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RESEARCH ARTICLES

Elementary Particles on p-Adic Spacetime and Temperedness of Invariant Measures*

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Abstract—We extend the method of L. Schwartz [1] to classify elementary scalar particles in p-adic space time. Schwartz obtained the states of the elementary particles over real spacetime as tempered distributions on spacetime itself. We obtain the analogous description in p-adic spacetime. We introduce a natural notion of temperedness similar to one introduced by Harish-Chandra in the p-adic case and show that the invariant measures corresponding to the elementary particles are tempered.

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1. INTRODUCTION

Planck length is the smallest measurable distance and measurements on a smaller scale are theoretically not possible. Planck length and Planck time arise naturally when quantum mechanics and general relativity are combined. In the 1970's, Beltrametti and his colleagues proposed that to overcome mathematical difficulties associated with physics on a sub Planck scale, one ought to consider alternative models of spacetime geometry based on non-archimedean or finite fields [4]. These ideas were solidified by Volovich who argued that at Planck scale one cannot compare distances or times and this naturally leads one to explore a non-Archimedean geometry as a possible geometry of spacetime [5]. Non-archimedean physics adopts the Volovich hypothesis and explores the consequences by studying well-known questions in physics in this new setting. The work of Dragovitch et al. [7] is a comprehensive survey of the field.

One of the fundamental questions is that of the classification and structure of elementary particles for a given symmetry group G. The elementary particles of a quantum system may be classified by irreducible projective unitary representations of G or subgroups of index 2 [10]. This method of classification has been very successfully used ever since it was developed by Wigner [3] in the 1930's to classify the particles of the real Poincaré group. The states of the particles correspond to functions on the orbits of the Lorentz group on the dual of spacetime. However, there is a lesser known method of Laurent Schwartz that uses the theory of distributions to arrive at the classification where the states are distributions directly on spacetime [1]. These distributions are the Fourier transforms on the orbits (treated as measures in the ambient momentum space) of the functions in the conventional picture, but Schwartz obtains them directly.

Schwartz defines a universe V to be a C^∞ manifold of dimension n. The space of test functions, $D(V)$, is the sheaf of complex valued infinitely differentiable functions on V with compact support and $D'(V)$ is the space of continuous linear functionals on $D(V)$. Schwartz then defines a scalar particle in the universe V as a set $\mathcal{H}$ satisfying the following:

1. $\mathcal{H}$ is a vector subspace of $D'(V)$.

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2. $\mathcal{H}$ is a Hilbert space.

3. The canonical imbedding of $\mathcal{H}$ into $D'(V)$ is continuous. That is, $\phi_j \to 0$ in $\mathcal{H}$ implies that $\phi_j \to 0$ in $D'(V)$.

Given a symmetry group G, Schwartz then defines the elementary scalar particles as certain scalar particles that are invariant under the action of G. The symmetry group G has an action on the universe V, hence on $D'(V)$. Schwartz shows that the Hilbert spaces describing elementary particles correspond to the distributions that are the Fourier transforms of the G-invariant measures supported on the orbits of G.

Schwartz's method has some advantages over the representation theory method. His method allows one to consider the universe as a finite dimensional analytic manifold whereas the algebraic methods force one to start with a flat spacetime. His method is also much closer to how the physicists think of elementary particles. His Hilbert space is less abstract since it is a space of distributions over spacetime. Finally, the distributions corresponding to scalar particles are certain weak solutions of the Klein-Gordon equation.

In this article, we extend the results of Schwartz to the p-adic setting. Our spacetime will be $V = \mathbb{Q}_p^n$ with a quadratic form $Q = a_0x_0^2 + \cdots + a_{n-1}x_{n-1}^2$ where the a_i are all non-zero. This form is invariant under the corresponding Lorentz group P. The set $D(V)$ will be the set of locally constant test functions with compact support. The Hilbert space associated to our quantum system is a set of linear functionals on $D(V)$. We follow Schwartz's model to show that the elementary particles correspond to certain positive measures defined on the orbits of G.

In the real setting, the notion of temperedness of the distributions is an essential technical tool. The measures are the Fourier transforms of the distributions and so the distributions must be tempered in order to define their Fourier transforms. In the p-adic setting, it is unnecessary because the Fourier transform is an automorphism on the space of locally constant test functions. Nevertheless, if we use the definition of slow growth to indicate temperedness, we show that the measures corresponding to elementary particles are tempered in this sense.

Most of the original proofs of Schwartz can be readily carried over to the p-adic setting, however, the proof of the temperedness of the measures, which is elementary in the real case [10], is somewhat more involved in the p-adic setting.

2. BASIC DEFINITIONS

Definition 2.1. *A* ***universe****, (V, G), is a totally disconnected space V which carries an action of a p-adic Lie group G. V is called a* ***spacetime*** *and G is said to be the* ***spacetime symmetry group*** *of V.*

Denote by $D(V)$ the set of locally constant complex valued functions on V with compact support and denote by $D'(V)$ the space of linear functionals on $D(V)$. The set $D(V)$ is often referred to as the space of ***test functions***, and the elements of $D'(V)$ as ***distributions*** on V. Unlike the real case, there are no continuity conditions in the definition of $D'(V)$.

Definition 2.2. *A* ***scalar particle*** *in the universe V is a pair $(\mathcal{H}, \langle\cdot|\cdot\rangle_{\mathcal{H}})$ such that $\mathcal{H}$ is a vector subspace of $D'(V)$, $\langle\cdot|\cdot\rangle_{\mathcal{H}}$ is an inner product on $\mathcal{H}$, and the pair satisfies the following two properties.*

(1) The pair $(\mathcal{H}, \langle\cdot|\cdot\rangle_{\mathcal{H}})$ is a Hilbert space.

(2) The canonical embedding, j, of $\mathcal{H}$ into $D'(V)$ is continuous, $D'(V)$ being given the weak- topology.*

Denote by $\mathfrak{H}$ the set of all scalar particles in V. Denote by $\|\cdot\|_{\mathcal{H}}$ the norm on $\mathcal{H}$ that $\langle\cdot|\cdot\rangle_{\mathcal{H}}$ induces.

Definition 2.3. *A scalar particle is called G-**covariant** if for all σ in G we have the following two properties.*

(1) We have the equality $\sigma\mathcal{H} = \mathcal{H}$.

(2) For all $\psi \in \mathcal{H}$, $\|\sigma\psi\|_{\mathcal{H}} = \|\psi\|_{\mathcal{H}}$.

