A natural introduction to basic category theory

SemiObServer

2023 at Edinburgh

Contents

Chapter 1

A natural introduction of Categorical fundations

“The art of doing mathematics consists in finding that special case which contains all the germs of generality.”

——David Hilbert

Category theory is powerful in many ways, and the content of Category theory([**?**], [**?**]) is far beyond this text. Here I only introduce the part we will use in TQFT.

I will try to introduce new things such as category and functor inherently by generalizing the concepts readers are already familiar with([**?**]).

Relation and graph are two examples of category, readers may review them in appendix A([**?**]).

1.1 Symmetric monoidal structure

The main goal of this part is to introduce the concept of symmetric monoidal category and monoidal functor in a natural way. The idea is compromised in the following diagram(1.46):

$$\begin{tikzcd}
(P,\leq) \arrow[d, "+\otimes"'] \arrow[drr, bend left, "\leq\Rightarrow f:a\to b"] & & (M,\otimes,I) \arrow[d, "\text{is a}"] \arrow[dll, bend left, "+\leq"] \\
(P,\leq,\otimes,I) \arrow[dr, "\leq\Rightarrow f:a\to b"'] & & \mathbf{C} \arrow[dl, "+\otimes"] \\
& (\mathbf{C},\otimes,I) &
\end{tikzcd} .$$

We begin with some trivial things.

1.1.1 “Less than equal to” and “addition” on natural numbers

Consider two relations $\leq_{\mathbb{N}}$ and addition $+$ on $\mathbb{N}$:

"+" is a function

$$\begin{aligned} + : \mathbb{N} \times \mathbb{N} &\longrightarrow \mathbb{N} \\ (a,b) &\longmapsto a+b \end{aligned}$$

satisfies the following properties:

- *associativity*: $\forall a,b,c \in \mathbb{N}$, $(a+b)+c = a+(b+c)$;

  Example 1.1. $(2+4)+5 = 2+(4+5) = 11$.

- *commutativity*: $\forall p,q \in \mathbb{N}$, $p+q = q+p$;
- *unit object*: an element $0 \in \mathbb{N}$, such that:
  - *left unity*: $\forall x \in \mathbb{N}$, $0+x = x$;
  - *right unity*: $\forall x \in \mathbb{N}$, $x+0 = x$.

    Example 1.2. $0+6 = 6+0 = 6, 3+5 = 5+3 = 8$.

These are the data of a monoid $(\mathbb{N},+,0)$.
$\leq_{\mathbb{N}} \subseteq \mathbb{N} \times \mathbb{N}$ is a relation satisfies the following properties:

- *transitivity*: $\forall a,b,c \in \mathbb{N}$, if $a \leq_{\mathbb{N}} b \leq_{\mathbb{N}} c$, then $a \leq_{\mathbb{N}} c$;

  Example 1.3. $2 \leq_{\mathbb{N}} 4 \leq_{\mathbb{N}} 5$, then $2 \leq_{\mathbb{N}} 5$;

- *reflexivity*: $\forall x \in \mathbb{N}$, $x \leq_{\mathbb{N}} x$.

  Example 1.4. $3 \leq_{\mathbb{N}} 3$.

These are the data of a preorder $(\mathbb{N}, \leq_{\mathbb{N}})$.
"+" preserves the preorder in the following way:

- *monotonicity*: $\forall a_1, b_1, a_2, b_2 \in M$, if $a_1 \leq_{\mathbb{N}} b_1$ and $a_2 \leq_{\mathbb{N}} b_2$, then $a_1 \cdot b_1 \leq_{\mathbb{N}} a_2 \cdot b_2$.

  Example 1.5.
$$\begin{tikzcd} (1,5) \arrow[r, mapsto, "(\leq_{\mathbb{N}},\leq_{\mathbb{N}})"] \arrow[d, mapsto, "+"] & (3,7) \arrow[d, mapsto, "+"] & \in & (\mathbb{N},\mathbb{N}) \arrow[d] \\ 6 \arrow[r, mapsto, "\leq_{\mathbb{N}}"] & 10 & \in & \mathbb{N}. \end{tikzcd}$$

The combination of all the data above is the data of
symmetric monoidal preorder $(\mathbb{N}, \leq_{\mathbb{N}}, +, 0)$:

1.1.2 “Less than equal to” and “multiplication” on positive real numbers

Let’s consider relation $\leq_{\mathbb{R}^+}$ and multiplication $\cdot$ in $\mathbb{R}$:

$\cdot$ is a function

$$\begin{aligned} \cdot : \mathbb{R}^+ \times \mathbb{R}^+ &\longrightarrow \mathbb{R}^+ \\ (a, b) &\longmapsto a \cdot b \end{aligned}$$

satisfies the following properties:

- *associativity*: $\forall a, b, c \in \mathbb{R}^+$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$;

  Example 1.6. $(e^2 \cdot e^4) \cdot e^5 = e^2 \cdot (e^4 \cdot e^5) = e^{11}$.

- *commutativity*: $\forall p, q \in \mathbb{R}^+$, $p \cdot q = q \cdot p$;

  Example 1.7. $1 \cdot e^6 = e^6 \cdot 1 = e^6, e^3 \cdot e^5 = e^5 \cdot e^3 = e^8$.

- *unit object*: an element $1 \in \mathbb{R}^+$, satisfies the following properties:
  - *left unity*: $\forall x \in \mathbb{R}^+$, $1 \cdot x = x$;
  - *right unity*: $\forall x \in \mathbb{R}^+$, $x \cdot 1 = x$.

These are the data of a monoid $(\mathbb{R}^+, \cdot, 1)$.

$\leq_{\mathbb{R}^+} \subseteq \mathbb{R}^+ \times \mathbb{R}^+$ is a relation satisfies the following properties:

- *transitivity*: $\forall a, b, c \in \mathbb{R}^+$, if $a \leq_{\mathbb{R}^+} b \leq_{\mathbb{R}^+} c$, then $a \leq_{\mathbb{R}^+} c$;

  Example 1.8. $e^2 \leq_{\mathbb{N}} e^4 \leq_{\mathbb{N}} e^5$, then $e^2 \leq_{\mathbb{N}} e^5$.

- *reflexivity*: $\forall x \in \mathbb{R}^+$, $x \leq_{\mathbb{R}^+} x$.

  Example 1.9. $e^3 \leq_{\mathbb{R}^+} e^3$.

These are the data of a preorder $(\mathbb{R}^+, \leq_{\mathbb{R}^+})$.

“$\cdot$” preserves the preorder.

- *monotonicity*: $\forall a_1, b_1, a_2, b_2 \in \mathbb{R}^+$, if $a_1 \leq_{\mathbb{R}} b_1$ and $a_2 \leq_{\mathbb{R}} b_2$, then $a_1 \cdot b_1 \leq_{\mathbb{R}} a_2 \cdot b_2$.

  Example 1.10.
$$\begin{tikzcd} (e, e^5) \arrow[r, mapsto, "(\leq_{\mathbb{R}}, \leq_{\mathbb{R}})"] \arrow[d, mapsto, "\cdot"] & (e^3, e^7) \arrow[d, mapsto, "\cdot"] & \in & (\mathbb{R}^+, \mathbb{R}^+) \arrow[d] \\ e^6 \arrow[r, mapsto, "\leq_{\mathbb{R}}"] & e^{10} & \in & \mathbb{R}^+. \end{tikzcd}$$

The combination of all the data above is the data of the symmetric monoidal preorder $(\mathbb{R}^+, \leq_{\mathbb{R}}, \cdot, 0)$.

1.1.3 Monoid and generator

Definition 1.11. A *commutative monoid* $\mathbf{M} = (M, \otimes, I)$ consists of:

- an underlying set M;
- a function $\otimes : M \times M \to M$(in convention, $\forall a, b \in M, \otimes(a, b)$ is written as $a \otimes b$),such that:
  - *associativity*: $\forall a, b, c \in M$, $(a \otimes b) \otimes c = a \otimes (b \otimes c)$;
  - *commutativity*: $\forall p, q \in M$, $p \otimes q = q \otimes p$.
- a unique element $I \in M$, such that:
  - *left unity*: $\forall x \in M$, $I \otimes x = x$;
  - *right unity*: $\forall x \in M$, $x \otimes I = x \otimes I = x$.

