A natural introduction to basic category theory

Semitransparent Observer

2023 at Edinburgh

Contents

1 Terminology explaination **3**

2 Basic set theory **4**

2.1 Introduction 4

2.2 Building blocks 4

2.2.1 Object 4

2.2.2 Set, intersection, and union 4

2.2.3 Union and intersection 5

2.2.4 Ordered pair and n-tupleple 6

2.3 Various relation 7

2.3.1 Graph and path 7

2.3.2 Cartesian product 7

2.3.3 Relation, domain, range, and codomain 8

2.3.4 Function and composition 8

3 Bibliography **13**

Chapter 1

Terminology explaination

i.e.: that is

e.g.: for example

etc.: and so on

UoE: University of Edinburgh

$\exists!$: exists a unique

ι: embedding

π: projection

Chapter 2

Basic set theory

2.1 Introduction

This part gives a brief review of basic set theory([1],[2]) for readers not familiar with standard math language.

2.2 Building blocks

Let's consider UoE and a family with 3 members Alex, Bob, and Carol. Alex and Bob are former students of UoE, they got married and their son Carol is now a student of UoE.

2.2.1 Object

Definition 2.1. An *object* is anything you wish to consider as an object.

Example 2.2. former students of UoE, UoE, this sentence is an object, •, you, and the word "you" are objects.

2.2.2 Set, intersection, and union

Definition 2.3. A *set* S is a collection of **distinct** objects called elements.

Example 2.4.

- $U = \{\text{UoE}\}$, UoE itself;
- $fs = \{\text{Alex, Bob}\}$, the couple/former students of UoE; $Us = \{\text{UoE, Carol}\}$, UoE and a current student; $f = \{\text{Alex, Bob, Carol}\}$, the family;
- $Uf = \{\text{UoE, Alex, Bob, Carol}\}$, UoE, its former students, and a current student.

Definition 2.5. *cardinality* of a set S is the number of elements in S, written as $\text{num}(S)$;

The order we write the elements in a set makes no difference.

i.e. $f = \{\text{Alex, Bob, Carol}\} = \{\text{Alex, Carol, Bob}\}$ etc.;

Definition 2.6. A *pair* is a set with 2 elements.

Example 2.7. $fs = \{\text{Alex, Bob}\}$, $Us = \{\text{UoE, Carol}\}$.

Definition 2.8. set A is a *subset* of set S if $\forall a \in A, a \in S$, written as $A \subseteq S$.

Example 2.9. a set Family $= \{\text{Alex, Bob, Carol}\} \subseteq Uf$.

Definition 2.10. *empty set* $\varnothing = \{\}$ is the unique set containing no object.

- for any set S, $\varnothing \subseteq S$.

2.2.3 Union and intersection

(Suppose we want to get a set containing all the elements of two sets, we put all the elements of them in one set:)

Definition 2.11. The *union* of two sets A and B is $\{\forall x | x \in A \text{ or } x \in B\}$, written as $A \bigcup B$.

Example 2.12. take a photo of UoE and Carol, with his family standing by, then we have figures of UoE, Alex, Bob, and Carol:
$\{\mathrm{UoE}, \mathrm{Carol}\} \cup \{\mathrm{Alex}, \mathrm{Bob}, \mathrm{Carol}\} = \{\mathrm{UoE}, \mathrm{Alex}, \mathrm{Bob}, \mathrm{Carol}\}$.

(Suppose we want to get a union of two sets containing all their information, we should first record the same element in different sets differently to make a difference, then we union them together.)

For a set S, S' is a set $\{(s,')| \forall s \in S\}$.

Definition 2.13. The *disjoint union* of two sets A and B is a set $A' \bigcup B''$, written as $A \bigsqcup B$.

Example 2.14. take a photo of UoE and Carol then another photo of his family, then we have figures of UoE, Carol, Alex, Bob, and another Carol:
$\{\mathrm{UoE}, \mathrm{Carol}\} \bigsqcup \{\mathrm{Alex}, \mathrm{Bob}, \mathrm{Carol}\} = \{\mathrm{UoE}', \mathrm{Carol}', \mathrm{Alex}'', \mathrm{Bob}'', \mathrm{Carol}''\}$.