*A G-covariant scalar particle, $\mathcal{H}$, is said to be **elementary** if it has no non-trivial, closed, G invariant subspaces.*

3. POSITIVE ANTI-KERNELS

Definition 3.1. *An **anti-kernel** is an anti-linear continuous map from $D(V)$ to $D'(V)$. An anti-kernel, L, is said to be **positive** if for all f in $D(V)$,*

$$(Lf)(f) \geq 0.$$

Theorem 3.2. *There is a bijection between $\mathfrak{H}$ and the set of positive anti-kernels on $D(V)$.*

Proof. Suppose that $\mathcal{H}$ is an element of $\mathfrak{H}$. Since the canonical imbedding j of $\mathcal{H}$ into $D'(V)$ is continuous we have, for each f in $D(V)$, the map $g \mapsto (j(g))(f)$ is a continuous linear function on $\mathcal{H}$. The Riesz Representation Theorem guarantees the existence of a unique element f' in $\mathcal{H}$ such that for any $g \in \mathcal{H}$,

$$j(g)(f) = \langle g|f'\rangle_{\mathcal{H}}.$$

Define $Lf = f'$ so that $j(g)(f) = \langle g|Lf\rangle_{\mathcal{H}}$ where $f \in D(V)$ and $g \in \mathcal{H}$). It is clear that L is anti-linear and positive. Therefore, L is the desired positive anti-kernel.

We now prove the converse: Suppose that L is a positive anti-kernel and let $\mathcal{H}_0 = L(D(V))$. Suppose that

$$u = Lf \quad \text{and} \quad v = Lg.$$

Define an inner product on $\mathcal{H}_0$ by

$$\langle u|v\rangle_{\mathcal{H}_0} = (Lg)(f).$$

We now prove that this definition is independent of the choice of f and g. It suffices to show that if either $Lf = 0$ or $Lg = 0$, then $(Lg)(f) = 0$. This is trivial if $Lg = 0$. Consider the case when $Lf = 0$. Notice that the sesquilinear form defined by

$$B(f, g) = (Lg)(f)$$

is actually a Hermitian form. This is because every sesquilinear form can be written as a sum of a Hermitian form H and a skew-Hermitian form S. The quadratic form associated to H is purely real and the quadratic form associated to S is purely imaginary. However, for all f in $D(V)$,

$$B(f, f) = (Lf)(f) \geq 0.$$

Therefore, S must be the zero form and B is actually Hermitian. Therefore,

$$(Lg)(f) = B(f, g) = \overline{B(g, f)} = \overline{(Lf)(g)}.$$

Hence, $Lf = 0$ implies that $(Lg)(f) = 0$. So $\langle u|v\rangle_{\mathcal{H}_0}$ is well defined, linear in v and anti-linear in u.

Since L is positive,

$$\langle u|u\rangle_{\mathcal{H}_0} = (Lf)(f) \geq 0.$$

Suppose that $u = Lf$, $\langle u|u\rangle_{\mathcal{H}_0} = 0$, and $g \in D(V)$. The Cauchy-Schwartz inequality implies that $\langle Lf|Lg\rangle_{\mathcal{H}_0} = 0$. Since $\langle Lf|Lg\rangle_{\mathcal{H}_0} = (Lf)(g) = 0$, we have that $u = Lf = 0$. Therefore, $\langle\cdot|\cdot\rangle_{\mathcal{H}_0}$ is positive definite.

Since $\langle LF|Lg\rangle_{\mathcal{H}_0} = (Lf)(g)$ and $|\langle Lf|Lg\rangle|_{\mathcal{H}_0} \leq ||Lf||_{\mathcal{H}_0}||Lg||_{\mathcal{H}_0}$, $Lf \to 0$ in $\mathcal{H}_0$ implies $(Lf)(g) \to 0$ for all g. Therefore, $\mathcal{H}_0$ is a pre-Hilbert space whose canonical injection into $D'(V)$ is continuous in the weak-* topology. We now have the following lemma.

Lemma 3.3. *If $\hat{\mathcal{H}}_0$ is the completion of $\mathcal{H}_0$, then the injection J of $\hat{\mathcal{H}}_0$ into $D'(V)$ extends uniquely to an injection $\hat{J}$ of $\hat{\mathcal{H}}$ into $D'(V)$.*

Proof. Since $D'(V)$ is sequentially complete, it follows that the canonical inclusion $J\colon \mathcal{H}_0 \to D'(V)$ extends to a continuous linear map $\hat{J}$ of the completion $\hat{\mathcal{H}}_0$ of $\mathcal{H}_0$ into $D'(V)$. We claim that $\hat{J}$ is an injection. Suppose that $u \in \hat{\mathcal{H}}_0$ is such that $\hat{J}u = 0$. We can find a sequence $u_n = Lf_n$, f_n in $D(V)$, such that $u_n \to u$. Then $J(u_n) \to 0$. If g in $D(V)$, then $\langle Lf_n|Lg\rangle_{\mathcal{H}_0} \to \langle u|Lg\rangle$. But $\langle Lf_n|Lg\rangle_{\mathcal{H}_0} = (Lf_n)(g) \to 0$. Hence $u \perp \mathcal{H}_0$, thus $u = 0$. □

The anti-kernel L then corresponds to the completion of $\mathcal{H}_0$ which has a continuous injection into $D'(V)$. □

Let V be $\mathbb{Q}_p^n$ and W be $\mathbb{Q}_p^m$. Let $D'(W \times V)$ denote the space of distributions on the product space $W \times V$.

Theorem 3.4 (Theorem of Kernels)**.** *The vector space $\mathcal{L}(D(V)\colon D'(W))$ of continuous linear maps from $D(V)$ into $D'(W)$ is isomorphic to $D'(W \times V)$*

Proof. The theorem is proved in [2] in the case where the distributions are taken to be continuous linear functionals on the test functions. In our case, there is no continuity requirement. Note that this theorem was stated without proof in [6] (48-50). Since $D'(V \times W)$ and $D'(W)$ are, respectively, the algebraic duals of $D(V \times W)$ and $D(W)$, it suffices to show that $D(V \times W) \cong D(V) \otimes D(W)$.