Example 1.12. $(\mathbb{N}, +, 0)$; $(\mathbb{R}^+, \times, 1)$.

Similar to subsets of a set, a monoid have submonoids.

For a monoid $\mathbf{M} = (M, \otimes)$,

Definition 1.13. A *submonoid* of $\mathbf{M}$ is a monoid $\mathbf{M'} = (M', \otimes)$ such that $M' \subseteq M$, written as $\mathbf{M'} \subseteq \mathbf{M}$.

Example 1.14. $(\{3n | \forall n \in \mathbb{N}\}, +, 0) \subseteq (\mathbb{N}, +, 0) \subseteq (\mathrm{Z}, +, 0) \subseteq (\mathbb{R}, +, 0)$.

Definition 1.15. A subset $G \subseteq \mathbf{M}$ is a *generating set* if $\forall x \in M$ is a product of some elements in G.

Example 1.16. For $(\{3n | \forall n \in \mathbb{N}\}, +, 0)$, $\{0, 3\}$; for $(\mathbb{N}, +, 0)$, $\{0, 1\}$; for $(\mathrm{Z}, +, 0)$, $\{1, -1\}$.

1.1.4 Preorder and category

Definition 1.17. A *preorder* $\mathbf{P} = (P, \leq)$ consists of:

- an *underlying set* P;
- a *preorder relation* $\leq \subseteq P \times P$, $(a, b) \in \leq$ is written as $a \leq b$ satisfies the following property:
  - *transitivity*: for $\forall a, b, c \in P$, if $a \leq b \leq c$, then $a \leq c$:

$$\begin{tikzcd} a \arrow[r, mapsto, "\leq"] \arrow[rr, bend left, "\leq"] & b \arrow[r, mapsto, "\leq"] & c, \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, mapsto, "r"] \arrow[rr, bend left, "sr"] & b \arrow[r, mapsto, "s"] & c \end{tikzcd};$$

    here $r = (a, b), s = (b, c)$, and $sr = (a, c)$.
    - when $a \leq b$ and $b \leq a$, a and b can be distinct objects.

      Example 1.18. for "married" on (family, family), Alice married Bob, Bob married Alice, but Alice is not Bob;
    - if $a \leq b \leq c \leq d$, we may get $a \leq d$ in 2 ways:

$$
\begin{tikzcd}
a \arrow[r, mapsto, "\leq"] \arrow[rr, bend left, "\leq"] \arrow[rrr, bend left=50, "\leq"] \arrow[rrr, bend right=50, "\leq"'] & b \arrow[r, mapsto, "\leq"] \arrow[rr, bend right, "\leq"'] & c \arrow[r, mapsto, "\leq"] & d,
\end{tikzcd}
\quad
\begin{tikzcd}
a \arrow[r, mapsto, "r"] \arrow[rr, bend left, "sr"] \arrow[rrr, bend left=50, "t(sr)"] \arrow[rrr, bend right=50, "(ts)r"'] & b \arrow[r, mapsto, "s"] \arrow[rr, bend right, "ts"'] & c \arrow[r, "t"] & d.
\end{tikzcd}
$$

upper: $\because a \leq b \leq c; \therefore a \leq c$, $\because a \leq c \leq d; \therefore a \leq d$;
lower: $\because b \leq c \leq d, \therefore b \leq d$, $\because a \leq b \leq d, \therefore a \leq d$;
The right diagram:let $r = (a, b), s = (b, c), t = (c, d)$,
then $sr = (b, d), ts = (a, c)$, finally $t(sr) = (ts)r = (a, d) \in \leq$.

– *reflexivity*: $\forall x \in P$, $x \leq x$(i.e. $=_P \subseteq \leq 0$. Reflexivity leads to the following two properties:
$\forall a, b \in P$, if $a \leq b$(i.e. $r = (a, b) \in \leq$), then:

 * *left unity*: $a \leq a \leq b$, $a \leq b$: $\begin{tikzcd} a \arrow[r, mapsto, "\leq"] \arrow[rr, bend left, "\leq"] & a \arrow[r, mapsto, "\leq"] & b \end{tikzcd}$,

 let $1_a = (a, a)$, then $1_a r = r$: $\begin{tikzcd} a \arrow[r, mapsto, "1_a"] \arrow[rr, bend left, "r"] & a \arrow[r, mapsto, "r"] & b \end{tikzcd}$;

 * *right unity*: $a \leq b \leq b$, $a \leq b$: $\begin{tikzcd} a \arrow[r, mapsto, "\leq"] \arrow[rr, bend left, "\leq"] & b \arrow[r, mapsto, "\leq"] & b \end{tikzcd}$,

 let $1_b = (b, b)$, then $r1_b = r$: $\begin{tikzcd} a \arrow[r, mapsto, "r"] \arrow[rr, bend left, "r"] & b \arrow[r, mapsto, "1_b"] & b \end{tikzcd}$.

If $a \leq b$, we say a is smaller than b, b is larger than a.

Example 1.19. For any set A, $(A, =_A)$; $(\mathbb{N} \times \mathbb{N}, \leq_{\mathbb{N}\times\mathbb{N}})$; (all the sets, $\subseteq$).

Similar to subset, for a preorder $\mathbf{P} = (P, \leq_P)$,

Definition 1.20. preorder $\mathbf{A} = (A, \leq_A)$ is a *subpreorder* of $\mathbf{P}$ if $A \subseteq P$ and $\leq_A \subseteq \leq_P$.

Example 1.21. $(\mathbb{N}, \leq_{\mathbb{N}}) \subseteq (\mathbb{R}^+, \leq_{\mathbb{R}^+})$.

For $\leq$ in $(P, \leq)$, the inverse relation(A.2) is $\geq$, $(P, \geq)$ is also a preorder.

Definition 1.22. A *category* $\mathbf{C} = (\mathrm{Obj}(\mathbf{C}), \mathrm{Mor}(\mathbf{C}))$ consists of:

- *objects*: a collection of objects $\mathrm{Obj}(\mathbf{C})$;
  – similar to underlying set P in preorder.
- *morphisms*: a collection of morphisms $\mathrm{Mor}(\mathbf{C})$. A morphism f is drawn as.

$$f : \mathrm{Obj}(\mathbf{C}) \xrightarrow{f} \mathrm{Obj}(\mathbf{C})$$
$$\mathrm{dom}(f) \xrightarrow{f} \mathrm{cod}(f)$$
$$a \xrightarrow{f} f(a)$$

dom(f)/cod(f) is called the domain/codomain of a morphism f.

- for a morphism $f : a \to b$, $a, b \in \mathrm{Obj}(\mathbf{C})$:
  - f is called a morphism from a to b;
  - similar to $a \le b$ in preorder.
- a category C is *small* if $\mathrm{Obj}(\mathbf{C})$ and $\mathrm{Mor}(\mathbf{C})$ are sets.

Similar to composition of functions, satisfy the following property:

- *composite rule*: $\forall f, g \in \mathrm{Mor}(\mathbf{C})$, if $\mathrm{cod}(f) = \mathrm{dom}(g)$, then we have a specify morphism gf: $\begin{tikzcd} a \arrow[r, "f"] \arrow[rr, bend left, "gf"] & b \arrow[r, "g"] & c \end{tikzcd}$;
  - similar to transitivity in preorder and composition of functions.

  Def. a sequence of n morphisms is *composable* if the codomain of one morphism is the domain of the next morphism.
- *associativity*: $\forall f, g, h \in \mathrm{Mor}(\mathbf{C})$, if (f, g, h) is composable, then $h(gf) = (hg)f = hgf$:

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[rr, bend left, "gf"] \arrow[rrr, bend left=50, "h(gf)"] \arrow[rrr, bend right=50, "(hg)f"'] & b \arrow[r, "g"] \arrow[rr, bend right, "hg"'] & c \arrow[r, "h"] & d \end{tikzcd}$$

  - similar to $a \le b \le c \le d$.

