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RESEARCH ARTICLES

Structure, Classification, and Conformal Symmetry of Elementary Particles over Non-Archimedean Space-Time*

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Abstract—It is well known that at distances shorter than Planck length, no length measurements are possible. The Volovich hypothesis asserts that at sub-Planckian distances and times, spacetime itself has a non-Archimedean geometry. We discuss the structure of elementary particles, their classification, and their conformal symmetry under this hypothesis. Specifically, we investigate the projective representations of the p-adic Poincaré and Galilean groups, using a new variant of the Mackey machine for projective unitary representations of semidirect products of locally compact and second countable (lcsc) groups. We construct the conformal spacetime over p-adic fields and discuss the imbedding of the p-adic Poincaré group into the p-adic conformal group. Finally, we show that the massive and the so called eventually masssive particles of the Poincaré group do not have conformal symmetry. The whole picture bears a close resemblance to what happens over the field of real numbers, but with some significant variations.

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1. INTRODUCTION

Divergences in quantum field theories led many physicists, notably Beltrametti and his collaborators, to propose in the 1970's the idea that one should include the structure of space-time itself as an unknown to be investigated [1–4]. In particular they suggested that the geometry of space-time might be based on a non-archimedean or even a finite field, and examined some of the consequences of this hypothesis. But the idea did not really take off until Volovich proposed in 1987 [6] that world geometry at sub-Planckian regimes might be non-archimedean because no length measurements are possible at such ultra-small distances and time scales. A huge number of articles have appeared since then, exploring this theme. Since no single prime can be given a distinguished status, it is even more natural to see if one could really work with an adelic geometry as the basis for space-time. Such an idea was first proposed by Manin [7]. For a definitive survey and a very inclusive set of references concerning p-adic mathematical physics see the article by Dragovich et al [8]. It is not our contention that there is sufficient experimental evidence for a non-archimedean or adelic spacetime. Rather we explore this question in the so-called *Dirac mode*, namely to do the mathematics first *and then* to seek the physical interpretation (see [9], p. 371).

In this paper we examine the consequences of the non-archimedean hypothesis for the classification of elementary particles. We consider both the Poincaré and the Galilean groups. Each of these is the group of k-points of a linear algebraic group defined over a local non-archimedean field k of characteristic $\neq 2$.

Beyond the classification of elementary particles with Poincaré and conformal symmetry lies the problem of constructing quantum field theories over p-adic spacetimes. For a deep study of this question see the paper of Kochubei and Sait-Ametov [10].

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It is a consequence of the basic principles of quantum mechanics (see [13]) that the symmetry of a quantum system with respect to a group G may be expressed by a projective unitary representation (PUR) of G (or at least of a normal subgroup of index 2 in G) in the Hilbert space of quantum states; this PUR may be lifted to an ordinary unitary representation (UR) of a suitable topological central extension (TCE) of it by the circle group T. Already in 1939, Wigner, in his great paper [11], proved that all PUR's of the Poincaré group P lift to UR's of the simply connected (2-fold) covering group P^* of the Poincaré group. In other words, $P^* = V \rtimes \mathrm{Spin}(V)$ is already the *universal* TCE of the Poincare group (UTCE). Here V is a *real* quadratic vector space, namely a real vector space with a quadratic form, of signature $(1, n)$ that defines the Minkowski metric, and $\mathrm{Spin}(V)$ is the spin group, which is the simply connected covering group of the orthogonal group $\mathrm{SO}(V)$. We note that for $n = 3$, $\mathrm{Spin}(V) = \mathrm{SL}(2, \mathbb{C})_{\mathbb{R}}$, the suffix $\mathbb{R}$ denoting the fact that we view $\mathrm{SL}(2, \mathbb{C})$ as a real group. For the real Galilean group, going to the simply connected covering group is not enough to unitarize all PUR's. One has to construct the UTCE (see [5]).

Not all groups have UTCE's. For a lcsc group to have a UTCE it is necessary that the commutator subgroup should be dense in it. Over a non-archimedean local field, the commutator subgroups of the Poincaré group and the orthogonal groups are open and closed *proper* subgroups and so they do not have UTCE's. The spin groups and the Poincaré groups associated to the spin groups *do have* UTCE's; for the spin groups this is a consequence of the work of Moore [14] and Prasad and Raghunathan [15] and for the corresponding Poincaré groups, of the work of Varadarajan [16]. However, the natural map from the spin group or the corresponding Poincaré group to the orthogonal group or the corresponding Poincaré group is *not* surjective over the local non-archimedean field k (even though they are surjective over the algebraic closure of k), and so replacing the orthogonal group by the spin group leads to a loss of information. So we have to work with the orthogonal group rather than the spin group. The following example, treated in [12], illustrates this.

