists $\delta > 0$ such that $\gamma_i \geq \delta$ for all $i \in \{1, \dots, d\}$. Under this condition, (1) should have a smooth density.

Suppose we have data in the form of a time series, \mathbf{x} , considered to be direct observations of $X(t)$ at discrete time points. For simplicity, let us assume the observations are collected at equispaced times, $j\Delta t$ for $0 \leq j \leq L$. Thus the observed data is $\mathbf{x} = x_0, x_1, \dots, x_L$. Each $x_j \in \mathbb{R}^d$.

Our goal is to use the data to estimate the functional form of f and the constant vector γ .

To achieve this goal, we propose to use EM. Here we regard \mathbf{x} as the incomplete data. The missing data \mathbf{z} is thought of as data collected at a time scale $h \ll \Delta t$ that is fine enough such that the transition density of (1) is approximately Gaussian. That is, if we discretize (1) in time via Euler-Maruyama method, we obtain

$$\tilde{X}_{n+1} = \tilde{X}_n + f(\tilde{X}_n; \beta)h + \gamma h^{1/2} Z_{n+1} \quad (3)$$

where Z_{n+1} is a standard normal, independent of X_n . Note that $\tilde{X}_{n+1} | \tilde{X}_n = v$ is multivariate Gaussian with mean vector $v + f(v)h$ and covariance matrix $h\Gamma$. Specifically, the density is

$$\left(\prod_{i=1}^d \frac{1}{\sqrt{2\pi h \gamma_i}} \right) \exp \left(-\frac{1}{2h} \left(x - v - h \sum_{k=1}^M \beta_k \phi_k(v) \right)^T \Gamma^{-1} \left(x - v - h \sum_{\ell=1}^M \beta_\ell \phi_\ell(v) \right) \right).$$

As h decreases, this Gaussian will better approximate the transition density

$$X((n+1)h) | X(nh) = v,$$

where $X(t)$ refers to the solution of (1), not its time-discretization.

EM. The EM algorithm consists of two steps, computing the expectation of the log likelihood function (on the completed data) and then maximizing it with respect to the parameters $\boldsymbol{\theta} = (\beta, \gamma)$.

1. Start with an initial guess for the parameters, $\boldsymbol{\theta}^{(0)}$.
2. For the expectation (or E) step,

$$Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(k)}) = \mathbb{E}_{\mathbf{z}|\mathbf{x}, \boldsymbol{\theta}^{(k)}}[\log p(\mathbf{x}, \mathbf{z} \mid \boldsymbol{\theta})] \quad (4)$$

Our plan is to evaluate this expectation via bridge sampling. That is, we will sample from diffusion bridges $\mathbf{z} \mid \mathbf{x}, \boldsymbol{\theta}^{(k)}$. Then (\mathbf{x}, \mathbf{z}) will be a combination of the original data together with sample paths.

3. For the maximization (or M) step, we start with the current iterate and a dummy variable $\boldsymbol{\theta}$ and define

$$\boldsymbol{\theta}^{(k+1)} = \arg \max_{\boldsymbol{\theta}} Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(k)}) \quad (5)$$

It will turn out that we can maximize this quantity without numerical optimization. All we will need to do is solve a least-squares problem.

4. Iterate Step 2 and 3 until convergence.

Details. With a fixed parameter vector $\boldsymbol{\theta}^{(k)}$, the SDE (1) is specified completely, i.e., the drift and diffusion terms have no further unknowns. For this SDE, we assume a diffusion bridge sampler is available. We take F diffusion bridge steps to march from x_i to x_{i+1} ; the time step will be $h = (\Delta t)/F$. We can think of this process as inserting $F - 1$ *new* samples, $\{z_{i,j}\}_{j=1}^{F-1}$ between x_i and x_{i+1} .

Let $\mathbf{z}^{(r)}$ denote the r^{th} diffusion bridge sample path:

$$z^{(r)} \sim z \mid x, \beta^{(k)} \quad (6)$$

The observed and sampled data can be interleaved together to create a time series (completed data)

$$\mathbf{y}^{(r)} = \{y_j^{(r)}\}_{j=1}^N$$

of length $N = LF + 1$. Suppose we form R such time series. The expected log likelihood can then be approximated by

$$\begin{aligned} Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(k)}) &= \mathbb{E}_{\mathbf{z}|\mathbf{x}, \boldsymbol{\theta}^{(k)}}[\log p(\mathbf{x}, \mathbf{z} \mid \boldsymbol{\theta})] \\ &\approx \frac{1}{R} \sum_{r=1}^R \left[\sum_{j=1}^N \left[\sum_{i=1}^d -\frac{1}{2} \log(2\pi h \gamma_i^2) \right] \right. \\ &\quad \left. - \frac{1}{2h} (y_j^{(r)} - y_{j-1}^{(r)} - h \sum_{k=1}^M \beta_k \phi_k(y_{j-1}^{(r)}))^T \Gamma^{-2} (y_j^{(r)} - y_{j-1}^{(r)} - h \sum_{\ell=1}^M \beta_\ell \phi_\ell(y_{j-1}^{(r)})) \right] \end{aligned}$$

To maximize Q over $\boldsymbol{\theta}$, we first assume $\Gamma = \text{diag } \gamma$ is known and maximize over β . This is a least squares problem. The solution is given by forming the matrix

$$\mathcal{M}_{k,\ell} = \frac{1}{R} \sum_{r=1}^R \sum_{j=1}^N h \phi_k^T(y_{j-1}^{(r)}) \Gamma^{-2} \phi_\ell^T(y_{j-1}^{(r)})$$

and the vector

$$\rho_k = \frac{1}{R} \sum_{r=1}^R \sum_{j=1}^N \phi_k^T(y_{j-1}^{(r)}) \Gamma^{-2} (y_j^{(r)} - y_{j-1}^{(r)}).$$

We then solve the system

$$\mathcal{M}\beta = \rho$$

for β . Now that we have β , we maximize Q over γ . The solution can be obtained in closed form:

$$\gamma_i^2 = \frac{1}{RNh} \sum_{r=1}^R \sum_{j=1}^N ((y_j^{(r)} - y_{j-1}^{(r)} - h \sum_{\ell=1}^M \beta_\ell \phi_\ell(y_{j-1}^{(r)})) \cdot e_i)^2$$

where e_i is the i^{th} canonical basis vector in \mathbb{R}^d .

Remarks

1. In $d = 1$ dimension, when Γ is fixed and known, the procedure appears to work well even with $R = 10$ sample paths. This is for a problem where we generate data from a known model and then try to recover the β coefficients.
2. It is of interest to prove that the alternating β, Γ maximization approach increases the Q function. As long as the Q function increases from one iteration to the next, up to the sampling error (which should decay like $R^{-1/2}$), the EM algorithm should converge monotonically. That is, both the completed log likelihood and the log likelihood of the original data \mathbf{x} should converge monotonically. The EM algorithm should yield a local maximizer.
3. If the alternating β, Γ maximization approach does not work, we could instead take a few gradient descent steps on the negative log likelihood. The gradients are simple to compute.