- *identity*: $\forall x \in \mathrm{Obj}(\mathbf{C})$, there's an unique morphism $1_x : x \to x$ called identity morphism,
  - similar to reflexivity, $1_{\mathbf{C}}$ is the collection of all identity morphisms in $\mathbf{C}$.

  such that $\forall f : a \to b \in \mathrm{Mor}(\mathbf{C})$, the following properties are satisfied:

  - *left unity*: $1_a f = f$: $\begin{tikzcd} a \arrow[r, "1_a"] \arrow[rr, bend left, "1_a f = f"] & a \arrow[r, "f"] & b \end{tikzcd}$, similar to $1_a r = r$;
  - *right unity*: $f 1_b = f$: $\begin{tikzcd} a \arrow[r, "f"] \arrow[rr, bend left, "f 1_b = f"] & b \arrow[r, "1_b"] & b \end{tikzcd}$, similar to $r 1_b = r$.

Example 1.23. A graph is a category, objects are vertices, morphisms are arrows, identity morphism is length 0 paths(A.4).

Example 1.24. A preorder $(P, \le)$ is a special category, there is at most 1 morphism between 2 objects. Conversely, for a category $\mathbf{C}$ and $\forall a, b \in \mathrm{Obj}(\mathbf{C})$, if we substitute all the morphisms from a to b by (a, b), then category $\mathbf{C}$ becomes a preorder $\mathbf{C}$.

Example 1.25. $\mathbf{S_n} = (s(n), f_P)$ is a category consists of:

- *object*: the set $s(n) = \{1, 2, \cdots, n\}$ with n elements(, $s(\mathbb{N}) = \{s(n) | \forall n \in \mathbb{N}\}$);
- *morphisms*: a set f_P of all the bijections from $s(n)$ to $s(n)$;
- *identity morphisms*: identity function $\mathrm{id}_{s(n)}$;
- this is the category of permutation group.

Example 1.26. $\mathbf{S}_{\mathbb{N}} = (s(\mathbb{N}), f_{\mathbb{N}})$ is a category consists of:

- *object*: $s(\mathbb{N}) = \{s(n) | \forall n \in \mathbb{N}\}$;
  f_{ij} is a set of all the functions from $s(i)$ to $s(j)$ for some $i, j \in \mathbb{N}$,
- *morphisms*: the union of functions $f_{\mathbb{N}} = \bigcup\limits_{\forall i,j \in \mathbb{N}} f_{ij}$;
- *identity morphisms*: $\mathrm{id}_{s(i)}$ is the identity function of $s(i) \in s(\mathbb{N})$.

Example 1.27. $\mathbf{\underline{Fin}Set} = (\underline{\text{finite}}\ \text{sets}, \text{functions})$ is a category consists of:

- *objects*: all the finite sets;
- *morphisms*: all the functions between finite sets;
- *identity morphisms*: identity functions.

Example 1.28. a monoid $\mathbf{M} = (M, \otimes, I)$ is a category consists of:

- *object*: the only object m;
- *morphisms*: $\forall x \in M$ is a morphism $x : m \to m$. composition of $(a, b) \in M \times M$ is $a \otimes b$;
- *identity morphisms*: I.

Similar to submonoid and subpreorder, for a category $\mathbf{C}$,

Definition 1.29. A category $\mathbf{S}$ is a *subcategory* of $\mathbf{C}$ if $\mathrm{Mor}(S) \subseteq \mathrm{Mor}(C)$.

Example 1.30. $\mathbf{S^n}$ is a subcategory of **FinSet**, **FinSet** is a subcategory of **Set**

Similar to a bijection between sets, isomorphism is a "bijective" morphism.

For a category $\mathbf{C}$, $a, b \in \mathrm{Obj}(\mathbf{C})$,

Definition 1.31. A morphism $f : a \to b$ is an *isomorphism* if $\exists f^{-1} : b \to a$ such that $f^{-1}f = 1_a$ and $ff^{-1} = 1_b$.

$$\begin{tikzcd} a \arrow[loop left, "ff^{-1}=1_a"] \arrow[r, bend left, "f"] & b \arrow[l, bend left, "f^{-1}"] \arrow[loop right, "f^{-1}f=1_b"] \end{tikzcd}$$

- a is *isomorphic* to b(, a and b are equal up to isomorphism(s)), written as $a \cong b$.

Definition 1.32. A *skeleton category* $\mathbf{S}$ is a subcategory of $\mathbf{C}$ such that $\forall c \in \mathrm{Obj}(\mathbf{C})$ is isomorphic to a unique object $s \in \mathrm{Obj}(\mathbf{C})$.

Example 1.33. Every finite set with n elements is isomorphic to $s(n)$(1.25) because there are bijections between them, hence $\mathbf{S}_{\mathbb{N}}$(1.26) is a skeleton category of **FinSet**(1.27).

Similar to generating set of monoid, for a category $\mathbf{C}$,

Definition 1.34. A subset of morphisms $G \subseteq \mathrm{Mor}(\mathbf{C})$ is a *generating set* if $\forall f \in \mathrm{Mor}(\mathbf{C})$ is a composition of some morphisms in G.

Example 1.35. All the permutation $\{P_i | i, i+1 \in \mathbb{N}\}$ between two adjoining elements in $s(n)$ is a generating set of $\mathbf{S_n}$(1.25).

$$P_i(j) = \begin{cases} i+1, & \text{if } j = i \\ i, & \text{if } j = i+1 \\ j, & \text{else} \end{cases}$$

Similar to the inverse of preorder relation(1.1.4), if we inverse all the morphisms in a category $\mathbf{C}$, we get the opposite category $\mathbf{C}^{op}$.

Definition 1.36. The *opposite* category $\mathbf{C}^{op}$ of a category $\mathbf{C}$ is a category consists of:

- *objects*: $\mathrm{Obj}(\mathbf{C^{op}}) = \mathrm{Obj}(\mathbf{C})$;
- *morphisms*: $\forall f : a \to b \in \mathrm{Mor}(\mathbf{C})$, a morphism $f^{op} : b \to a \in \mathrm{Mor}(\mathbf{C}^{op})$.

Example 1.37. $\mathbf{S_n}$ is the opposite category of $\mathbf{S_n}$.

Similar to the cartesian product of two sets,

Definition 1.38. The *product* $\mathbf{C} \times \mathbf{D}$ of two categories $\mathbf{C}$ and $\mathbf{D}$ consists of:

- *objects*: $\mathrm{Obj}(\mathbf{C} \times \mathbf{D}) = \mathrm{Obj}(\mathbf{C}) \times \mathrm{Obj}(\mathbf{D})$,

  Example 1.39. $(c, d) \in (\mathrm{Obj}(\mathbf{C}), \mathrm{Obj}(\mathbf{D}))$.

- *morphisms*: $\mathrm{Mor}(\mathbf{C} \times \mathbf{D}) = \mathrm{Mor}(\mathbf{C}) \times \mathrm{Mor}(\mathbf{D})$,

  Example 1.40. $(f, g) \in (\mathrm{Mor}(\mathbf{C}), \mathrm{Mor}(\mathbf{D}))$.

1.1.5 Monotone function and functor

Let's consider a function f from preorder $(\mathbb{N}, \leq_{\mathbb{N}})$ to preorder $(\mathbb{R}^+, \leq_{\mathbb{R}^+})$:

$$\begin{aligned} f : \mathbb{N} &\longrightarrow \mathbb{R}^+ \\ x &\longmapsto e^x \end{aligned}$$

- *preservation of preorder*: If $a \leq_{\mathbb{N}} b$ in $\mathbb{N}$, then $e^a \leq_{\mathbb{R}^+} e^b$ in $\mathbb{R}^+$.

Example 1.41. $2 \leq_{\mathbb{N}} 3$, $e^2 \leq_{\mathbb{R}^+} e^3$:

$$\begin{tikzcd}
2 \arrow[r, mapsto, "(a,b)\in\leq_P"] \arrow[d, "f"] & 3 \arrow[d, "f"] & (\mathbb{N}, \leq_{\mathbb{N}}) \arrow[d, "f"] \\
e^2 \arrow[r, mapsto, "(e^2,e^3)\in\leq_Q"'] & e^3 & (\mathbb{R}^+, \leq_{\mathbb{R}^+}) .
\end{tikzcd}$$

f also "takes" $r \in \leq_{\mathbb{N}}$ to $s \in \leq_{\mathbb{N}}$.