Let $G = \mathrm{SL}(2, \mathbb{Q}_p)$. The adjoint representation exhibits G as the spin group corresponding to the quadratic vector space $\mathfrak{g}$ which is the Lie algebra of G equipped with the Killing form. The adjoint map $G \longrightarrow G_1 = \mathrm{SO}(\mathfrak{g})$ is the spin covering for $\mathrm{SO}(\mathfrak{g})$ but this is *not surjective*; in the standard basis

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

the spin covering map is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a^2 & -2ab & -b^2 \\ -ac & ad+bc & bd \\ -c^2 & 2cd & d^2 \end{pmatrix}.$$

The matrix

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha^{-1} \end{pmatrix}$$

is in $\mathrm{SO}(\mathfrak{g})$; if it is the image of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $b = c = 0, d = a^{-1}$, and $\alpha = a^2$, so that unless $\alpha \in \mathbb{Q}_p^{\times 2}$, this will not happen.

So in this paper we work with the orthogonal groups rather than the spin groups. This means that we have to deal with projective UR's of the Poincaré and Galilean groups directly.

An announcement containing the main results of this paper (without proofs) has appeared in the Letters in Mathematical Physics [12]. The present article is an elaboration of this announcement, with proofs.

2. MULTIPLIERS AND PURS FOR SEMIDIRECT PRODUCT GROUPS

2.1. Multipliers for semidirect products

We assume that the reader is familiar with the basic facts regarding multipliers [16]. We begin by discussing the multiplier group of a semidirect product. We work in the category of locally compact second countable (lcsc) groups.

Multipliers of a group K form a group $Z^2(K)$. The subgroup of trivial multipliers is denoted by $B(K)$. We define $H^2(K) = Z^2(K)/B(K)$ to be the *multiplier group of* K. If $m \in Z^2(K)$ we define a m-representation of K to be a Borel map $x \longmapsto U(x)$ of K into the unitary group of a Hilbert space $\mathcal{H}$ (which is a standard Borel group) such that $U(x)U(y) = m(x,y)U(xy) \qquad (x, y \in K)$. If K is totally disconnected, every multiplier is equivalent to a continuous one and the subgroup of continuous multipliers has the property that the natural inclusion map induces an isomorphism with $H^2(K)$. This is true for the p-adic groups.

Let $H = A \rtimes G$, where A and G are lcsc groups and A is abelian. Let A^* be the character group of A. Our starting point is to investigate the subgroup of multipliers of H that are trivial when restricted to A, denoted by $M_A(H)$. Let $H^2_A(H)$ be its image in $H^2(H)$. Let $M'_A(H)$ be the group of multipliers m for H with $m|_{A\times A} = m|_{A\times G} = 1$. Results from [16] and [18] tell us that any element of $M_A(H)$ is equivalent to one in $M'_A(H)$. We define a 1-cocycle for G with coefficients in A^* as a Borel map $f(G \to A^*)$ such that $f(gg') = f(g) + g[f(g')] \;\; (g, g' \in G)$. This is equivalent to saying that $g \mapsto (f(g), g)$ is a Borel homomorphism of G into the semidirect product $A^* \rtimes G$. Hence any 1-cocycle is continuous and defines a continuous map of $G \times A$ into T. We denote the abelian group of 1-cocycles by $Z^1(G, A^*)$. The coboundaries are the cocycles of the form $g \mapsto g[a] - a$ for some $a \in A^*$. The coboundaries form a subgroup $B^1(G, A^*)$ of $Z^1(G, A^*)$. We now form the cohomology group $H^1(G, A^*) = Z^1(G, A^*)/B^1(G, A^*)$. The following theorem due to Mackey describes the multipliers of H. Full details can be found in [16].