Suppose that f is in $D(V \times W)$ and that $\operatorname{supp}(f) \subseteq K \times L$ where K and L are compact open subsets of V and W respectively. At each point x of $K \times L$ there is a neighborhood $K_x \times L_x$ of x where K_x and L_x are compact and open, $K_x \times L_x \subseteq K \times L$, and f is constant on $K_x \times L_x$. We can therefore find a finite collection, $\{K_t \times L_t\}_{1 \le t \le N}$, of compact open subsets of $K \times L$ that cover $K \times L$ and such that f is a constant c_t on each $K_t \times L_t$. Write

$$\begin{cases} K'_1 = K_1 \\ K'_i = K_i \setminus \bigcup_{1 \le j \le i} K'_j \quad \text{for } i \ge 2, \end{cases} \quad \text{and} \quad \begin{cases} L'_1 = L_1 \\ L'_i = L_i \setminus \bigcup_{1 \le j \le i} L'_j \quad \text{for } i \ge 2. \end{cases}$$

The sets $\{K'_i\}$ and $\{L'_i\}$ are sets of disjoint elements and

$$K_j = \coprod_{1 \le i \le j} K'_i \quad \text{and} \quad L_j = \coprod_{1 \le i \le j} L'_i.$$

If $1 \le r, s \le N$ and $t = \max(r, s)$, then $K'_r \subseteq K_t$, $L'_s \subseteq L_t$ so that $K'_r \times L'_s \subseteq K_t \times L_t$. The $K'_r \times L'_s$ are disjoint and so

$$\coprod_{r,s} K'_r \times L'_s = K \times L, \quad \text{but} \quad K'_r \times L'_s \subseteq K \times L \quad \text{and} \quad K \times L = \coprod K'_r \times L'_s.$$

For a given pair (r, s), it is clear that f is a constant $c_{r,s}$ on $K'_r \times L'_s$ because if $t = \max(r, s)$, then $K'_r \times L'_s \subseteq K_t \times L_t$ and $f = c_t$ on $K_t \times L_t$. Hence

$$f = \sum_{r,s} c_t \chi_{K'_r} \otimes \chi_{L'_s},$$

where $\chi_{K'_r}$ and $\chi_{L'_s}$ are the characteristic functions on K'_r and L'_s respectively. This shows that $f \in D(V) \otimes D(W)$.

If $f \in D(V) \otimes D(W)$, then it is certainly in $D(V \times W)$.

□

We will consider only the positive anti-kernels. Suppose that v and w are in $D(V)$. A distribution of two variables K in $D'(V \times V)$ acts via the dual paring $\langle K, v \otimes w\rangle$ and defines a continuous linear map from $D(V)$ into $D'(V)$ in the following way: The function

$$v \mapsto K(v)\colon w \mapsto \langle K, v \otimes w\rangle$$

maps v to an element of $D'(V)$. Since we consider the anti-linear maps, we take

$$v \mapsto K(\overline{v})\colon w \mapsto \langle K, \overline{v} \otimes w\rangle.$$

Definition 3.5. *An anti-kernel defined by a distribution of two variables, $K_{x,y}$, is called **positive** if for every $\psi \in D(V)$*

$$\langle K_{x,y}, \psi(x) \otimes \overline{\psi(y)}\rangle \geq 0.$$

Corollary 3.6. *The space $\mathfrak{H}$ of Hilbert spaces with continuous injection in $D'(V)$ is in bijective correspondence with the space of positive anti-kernels defined by elements of $D'(V \times V)$.*

Corollary 3.7. *The G-covariant scalar particles in $\mathfrak{H}$ are in bijective correspondence with G-invariant positive anti-kernels defined by elements of $D'(V \times V)$.*

Proof. Suppose that σ is an automorphism of V. The automorphism σ has a natural action on $D(V)$ and $D'(V)$. Suppose that σ leaves $\mathcal{H}$ invariant and is unitary on $\mathcal{H}$, that is for all g in $\mathcal{H}$, $||g|| = ||\sigma g||$. Recall that j is the canonical imbedding of $\mathcal{H}$ into $D'(V)$. We have from

$$j(g)(f) = \langle g | Lf\rangle_{\mathcal{H}} \qquad (f \in D(V),\ g \in \mathcal{H})$$

the equation

$$j(\sigma g)(\sigma f) = \langle \sigma g | L\sigma f\rangle_{\mathcal{H}} = \langle g | \sigma^{-1} L \sigma f\rangle_{\mathcal{H}}.$$

Since $j(\sigma g)(\sigma f) = \langle j(g) | f\rangle_{\mathcal{H}}$, we see that $L = \sigma^{-1} L \sigma$. Therefore

$$\langle \sigma g | L\sigma f\rangle_{\mathcal{H}} = \langle g | Lf\rangle_{\mathcal{H}}.$$

□

4. TRANSLATION INVARIANCE

We now specialize to the case where the universe is an affine space E and the underlying vector space, $\mathbf{E}$, is $\mathbb{Q}_p^n$. The vector space $\mathbf{E}$ acts as the group of translations on E. We assume that G contains the translations G_0. Consider the isomorphism

$$E \times E \to E \times \mathbf{E}$$

defined by

$$(x, \zeta) \to (x, x - \zeta) = (x, \mathbf{u}),$$

where we denote $x - \zeta$ by $\mathbf{u}$. Further, $\zeta = x - \mathbf{u}$. The commutativity of the diagram below describes the action on $E \times \mathbf{E}$ that is induced by the action of $\mathbf{E}$ on spacetime. Note that translation by an element of $\mathbf{E}$ effects only the first component of an element of $E \times \mathbf{E}$.

$$\begin{tikzcd}
E \times E \arrow[r, "{(x,\zeta) \mapsto (x,\mathbf{u})}"] \arrow[d, "{(x,\zeta) \mapsto (x+\mathbf{a},\zeta+\mathbf{a})}"'] & E \times \mathbf{E} \arrow[d, "{(x,u) \mapsto (x+\mathbf{a},\mathbf{u})}"] \\
E \times E \arrow[r, "{(x+\mathbf{a},\zeta+\mathbf{a}) \mapsto (x+\mathbf{a},\mathbf{u})}"'] & E \times \mathbf{E}
\end{tikzcd}$$

The isomorphism between $D'(E \times E)$ and $D'(E \times \mathbf{E})$ given by

$$\langle H_{x,\mathbf{u}}, \psi(x, \mathbf{u})\rangle = \langle K_{x,\zeta}, \psi(x, x-\zeta)\rangle, \quad \psi \in D'(E, \mathbf{E})$$

and

$$\langle K_{x,\zeta}, \varphi(x, \zeta)\rangle = \langle H_{x,\mathbf{u}}, \varphi(x, x-\mathbf{u})\rangle \quad \varphi \in D'(E, E)$$

give us a transport of structure. Instead of looking for the elements of $D'(E \times E)$ that are invariant under all translations acting on two variables, it suffices to look for the elements of $D'(E \times \mathbf{E})$ that are invariant under all translations acting on the first variable only. The Theorem of Kernels gives us the isomorphism

$$D'(E \times \mathbf{E}) \cong \mathcal{L}(D(E), D'(\mathbf{E})) = D'(E : D'(\mathbf{E})).$$

An element $H_{x,u}$ of $D'(E : D'(\mathbf{E}))$ is a distribution on E taking values in the space of distributions $D'(\mathbf{E})$. We will find all H that are invariant under all translations acting on E.