Example 1.42. $(f, f)(2, 3) = (e^2, e^3) \in \leq_{\mathbb{R}^+}$, f is called a monotone function:

For two preorders $(P, \leq_P)$ and $(Q, \leq_Q)$,

Definition 1.43. A function $f : P \to Q$ is *monotone* if the following property is satisfied:

- *monotonicity*: $a \leq_P b$ then $f(a) \leq_Q f(b)$ for all elements $a, b \in P$, or equivalently
- $\forall (a, b) \in \leq_P$, $(f, f)(a, b) = (f(a), f(b)) \in \leq_Q$:

$$\begin{tikzcd}
a \arrow[r, mapsto, "(a,b)\in\leq_P"] \arrow[d, "f"] & b \arrow[d, "f"] & (P, \leq_P) \arrow[d, "f"] \\
f(a) \arrow[r, mapsto, "(f(a),f(b))\in\leq_Q"'] & f(b) & (Q, \leq_Q) .
\end{tikzcd}$$

A monotone function f naturally preserves transitivity and reflexivity:

- *transitivity*: if $a \leq_P b \leq_P c$, then $f(a) \leq_Q f(b) \leq_Q f(c)$:

$$\begin{tikzcd}
a \arrow[r, mapsto, "\leq_P"] \arrow[d, "f"] & b \arrow[r, mapsto, "\leq_P"] \arrow[d, "f"] & c \arrow[d, "f"] & (P, \leq_P) \arrow[d, "f"] \\
f(a) \arrow[r, mapsto, "\leq_Q"] & f(b) \arrow[r, mapsto, "\leq_Q"] & f(c) & (Q, \leq_Q) ;
\end{tikzcd}$$

- *reflexivity*: $x \leq_P x$ and $f(x) \leq_Q f(x)$ always hold:

$$\begin{tikzcd}
x \arrow[loop left, "\leq_P"] \arrow[d, "f"] & \in & P \arrow[d, "f"] \\
f(x) \arrow[loop left, "\leq_Q"] & \in & Q .
\end{tikzcd}$$

If we generalize these into category, we can define a "function" from one category to another called functor.

Definition 1.44. A *functor* $F : \mathbf{C} \to \mathbf{D}$ consists of:

- $\forall a \in \mathrm{Obj}(\mathbf{C})$, an object $F(a) \in \mathrm{Obj}(\mathbf{D})$;
  - similar to monotone function(1.43) $f : P \to Q$;

- $\forall f \in \mathrm{Mor}(\mathbf{C})$, an object $F(f) \in \mathrm{Mor}(\mathbf{D})$;
  - similar to $(f, f) : \leq_P \to \leq_Q$.

  such that $F(\mathbf{C}) = (F(\mathrm{Obj}(\mathbf{C})), F(\mathrm{Mor}(\mathbf{C})))$ is a subcategory of D satisfies the following properties:

- *compatibility*: $\forall f \in \mathrm{Mor(C)}$ and $F(f) \in \mathrm{Mor(D)}$,

  $\mathrm{dom}(F(f)) = F(\mathrm{dom}(f)) \in \mathrm{Obj(D)}, \mathrm{cod}(F(f)) = F(\mathrm{cod}(f)) \in \mathrm{Obj(D)}$;

$$\begin{tikzcd}
a \arrow[rr, "f"] \arrow[d, "F"] & {} \arrow[d, "F"] & b \arrow[d, "F"] & \mathbf{C} \arrow[d, "F"] \\
F(a) \arrow[rr, "F(f)"'] & {} & F(b) & \mathbf{D}
\end{tikzcd}$$

  - similar to monotonicity.

- *functoriality*:
  - if (f, g) is composable in $\mathbf{C}$, then $(F(f), F(g))$ is composable in $\mathbf{D}$, such that $F(gf) = F(g)F(f)$:

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[rr, bend left, "gf"] \arrow[d, "F"] & b \arrow[r, "g"] \arrow[d, "F"] & c \arrow[d, "F"] & \mathbf{C} \arrow[d] \\
F(a) \arrow[r, "F(f)"] \arrow[rr, bend right, "F(gf)=F(g)F(f)"'] & F(b) \arrow[r, "F(g)"] & F(c) & \mathbf{D};
\end{tikzcd}$$

    * similar to the preservation of transitivity.

  - for each object $x \in \mathrm{C}, F(1_x) = 1_{F(x)}$:

$$\begin{tikzcd}
x \arrow[loop left, "1_x"] \arrow[d, "F"] & \mathbf{C} \arrow[d] \\
F(x) \arrow[loop left, "1_{F(x)}"] & \mathbf{D}.
\end{tikzcd}$$

    * similar to the preservation of reflexivity.

Similar to identity function,

Definition 1.45. $\mathrm{id}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$ is the *identity functor* such that:

- $\forall a \in \mathrm{Obj(C)}$, $\mathrm{id}_{\mathbf{C}}(a) = a$;
- $\forall a \in \mathrm{Obj(C)}$, $\mathrm{id}_{\mathbf{C}}(a) = a$ and $\forall f \in \mathrm{Mor(C)}$, $\mathrm{id}_{\mathbf{C}}(f) = f$.

For a graph G and a category $\mathbf{C}$,

Definition 1.46. A *diagram* in $\mathbf{C}$ is a functor $D : G \to \mathbf{C}$.

Definition 1.47. A diagram D *commutes* if for any parallel paths p and p' in G, $D(f) = D(f')$.

The diagrams in this text are commutative in default.

Similar to **Set**(1.27), if we view functors as a generalization of functions, we may define the category of categories:a

Example 1.48. **Cat** is a category, Obj(**Cat**) is the collection of all categories, Mor(**Cat**) is the collection of all the functors, identity morphisms are identity functors.

With the product of categories and functors, we may generalize symmetric monoidal preorder into strict symmetric monoidal category, a core concept in TQFT.

1.1.6 Monoidal preorder and category

A function that maps elements of monoid A to elements of another monoid B preserving the multiplicative structure is called a homomorphism between monoids.

If we combine the data of "$\leq$" and "$\otimes$" together, we get a new structure $(PM, \leq, I, \otimes)$, where commutativity is replaced by symmetry.

Example 1.49. $(\mathbb{N}, \leq_{\mathbb{N}}, 0, +)$, $(\mathbb{N}, \leq_{\mathbb{R}^+}, 0, \times)$.

Definition 1.50. A *symmetric monoidal preorder* $\mathbf{PM} = (PM, \leq, \otimes, I)$ consists of:

- a preorder $(PM, \leq)$;
- a monotone function $\otimes : PM \times PM \to PM$, satisfies the following properties:
  - *associativity*: $\forall a, b, c \in PM$, $(a \otimes b) \otimes c = a \otimes (b \otimes c)$;
  - *symmetry*: $\forall p, q \in M$, $p \otimes q = q \otimes p$.
- a unit element $I \in PM$, such that $\forall x \in PM$, the following properties are satisfied:
  - *left unity*: $I \otimes x = x$;
  - *right unity*: $x \otimes I = x$.

These data agree with the ones in $(\mathbb{N}, \leq, +, 0)$.

In order to generalize the underlying set PM into a category, we need to generalize the function $\otimes : PM \times PM \to PM$ to a special functor $\otimes : \mathrm{C} \times \mathrm{C} \to \mathrm{C}$. Before that, we need to define what is the product of categories and what is a functor between categories.

For a category $\mathbf{C}$,

Definition 1.51. A *strict symmetric monoidal category* $(\mathbf{C}, \otimes, I, \sigma)$ consists of:

- a category $\mathbf{C}$;
- *monoidal product*: a functor $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$,

  functoriality: for two morphisms $f_1 : a_1 \to b_1$ and $f_2 : a_2 \to b_2$, then there exist a morphism $f_1 \otimes f_2 : a_1 \otimes b_1 \to a_2 \otimes b_2$:

$$\begin{tikzcd}
(a_1, a_2) \arrow[r, "(f_1, f_2)"] \arrow[d, "\otimes"] & (b_1, b_2) \arrow[d, "\otimes"] & (\mathbf{C}, \mathbf{C}) \arrow[d, "\otimes"] \\
a_1 \otimes b_2 \arrow[r, "f_1 \otimes f_2"'] & b_1 \otimes b_2 & \mathbf{C};
\end{tikzcd}$$

satisfies the following properties:

- *associativity*: $\forall a, b, c \in \mathrm{Obj}(\mathbf{C})$, $(a \otimes b) \otimes c = a \otimes (b \otimes c)$;
- *braid*: $\forall a, b \in \mathrm{Obj}(\mathbf{C})$, $\exists \sigma_{a,b} : a \otimes b \approxeq b \otimes a \in \mathrm{Mor}(\mathbf{C})$, $\underline{\text{such that } \sigma_{a,b}\sigma_{b,a} = 1_{a \otimes b}}$.