Theorem 1. *Any element in $M_A(H)$ is equivalent to one in $M'_A(H)$. If $m \in M'_A(H)$ and $m_0 = m|_{G\times G}$ and $\theta_m(g^{-1})(a') = m(g, a')$, then $m \mapsto (m_0, \theta_m)$ is an isomorphism $M'_A(H) \simeq Z^2(G) \times Z^1(G, A^*)$, which is well defined in cohomology and gives the isomorphisms $H^2_A(H) \simeq H^2(G) \times H^1(G, A^*)$. Moreover, $m(ag, a'g') = m_0(g, g')\theta_m(g^{-1})(a')$.*

Corollary 12. *If $m_0 = 1$, then m is a continuous multiplier and $m(ag, a'g') = \theta(g^{-1})(a')$.*

Remark: A multiplier m for H is said to be *standard* if $m|_{A\times A} = m|_{A\times G} = m|_{G\times A} = 1$. It follows from the above that a multiplier for H is standard if and only if it is the lift to H of a multiplier for G via $H \to H/A \simeq G$.

2.2. m-Systems of imprimitivity

Classically, systems of imprimitivity are the key to finding UIR's of semidirect products. In this section we utilize m-systems of imprimitivity to describe the PUIR's of semidirect product groups. This is a straightforward variation of the corresponding theory for ordinary systems of imprimitivity.

We assume the following setup. Let G be a lcsc group. Let X be a G-space that is also a standard Borel space. Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{U}$ the group of unitary transformations of $\mathcal{H}$. An *m-system of imprimitivity* is a pair (U, P), where $P(E \to P_E)$ is a projection valued measure (pvm) on the class of Borel subsets of X, the projections being defined in $\mathcal{H}$, and U is an m-representation of G in $\mathcal{H}$ such that

$$U(g)P(E)U(g)^{-1} = P(g[E]) \;\; \forall\, g \in G \text{ and all Borel } E \subset X.$$

The pair (U, P) is said to be *based on* X. For what follows we take X to be a *transitive* G-space. We fix some $x_0 \in X$ and let G_0 be the stabilizer of x_0 in G, so that $X \simeq G/G_0$. We will also fix a multiplier m for G and let $m_0 = m|_{G_0\times G_0}$.

Theorem 2. *There is a natural one to one correspondence between m_0-representations μ of G_0 and m-systems of imprimitivity $S_\mu := (U, P)$ of G based on X. Under this correspondence, we have a ring isomorphism of the commuting ring of μ with that of S_μ, so that irreducible μ correspond to irreducible S_μ.*

The proof is straightforward and we have relegated it to the Arxiv version of this paper for brevity. See [23].

2.3. The Mackey machine for projective unitary irreducible representations of semidirect products

We now turn our attention to the Mackey treatment of lcsc groups with a semidirect product structure. In this section we are going to consider a group $H = A \rtimes G$ where G and A are lcsc groups and A is abelian. We concern ourselves only with multipliers of H that are trivial when restricted to $A \times A$. We recall that these multipliers are completely described by Theorem 1. The following lemma introduces the key idea that is needed for the variant of the Mackey machine for semidirect products when we deal with projective unitary representations.

Lemma 1. *Let $\phi : G \to A^*$ be a continuous map with $\phi(1) = 0$. Define $g\{\chi\} = g_\phi\{\chi\} = g[\chi] + \phi(g)$, for $g \in G, \chi \in A^*$. Then $a_\phi : (g, \chi) \mapsto g_\phi\{\chi\}$ defines an action of G on A^* if and only if $\phi \in Z^1(G, A^*)$.*

Proof. If a_ϕ is to be an action on A^*, then $g_2\{g_1\{\chi\}\} = g_2g_1\{\chi\}$ for all $\chi \in A^*$. Now, $g_2\{g_1\{\chi\}\} = g_2[g_1[\chi] + \phi(g_1)] + \phi(g_2) = g_2[g_1[\chi]] + g_2[\phi(g_1)] + \phi(g_2)$. On the other hand, $g_2g_1\{\chi\} = g_2g_1[\chi] + \phi(g_2g_1)$. Equating the two we see that the condition on ϕ is $\phi(g_2g_1) = g_2[\phi(g_1)] + \phi(g_2)$, that is, $\phi \in Z^1(G, A^*)$. □

If $\phi' \in Z^1(G, A^*)$ defines the same element as ϕ in $H^1(G, A^*)$, then $\phi'(g) = \phi(g) + g[\chi_0] - \chi_0$ for some $\chi_0 \in A^*$. So,

$$g_{\phi'}\{\chi\} = g[\chi] + \phi'(g) = g[\chi] + g[\chi_0] - \chi_0 = g[\chi + \chi_0] - \chi_0.$$

Let $\tau : \chi \mapsto \chi + \chi_0$ be the translation by χ_0 in A^*. Then, $g_{\phi'} = \tau^{-1} g_\phi \tau$. So the actions defined by ϕ and ϕ' are equivalent in this strong sense.