(Suppose we want to get a set containing all the common elements of two sets, we fetch every element and check if it appears in both of them:)

Definition 2.15. The *intersection* of two sets A and B is $\{x | x \in A \text{ and } x \in B\}$,

written as $A \bigcap B$.

Example 2.16. take a photo of UoE and Carol then another photo of his family, the figure of Carol is in both of them:
$\{\mathrm{UoE}, \mathrm{Carol}\} \cap \{\mathrm{Alex}, \mathrm{Bob}, \mathrm{Carol}\} = \{\mathrm{Carol}\}$.

2.2.4 Ordered pair and n-tupleple

(If Alex, Bob, and Carol are singing "Ode to Joy", the tone is like 3345 5432. The notes are sung in order and the same note can appear repeatedly. We may construct a set called "n-tupleple" to describe it, which is defined recursively from ordered pair(2-tuple).)

For two objects a_1 and a_2,

Definition 2.17. A *ordered pair* (a_1, a_2) is a set $\{\{a_1\}, \{a_1, a_2\}\}$.

- $(a_2, a_1) = \{\{a_2\}, \{a_2, a_1\}\} \neq \{\{a_1\}, \{a_1, a_2\}\} = (a_1, a_2)$, hence the order do matters;
- $(\bullet, \bullet) = \{\{\bullet\}, \{\bullet\}\} = \{\{\bullet\}\}$;

For a series of n objects $a_1, a_2, \cdots, a_n$,

Definition 2.18. The *n-tuple* $A = (a_1, a_2, \cdots, a_n)$ is an ordered pair $((a_1, a_2, \cdots, a_{n-1}), a_n)$.

- for $a_1, a_2, \cdots, a_n$, the n-tupleples is:

  $$(a_1, a_2, \cdots, a_n) = \{(a_1, a_2, \cdots, a_{n-1}), \{(a_1, a_2, \cdots, a_{n-1}), a_n\}\}$$

  ;
- the index of element of A is $I = (1, 2, \cdots, n)$,
  $\forall i \in I$, $A_i = a_i$ is the ith element of A;
- a length n *sequence* is an n-tuple such that all the elements are in another set.

Example 2.19. the tone of "Ode to Joy" is $(3, 3, 4, 5, 5, 4, 3, 2)$ is a sequence, all the elements are notes;

Example 2.20. $(\bullet, \bullet, \bullet) = ((\bullet, \bullet), \bullet) = (\{\{\bullet\}\}, \bullet) = \{\{\{\{\bullet\}\}\}, \{\{\{\bullet\}\}, \bullet\}\}$.

(There can be some "relation" between elements of sets:

Example 2.21. "$\in$", Alice $\in f$, Bob $\in f$, Bob $\in fs$;

Example 2.22. "$\subseteq$", $fs \subseteq f$, $s \subseteq f$, $f \subseteq f$;

Example 2.23. "$=$" on $\mathbb{R}$, $1 = 1$, $2.0 = 2$, $0.999\cdots = 1$ etc;

Example 2.24. "$\neq$" on $\mathbb{R}$, whenever "$a = b$" doesn't hold;

Example 2.25. "$\leq_{\mathbb{N}}$" on $\mathbb{N}$: $0 \leq_{\mathbb{N}} 2$, "$\leq_{\mathbb{R}^+}$" on $\mathbb{R}^+$: $1 \leq_{\mathbb{R}^+} e^2$.

Example 2.26. in the set Family, there are many *relations* between the family members: Alice *is the mother of* Carol, Bob *is the husband of* Alice, Alice *married* Bob, Bob *married* Alice, Carol *is not married to* Bob, Carol *is* Carol, they *love* each other, and there's no *hate* between them.

2.3 Various relation

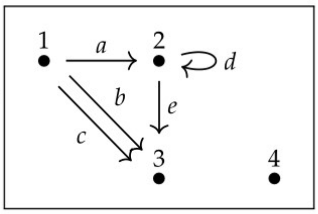
2.3.1 Graph and path

Before introducing the rigid definition of relation, we may introduce

- "graph", a way to represent relations and other concepts intuitively;
- "cartesian product", the "universe" relations live in.

