Exercises

1. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ and let $f\colon \mathbb{N} \to \mathbb{N}$ be a function given by
$$f(x) = 2x + 3.$$
   (i) Find $f(f(4) + 1)$.
   (ii) Find $f(\{1, 2, 3, 4, 5\})$.
   (iii) Find $f^{-1}(\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\})$.
   (iv) Is f injective?
   (v) Is f surjective?

2. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$. Are the following functions injective, surjective and/or bijective?
   (i) $f\colon \mathbb{N} \to \mathbb{N}, f(n) = 2n$;
   (ii) $f\colon \mathbb{N} \to \mathbb{N}, f(n) = 2^n$;
   (iii) $f\colon \mathbb{N} \to \mathbb{N}, f(n) =$ number of all positive divisors of n.

3. Let $S = \{1, 2, 3, \ldots\}$. Let R be a relation on he set $S \times S$ defined by
$$(a, b)\, R\, (c, d) \Leftrightarrow 2a - b = 2c - d.$$
   (i) Show that R is an equivalence relation.
   (ii) If $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$, find $R[(2, 5)]$, the equivalence class of $(2, 5)$.

4. Let R_1 be a partial order on X, and let R_2 a partial order on Y. Let R be a relation on $X \times Y$ defined by:
$$(x_1, y_1)\, R\, (x_2, y_2) \Leftrightarrow x_1\, R_1\, x_2 \wedge y_1\, R_2\, y_2.$$
   Show that R is a partial order on $X \times Y$.

5. Let $\mathbb{R}^+ = \{x \,|\, x \in \mathbb{R} \wedge x > 0\}$, and let $\mathbb{N} = \{1, 2, 3, \ldots\}$. Are the following functions injective, surjective and/or bijective?
   (i) $f\colon \mathbb{R}^+ \to \mathbb{R}^+, f(x) = |x|$;
   (ii) $f\colon \mathbb{N} \to \mathbb{R}^+, f(x) = 2x + 7$;
   (iii) $f\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, f(x, y) = 2x - y$.

6. Let $S = \{x \in \mathbb{N} \,|\, 2 \leq x \leq 16\}$. Let R be the relation of division on S, that is,
$$x\, R\, y \Leftrightarrow x | y.$$

(i) Draw the Hasse diagram with respect to R.

(i) Find all R-maximal elements, if they exist.

(iii) Find all R-minimal elements, if they exist.

(iv) Does R define a lattice on S?

7. Let $S = \mathbb{R}$. Let R be a relation on S defined by

$$x\,R\,y \Leftrightarrow x - y \in \mathbb{Z}.$$

(i) Show that R is an equivalence relation.

(ii) Find $R[\frac{1}{2}]$, the equivalence class of $\frac{1}{2}$.

8. Let $S = \{1, 2, \ldots, 10\}$. Let R be a relation on S defined by

$$x\,R\,y \Leftrightarrow x + y \text{ is even} \ \text{ and } \ x \leq y.$$

(i) Show that R is a partial order on S.

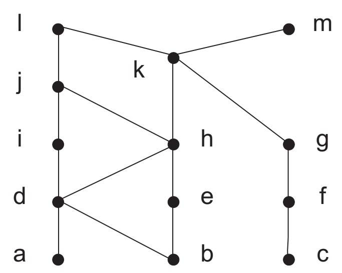
(ii) Draw the Hasse diagram with respect to R.

(i) Find all R-maximal elements, if they exist.

(iii) Find all R-minimal elements, if they exist.

(iv) Does R define a lattice on S?

9. A partial order R is given by the following Hasse diagram:



(i) Find all R-maximal elements, if they exists.

(iii) Find all R-minimal elements, if they exists.

(iii) Does there exist an R-least element?

(iv) Does there exist the R-greatest element?

(v) Find all R-upper bounds of $\{a, b, c\}$, if they exist.

(vi) Find the R-lowest upper bound of $\{a, b, c\}$, if it exists.

(vii) Find all R-lower bounds of $\{f, g, h\}$, if they exist.

(viii) Find the R-greatest lower bound of $\{f, g, h\}$, if it exists.

10. Let $S = \mathbb{N}$. Let R be a relation on S defined by

$$x \, R \, y \Leftrightarrow (\exists a)(a \in \mathbb{N} \wedge xy = a^2).$$

   (i) Show that R is an equivalence relation.

   (ii) If $S = \{1, 2, 3, \dots, 10\}$, find$R[1]$, the equivalence class of 1.

   (iii) If $S = \{1, 2, 3, \dots, 10\}$, find all equivalence classes with more then one element.

11. Let $n \geq 2$ and $M = \{1, 2, \dots, n\} \subset \mathbb{N}$. Let R be a relation on the power set $\mathcal{P}(M)$ of M defined by:

$$A \, R \, B \Leftrightarrow A \cup \{1\} = B \cup \{n\}.$$

   (i) Show that R is neither irreflexive, nor symmetric, nor strict total.

   (ii) Show that R is transitive.

12. Let $S = \{[a, b] \,|\, a, b \in \mathbb{R}\} \cup \{[a, \infty) \,|\, a \in \mathbb{R}\} \cup \{(-\infty, b] \,|\, b \in \mathbb{R}\}$ be the set of all bounded or unbounded closed intervals on $\mathbb{R}$. Let R be a relation on S defined by

$$A \, R \, B \Leftrightarrow A \subseteq B,$$

   and let $U = \{[1, 10], [3, 20], [4, 15]\} \subset S$.

   (i) Find an R-lower bound for U that is not the R-greatest lower bound for U.

   (ii) Find the R-greatest lower bound for U.

   (iii) Find an R-upper bound for U that is not the R-least upper bound for U.

   (iv) Find the R-least upper bound for U.

   (v) Is R a linear order on S?

   (vi) Find a subset $V \subseteq S$ that has no R-lower bound.

13. Find a bijection between the interval $(-3, \infty)$ and the set $\mathbb{R}$.

14. Show that the intervals $[-5, \infty)$ and $[-1, 1)$ are equipolent.