Тема „Функция нескольких переменных"

Исследовать функцию на условный экстремум

1. $U = 3 - 8x + 6y$, если $x^2 + y^2 = 36$

$$L(\lambda, x, y) = 3 - 8x + 6y + \lambda(x^2 + y^2 - 36)$$

$$\begin{cases} L'_x = -8 + 2\lambda x = 0 \\ L'_y = 6 + 2\lambda y = 0 \\ L'_\lambda = x^2 + y^2 - 36 = 0 \end{cases} \Rightarrow \begin{cases} x = \frac{4}{\lambda} \\ y = -\frac{3}{\lambda} \\ \frac{16}{\lambda^2} + \frac{9}{\lambda^2} = 36 \end{cases} \Rightarrow \begin{cases} x = \frac{4}{\lambda} \\ y = -\frac{3}{\lambda} \\ \lambda = \pm\frac{5}{6} \end{cases}$$

$$\left(\frac{5}{6}, \frac{24}{5}, -\frac{18}{5}\right) \left(-\frac{5}{6}, -\frac{24}{5}, \frac{18}{5}\right)$$

$$L''_{xx} = 2\lambda, \quad L''_{yy} = 2\lambda, \quad L''_{\lambda\lambda} = 0$$

$$L''_{xy} = 0, \quad L''_{x\lambda} = 2x, \quad L''_{y\lambda} = 2y$$

$$\begin{pmatrix} L''_{\lambda\lambda} & L''_{\lambda x} & L''_{\lambda y} \\ L''_{x\lambda} & L''_{xx} & L''_{xy} \\ L''_{yx} & L''_{yx} & L''_{yy} \end{pmatrix} = \begin{pmatrix} 0 & 2x & 2y \\ 2x & 2\lambda & 0 \\ 2y & 0 & 2\lambda \end{pmatrix}$$

$$\begin{vmatrix} 0 & 2x & 2y \\ 2x & 2\lambda & 0 \\ 2y & 0 & 2\lambda \end{vmatrix} = 0 \cdot \begin{vmatrix} 2\lambda & 0 \\ 0 & 2\lambda \end{vmatrix} - 2x \begin{vmatrix} 2x & 0 \\ 2y & 2\lambda \end{vmatrix} + 2y \begin{vmatrix} 2x & 2\lambda \\ 2y & 0 \end{vmatrix} = -8\lambda(x^2 + y^2)$$

$\left(\frac{5}{6}, \frac{24}{5}, -\frac{18}{5}\right)$ – минимум

$\left(-\frac{5}{6}, -\frac{24}{5}, \frac{18}{5}\right)$ – максимум

2. $U = 2x^2 + 12xy + 32y^2 + 15$, если $x^2 + 16y^2 = 64$

$L(\lambda, x, y) = 2x^2 + 12xy + 32y^2 + 15 + \lambda(x^2 + 16y^2 - 64)$

$$\begin{cases} L'_x = 4x + 12y + 2\lambda x = 0 \\ L'_y = 12x + 64y + 32\lambda y = 0 \\ L'_\lambda = x^2 + 16y^2 - 64 = 0 \end{cases} \Rightarrow \begin{cases} x(4+2\lambda) = -12y \\ y(64+32\lambda) = -12x \\ x^2 + 16y^2 = 64 \end{cases} \Rightarrow \begin{cases} x = -\frac{12y}{4+2\lambda} \\ y = -\frac{12x}{64+32\lambda} \\ x^2 + 16y^2 = 64 \end{cases}$$

$$y = \frac{144y}{(4+2\lambda)(64+32\lambda)} = \frac{144y}{64\lambda^2 + 256\lambda + 256} = \frac{9y}{4\lambda^2 + 16\lambda + 16}$$

$4\lambda^2 + 16\lambda + 16 = 9$

$4\lambda^2 + 16\lambda + 7 = 0$

$D = 144$

$\lambda_{1,2} = \frac{-16 \pm 12}{2 \cdot 4} \Rightarrow \lambda_1 = -3{,}5 \ , \ \lambda_2 = -0{,}5$