Definition 2.27. A *directed graph* $G = (V, A, s, t)$ consists of:

- a set V of vertices, drawn as $\bullet$;
- a set A of arrows, drawn as $\to$.
  - an arrow $\mathbf{a} \in A$ is a 3-tuple (s, a, t), $s, t \in V$. s/t is the source/target of a.

$G =$ 

arrow a	**source** $s(a) \in V$	**target** $t(a) \in V$
a	1	2
b	1	3
c	1	3
d	2	2
e	2	3

Figure 2.1: graph G

Definition 2.28. A *path* p in a graph G is a sequence of arrows such that the target of one arrow is the source of the next arrow.

The source/target of the first/last arrow is the source/target of a path.

Example 2.29. eda, drawn as $1 \xrightarrow{a} 2 \xrightarrow{d} 2 \xrightarrow{e} 3$ is a length 3 path from 1 to 3.

A length 0 path 0_v start and end at the same vertice v. Adding it to a path makes no difference.

Definition 2.30. Two paths p, p' are *parallel* if they have the same source and target, written as $p \parallel p'$.

Example 2.31. $b \parallel b0_1 \parallel c \parallel ea \parallel e0_2d0_2a$.

2.3.2 Cartesian product

For a 2 sets A and B,

Definition 2.32. The *cartesian product* $A \times B$ is a set consists of 2-tuples:

$$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}.$$

Example 2.33. 2-dimensional Cartesian coordinate system $(x, y) \in \mathbb{R} \times \mathbb{R}$.

2.3.3 Relation, domain, range, and codomain

$\mathrm{dom}(R) \times \mathrm{cod}(R)$ is the "universe" a relation R lives in:

Definition 2.34. A *relation* R is a set of ordered pairs $R \subseteq \mathrm{dom}(R) \times \mathrm{cod}(R)$, $(a, b) \in R$ is written as aRb.

Example 2.35. $\leq_{\mathbb{N}} \subseteq \mathbb{N} \times \mathbb{N}$, $(1, 2) \in \leq_{\mathbb{N}}$, written as $1 \leq_{\mathbb{N}} 2$;

Example 2.36. For any set A, $=_A$ is a relation $\{(a, a) | a \in A\}$, written as $a =_A a$.

Definition 2.37. The *domain* of R is a set $\mathrm{dom}(R) = \{a | \exists b, (a, b) \in R\}$.

Definition 2.38. The *range* of R is a set $\mathrm{ran}(R) = \{b | \exists a, (a, b) \in R\}$.

Definition 2.39. A *codomain* of R is a set $\mathrm{cod}(R) \supseteq \mathrm{ran}(R)$.

- an element $r = (a, b) \in R$ is drawn as $a \xmapsto{R} b$, $a \xmapsto{r} b$, or $a \xmapsto{(a,b)} b$;
- If $s = (b, c) \in R$, aRb, bRc is written as $aRbRc$, drawn as $a \xmapsto{r} b \xmapsto{s} c$.

Example 2.40.

- $\leq_{\mathbb{N}}$ is a relation, $f = (0, 2) \in \leq_{\mathbb{N}}$, $0 \leq_{\mathbb{N}} 2$, drawn as $0 \xmapsto{\leq_{\mathbb{N}}} 2$, $0 \xmapsto{(0,2)} 2$, or $0 \xmapsto{f} 2$.
- $2 \leq_{\mathbb{N}} 4$, $4 \leq_{\mathbb{N}} 5$ is written as $2 \leq_{\mathbb{N}} 4 \leq_{\mathbb{N}} 5$, drawn as: $2 \xmapsto{(2,4)} 4 \xmapsto{(4,5)} 5$;
- $\mathrm{dom}(\leq_{\mathbb{N}}) = \mathrm{ran}(\leq_{\mathbb{N}}) = \mathbb{N}$, $\mathrm{cod}(\leq_{\mathbb{N}})$ can be any set containing $\mathbb{N}$, such as Z, $\mathbb{R}^+$, and $\mathbb{C}$;
- $\leq_{\mathbb{N}} \subset \leq_{\mathbb{R}^+}$.