- *unit object*: an object $I \in \mathrm{Obj}(\mathbf{C})$, such that $\forall x \in \mathrm{Obj}(\mathbf{C})$, the following properties are satisfied:
  - *left unity*: $I \otimes x = x$;
  - *right unity*: $x \otimes I = x$.

Example 1.52. $(\mathbf{Set}, \bigsqcup, \varnothing), (\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk)$.

1.1.7 Monoidal monotone and monoidal functor

f also takes $+$ to $\cdot$, we may write it as $f(+) = \cdot$:

- *preservation of multiplication*: $\forall a, b \in \mathbb{N}, f(a + b) = f(a) \cdot f(b)$.

Example 1.53.
$$\begin{tikzcd}
(1, 3) \arrow[r, mapsto, "+"] \arrow[d, mapsto, "(f,f)"] & 4 \arrow[d, mapsto, "f"] \\
(e, e^3) \arrow[r, mapsto, "\cdot"'] & e^4;
\end{tikzcd}$$

These are data of monoidal monotone.

For two symmetric monoidal preorders $\mathbf{P} = (P, \leq_P, \otimes_P, I_P)$ and $\mathbf{Q} = (Q, \leq_Q, \otimes_Q, I_Q)$,

Definition 1.54. A *monoidal monotone* $f : \mathbf{P} \to \mathbf{Q}$ is a monotone function $f : P \to Q$, satisfies the following conditions:

- *preservation of multiplication*: $\forall a, b \in P, f(a) \otimes_Q f(b) \leq_Q f(a \otimes_P b)$:

$$\begin{tikzcd}
(a, b) \arrow[r, mapsto, "\otimes_P"] \arrow[d, mapsto, "(f,f)"] & a \otimes_P b \arrow[d, mapsto, "f"] \\
(f(a), f(b)) \arrow[r, mapsto, "\otimes_Q"'] & f(a) \otimes_Q f(b) \leq_Q f(a \otimes_P b);
\end{tikzcd}$$

- *preservation of unit element*: $f(I_P) = I_Q$;

Example 1.55.

$$\begin{aligned} f : \mathbb{N} &\longrightarrow \mathbb{N} \\ x &\longmapsto x^2 \end{aligned}$$

is a monoidal monotone from $(\mathbb{N}, \leq_{\mathbb{N}}, +, 0)$ to $(\mathbb{N}, \leq_{\mathbb{N}}, +, 0)$, $2^2 + 3^2 \leq_{\mathbb{N}} (2+3)^2$.

Then we substitute preorders by categories.

For two strict symmetric monoidal categories $(\mathbf{C}, \otimes_{\mathbf{C}}, I_{\mathbf{C}})$ and $(\mathbf{D}, \otimes_{\mathbf{D}}, I_{\mathbf{D}})$,

Definition 1.56. A *symmetric monoidal functor* $F : (\mathbf{C}, \otimes_{\mathbf{C}}, I_{\mathbf{C}}) \to (\mathbf{D}, \otimes_{\mathbf{D}}, I_{\mathbf{D}})$ is a functor satisfies the following conditions:

- *preservation of multiplication*: $\forall a, b \in \mathrm{Obj}(\mathbf{C})$, a morphism
  $f : F(a) \otimes_{\mathbf{D}} F(b) \to F(a \otimes_{\mathbf{C}} b)$:

$$\begin{tikzcd}
(a,b) \arrow[rr, "\otimes_{\mathbf{C}}"] \arrow[d, "(F,F)"] & & a \otimes_{\mathbf{C}} b \arrow[d, "F"] \\
(F(a), F(b)) \arrow[r, "\otimes_{\mathbf{D}}"'] & F(a) \otimes_{\mathbf{D}} F(b) \arrow[r] & F(a \otimes_{\mathbf{C}} b);
\end{tikzcd}$$

- *preservation of unit element*: $F(I_{\mathbf{C}}) = I_{\mathbf{D}}$;

1.2 From extremum to universal property

1.2.1 Largest and smallest, terminal and initial

A subset of $\mathbb{R}$ has infimum(the largest element of lower bounds) and supremum(the smallest element of upper bounds).

For $A \subseteq \mathbb{R}$,

Definition 1.57. The *lower class* is a set $low(A) \subseteq \mathbb{R}$ such that $\forall a \in A$ and $x \in L_b$, $a \leq_{\mathbb{R}^+} x$.

Definition 1.58. The *infimum* is an element $\inf(A) \in \mathbb{R}$ such that $\forall x \in Low(A)$, $x \leq_{\mathbb{R}^+} \inf(A)$.

Example 1.59. For $A = [1, 4)$, $low(A) = (-\infty, 1]$, $\inf(A) = 1$.

If we substitute $\leq_{\mathbb{R}^+}$ with the inverse relation $\geq_{\mathbb{R}^+}$, we get the upper bound and supremum.

Definition 1.60. An *upper class* is a set $upp(A) \subseteq \mathbb{R}$ such that $\forall a \in A$ and $x \in upp(A)$, $x \leq_{\mathbb{R}^+} a$ $(a \geq_{\mathbb{R}^+} x)$.

Definition 1.61. A *supremum* is an element $\sup(A) \in \mathbb{R}$ such that $\forall x \in upp(A)$, $\sup(A) \leq_{\mathbb{R}^+} x$ $(x \geq_{\mathbb{R}^+} \sup(A))$.

Example 1.62. For $A = [1, 4)$, $upp(A) = [4, \infty)$, $\sup(A) = 4$.

We may generalize the definition of "largest/smallest", and then "infimum/supremum" into preorders and categories.

For a preorder $(P, \leq)$,

Definition 1.63. An element $\max(P)$ is *largest* if $\forall a \in P$, $a \leq \max(P)$.

Definition 1.64. An element $\min(P)$ is *smallest* if $\forall a \in P$, $\max(P) \leq a$.

For a category $\mathbf{C}$,

Definition 1.65. An object $tmn(\mathbf{C}) \in \text{Obj}(\mathbf{C})$ is *terminal* if $\forall c \in \text{Obj}(\mathbf{C})$, $\exists! f : c \to tmn(\mathbf{C})$.

- "if proposition $A = true$, then $\exists!$ B" is the *universal property*;
- the "largest element" of $\mathbf{C}$.

Definition 1.66. An object $ini(\mathbf{C}) \in \text{Obj}(\mathbf{C})$ is *initial* if $\forall c \in \text{Obj}(\mathbf{C})$, $\exists! f : ini(\mathbf{C}) \to c$.

- the "smallest element" of $\mathbf{C}$.

Example 1.67. $\varnothing$ is the initial object of **Set**; $\forall a$,$\{a\}$ is a terminal object of **Set**.

Example 1.68. An initial/terminal object of $\mathbf{C}$ is the terminal/initial object of $\mathbf{C}^{op}$.

A terminal/initial object may not exist.

Example 1.69. A category $(\{a, b\}, \{1_a, 1_b\})$ has no morphism $f : a \to b$ or $g : b \to a$, hence it's impossible to tell which object is "smaller".

1.2.2 Meet, join, and cone

For a preorder $(P, \leq)$, and a subset $A \subseteq P$:

Definition 1.70. The *lower class* of A is a set $low(A) = \{\forall l \in P | \forall a \in A, l \leq a\}$.

Definition 1.71. A *meet* of A is $m(A) = max(low(A))$, a "largest" element of the lower class.

Example 1.72. $\{2\}$ is the meet of $\{\{1, 2\}, \{0, 2\}\} = \{1, 2\} \bigcap \{0, 2\} \subset X$, i.e. the intersection of $\{1, 2\}$ and $\{0, 2\}$.

Definition 1.73. The *upper class* of A is a set $upp(A) = \{\forall u \in P | \forall a \in A, a \leq u(u \geq_{\mathbb{R}^+} a)\}$.

Definition 1.74. A *join* of A is $j(A) = min(upp(A))$, a "smallest" element of the upper class.