Let $\mathcal{E}(V)$ be the set of locally constant functions on V. The Schwartz-Bruhat space on V, $D(V)$, is the subset of functions in $\mathcal{E}(V)$ with compact support. For any fixed $x \in V$, we define the function

$$S_x \colon D(V) \to D(V) \quad \text{by} \quad S_x(\psi)(y) = \psi(x-y).$$

Define the convolution of a distribution $f \in D'(V)$ with a test function $\psi \in D(V)$ by

$$(f * \psi)(x) = \langle f, S_x(\psi)\rangle.$$

Lemma 4.1. *If $\psi \in D$ and $f \in D'$ then $f * \psi \in \mathcal{E}(V)$.*

Lemma 4.1 is proven in [2], page 97.

Proposition 4.2. *If T is an element of $D'(E)$ and T is invariant under translation, then T is constant.*

Proof. Let $\alpha \in D(V)$ be a locally constant function with $\int \alpha \, \mathrm{d}x = 1$. Then for every $\psi \in D(V)$,

$$\begin{aligned}\langle T * \alpha, \psi\rangle &= \langle T_\zeta \otimes \alpha(\eta), \psi(\zeta+\eta)\rangle \\ &= \int \alpha(\eta)\, \langle T_\zeta, \psi(\zeta+\eta)\rangle \, \mathrm{d}\eta \\ &= \int \alpha(\eta)\, \langle T_t, \psi(t)\rangle \, \mathrm{d}\eta = \langle T, \psi\rangle.\end{aligned}$$

In the last step the invariance of T was used. Thus $T * \alpha = T$. Now by Lemma 4.1, T is a locally constant function. The fact that T is invariant under all translations implies that it is constant. □

Corollary 4.3. *Every distribution $T \in D'(E : D'(\mathbf{E}))$ which is invariant under translations is constant, that is,*

$$T = 1_{(x)} \otimes \lambda \quad \textit{where} \quad \lambda \in D'(\mathbf{E}).$$

We make the appropriate identification of $\mathbf{E}$ with G_0. We have shown that the anti-kernels $K_{x,\zeta} \in D'(E \times E)$ which are invariant under the group G_0 of translations are in one-to-one correspondence with distributions $1 \otimes H_{\mathbf{u}}$, where $H_{\mathbf{u}} \in D'(G_0)$. We have that $\mathbf{u} = x - \zeta$ or equivalently that $\zeta = x - \mathbf{u}$. If $H_{\mathbf{u}}$ is given, then by the transport of structure,

$$K_{x,\zeta} = H_{x,x-\zeta} = 1_{(x)} \otimes H_{x-\zeta}.$$

For every $\psi \in D(E \times E)$,

$$\begin{aligned}\langle K_{x,\zeta}, \psi(x,\zeta)\rangle &= \left\langle 1_{(x)} \otimes H_{\mathbf{u}}, \psi(x, x-\mathbf{u})\right\rangle \\ &= \left\langle H_{\mathbf{u}}, \int \psi(x, x-\mathbf{u})\, \mathrm{d}x \right\rangle\end{aligned}$$

$$= \int \langle H_{\mathbf{u}}, \psi(x, x - \mathbf{u}) \rangle \, \mathrm{d}x.$$

The anti-kernel $K_{x,\zeta}$ is considered positive if

$$\left\langle K_{x,\zeta}, \psi(x) \otimes \bar{\psi}(\zeta) \right\rangle \geq 0.$$

For $H_{\mathbf{u}}$ the condition becomes

$$\left\langle H_{\mathbf{u}}, \int \psi(x)\bar{\psi}(x - \mathbf{u}) \, \mathrm{d}x \right\rangle \geq 0$$

for every $\psi \in D(E)$. We can eliminate references to the affine space by fixing an origin. We have then have that

$$\left\langle H_{\mathbf{u}}, \int \psi(\mathbf{x})\bar{\psi}(\mathbf{x} - \mathbf{u}) \, \mathrm{d}x \right\rangle \geq 0.$$

The above condition is almost a convolution. For convolution we need to have $\mathbf{u} - \mathbf{x}$ instead of $\mathbf{x} - \mathbf{u}$. We set the notation

$$\tilde{\psi}(\mathbf{x}) = \bar{\psi}(-\mathbf{x}).$$

The positivity condition on $K_{x,\zeta}$ corresponds to the positivity condition on H given by

$$\langle H, \psi * \tilde{\psi} \rangle \geq 0.$$

5. BOCHNER-SCHWARTZ

Definition 5.1. *A distribution $T \in D'(\mathbf{E})$ is of positive type if $\langle T, \varphi * \tilde{\varphi} \rangle \geq 0$ for all $\varphi \in D(\mathbf{E_n})$.*

Theorem 5.2. *(Bochner-Schwartz, p-adic case) A distribution $T \in D'(\mathbf{E})$ is of positive type if and only if its Fourier transform is a regular measure.*

The Fourier transform of a distribution of positive type is a positive distribution and every positive distribution is the Fourier transform of a distribution of positive type. Therefore, the Riesz-Markov-Kakutani Theorem implies Theorem 5.2. A proof of Theorem 5.2 along these lines can be found in [11]. Together with the correspondences we have already established, Theorem 5.2 implies that the G-covariant scalar particles in $\mathfrak{H}$ are in bijective correspondence with the G invariant regular measures on $\mathbf{E}$.

6. ORDER RELATIONS

Definition 6.1. *One may define a partial ordering, $\leq$ on $\mathfrak{H}$. We say that for $\mathcal{H}_1, \mathcal{H}_2 \in \mathfrak{H}$, $\mathcal{H}_1 \leq \mathcal{H}_2$ if $\mathcal{H}_1 \subset \mathcal{H}_2$ and the norm in $\mathcal{H}_1$ is greater than or equal to norm in $\mathcal{H}_2$.*

Let $\mathcal{H}$ be an elementary particle, then the following descriptions hold:

1. If K_1 is a G invariant positive anti-kernel corresponding to $\mathcal{H}$ and if $K_2 \leq K_1$ is another G invariant positive anti-kernel corresponding to $\mathcal{H}$ then $K_2 = \lambda K_1$ for some $0 \leq \lambda \leq 1$.

2. If H_1 is a G invariant positive distribution corresponding to $\mathcal{H}$ and H_2 is another G invariant positive distribution corresponding to $\mathcal{H}$ and $H_2 \leq H_1$ then $H_2 = \lambda H_1$.

3. If μ_1 is a G invariant measure and μ_2 is another one then $\mu_2 = \lambda \mu_1$.

Remark: The collection $\mathfrak{H}$ of scalar particles form a convex cone. The G invariant scalar particles form a subcone. The elementary scalar particles form the extremal generatrices of this subcone.

Proofs of the assertions above follow very closely to their real case counterparts and we omit the discussion here. We refer the reader to [1].

7. ORBITS AND INVARIANT MEASURES

For an orbit to admit an invariant measure it is sufficient that G is unimodular and that the stabilizer of the orbit is a closed unimodular subgroup of G. This result is well known for the Poincare group. See for example [9]. It is also well known that the orbits are given by the level sets of the quadratic form and each orbit has an invariant measure that is unique up to a positive constant [8].