For two relations R_1 and R_2,

Definition 2.41. The *product* $R_1 \times R_2$ is a relation
$(R_1, R_2) \in (\mathrm{dom}(R_1) \times \mathrm{dom}(R_2)) \times (\mathrm{cod}(R_1) \times \mathrm{cod}(R_2))$, such that:

$$(x, y)(R_1, R_2)(x', y') \text{ if and only if } xR_1x' \text{ and } yR_2y'.$$

For a relation $R \subseteq A \times B$,

Definition 2.42. The *inverse relation* $R^{-1} \in B \times A$ is a relation $\{\forall(b, a) | (a, b) \in B\}$.

Example 2.43. inverse of $\leq_{\mathbb{N}}$ is $\geq_{\mathbb{N}}$. $\because 1 \leq_{\mathbb{N}} 2, \therefore 2 \geq_{\mathbb{N}} 1$.

2.3.4 Function and composition

There's a special type of relation, namely "function". A function f assigns every element d in its domain $\mathrm{dom}(f)$ only one element c in its codomain $\mathrm{cod}(f)$.

Example 2.44. in the alphabet, lower-case "a" has its upper-case "A", written as (a, A);

Example 2.45. *all the cities* is a set, and the cities' location is described by latitude and longitude. Location of Edinburgh is $(55°57'\text{N}, 3°11'\text{W})$, written as
$(\text{Edinburgh}, (55°57'\text{N}, 3°11'\text{W}))$.

For two sets S and T,

Definition 2.46. A *function* $f : S \to T$ is a subset $f \subseteq S \times T$, such that $\forall x \in S$, $\exists! y \in T$ and $(x, y) \in f$.

- $(x, y) \in f$ is written as $f(x) = y$, y is called the *value* of x;
- function f is written as:

$$\begin{aligned} f : S &\longrightarrow T \\ x &\longmapsto f(x) \end{aligned}$$

  drawn as an arrow (f, S, T): $S \xrightarrow{f} T$

  and $(f, x, f(x))$: $x \xmapsto{f} f(x)$ for an element $x \in S$.

The range of f is written as $f(S)$.

Example 2.47. $\text{dom}(f) = \mathbb{R}$, $\text{cod}(f) = \mathbb{R}$, and $\text{ran}(f) = [-1, 1]$.

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \sin(x). \end{aligned}$$

For functions $f : A \to B'$ and $g : B \to C$, if $\text{ran}(f) \subseteq \text{dom}(g) = B$(i.e., $\text{dom}(g)$ is a codomain of f) we may construct a new function $gf : A \to C$ in a natural way:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend left, "gf"] & B \arrow[r, "g"] & C \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, mapsto, "f"] \arrow[rr, bend left, "gf"] & f(a) \arrow[r, mapsto, "g"] & g(f(a)) \end{tikzcd} .$$

Definition 2.48. *compsite* is a function $\circ$:

$$\begin{aligned} \circ : \{\forall (f, g) | \text{ran}(f) \subseteq \text{dom}(g)\} &\longrightarrow \{\forall h : \text{dom}(f) \to \text{cod}(g)\} \\ (f, g) &\longmapsto gf(\text{or } g \circ f) \end{aligned}$$

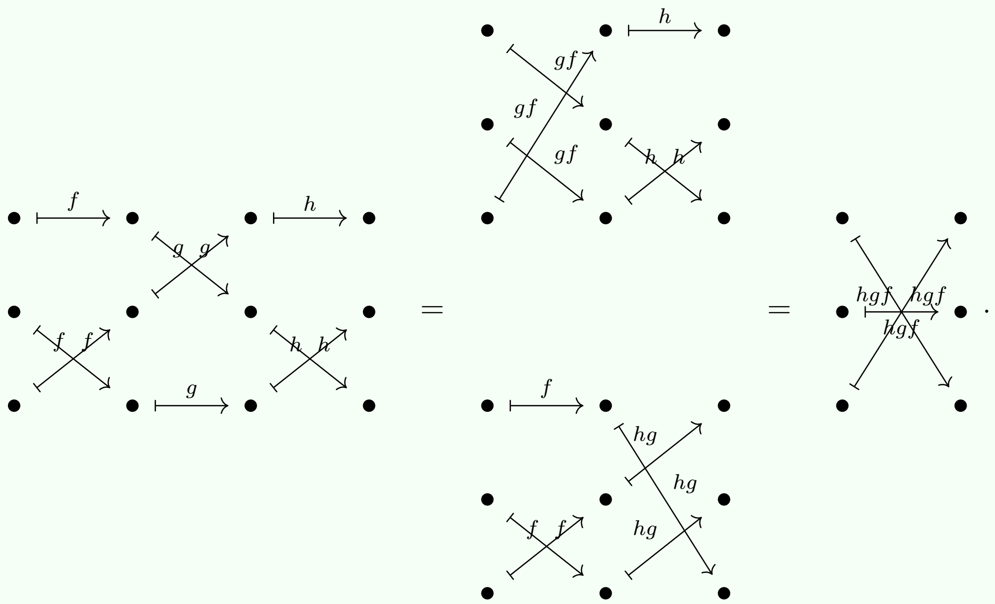
- the *composition* of (f, g) is gf;