Definition 29. The action $a_\phi : (g, \chi) \mapsto g_\phi\{\chi\}$ is called the affine action of G on A^* determined by ϕ.

Theorem 3. *Fix $\theta \in Z^1(G, A^*)$ and $m \in M'_A(H)$, $m \simeq (m_0, \theta)$. Then there is a natural bijection between m-representations V of $H = A \rtimes G$ and m_0-systems of imprimitivity (U, P) on A^* for the affine action $g, \chi \mapsto g_\theta\{\chi\}$, defined by θ. The bijection is given by:*

$$V(ag) = U(a)U(g), \quad U(a) = \int_{A^*} \langle a, \chi\rangle dP(\chi).$$

Proof. The assumption $m \simeq (m_0, \theta)$ means that $m(ag, a'g') = m_0(g, g')\theta(g^{-1})(a')$, where m_0 is a multiplier for G and θ is a cocycle in $Z^1(G, A^*)$. Let V be a m-representation for H and let us write U for the restriction of V to A and G. Then U is an ordinary representation of A as well as a m_0-representation of G and $V(ag) = U(a)U(g)$. Moreover, $U(g)U(a)U(g)^{-1} = \theta(g^{-1})(a)U(g[a])$. Indeed, we have $\theta(g^{-1})(a) = m(g, a)$ and

$$U(g)U(a) = m(g, a)V(ga) = m(g, a)V(g[a]g) = m(g, a)U(g[a])U(g).$$

Since U is an ordinary representation on A, there exists a unique pvm (projection valued measure) P on A^* such that:

$$U(a) = \int_{A^*} \langle a, \chi\rangle dP(\chi) \quad (a \in A).$$

Thus,

$$U(g)U(a)U(g)^{-1} = \int_{A^*} \langle a, \chi\rangle dQ_g(\chi).$$

Here, Q_g is the pvm defined by $Q_g(E) = U(g)P(E)U(g)^{-1}$. On the other hand,

$$U(g)U(a)U(g)^{-1} = \theta(g^{-1})(a)U(g[a]) = \theta(g^{-1})(a) \int_{A^*} \langle g[a], \chi\rangle dP(\chi)$$

and the right side can be rewritten as:

$$\int_{A^*} \langle a, g^{-1}[\chi] + \theta(g^{-1})\rangle dP(\chi) = \int_{A^*} \langle a, g^{-1}\{\chi\}\rangle dP(\chi)$$

so that

$$U(g)U(a)U(g)^{-1} = \int_{A^*} \langle a, g^{-1}\{\chi\}\rangle dP(\chi).$$

Now, if t is a Borel automorphism of A^* as a Borel space and f is a bounded Borel function on A^*, then

$$\int_{A^*} f(t^{-1}(\chi))dP(\chi) = \int_{A^*} f(\chi)dP_t(\chi) \qquad (*),$$

where P_t is the pvm defined by: $P_t(E) = P(t[E])$. To see this, observe that $(*)$ is true if $f = 1_E$, the characteristic function of a Borel set $E \subset A^*$; hence $(*)$ is true for f which are finite linear combinations of such characteristic functions, and hence also for all their uniform limits, which are precisely all bounded Borel functions. Hence we get:

$$U(g)U(a)U(g)^{-1} = \int_{A^*} \langle a, g^{-1}\{\chi\}\rangle dP(\chi) = \int_{A^*} \langle a, \chi\rangle dP_g(\chi),$$

where P_g is the pvm defined by: $P_g(E) = P(g\{E\})$. But we had seen that

$$U(g)U(a)U(g)^{-1} = \int_{A^*} \langle a, \chi\rangle dQ_g(\chi).$$

Hence,

$$\int_{A^*} \langle a, \chi\rangle dQ_g(\chi) = \int_{A^*} \langle a, \chi\rangle dP_g(\chi),$$

showing that $Q_g = P_g$ or $U(g)P(E)U(g)^{-1} = P(g\{E\})$. We have thus shown that for the action of G on A^* by $g, \chi \mapsto g\{\chi\}$, (U, P) is an m_0-system of imprimitivity. Conversely, suppose (U, P) is an m_0-system of imprimitivity for this action. Then, by retracing the steps in the above calculation with $U(a) = \int_{A^*} \langle a, \chi\rangle dP(\chi)$ we find:

$$U(g)U(a)U(g)^{-1} = \theta(g^{-1})(a)U(g[a]).$$

If we define $V(ag) = U(a)U(g)$, then V becomes an m-representation, where

$$m(ag, a'g') = m_0(g, g')\theta(g^{-1})(a').$$

□

We need the following definition.