Theorem 7.1. *The invariant measures on the orbit*

$$x_0^2 + a_1 x_1^2 + ... + a_n x_n^2 = m$$

are proportionate to the measure given by

$$\frac{1}{\left|m - a_1 x_1^2 - \cdots - a_n x_n^2\right|_p^{\frac{1}{2}}} dx.$$

Although the above result is well known, we give a proof here since it uses a novel approach. We begin with a lemma.

Lemma 7.2. *Let A be a matrix with entries $\{x_i x_j\}_{ij}$ and let X be the column vector $X = (x_1 \quad x_2 \quad \cdots \quad x_n)$. We have that $A = XX^T$ and that*

$$\det\left(x_0^2 I + XX^T\right) = x_0^{2n-2}\left(x_0^2 + x_1^2 + \cdots + x_n^2\right).$$

Proof. We begin with the observation that XX^T is a projection onto the vector X combined with a scaling by the quantity $\left(x_1^2 + x_2^2 + \cdots + x_n^2\right)$. Hence, there exists a change of basis B so that BAB^{-1} has the form

$$BAB^{-1} = \left(\begin{array}{c|c} x_1^2 + x_2^2 + \cdots + x_n^2 & 0 \ \cdots \ 0 \\ \hline 0 & \\ \vdots & \text{\Large 0} \\ 0 & \end{array}\right).$$

The result now follows. ∎

Proof. Let

$$Q = a_0 x_0^2 + a_1 x_1^2 + ... + a_n x_n^2.$$

This quadratic form is invariant under the Poincaré group, G. It defines a pseudo-Riemannian metric S on V. The metric in turn induces a volume form on V which is invariant under G. We may restrict S to the level set

$$a_0 x_0^2 + a_1 x_1^2 + \cdots + a_n x_n^2 = m,$$

where $m \in \mathbb{Q}_p$ is fixed. The restriction of S to this level set defines another pseudo-Riemannian metric S' that is invariant under the action of G. The restriction S' then induces a volume form F on the level set. We may assume that a_0 equals 1. We have that

$$ds^2 = dx_0^2 + a_1^2 dx_1^2 + \cdots + a_n dx_n^2.$$

On the level set, we have that

$$x_0 dx_0 + a_1 x_1 dx_1 + \cdots + a_n x_n dx_n = 0$$

and so

$$x_0^2 dx_0^2 = \left(a_1 x_1 dx_1 + \cdots + a_n x_n dx_n\right)^2.$$

Therefore,

$$x_0^2 ds^2 = (a_1 x_1 dx_1 + \cdots + a_n x_n dx_n)^2 + x_0^2 \left(a_1^2 dx_1^2 + \cdots + a_n dx_n^2\right).$$

We now write the metric tensor g for this metric. We have that

$$x_0^2 g = x_0^2 \mathrm{diag}\big(a_i^2\big) + A$$

where A is the matrix whose entry $A_{i,j}$ is given by $a_i a_j x_i x_j$. We now compute the determinant of g to find the invariant volume form. Since our equations will involve only rational functions with p-adic coefficients, we may work in an extension field of $\mathbb{Q}_p$ so any square roots we take exist. We begin by noting that the matrix A can be written as a product XX^T where X is the column vector

$$\begin{pmatrix} a_1 x_1 \\ \vdots \\ a_n x_n \end{pmatrix}.$$

We write $D = \mathrm{diag}\big(a_i^2\big)$ and $F^2 = D^{-1}$. One may easily verify the following identity,

$$\det\big(XX^T + x_0^2 D\big) = \det(D) \det\big(FXFX^T + x_0^2 I\big).$$

We give the details of the calculation here. We have that

$$\det\big(XX^T + x_0^2 D\big) = \det\big(XX^T D^{-1} + x_0^2 I\big) \det(D).$$

Since F is diagonal, we have that

$$(FX)^T = X^T F^T = X^T F.$$

Therefore,

$$XX^T D = XX^T F^2 = F^{-1} FX (FX)^T F$$

and so we have the equalities

$$\begin{aligned} XX^T D^{-1} + x_0^2 I &= XX^T F^2 + F^{-1} x_0^2 F \\ &= \big(F^{-1} FX(FX)^T F + F^{-1} x_0^2 F\big) = F^{-1} \left(FX(FX)^T + x_0^2 I\right) F. \end{aligned}$$

Letting $y = FX$, we obtain the equality

$$XX^T D^{-1} + x_0^2 I = F^{-1}(yy^T + x_0^2 I)F.$$

We therefore have the equalities

$$\det\big(XX^T + x_0^2 D\big) = \det\big(F^{-1} \left(yy^T + x_0^2 I\right) F\big) \det(D) = \det\big(yy^T + x_0^2 I\big) \det(D).$$

By Lemma 7.2, this determinant is

$$(a_0 a_1 \cdots a_n) \left(x_0^2 + a_1 x_1^2 + a_2 x_2^2 + \cdots + a_n x_n^2\right) x_0^{2n-2} = (a_0 a_1 \ldots a_n)\, m x_0^{2n-2}.$$

We have

$$\det(g) = x_0^{-2n} \det\big(x_0^2 \mathrm{diag}\big(a_i^2\big) + A\big) = (a_0 a_1 \cdots a_n)\, m x_0^{-2},$$

hence the invariant measure on the orbit

$$x_0^2 + a_1 x_1^2 + \cdots + a_n x_n^2 = m$$

is proportionate to the measure given by

$$\frac{1}{\big|m - a_1 x_1^2 - \cdots - a_n x_n^2\big|_p^{\frac{1}{2}}} dx.$$

□

Theorem 7.3. *Let μ be a measure on $\mathbb{Q}_p^n$. If μ is positive G-invariant and extremal among measures having these properties, then the support of μ is the closure of one G-orbit.*

The proof in [1] carries over to our case with the obvious modifications.

By direct calculation we verify below that measures corresponding to elementary particles grow at most like the volume of a ball in Q_p^n. They are therefore tempered in this sense.

8. VERIFICATION OF TEMPEREDNESS

Let p be a fixed prime and denote by $|\cdot|$ the p-adic norm. If $x \in \mathbb{Q}_p^k$, then denote $||x|| = \max\{|x_1|, \ldots, |x_k|\}$. For each i with $1 \leq i \leq k$ suppose that $a_i \neq 0$. Suppose that $m \in \mathbb{Q}_p$ and consider the integral

$$I_{k,a_1,\ldots,a_k}(m,n) = \int\limits_{||x|| \leq p^n} \frac{1}{\left|m + a_1 x_1^2 + \cdots + a_k x_k^2\right|^{\frac{1}{2}}}\, \mathrm{d}x^k.$$

The measure is taken to be the Haar measure on $\mathbb{Q}_p^k$ normalized so that the unit ball has measure 1. *Massless particle cases* are when $m = 0$, otherwise we say we are in a *massive particle case*.