Example 1.75. For a preorder $(\mathbb{N} \times \mathbb{N}, (\leq_{\mathbb{N}}, \leq_N))$ and two elements $(3, 5)$, $(7, 4) \in \mathbb{N} \times \mathbb{N}$, , the position of (a, b) match with the XOY coordinate system.

The lower class is $\{\forall(m, n) | m, n \in \mathbb{N}, m \leq 3, n \leq 4\}$, the meet is $(3, 5)$;

The upper class $upp(P) = \{\forall(m, n) | m, n \in \mathbb{N}, 3 \leq m, 4 \leq n\}$, the join is $(7, 4)$.

$$\begin{tikzcd}
& & & (9,8) \\
& (3,5) \arrow[r, mapsto, "\leq"] \arrow[urr, mapsto, bend left, "\leq"] & (7,5) \arrow[ur, mapsto, "\leq"] & \\
& (3,4) \arrow[u, mapsto, "\leq"] \arrow[r, mapsto, "\leq"] & (7,4) \arrow[u, mapsto, "\leq"] \arrow[uur, mapsto, bend right, "\leq"'] & \\
(2,4) \arrow[ur, mapsto, "\leq"] \arrow[uur, mapsto, bend left, "\leq"] \arrow[urr, mapsto, bend right, "\leq"'] & & &
\end{tikzcd}$$

Example 1.76. For a set $AB = \{A, B\}$ and preorder $(\text{sets}, \subseteq)$, arrows represent $\subseteq$.

The lower class is $\{\forall S | S \subseteq A \cap B\}$, the meet is $A \cap B$;

The upper class is $\{S \forall | S, A \cup B \subseteq S\}$, the join is $A \cup B$.

$$\begin{tikzcd}
& & & A \cup B \cup \cdots \\
& A \arrow[r, mapsto, "\subseteq"] \arrow[urr, mapsto, bend left, "\subseteq"] & A \cup B \arrow[ur, mapsto, "\subseteq"] & \\
& A \cap B \arrow[u, mapsto, "\subseteq"] \arrow[r, mapsto, "\subseteq"] & B \arrow[u, mapsto, "\subseteq"] \arrow[uur, mapsto, bend right, "\subseteq"'] & \\
\{\} \arrow[ur, mapsto, "\subseteq"] \arrow[uur, mapsto, bend left, "\subseteq"] \arrow[urr, mapsto, bend right, "\subseteq"'] & & &
\end{tikzcd}$$

To generalize meet into categories, let's state lower bound and meet differently.

For a preorder $(P, \leq)$, a subset $A \in P$, and $\leq_A = \{\forall (a, b) \in \leq | a, b \in A\}$

Definition 1.77. A *cone* $(x, C(x))$ of A consists of:

- an element $x \in P$;
- a relation $C(x) = \{\forall (x, a) | a \in A\}$, such that $C(x) \in \leq$.
  - the cone and A forms a preorder $A_x = (x \cap A, C(x) \cap \leq_A)$. $\forall a_x \in A_x, x \leq a_x$.

    Example 1.78. $((2,4), C((2,4)) = ((2,4), \{((2,4),(3,5)), ((2,4),(7,3))\}),$
    $((3,4), C((3,4)) = ((3,4), \{((3,4),(3,5)), ((3,4),(7,3))\}).$

$$\begin{tikzcd}
(3,5) & & (3,5) & \\
(2,4) \arrow[u, mapsto, "((2,4),(3,5))"'] \arrow[r, mapsto, "((2,4),(7,3))"'] & (7,3), & (3,4) \arrow[u, mapsto, "((3,4),(3,5))"'] \arrow[r, mapsto, "((3,4),(7,3))"'] & (7,3).
\end{tikzcd}$$

Hence $(x, C(x))$ is a cone if and only if $x \in low(A)$.

The collection of all the cones of A forms a set $Cone(A)$.

For two cones $(x, C(x))$ and $(y, C(y))$, if $x \leq y$, then $\forall a \in A$, $x \leq y \leq a$, hence $x \leq a$.

We may also interpret $x \leq y$ as a preorder relation $\leq_{cone} \subseteq Cone(A) \times Cone(A)$

$$(x, C(x)) \xmapsto{(x,y)} (y, C(y)) \ .$$

We may construct a preorder $\mathbf{Cone}(A) = (Cone(A), \leq_{cone})$.

Example 1.79. $((2,4), C((2,4))) \xmapsto{((2,4),(3,4))} ((3,4), C((3,4)))$,

$$\begin{tikzcd}
 & (3,5) & \\
 & (3,4) \arrow[u, mapsto, "{((3,4),(3,5))}"'] \arrow[r, mapsto, "{((3,4),(7,3))}"'] & (7,3) \\
(2,4) \arrow[uur, mapsto, bend left, "{((2,4),(3,5))}"] \arrow[ur, mapsto, "{((2,4),(3,4))}"'] \arrow[urr, mapsto, bend right, "{((2,4),(7,3))}"'] & &
\end{tikzcd} \ .$$

A largest element of $\mathbf{Cone}(A)$ is $max(Cone(A)) = (m(A), C\left(m(A)\right))$

Definition 1.80. A *meet* of A is an element $m(A)$ of $max(Cone(A)) = (m(A), C\left(m(A)\right))$.

1.2.3 Limit and product

To generalize meet into limit of functors, we need to generalize the concepts we used in the following way:

- "subset" becomes $F(\mathrm{Obj}(\mathbf{I}))$, a subcategory of a category $\mathbf{D}$ given by a functor $F : \mathbf{I} \to \mathbf{D}$;
- "smaller" (a, b) becomes a morphism $c(a, b)$;

  For cone $(x, C(x))$,
- $C(x)$ becomes a collection of morphisms (compatible with morphisms in $F(\mathbf{I})$);
- "meet" becomes limit, "largest element" becomes terminal object.

1.Use functor to Define a "subpreorder":

For a small category $\mathbf{I}$ called *index category* and a functor $F : \mathbf{I} \to \mathbf{D}$, $F(\mathbf{I})$ is a subcategory of $\mathbf{D}$;

2. Generalize cone(equip a object x with a selection of morphisms $x \to F(\mathbf{I})$):

Definition 1.81. A *cone* $(x, C(x))$ of F consists of:

- an object $x \in \mathbf{D}$, called the base object;
- a collection of morphisms $C(x)$ consists of $c(x, F(a))$:

$\forall a \in \mathrm{Obj}(\mathbf{I})$, a morphism $c(x, F(a)) : x \to F(a) \in \mathrm{Mor}(\mathbf{D})$.

- $c(x, F(a))$ is similar to (x, a), x is "smaller" than all the objects in $F(\mathrm{Obj}(\mathbf{I}))$.

such that

- $\forall f : a \to b \in \mathrm{Mor}(\mathbf{I})$ and $F(f) : F(a) \to F(b) \in \mathrm{Mor}(\mathbf{D})$, $F(f)c(x, F(a)) = c(x, F(b))$:

$$\begin{tikzcd} x \arrow[r, "{c(x,F(a))}"] \arrow[rr, bend left, "{F(f)c(x,F(a))=c(x,F(b))}"] & F(a) \arrow[r, "F(f)"] & F(b). \end{tikzcd}$$

  - "smaller" is produced uniformly and "$\leq$" in "subset" $F(\mathbf{I})$ preserves "transitivity";
  - $(x, C(x))$ and $F(\mathbf{I})$ forms a category $\mathbf{F_{cone}} = (x \cup \mathrm{Obj}\,(F(\mathbf{I}))\,, C(x) \cup \mathrm{Mor}\,(F(\mathbf{I})))$, x is a initial object of $\mathbf{F_{cone}}$, a "lower bound" of $F(\mathbf{I})$;
  - different collection of morphisms $C(x), C'(x), C''(x), \cdots$ define different cones, hence an object x is "decomposed" into many cones, each of them equips with a unique collection of morphisms.