Definition 30. If U is a UR of A and $E \mapsto P(E)$ is its associated pvm, we say that $Spec(U) \subset F$, if $P(F) = I$. Here F is some Borel set in A^*. We extend this terminology to any PUR of $H = A \rtimes G$ that is a UR on A.

By combining theorems 2 and 3 we obtain the basic theorem of irreducible m-representations of H.

Theorem 4. *Fix $\chi_0 \in A^*$, $m \simeq (m_0, \theta)$. Then there is a natural bijection between irreducible m-representations V of $H = A \rtimes G$ with $Spec(V) \subset G\{\chi_0\}$ (the orbit of χ_0 under the affine action) and irreducible m_0-representations of G_{χ_0}, the stabilizer of χ_0 in G for the affine action. If the affine action is regular, every irreducible m-representation of H, up to unitary equivalence, is obtained by this procedure. Let $X = G\{\chi_0\}$ and λ be a σ-finite quasi-invariant measure for the action of G. Then, for any irreducible m_0-representation μ of G_{χ_0} in the Hilbert space $\mathcal{K}$, the corresponding m-representation V acts on $L^2(X, \mathcal{K}, \lambda)$ and has the following form:*

$$(V(ag)f)(\chi) = \langle a, \chi\rangle \rho_g(g^{-1}\{\chi\})^{\frac{1}{2}})\delta(g, g^{-1}\{\chi\})f(g^{-1}\{\chi\}),$$

where δ is any strict m_0-cocyle for (G, X) with values in $\mathcal{U}$, the unitary group of $\mathcal{K}$, such that $\delta(g, \chi_0) = \mu(g),\ g \in G_{\chi_0}$.

We note that the ρ_g-factors drop out if λ is an invariant measure.

Corollary 13. *Suppose $H^1(G, A^*) = 0$. Then we can take $\theta(g) = 1$ so that $m(ag, a'g') = m_0(g, g')$. In this case, the affine action reduces to the ordinary action.*

3. PUIR'S OF THE p-ADIC POINCARÉ GROUP AND PARTICLE CLASSIFICATION

3.1. Preliminaries

We shall now discuss the PUIRs of the p-adic Poincaré group. For all that follows we work over the field $\mathbb{Q}_p$. All of the groups described will be algebraic groups defined over $\mathbb{Q}_p$, so that the groups of their $\mathbb{Q}_p$-points are p-adic Lie groups, in particular lcsc. By the Poincaré group we mean the group $P_V = V \rtimes \mathrm{SO}(V)$ where V is a finite-dimensional quadratic vector space over $\mathbb{Q}_p$. Elementary particles correspond to PUIRs of the Poincaré group. Our aim is to describe these PUIRs and thus classify the elementary particles associated to the p-adic Poincaré group.

We want to establish first that the PUIRs of the Poincaré group are indeed described by Theorem 4. This means we must establish that P_V satisfies the criteria required for Theorem 4. We shall replace V^* by the algebraic dual V' of V since V^*, the topological dual of V, is isomorphic to the algebraic dual V', the isomorphism being natural and compatible with actions of $\mathrm{GL}(V)$. See for example [17]. The isomorphism is easy to set up but depends on the choice of a non-trivial additive character on $\mathbb{Q}_p$, say ψ. Once we choose ψ, then, for any $p \in V'$, $\chi_p : a \longmapsto \psi(\langle a, p\rangle)$ is in V^*, and $p \longmapsto \chi_p$ is a topological group isomorphism of V' with V^*.

We note that the cohomology $H^1(\mathrm{SO}(V), V')$ is trivial. This is because $\mathrm{SO}(V)$ is semisimple, and so, by theorem 3 of [16], $H^1(\mathrm{SO}(V), V') = 0$. Hence, by earlier remark, every multiplier of P_V is equivalent to the lift of a multiplier of $\mathrm{SO}(V)$.