8.1. The Weighted Massive Case

8.1.1. The One Dimensional Case. Suppose that $a_1 \neq 0$. We consider the case

$$I_{1,a_1}(m,n) = \int\limits_{|x_1| \leq p^n} \frac{1}{\left|m + a_1 x_1^2\right|^{\frac{1}{2}}}\, \mathrm{d}x_1.$$

We have that

$$I_{1,a_1}(m,n) = \int\limits_{\substack{|x_1| \leq p^n \\ |a_1 x_1^2| < |m|}} \frac{1}{\left|m + a_1 x_1^2\right|^{\frac{1}{2}}}\, \mathrm{d}x_1 + \int\limits_{\substack{|x_1| \leq p^n \\ |a_1 x_1^2| = |m|}} \frac{1}{\left|m + a_1 x_1^2\right|^{\frac{1}{2}}}\, \mathrm{d}x_1$$
$$+ \int\limits_{\substack{|x_1| \leq p^n \\ |a_1 x_1^2| > |m|}} \frac{1}{\left|m + a_1 x_1^2\right|^{\frac{1}{2}}}\, \mathrm{d}x_1. \tag{8.1}$$

We consider independently the three integrals in the above sum. For the first summand in (8.1), we have that

$$\int\limits_{\substack{|x_1| \leq p^n \\ |a_1 x_1^2| < |m|}} \frac{1}{\left|m + a_1 x_1^2\right|^{\frac{1}{2}}}\, \mathrm{d}x_1 = \int\limits_{\substack{|x_1| \leq p^n \\ |a_1 x_1^2| < |m|}} \frac{1}{|m|^{\frac{1}{2}}}\, \mathrm{d}x_1$$
$$\leq \int\limits_{|a_1 x_1^2| < |m|} \frac{1}{|m|^{\frac{1}{2}}}\, \mathrm{d}x_1 \leq \frac{1}{|m|^{\frac{1}{2}}} \int\limits_{|x_1| < \left(\frac{|m|}{|a_1|}\right)^{\frac{1}{2}}} \mathrm{d}x_1 = \frac{1}{|a_1|^{\frac{1}{2}}}.$$

For the second summand in (8.1), we begin by taking a change in variables. For some e with $|e| = 1$, we have that

$$m = \frac{e}{|m|}.$$

This is because $|m|$ is a rational number and can be viewed as a p-adic number. Furthermore, $\frac{1}{|m|} = |m|$. For each x_1, there is an η with $|\eta| = \left(\frac{1}{|a_1|}\right)^{\frac{1}{2}}$ such that $x_1 = \frac{\eta}{|m|^{\frac{1}{2}}}$. We therefore have that

$$\int\limits_{\substack{|x_1|\leq p^n \\ |a_1x_1^2|=|m|}} \frac{1}{|m+a_1x_1^2|^{\frac{1}{2}}}\,\mathrm{d}x_1 \leq \int\limits_{|x_1^2|=\frac{|m|}{|a_1|}} \frac{1}{|m+a_1x_1^2|^{\frac{1}{2}}}\,\mathrm{d}x_1$$

$$\leq \int\limits_{|\eta|=\left(\frac{1}{|a_1|}\right)^{\frac{1}{2}}} \frac{1}{\left|\frac{e}{|m|}+a_1\frac{\eta^2}{|m|}\right|^{\frac{1}{2}}}\,\mathrm{d}\left(\frac{\eta}{|m|^{\frac{1}{2}}}\right)$$

$$= \int\limits_{|\eta|=\left(\frac{1}{|a_1|}\right)^{\frac{1}{2}}} \frac{1}{|m|^{\frac{1}{2}}\,|e+a_1\eta^2|^{\frac{1}{2}}}|m|^{\frac{1}{2}}\,\mathrm{d}\left(\eta\right) = \int\limits_{|\eta|=\left(\frac{1}{|a_1|}\right)^{\frac{1}{2}}} \frac{1}{|e+a_1\eta^2|^{\frac{1}{2}}}\,\mathrm{d}\eta.$$

Since $|e| = |a_1\eta^2| = 1$, we have that $|e+a_1\eta^2| \leq 1$. If A is a subset of $\mathbb{Q}_p$, denote by $\mu\left(A\right)$ the measure of A. Note that if $|e+a_1\eta^2| = p^{-n}$ holds for a given η, then the same equality will hold for η' only if $|a_1|\,|\eta-\eta'| \leq p^{-n}$. If $|a_1||\eta'-\eta| > p^{-n}$, then the formal Laurent expansion of η' and η have coefficients that differ at the $\frac{p^k}{|a_1|}$ place in the expansion for some $k < n$. We take k to be lowest power of p where the two expansions differ at the $\frac{p^k}{|a_1|}$ place. Then η^2 and $(\eta')^2$ will also differ at the $\frac{p^k}{|a_1|}$ place of their respective expansions and so the expansions of $a_1\left(\eta'\right)^2$ and e will differ at the p^k place. Therefore, $\left|e+a_1\left(\eta'\right)^2\right| > p^{-n}$. Therefore,

$$\mu\left(\left\{\eta\colon\ \left|e+a_1\eta^2\right| = p^{-n}\right\}\right) \leq \frac{p^{-n}}{|a_1|}.$$

Since

$$\mu\left(\left\{\varepsilon\colon\ \left|e+a_1\eta^2\right| = 0\right\}\right) = 0$$

and, in fact, the set may be empty depending on the choices of e, a_1 and p, we therefore have the inequality

$$\int\limits_{|\eta|=\left(\frac{1}{|a_1|}\right)^{\frac{1}{2}}} \frac{1}{|e+a_1\eta^2|^{\frac{1}{2}}}\,\mathrm{d}\eta \leq \sum_{n=0}^{\infty}\frac{1}{(p^{-n})^{\frac{1}{2}}}\mu\left(\left\{\varepsilon\colon\ \left|e+a_1\eta^2\right| = p^{-n}\right\}\right)$$

$$\leq \sum_{n=0}^{\infty}\frac{1}{(p^{-n})^{\frac{1}{2}}}\frac{p^{-n}}{|a_1|} = \frac{1}{|a_1|}\sum_{n=0}^{\infty}p^{-\frac{n}{2}} = \frac{1}{|a_1|}\frac{1}{1-\frac{1}{p^{\frac{1}{2}}}} \leq K_1$$

for some $K_1 \in \mathbb{R}$ that is independent of n.