Example 1.82. for index category $\mathbf{I} = (\{a_1, a_2\}, \{f, 1_{a_1}, 1_{a_2}\})$,
$fc_1 = c_2$, $fc_1' = c_2'$, $fc_1'' = c_2''$, a **NON-commutative** diagram

$$\begin{tikzcd} a_1 \arrow[rr, bend left, "f"] & & a_2 \\ & x \arrow[ul, bend right, "c_1"'] \arrow[ul, "c_1'"] \arrow[ul, bend left, "c_2''"] \arrow[ur, bend left, "c_2"] \arrow[ur, "c_2'"'] \arrow[ur, bend right, "c_2''"'] \arrow[dl, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & \\ c(x, \{c_1, c_2\}) & c(x, \{c_1', c_2'\}) & c(x, \{c_1'', c_2''\}) \end{tikzcd}$$

3. Generalize **Cone**(A)(cones form a "lower class"):

Definition 1.83. Cone(F) is a category consists of:

- *objects*: the cones of F; For a morphism $f : x \to y \in \mathrm{Mor}(\mathbf{D})$,
- *morphisms*: a morphism $f_{cone} : (x, C(x)) \to (y, C(y)) \in \mathrm{Mor}(\mathbf{Cone}(F))$ such that $\forall s \in \mathrm{Obj}(\mathbf{I})$, $c(y, F(a))f = c(x, F(a))$:

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[rr, bend left, "{c(y,F(a))f=c(x,F(a))}"] & y \arrow[r, "{c(y,F(a))}"] & F(a)\ . \end{tikzcd}$$

  - similar to $\leq_{cone}$.

4. Fetch the "maximal" of **Cone**(F):

For a functor $F : \mathbf{I} \to \mathbf{D}$, the terminal object of **Cone**(F) is $\big(\lim(F), c_{\lim(F)}\big)$,

Definition 1.84. The *limit* of F is $\lim(F)$.

Therefore, meet is a special limit where functor F is replaced by some embedding of subset, and the morphisms in $\mathbf{D}$ are replaced by $\leq$ in P.

Product, an important limit, is another example.

For a category $\mathbf{P} = (\{a, b\}, \{1_a, 1_b\})$, a category $\mathbf{D}$, and a functor $F : \mathbf{P} \to \mathbf{D}$,

Definition 1.85. The *product* of $F(a), F(b) \in \mathrm{Obj}(\mathbf{D})$ is $\lim(F)$.

$\forall c(x, F(a)), c(x, F(b) \in \mathbf{D})$, $1_a c(x, F(a)) = c(x, F(a))$ and $1_b c(x, F(b)) = c(x, F(b))$,

hence $\forall x \in \mathrm{Obj}(\mathbf{D})$, $(x, \{c(x, F(a)), c(x, F(b))\}$ forms a cone in $Cone(F)$:
$$\begin{tikzcd} F(a) \\ X \arrow[u, "c(X,F(a))"'] \arrow[r, "c(X,F(b))"'] & F(b) \end{tikzcd}.$$

Example 1.86. product of two sets A and B: a set X and any two functions $c(X, A)$, $c(X, B)$ forms a cone $(X, \{c(X, A), c(X, B)\})$:
$$\begin{tikzcd} A \\ X \arrow[u, "c(X,A)"'] \arrow[r, "c(X,B)"'] & B \end{tikzcd}.$$

Definition 1.87. The *product* of two sets (A, B) is the the set $A \times B$ such that
for a set X with any two functions $c(X, A) : S \to A$ and $c(X, B) : X \to B$,
$\exists! T : S \to A \otimes B$ so that $\pi_A T = c(X, A)$ and $\pi_B T = c(X, B)$:

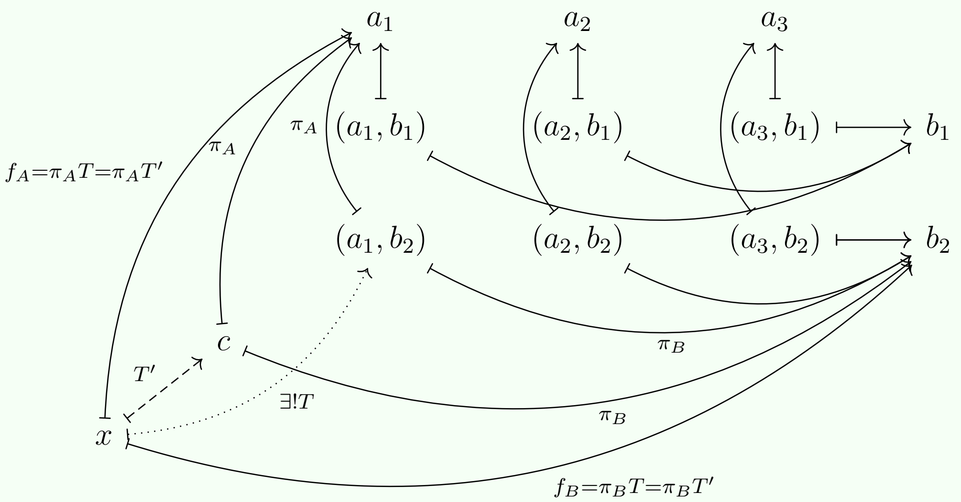
$$\begin{tikzcd} & A \\ & A \otimes B \arrow[u, "\pi_A"'] \arrow[r, "\pi_B"'] & B \\ X \arrow[uur, bend left, "c(X,A)"] \arrow[ur, dotted, "\exists! T"] \arrow[urr, bend right, "c(X,B)"'] \end{tikzcd} \qquad \begin{tikzcd} & a \\ & (a,b) \arrow[u, mapsto] \arrow[r, mapsto] & b \\ x \arrow[uur, bend left, mapsto] \arrow[ur, dotted, mapsto, "\exists!"] \arrow[urr, bend right, mapsto] \end{tikzcd}$$

The definition of Cartesian product $\times_c$ and product $\times$ are compatible, i.e. $A \times_c B = A \times B$. We will use a trivial example to explain the following statements:

- larger sets have extra elements that will kill the uniqueness.
- smaller sets may fall to reach some value.

Proof. $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2\}$, $A \times_c B = (A, B)$.

Example 1.88.



- if there are extra elements in $A \times_c B$ such as c, its value must be the same with an element in (A, B).

  Example 1.89. if $f_A(x) = a_1$, $f_B(x) = b_2$, there are two ways to get them:

  - $T(x) = (a_1, b_2)$, then $\pi_A(a_1, b_2) = a_1, \pi_B(a_1, b_2) = b_2$;
  - $T'(x) = c$, then $\pi_A(c) = a_1, \pi_B(c) = b_2$.

  This breaks the uniqueness of L, hence $A \times B \bigcup \{c\}$ is not a terminal object;

- if we delete an element in $A \times_c B$, the remains fail to reach the value it represent.

  Example 1.90. delete (a_1, b_2), then there's no way to get to $h_A(x) = a_1$, $h_B(x) = b_2$, hence $A \times B$ without (a_1, b_2) is not a terminal object.

Hence $(A \times_c B, C(A \times B))$ is indeed a terminal object of $\mathbf{Cone}(F)$,
$A \times B = A \times_c B$ is a limit. □

1.2.4 Colimit and coproduct

On the opposite, we may define colimit(i.e., the limit of $F : \mathbf{I}^{op} \to \mathbf{D}^{op}$).

1.For a small category $\mathbf{I}$ such that $\mathrm{Obj}(\mathbf{I})$ is a set and functor $F : \mathbf{I} \to \mathbf{D}$, we get a subcategory $F(\mathbf{I})$ of $\mathbf{D}$;

2. "upper bound":

Definition 1.91. A *cocone* $(x, Co(x))$ of F consists of:

- an object $x \in \mathbf{D}$, called the base object;
- a collection of morphisms $C(x)$ consists of $c(F(a), x)$:
  $\forall a \in \mathrm{Obj}(\mathbf{I})$, $\exists! c(F(a), x) : F(a) \to x \in \mathrm{Mor}(\mathbf{D})$.

3. “upper class”

Definition 1.92. $\mathbf{Cocone}(F)$ is a category consists of:

- *objects*: the cocones of F;

  For a morphism $f : x \to y \in \text{Mor}(\mathbf{D})$,

- *morphisms*: a morphism $f_{coc} : (x, Co(x)) \to (y, Co(y)) \in \text{Mor}(\mathbf{Cone}(F))$ such that $\forall a \in \text{Obj}(\mathbf{I})$, $f co(F(a), x) = co(F(a), y)$:

$$\begin{tikzcd} F(a) \arrow[r, "c(F(a){,}x)"] \arrow[rr, bend left, "fco(F(a){,}x)=co(F(a){,}y)"] & x \arrow[r, "f"] & y \end{tikzcd} .$$

4. Fetch the “minimal” of $\mathbf{Cocone}(F)$:

For a functor $F : \mathbf{I} \to \mathbf{D}$, an initial object of $\mathbf{Cocone}(F)$ is $\big(\text{col}(F), Co_{\text{col}(F)}\big)$,

Definition 1.93. The *colimit* of F is $\text{col}(F)$.