Theorem 4 requires that the action of $\mathrm{SO}(V)$ on V' is regular. Since the quadratic form on V is nondegenerate, we have, canonically, $V \simeq V'$. We transfer the quadratic form in V to V', denoting it again by $(\cdot, \cdot)$. The action of $\mathrm{SO}(V)$ on V^* then goes over to the action of $\mathrm{SO}(V)$ on V'. Since the quadratic form on V' is invariant under $\mathrm{SO}(V)$, the level sets of the quadratic form are invariant sets. Under $\mathrm{SO}(V')$, V' decomposes into invariant sets of the following types.

(a) The sets $M_a = \{p \in V' \mid (p, p) = a \neq 0\}$.

(b) The set $M_0 = \{p \in V' \mid (p, p) = 0,\ p \neq 0\}$.

(c) The set $\{0\}$.

We may think of the elements of V' as momenta although this is just formal.

Lemma 2. *If $\dim(V) \geq 3$, the sets M_a and $\{0\}$ are all the orbits. Moreover, the action is regular.*

Proof. First, take $a \neq 0$ and $p, p' \in M_a$. The map that takes p to p' is an isometry between their one dimensional spans, and so, we can extend it to an isometry t of V with itself. If $\det(t) = 1$, we are done, as $t \in \mathrm{SO}(V)$. Suppose $\det(t) = -1$. If we can find an isometry s fixing p with $\det(s) = -1$, then $u = ts$ will be in $\mathrm{SO}(V)$ and take p to p'. To see that we can find such an s, notice that for $U = p^\perp$, we have $V = U \oplus \langle p\rangle$; as $\dim(U) \geq 1$ we can find $s' \in \mathrm{O}(U)$ with $\det(s') = -1$. Then s can be defined as s' on U and $sp = p$, and we are done. Let $a = 0$ and $p, p' \in M_0$. The argument is the same as before and we are reduced to finding s as before. We can find $q \in V$ such that $(q, q) = 0$ and $(p, q) = 1$. Let W be the span of p and q. Then the quadratic form of V is non-degenerate when restricted to W and so $V = W \oplus W^\perp$. We have $\dim(W^\perp) \geq 1$ and so we can find $s' \in \mathrm{O}(W^\perp)$ with $\det(s') = -1$. Then s is defined as s' on $W^\perp$ and identity on W, and we are done. Since $\{0\}$ is trivially an orbit, we are finished. The regularity follows from the theorem of Effros [21] as all orbits are obviously either closed or locally closed. $\square$

Definition 31. We will call the orbits $M_a (a \neq 0)$ massive, the orbit M_0 massless, and $\{0\}$ trivial-massless.

Next, Theorem 4 requires that the multipliers of P_V be trivial when restricted to V. To see this we use Corollary 2 to Proposition 2 of Section 4 of [16] and reduce the proof to showing that 0 is the only skew symmetric invariant bilinear form on V. But V is irreducible under the $\mathrm{SO}(V)$ and admits a symmetric invariant bilinear form, namely $(\cdot,\cdot)$. Hence, any invariant bilinear form must be a multiple of this, and so, a skew symmetric invariant bilinear form must be 0.

Finally, we shall show that all of the orbits admit invariant measures.

Lemma 3. *For V of any dimension ≥ 1, all the orbits of $\mathrm{SO}(V)$ admit invariant measures.*

Proof. Let G be a unimodular lcsc group, and H is a closed subgroup of G; then for G/H to admit a G-invariant measure it is well known that the unimodularity of H is a sufficient condition. We apply this to our present situation. For $p \in V$, let L_p be its stabilizer. We shall check that L_p is unimodular for all p. We also check that the Poincaré group is unimodular, as it is needed in the proof.

Poincaré group. Here $P = V \rtimes G$ where $G = \mathrm{SO}(V)$. The group G acts on V with determinant 1 and so the action of the corresponding p-adic group G_p on V_p preserves the Haar measure on V_p. It is then easy to see that the product measure $dvdg$ is invariant under both left and right translations of P, dv, dg being the respective Haar measures on V, G, provided we know that G_p is unimodular. If $\dim(V) = 1$, $G = \{e\}$ and there is nothing to prove. If $\dim(V) = 2$, then G is abelian and so G_p is unimodular. Let $\dim(V) \geq 3$. Then G is semisimple. For G_p to be unimodular it is enough to check that its action on its Lie algebra has determinant with p-adic absolute value 1. Actually, its determinant itself is 1. It is enough to verify this last statement at the level of the algebraic closure of $\mathbf{Q}_p$, where it follows from the fact that over the algebraic closure G is its own commutator group and so any morphism into an abelian algebraic group is trivial.