For the third summand in (8.1), we have that

$$\int\limits_{\substack{|x_1|\leq p^n \\ |a_1x_1^2|>|m|}} \frac{1}{|m+a_1x_1^2|^{\frac{1}{2}}}\,\mathrm{d}x_1 \leq \int\limits_{\substack{|x_1|\leq p^n \\ |a_1x_1^2|>|m|}} \frac{1}{|m|^{\frac{1}{2}}}\,\mathrm{d}x_1 \leq \int\limits_{|x_1|\leq p^n} \frac{1}{|m|^{\frac{1}{2}}}\,\mathrm{d}x_1 = \frac{p^n}{|m|^{\frac{1}{2}}}.$$

There is therefore a $K \in \mathbb{N}$ that is independent of n and m such that

$$I_1(m,n) \leq \frac{Kp^n}{|m|^{\frac{1}{2}}}.$$

8.1.2. In Arbitrary Dimension. We now find an upper bound for $I_{k,a_1,\dots,a_k}(m,n)$. Suppose as the inductive hypothesis that given $k>1$ there is an $L\in\mathbb{R}$ independent of n such that

$$I_{k-1,a_1,\dots,a_{k-1}}(m,n)\le \frac{Lp^{(k-1)n}}{|m|^{\frac{1}{2}}}.$$

The base case of the induction hypothesis is verified in the one variable case above. Notice first that if $k>1$, then the positivity of the integrand together with Tonelli's Theorem gives us the equality

$$\begin{aligned} I_{k,a_1,\dots,a_k}(m,n) &= \int\limits_{||x||\le p^n} \frac{1}{\left|m+a_1x_1^2+\dots+a_kx_k^2\right|^{\frac{1}{2}}}\,\mathrm{d}x^k \\ &= \int\limits_{|x_1|<p^n} I_{k-1,a_2,\dots,a_k}\left(m+a_1x_1^2,n\right)\,\mathrm{d}x_1. \end{aligned}$$

Since $m+a_1x_1^2=0$ on a set of measure 0,

$$\begin{aligned} I_{k,a_1,\dots,a_k}(m,n) &= \int\limits_{|x_1|<p^n} I_{k-1,a_2,\dots,a_k}\left(m+a_1x_1^2,n\right)\mathrm{d}x_1 \\ &\le \int\limits_{|x_1|\le p^n} \frac{Lp^{(k-1)n}}{\left|m+a_1x_1^2\right|^{\frac{1}{2}}}\,\mathrm{d}x_1 \\ &\le Lp^{(k-1)n}\int\limits_{|x_1|\le p^n} \frac{1}{\left|m+a_1x_1^2\right|^{\frac{1}{2}}}\,\mathrm{d}x_1 \\ &\le \frac{Kp^n}{|m|^{\frac{1}{2}}}Lp^{(k-1)n} \le \left(\frac{KL}{|m|^{\frac{1}{2}}}\right)\mathrm{vol}_k(B_{p^n}) \le M\mathrm{vol}_k(B_{p^n}) \end{aligned}$$

for some real constant M that is not dependent on n.

We have proved the following theorem.

Theorem 8.1. *For each $m\in\mathbb{Q}_p$ with $m\ne 0$ and for each $k\in\mathbb{N}$, there is a constant K_m such that*

$$I_{k,a_1,\dots,a_k}(m,n)\le K_m\mathrm{vol}_k(B_{p^n}).$$

8.2. The Weighted Massless Case

8.2.1. The Weighted Quadratic Form in Dimension 2. Suppose that $a_1\ne 0$. Note that the integral $I_{1,a_1}(0,n)$ is not convergent. We therefore begin by finding bounds for $I_{2,a_1,a_2}(0,n)$ where $a_1,a_2\in\mathbb{Q}_p$ and $a_1,a_2\ne 0$. Notice that we can partition the set

$$E_n=\left\{x\in\mathbb{Q}_p^2\colon ||x||\le p^n\right\}\cap\left\{x\in\mathbb{Q}_p^2\colon x_1\ne 0\right\}\cap\left\{x\in\mathbb{Q}_p^2\colon x_2\ne 0\right\}$$

into three disjoint subsets, namely,

$$E_n=E_n'\cup E_n(1,2)\cup E_n(2,1).$$

We take

$$E_n'=E_n\cap\left\{x\in\mathbb{Q}_p^2\colon \left|a_1x_1^2\right|=\left|a_2x_2^2\right|\right\},\qquad E_n(1,2)=E_n\cap\left\{x\in\mathbb{Q}_p^2\colon \left|a_1x_1^2\right|<\left|a_2x_2^2\right|\right\},$$
$$\text{and}\qquad E_n(2,1)=E_n\cap\left\{x\in\mathbb{Q}_p^2\colon \left|a_2x_2^2\right|<\left|a_1x_1^2\right|\right\}.$$

We therefore have that

$$I_{2,a_1,a_2}(0,n)=\int\limits_{x\in E_n(1,2)}\frac{1}{\left|a_1x_1^2+a_2x_2^2\right|^{\frac{1}{2}}}\,\mathrm{d}x^2+\int\limits_{x\in E_n(2,1)}\frac{1}{\left|a_1x_1^2+a_2x_2^2\right|^{\frac{1}{2}}}\,\mathrm{d}x^2$$

$$+\int\limits_{x\in E'_n} \frac{1}{\left|a_1x_1^2+a_2x_2^2\right|^{\frac{1}{2}}}\,\mathrm{d}x^2.$$

We now compute the three integrals. Once again by the positivity of the integrand, Tonelli's Theorem implies that

$$\int\limits_{x\in E_n(1,2)} \frac{1}{\left|a_1x_1^2+a_2x_2^2\right|^{\frac{1}{2}}}\,\mathrm{d}x^2 = \int\limits_{\substack{|x_1|\le p^n\\|x_1|\neq 0}} \left\{ \int\limits_{\left|a_2x_2^2\right|<\left|a_1x_1^2\right|} \frac{1}{\left|a_1x_1^2+a_2x_2^2\right|^{\frac{1}{2}}}\,\mathrm{d}x_2 \right\} \mathrm{d}x_1$$

$$= \int\limits_{\substack{|x_1|\le p^n\\|x_1|\neq 0}} \left\{ \int\limits_{\left|a_2x_2^2\right|<\left|a_1x_1^2\right|} \frac{1}{|a_1|^{\frac{1}{2}}\,|x_1|}\,\mathrm{d}x_2 \right\} \mathrm{d}x_1$$

$$\le \int\limits_{\substack{|x_1|\le p^n\\|x_1|\neq 0}} \left\{ \int\limits_{\left|a_2x_2^2\right|\le\left|a_1x_1^2\right|} \frac{1}{|a_1|^{\frac{1}{2}}\,|x_1|}\,\mathrm{d}x_2 \right\} \mathrm{d}x_1$$

$$= \int\limits_{\substack{|x_1|\le p^n\\|x_1|\neq 0}} \left\{ \int\limits_{|x_2|\le\left|\frac{a_1}{a_2}\right|^{\frac{1}{2}}|x_1|} \frac{1}{|a_1|^{\frac{1}{2}}\,|x_1|}\,\mathrm{d}x_2 \right\} \mathrm{d}x_1 \le \int\limits_{\substack{|x_1|\le p^n\\|x_1|\neq 0}} \frac{1}{|a_2|^{\frac{1}{2}}}\,\mathrm{d}x_1 = \frac{p^n}{|a_2|^{\frac{1}{2}}}.$$