Definition 2.49. A sequence of functions $fun = (f, g, h, \cdots)$ is *composable* if $\forall i, i + 1 \in I, \text{ran}(fun_i) \subseteq \text{dom}(fun_{i+1})$.

Example 2.50. for functions $f : A \to B, g : B \to C$, and $h : C \to D$, 3-tuple (f, g, h) is composable:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[rr, bend left, "gf"] \arrow[rrr, bend left=50, "h(gf)"] \arrow[rrr, bend right=50, "(hg)f"'] & B \arrow[r, "g"] \arrow[rr, bend right, "hg"'] & C \arrow[r, "h"] & D
\end{tikzcd}$$

the function $h(gf) = (hg)f = hgf : A \to D$ is the composition of (f, g, h).
for functions $f, g, h : (\bullet, \bullet, \bullet) \to (\bullet, \bullet, \bullet)$:



We may classify functions by their properties:

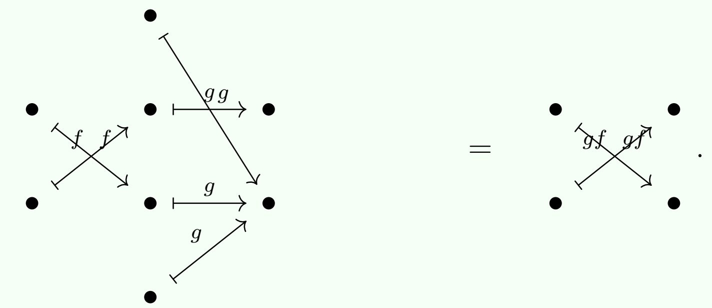
For two sets S and T,

Definition 2.51. A function $f_i : S \to T$ is *injective* if $\forall y \in \mathrm{ran}(f_i) \subseteq T$, $\exists! x \in S$, such that $(x, y) \in f_i$.

Definition 2.52. A function $f_s : S \to T$ is *surjective* if $\forall y \in T, \exists x \in S$, such that $(x, y) \in f_s$ (i.e. $\mathrm{ran}(f_s) = \mathrm{cod}(f_s)$).

Definition 2.53. A function f_b is *bijective* if it's injective and surjective (e.g. f, g, h(2.50)). (i.e. $\forall y \in T, \exists! x \in S$ such that $(x, y) \in f_b$.)

Example 2.54. injection $f : (\bullet, \bullet) \to (\bullet, \bullet, \bullet, \bullet)$, surjection $g : (\bullet, \bullet, \bullet, \bullet) \to (\bullet, \bullet)$, and bijection $gf : (\bullet, \bullet) \to (\bullet, \bullet)$:



Example 2.55.

injection :
$$f : \mathbb{N} \longrightarrow \mathbb{R}^+$$
$$x \longmapsto e^x;$$

surjection :
$$f : \mathbb{R}^+ \longrightarrow [1, e)$$
$$x \longmapsto e^{x-[x]};$$

bijection :
$$f : \mathbb{R} \longrightarrow \mathbb{R}^+$$
$$x \longmapsto e^x.$$