The stabilizer of a massive point. First consider $p \in M_a,\ a \neq 0$. Then as we saw in the proof of the previous lemma, $V = U \oplus \langle p \rangle$, and $s \in \mathrm{SO}(V)$ fixes p if and only if it leaves U invariant and restricts to an element of $\mathrm{SO}(U)$ on U. Hence $L_p \simeq \mathrm{SO}(U)$, which is thus unimodular as observed above.

Stabilizer of a massless point. Let $p \in M_0$. We shall show in Theorem 6 that $L_p \simeq P_W$, where P_W is the Poincaré group of a quadratic vector space W (with $\dim(W) = \dim(V) - 2$ and W is Witt equivalent to V). Hence, P_W is unimodular from above. □

We now see that the Theorem 4 applies to the p-adic Poincaré group and we summarize our results in the following theorem that completely describes the particles of the p-adic Poincaré group. Recall that every multiplier for P_V is the lift to P_V of a multiplier for $\mathrm{SO}(V)$, up to equivalence. For any $p \in V$ we denote by λ_p an invariant measure on the orbit of V. If m_0 is a multiplier for $\mathrm{SO}(V)$ and m its lift to P_V, we write m_p for the restriction of m to the stabilizer of p in $\mathrm{SO}(V)$.

Theorem 5. *Let $P_V = V \rtimes SO(V)$ be the p-adic Poincaré group. Fix $p_0 \in V'$ and let m_0 be a multiplier of $SO(V)$ and m its lift to P_V. Then there is a natural bijection between irreducible m-representations of $P_V = V \rtimes SO(V)$ with $Spec(V) \subset SO(V)[p_0]$, the orbit of p_0 under the ordinary action of $SO(V)$ and irreducible m_{p_0}-representations of $SO(V)_{p_0}$, the stabilizer of p_0 in $SO(V)$. Every PUIR of P_V, up to unitary equivalence, is obtained by this procedure. Let $X = SO(V)[p_0]$, λ_{p_0} a σ-finite invariant measure on X for the action of $SO(V)$. Then, for any irreducible m_{p_0}-representation μ of $SO(V)_{p_0}$ in the Hilbert space $\mathcal{K}$, the corresponding m-representation U acts on $L^2(X, \mathcal{K}, \lambda_{p_0})$ and has the following form:*

$$(U(ag)f)(p) = \psi(\langle a, p\rangle)\delta(g, g^{-1}\{p\})f(g^{-1}\{p\})$$

where δ is any strict m_{p_0}-cocyle for $(SO(V)), X)$ with values in $\mathcal{U}$, the unitary group of $\mathcal{K}$, such that $\delta(g, p_0) = \mu(g),\ g \in SO(V)_{p_0}$.

Thus, to determine the PUIRs of the Poincaré group, one must determine the multipliers of $L = \mathrm{SO}(V)$ and for each given multiplier m, determine the irreducible m-representations. The PUIRs then correspond to $p \in V'$ and m_p-representations of the stabilizer L_p of p in L, m_p being $m|_{L_p \times L_p}$.

We now define massless and massive particles.

Definition 32. A PUIR of the Poincaré group is called an elementary particle. A particle which corresponds to the orbit of a vector $p \in V'$ is called massless if $p \neq 0$ and is massless ($(p,p) = 0$), trivial if $p = 0$, and massive if p is massive ($(p,p) \neq 0$).

4. GALILEAN GROUP

4.1. Galilean group over $\mathbb{R}$

Classically, the Galilean group is the group of translations, rotations, and boosts, of spacetime consistent with Newtonian mechanics. Let $V_0 = \mathbb{R}^3$ be space with $x = (x_1, x_2, x_3)$ as space coordinates and $V_1 = \mathbb{R}$ be time with t as the time coordinate. We define spacetime as $V = V_0 \oplus V_1$, and we write for $w \in V$, $w = (x, t)$. Then a Galilean transformation $g : w = (x, t) \mapsto w' = (x', t')$ is defined by

$$g : w \mapsto w' \qquad x' = Wx + tv + u, \quad t' = t + \eta.$$

Here $W \in \mathrm{SO}(3)$, u and v are vectors in 3-space and η is a real number. In this transformation u is a spatial translation, η is a time translation, v is a boost. We may think of v as a velocity vector and W as a rotation in the 3-space. The set of all such transformations forms the Galilean group. The Galilean group is a semidirect product $V \rtimes R$ of the group V of all translations in spacetime and the group $R = V_0 \rtimes R_0$. Here $R_0 = \mathrm{SO}(V_0)$. The subgroup R is not semisimple. This creates some subtle differences between the theory involving the Poincaré group and the theory involving the Galilean group [13], p. 283–284.