Similarly, we have that

$$\int\limits_{x\in E_n(2,1)} \frac{1}{\left|a_1x_1^2+a_2x_2^2\right|^{\frac{1}{2}}}\,\mathrm{d}x^2 \le \frac{p^n}{|a_1|^{\frac{1}{2}}}.$$

Again by the positivity of the integrand, Tonelli's Theorem, together with the computation performed in the massive one dimensional case, imply that

$$\int\limits_{x\in E'} \frac{1}{\left|a_1x_1^2+a_2x_2^2\right|^{\frac{1}{2}}}\,\mathrm{d}x^2 = \int\limits_{\substack{|x_1|\le p^n\\|x_1|\neq 0}} \left\{ \int\limits_{\left|a_2x_2^2\right|=\left|a_1x_1^2\right|} \frac{1}{\left|a_1x_1^2+a_2x_2^2\right|^{\frac{1}{2}}}\,\mathrm{d}x_2 \right\} \mathrm{d}x_1$$

$$\le \int\limits_{\substack{|x_1|\le p^n\\|x_1|\neq 0}} \left\{ \frac{1}{|a_2|\left(1-\frac{1}{p^{\frac{1}{2}}}\right)} \right\} \mathrm{d}x_1 = \frac{p^n}{|a_2|}\frac{1}{1-\frac{1}{p^{\frac{1}{2}}}}.$$

By the symmetry of the equation, we have the inequality

$$\int\limits_{x\in E'} \frac{1}{\left|a_1x_1^2+a_2x_2^2\right|^{\frac{1}{2}}}\,\mathrm{d}x^2 \le \left(\frac{1}{1-\frac{1}{p^{\frac{1}{2}}}}\right)\frac{p^n}{\max\{|a_1|,|a_2|\}}.$$

We therefore have that there exists a K independent of n such that

$$I_{2,a_1,a_2}(0,n) \le Kp^n.$$

8.2.2. In Arbitrary Dimension. Take $1 \le i \le k$ and suppose that for each i, $a_i \neq 0$. We now find an upper bound for $I_{k,a_1,\ldots,a_k}(0,n)$. Suppose as the inductive hypothesis that given $k > 1$ there is an $L \in \mathbb{R}$ such that independent of n such that

$$I_{k-1,a_1,\ldots,a_{k-1}}(0,n) < L\mathrm{vol}_{k-1}(B_{p^n})\,.$$

Notice that since the integrand of $I_{k,a_1,\ldots,a_k}(0,n)$ is positive, Tonelli's theorem implies that $I_{k,a_1,\ldots,a_k}(0,n)$ is defined in the following way. We have that

$$\begin{aligned} I_{k,a_1,\ldots,a_k}(0,n) &= \int\limits_{||x||<p^n} \frac{1}{\left|a_1x_1^2+\cdots+a_kx_k^2\right|^{\frac{1}{2}}}\,\mathrm{d}x^k \\ &= \int\limits_{|x_1|\le p^n}\cdots\int\limits_{|x_k|\le p^n} \frac{1}{\left|a_1x_1^2+\cdots+a_kx_k^2\right|^{\frac{1}{2}}}\,\mathrm{d}x_k\cdots\mathrm{d}x_1 \end{aligned}$$

$$\begin{aligned} = \int\limits_{|x_1|\le p^n}\cdots\int\limits_{|x_{k-1}|\le p^n} \Bigg\{ &\int\limits_{\substack{|x_k|\le p^n\\ \left|a_kx_k^2\right|<\left|a_1x_1^2+\cdots+a_{k-1}x_{k-1}^2\right|}} \frac{1}{\left|a_1x_1^2+\cdots+a_kx_k^2\right|^{\frac{1}{2}}}\,\mathrm{d}x_k \\ &+ \int\limits_{\substack{|x_k|\le p^n\\ \left|a_kx_k^2\right|=\left|a_1x_1^2+\cdots+a_{k-1}x_{k-1}^2\right|}} \frac{1}{\left|a_1x_1^2+\cdots+a_kx_k^2\right|^{\frac{1}{2}}}\,\mathrm{d}x_k \\ &+ \int\limits_{\substack{|x_k|\le p^n\\ \left|a_kx_k^2\right|>\left|a_1x_1^2+\cdots+a_{k-1}x_{k-1}^2\right|}} \frac{1}{\left|a_1x_1^2+\cdots+a_kx_k^2\right|^{\frac{1}{2}}}\,\mathrm{d}x_k \Bigg\}\,\mathrm{d}x_{k-1}\cdots\mathrm{d}x_1 \end{aligned}$$

$$\begin{aligned} &\le \int\limits_{|x_1|\le p^n}\cdots\int\limits_{|x_{k-1}|\le p^n} \left\{\frac{2p^n}{\left|a_1x_1^2+\cdots+a_kx_{k-1}^2\right|^{\frac{1}{2}}} + \frac{1}{|a_k|}\frac{1}{1-\frac{1}{p^{\frac{1}{2}}}}\right\}\mathrm{d}x_{k-1}\cdots\mathrm{d}x_1 \\ &= 2p^n\int\limits_{|x_1|\le p^n}\cdots\int\limits_{|x_{k-1}|\le p^n} \left\{\frac{1}{\left|a_1x_1^2+\cdots+a_{k-1}x_{k-1}^2\right|^{\frac{1}{2}}}\right\}\mathrm{d}x_{k-1}\cdots\mathrm{d}x_1 + \left(\frac{1}{1-\frac{1}{p^{\frac{1}{2}}}}\right)\frac{1}{|a_k|}p^{(k-1)n} \\ &= 2p^nI_{k-1,a_1,\ldots,a_{k-1}}(0,n) + \left(\frac{1}{1-\frac{1}{p^{\frac{1}{2}}}}\right)\frac{1}{|a_k|}p^{(k-1)n} \le Mp^{nk} = M\mathrm{vol}_k(B_{p^n}) \end{aligned}$$

for some real number M independent of n.

We have therefore proved the following theorem.

Theorem 8.2. *For each $k \in \mathbb{N}$, and for each collection of nonzero elements $a_1,\ldots,a_k$ of $\mathbb{Q}_p$, there is a constant M such that*

$$I_{k,a_1,\ldots,a_k}(0,n) \le M\,\mathrm{vol}_k\left(B_{p^n}\right).$$

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