Therefore, join is a special colimit where functor F is replaced by some embedding of subset, and the morphisms in $\mathbf{D}$ are replaced by $\leq$ in P.

Coproduct, an important colimit, is another example.

For a category $\mathbf{P} = (\{a, b\}, \{1_a, 1_b\})$, a category $\mathbf{D}$, and a functor $F : \mathbf{P} \to \mathbf{D}$,

Definition 1.94. The *coproduct* of $F(a), F(b) \in \text{Obj}(\mathbf{D})$ is $\text{col}(F)$.

Example 1.95. coproduct of two sets A and B: a set Y and any two functions $c(A, Y)$, $c(B, Y)$ forms a cocone:$(Y, \{co(A, Y), co(B, Y)\})$

$$\begin{tikzcd} A \arrow[r, "co(A{,}Y)"] & Y \\ & B \arrow[u, "co(B{,}Y)"'] \end{tikzcd} .$$

Definition 1.96. The *coproduct* of two sets A, B is the set $A + B$ such that $\forall S$ and two functions $co(A, S) : A \to S$ and $co(B, S) : B \to S$, $\exists! T : A + B \to S$ such that $T\iota_A = co(A, S)$ and $T\iota_B = co(B, S)$.

Example 1.97. $y_a = T(a') = co(A, S)(a), y_b = T(b'') = co(B, S)(b)$:

$$\begin{tikzcd} & & Y \\ A \arrow[r, "\iota_A"] \arrow[urr, bend left, "co(A{,}Y)"] & A+B \arrow[ur, dotted, "\exists!T"] & \\ & B \arrow[u, "\iota_B"'] \arrow[uur, bend right, "co(B{,}Y)"'] & \end{tikzcd} \qquad \begin{tikzcd} & & y_a/y_b \\ a \arrow[r, mapsto] \arrow[urr, mapsto, bend left] & a'/b'' \arrow[ur, dotted, mapsto, "\exists!"] & \\ & b \arrow[u, mapsto] \arrow[uur, mapsto, bend right] & \end{tikzcd}$$

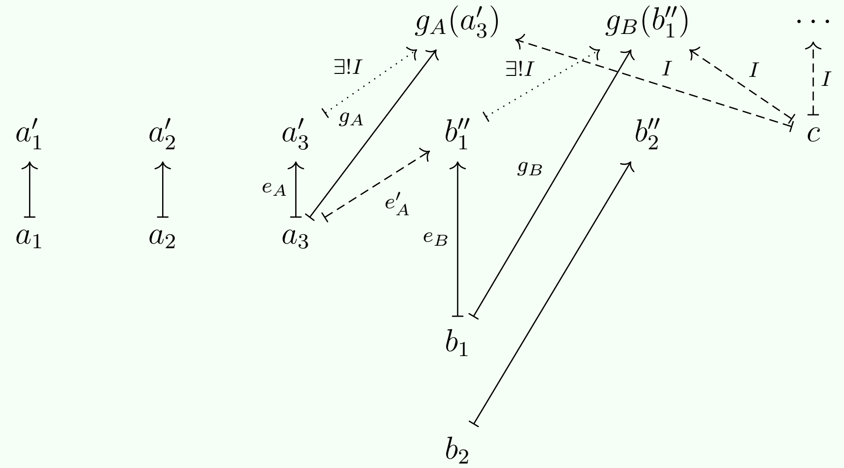
The definition of disjoint union and coproduct are compatible, i.e. $A + B = A \bigsqcup B$.

We will use a trivial example to explain the following arguments:

- larger sets have extra elements that will kill the uniqueness;
- smaller sets may lose results and fall to be initial.

Proof. $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2\}$, $A \bigsqcup B = a'_1, a'_2, a'_3, b''_1, b''_2$(A.8).

Example 1.98. $g_A(a_3), g_B(b_1) \in Y$, $g_A(a_3) \neq g_B(b_1)$,



- if there are extra elements in $A + B$ such as c, then $I(c)$ can be any value in Y, hence breaks the uniqueness of I, $A \bigsqcup B, \{c\}$ is not a initial object;

  Example 1.99. $I(c) = g_A(a'_3)$ or $g_b(b''_1)$ or elements in $\cdots$.

- if we delete an element in $A \bigsqcup B$,

  Example 1.100. a'_3, then $e_A(a_3)$ have to go eleswhere. If $e_A(a_3) = b''_1$, then $Ie_A(a_3) = Ie_B(b_1)$, contridict to
  $Ie_A(a_3) = g_A(a_3) \neq g_B(b_1) = Ie_B(b_1)$, hence $A \bigsqcup B$ without a''_3 is not an initial object.

Hence $(A \bigsqcup B, Co(A \times B))$ is indeed the initial object of $\mathbf{Cocone}(F)$,
$A + B = A \bigsqcup B$ is a colimit. ☐

In conclusion, $A \bigsqcup B$ is a coproduct of A and B.

We may put product and coproduct into a **NON-commutative** diagram diagram:

$$\begin{tikzcd}
& & & Y \\
& A \arrow[r, "\iota_A"] \arrow[urr, bend left, "g_A"] & A \bigsqcup B \arrow[ur, dotted, "\exists! I"] & \\
& A \times B \arrow[u, "\pi_A"] \arrow[r, "\pi_B"] & B \arrow[u, "\iota_B"'] \arrow[uur, bend right, "g_B"'] & \\
X \arrow[uur, bend left, "f"] \arrow[ur, dotted, "\exists! T"] \arrow[urr, bend right, "f"'] & & &
\end{tikzcd} .$$

1.2.5 Pushout

Appendix A

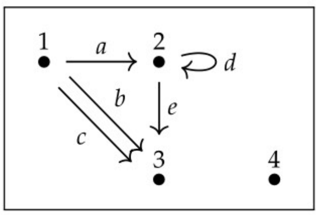
Set theory preliminaries

Definition A.1. A *relation* R is a set of ordered pairs $R \subseteq \mathrm{dom}(R) \times \mathrm{cod}(R)$, $(a, b) \in R$ is written as aRb.

Definition A.2. The *inverse relation* $R^{-1} \in B \times A$ is a relation $\{\forall(b, a)|(a, b) \in B\}$.

Definition A.3. A *directed graph* $G = (V, A, s, t)$ consists of:

- a set V of vertices, drawn as $\bullet$;
- a set A of arrows, drawn as $\rightarrow$.
    - an arrow $\mathbf{a} \in A$ is a 3-tuple (s, a, t), $s, t \in V$. s/t is the source/target of a.

$G =$ 

arrow a	source $s(a) \in V$	target $t(a) \in V$
a	1	2
b	1	3
c	1	3
d	2	2
e	2	3

Figure A.1: graph G

Definition A.4. A length n *path* p in a graph G is a sequence of n narrows such that the target of one arrow is the source of the next arrow.

- a length 0 path 0_v start and end at the same vertice v. Adding it to a path makes no difference.

The source/target of the first/last arrow is the source/target of a path.

Example A.5. eda, drawn as $1 \xrightarrow{a} 2 \xrightarrow{d} 2 \xrightarrow{e} 3$ is a length 3 path from 1 to 3.

Definition A.6. Two paths p, p' are *parallel* if they have the same source and target, written as $p \parallel p'$.

Example A.7. $b \parallel b0_1 \parallel c \parallel ea \parallel e0_2d0_2a$.

For a set S, S' is a set $\{(s,')|\forall s \in S\}$.

Definition A.8. The *disjoint union* of two sets A and B is a set $A' \bigcup B''$, written as $A \bigsqcup B$.

Example A.9. $\{\mathrm{UoE}, \mathrm{Carol}\} \bigsqcup \{\mathrm{Alex}, \mathrm{Bob}, \mathrm{Carol}\} = \{\mathrm{UoE}', \mathrm{Carol}', \mathrm{Alex}'', \mathrm{Bob}'', \mathrm{Carol}''\}$.