Example 2.56. injections ι_A, ι_B(embedding), surjections π_A, π_B(projection):

$$\iota_A : A \longrightarrow A \bigsqcup B \qquad \iota_B : B \longrightarrow A \bigsqcup B$$
$$a \longmapsto a', \qquad b \longmapsto b'';$$

$$\begin{tikzcd} A \arrow[r, "\iota_A"] & A \bigsqcup B \\ A \times B \arrow[u, "\pi_A"] \arrow[r, mapsto, "\pi_B"] & B \arrow[u, "\iota_B"'] \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "\iota_A"] & a'/b'' \\ (a, b) \arrow[u, "\pi_A"] \arrow[r, mapsto, "\pi_B"] & b \arrow[u, "\iota_B"'] \end{tikzcd}$$

$$\pi_A : A \times B \longrightarrow A \qquad \pi_2 : A \times B \longrightarrow B$$
$$(a, b) \longmapsto a, \qquad (a, b) \longmapsto b.$$

For a set S,

Definition 2.57. An *identity function* is a bijection:

$$\mathrm{id}_S : S \longrightarrow S$$
$$x \longmapsto x.$$

For a bijection $f \subseteq S \times T$,

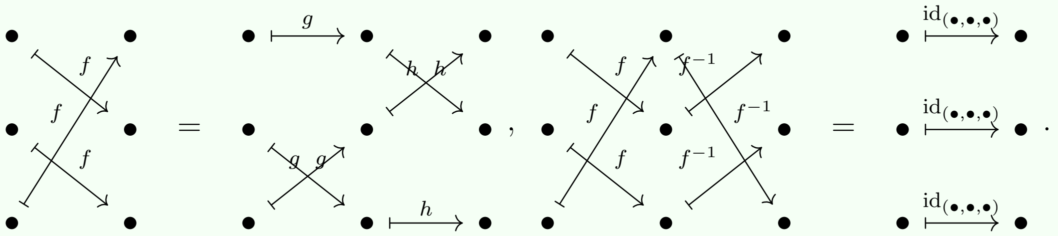
Definition 2.58. The *inverse function* $f^{-1} \in T \times S$ is the inverse relation(2.42) of f (, i.e. $f^{-1} = \{\forall(y, x) | (x, y) \in f\}$).

- $\because f$ is a bijection, $\therefore \forall y \in T, \exists! x \in S$ such that $(x, y) \in f$,
  $\therefore \forall y \in T, \exists! x \in S$ such that $(y, x) \in f^{-1}$, $\therefore f^{-1} : T \to S$ is a function;
- $\because f$ is a function, $\therefore \forall x \in S, \exists! y \in T$ such that $(x, y) \in f$,
  $\therefore \forall x \in S, \exists! y \in T$ such that $(y, x) \in f^{-1}$, $\therefore f^{-1}$ is also a bijection.

Example 2.59. For $f : S \to T$ and $f^{-1} : T \to S$,

- $\forall x \in S$, $(f^{-1}f)(x) = f^{-1}(f(x)) = x$, $f^{-1}f = \mathrm{id}_S$;
- $\forall y \in T$, $(ff^{-1})(y) = f(f^{-1}(y)) = y$, $ff^{-1} = \mathrm{id}_T$.

Example 2.60. bijections $f, g, h : (\bullet, \bullet, \bullet) \to (\bullet, \bullet, \bullet)$, and $\mathrm{id}_{(\bullet,\bullet,\bullet)}$, $f = gh$, $ff^{-1} = \mathrm{id}_{(\bullet,\bullet,\bullet)}$:



Notation: sometimes $f\left((x_1, x_2)\right)$ is written as $x_1 f x_2$.

Example 2.61. $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $+(1, 3) = 4$ is written as $1 + 3 = 4$.

Similar to product of relations, for two functions

$$\begin{aligned} f : A &\longrightarrow B \\ x &\longmapsto f(x) \end{aligned} \qquad\qquad \begin{aligned} g : C &\longrightarrow D \\ y &\longmapsto g(y), \end{aligned}$$

Definition 2.62. the *product* $f \times g$ is a function

$$\begin{aligned} (f, g) : A \times C &\longrightarrow B \times D \\ (x, y) &\longmapsto (f(x), g(y)) \end{aligned}$$

Chapter 3

Bibliography

[1] H.B. Enderton. *Elements of Set Theory*. Elsevier Science, 1977.

[2] H.B. Enderton. *A Mathematical Introduction to Logic*. Elsevier Science, 2001.