TCS I – Discrete mathematics

– Week 12 –

1. Let $S = \mathbb{R}^2$ and let R be a binary relation on S defined by
$$(x_1, y_1)R(x_2, y_2) \Leftrightarrow x_1^2 + y_1^2 = x_2^2 + y_2^2.$$
Show that R is equivalence relation and find $R[(7, 1)]$.

2. Show that $f = \{(x, y) \in \mathbb{R} \times \mathbb{R}; x = y^3\}$ is a function from $\mathbb{R}$ to $\mathbb{R}$.

3. Is $g = \{(x, y) \in [-5, 5] \times \mathbb{R} : x^2 + y^2 = 25\}$ a function from $[-5, 5]$ to $\mathbb{R}$.

4. Show that the following relations are not functions on $\mathbb{R}$:
   (i) $\{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 = y^2\}$,
   (ii) $\{(x, y) \in \mathbb{R} \times \mathbb{R} : x = \cos(y)\}$,
   (iii) $\{(x, y) \in \mathbb{R} \times \mathbb{R} : y^2 = \sqrt{x}\}$.

5. Which of the following functions are injective or surjective:
   (i) $f\colon \mathbb{R} \to \mathbb{R}; f(x) = x^2$,
   (ii) $f\colon \mathbb{R} \to \mathbb{Z}; f(x) = \lfloor x \rfloor$,
   (iii) $f\colon \mathbb{R}^+ \to \mathbb{R}^+; f(x) = \sqrt{x}$,
   (iv) $f\colon \mathbb{Z} \to \mathbb{N}; f(x) = |x|$.

6. We are given the following functions
   (i) $f\colon \mathbb{N} \to \mathbb{N}, f(n) = \begin{cases} 2n, & n \text{ even} \\ 3n - 1, & n \text{ odd}, \end{cases}$
   (ii) $f\colon \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}, f(x) = (x^2, x^3)$.

   (a) Find images $f(\{1, 2, 3\})$ and $f(\mathbb{N})$.
   (b) Find those preimages $f^{-1}(\{1, 2, 3, 4\})$ and $f^{-1}(\{(1, -1), (4, 8)\})$ that are well-defined.

7. Let $f\colon X \to Y$ and $g\colon X \to Z$ be two bijective functions. Is the function $h\colon X \to Y \times Z$ defined by
$$h(x) = (f(x), g(x)),$$
(i) injective, (ii) surjective?

8. Let A, B and C be arbitrary sets, and let $g\colon A \to B$ and $f\colon B \to C$ be functions. Show:

   (i) if f, g are injective, then $f \circ g$ is injective,

   (ii) if f, g are surjective, then $f \circ g$ is surjective.