$x_1 = -\frac{12y}{4 - 2 \cdot 3{,}5} = 4y$

$x_2 = -\frac{12y}{4 - 2 \cdot 0{,}5} = -4y$

$16y_{1,2}^2 + 16y_{1,2}^2 = 64$

$32y_{1,2}^2 = 64$

$y_{1,2} = \pm\sqrt{2}$

$(-3{,}5, 4\sqrt{2}, \sqrt{2}), (-3{,}5, -4\sqrt{2}, -\sqrt{2}), (-0{,}5, -4\sqrt{2}, \sqrt{2}), (-0{,}5, 4\sqrt{2}, -\sqrt{2})$

$$\begin{pmatrix} L''_{\lambda\lambda} & L''_{\lambda x} & L''_{\lambda y} \\ L''_{x\lambda} & L''_{xx} & L''_{xy} \\ L''_{y\lambda} & L''_{yx} & L''_{yy} \end{pmatrix} = \begin{pmatrix} 0 & 2x & 32y \\ 2x & 4+2\lambda & 12 \\ 32y & 12 & 64+32\lambda \end{pmatrix}$$

$$\begin{vmatrix} 0 & 2x & 32y \\ 2x & 4+2\lambda & 12 \\ 32y & 12 & 64+32\lambda \end{vmatrix} = 0 \cdot \begin{vmatrix} 4+2\lambda & 12 \\ 12 & 64+32\lambda \end{vmatrix} - 2x \begin{vmatrix} 2x & 12 \\ 32y & 64+32\lambda \end{vmatrix} + 32y \begin{vmatrix} 2x & 4+2\lambda \\ 32y & 12 \end{vmatrix} =$$

$= -2x(2x(64+32\lambda) - 12 \cdot 32y) + 32y(2x \cdot 12 - 32y(4+2\lambda)) = -256x^2 - 128\lambda x^2 + 768xy +$
$+ 768xy - 4096y^2 - 2048\lambda y^2 = -(2+\lambda)(128x^2 + 2048y^2) + 1536xy$

$(-3{,}5, 4\sqrt{2}, \sqrt{2})$ – в точке значение $24576 > 0$ – максимум

$(-3{,}5, -4\sqrt{2}, -\sqrt{2})$ – в точке значение $24576 > 0$ – максимум

$(-0{,}5, -4\sqrt{2}, \sqrt{2})$ – в точке значение $-24576 < 0$ – минимум

$(-0{,}5, 4\sqrt{2}, -\sqrt{2})$ – в точке значение $-24576 < 0$ – минимум

3. Найти производную функции $U = x^2+y^2+z^2$ по направлению вектора $\vec{e}(-9,8,-12)$ в точке $M(8;-12;9)$

$$\frac{\partial U}{\partial x} = 2x, \quad \frac{\partial U}{\partial y} = 2y, \quad \frac{\partial U}{\partial z} = 2z$$

$$\left.\frac{\partial U}{\partial x}\right|_{(8,-12,9)} = 16, \quad \left.\frac{\partial U}{\partial y}\right|_{(8,-12,9)} = -24, \quad \left.\frac{\partial U}{\partial z}\right|_{(8,-12,9)} = 18$$

$$|\bar{e}| = \sqrt{(-9)^2+8^2+(-12)^2} = 17$$

$$\cos\alpha = -\frac{9}{17}, \quad \cos\beta = \frac{8}{17}, \quad \cos\gamma = -\frac{12}{17}$$

$$\left.\frac{\partial U}{\partial l}\right|_M = 16\cdot\left(-\frac{9}{17}\right) - 24\frac{8}{17} - 18\frac{12}{17} = -32\frac{8}{17}$$

4. Найти производную функции $U = e^{x^2+y^2+z^2}$ по направлению вектора $\vec{d} = (4,-13,-16)$ в точке $L(-16;4;-13)$

$$\frac{\partial U}{\partial x} = 2xe^{x^2+y^2+z^2}, \quad \frac{\partial U}{\partial y} = 2ye^{x^2+y^2+z^2}, \quad \frac{\partial U}{\partial z} = 2ze^{x^2+y^2+z^2}$$

$$\left.\frac{\partial U}{\partial x}\right|_{(-16;4;-13)} = -32\cdot e^{441}, \quad \left.\frac{\partial U}{\partial y}\right|_{(-16;4;-13)} = 8\cdot e^{441}, \quad \left.\frac{\partial U}{\partial z}\right|_{(-16;4;-13)} = -26\cdot e^{441}$$

$$|\bar{e}| = \sqrt{4^2+(-13)^2+(-16)^2} = 21$$

$$\cos\alpha = \frac{4}{21}, \quad \cos\beta = -\frac{13}{21}, \quad \cos\gamma = -\frac{16}{21}$$

$$\left.\frac{\partial U}{\partial l}\right|_L = -32e^{441}\cdot\frac{4}{21} + 8e^{441}\left(-\frac{13}{21}\right) + (-26)e^{441}\left(-\frac{16}{21}\right) = 8\frac{16}{21}e^{441}$$