4.2. Galilean group over $\mathbb{Q}_p$

We now define the analogue of the Galilean group over $\mathbb{Q}_p$. Let V be a finite-dimensional vector space over $\mathbb{Q}_p$ such that $V = V_0 \oplus V_1$ where V_0 is an isotropic quadratic vector space and V_1 has dimension 1, which we identify with $\mathbb{Q}_p$. The Galilean group is now defined as $G = V \rtimes R$ where $R = V_0 \rtimes \mathrm{SO}(V_0)$. Technically, one should think of this as a pseudo-Galilean group since in the real case V_0 is anisotropic. We need the presence of isotropic vectors in V_0 as a technical requirement that we cannot do away with. As before, the action of $((u, \eta), (v, W)) \in G$ on V is given by $((u, \eta), (v, W)) : (x, t) \mapsto (Wx + tv + u, t + \eta)$. Let $(\cdot, \cdot)$ be the bilinear form on V_0 that describes its quadratic structure. Given a pair $(\xi, t) \in V$, we define a linear form $\langle(\xi, t), \cdot\rangle$ on V by $\langle(\xi, t), (u, \eta)\rangle = (\xi, u) + t\eta$. We identify the algebraic dual V' with the set of all such pairs (ξ, t). We now describe the action of R on V. $(v, W) : (u, \eta) \mapsto (Wu + \eta v, \eta)$. The action of R on V' is given by $(v, W) : (\xi, t) \mapsto (W\xi, t - (W\xi, v))$.

4.3. Particle classification of the p-adic Galilean group

The study of particles of the real Galilean group corresponds to the study of particles of ordinary non-relativistic quantum mechanics. A natural question that arises is that of classifying particles of the p-adic Galilean group. This classification is a consequence of Theorem 4. It is noteworthy that in the presence of a nontrivial affine action the theorem differs from the usual Mackey theorem.

4.4. Multipliers of the Galilean group

To determine $H^2(G)$, we must first show that the multipliers of G are trivial when restricted to V. This reduces to showing that 0 is the only R-invariant skew symmetric bilinear form on V (Corollary 2 to Proposition 2 of Section 4 of [16]). Let B be one such. As V_0 is invariant under R with the action $(v, W), x \mapsto Wx$, the restriction B_0 of B to V_0 is also skew symmetric and R_0-invariant. But V_0 already has a R_0-invariant *symmetric* form, namely $(\cdot, \cdot)$, which is non-degenerate. Since V_0 is irreducible under R_0, *any* R_0-invariant bilinear form has to be a multiple of this, and so, B_0 being skew symmetric, we may conclude that $B_0 = 0$. Now $V_1 = \mathbb{Q}_p$ and we take $\beta = 1$ as the basis vector for V_1. Let $f(x) = B(x, \beta), x \in V_0$. Now (v, W) acts on V_0 as $x \mapsto Wx$ and on $\beta \in V_1$ as $\beta \mapsto v + \beta$ and so the condition for invariance is $B(x, \beta) = B(Wx, v + \beta)$ for all $x, v \in V_0$. Thus, as we have already seen that $B_0 = 0$, we have $f(x) = f(Wx)$ or that f is an R_0-invariant linear form. By irreducibility of V_0 under R_0 we now have $f = 0$. Since $B(\beta, \beta) = 0$ as B is skew symmetric, we have proved that $B = 0$.

From [16] we know that $H^1(R, V')$ is a vector space over $\mathbb{Q}_p$ and is isomorphic to $\mathbb{Q}_p$:

$$H^1(R, V') \simeq \mathbb{Q}_p.$$

In [16] the cocycles that describe this one-dimensional cohomology were explicitly given. For $\tau \in \mathbb{Q}_p$ let $\theta_\tau(v, W) = (2\tau v, -\tau(v, v))$, where the right side is interpreted as an element of V' according to the conventions established earlier. It is then directly verifiable that the θ_τ are in $Z^1(R, V')$, and the result of [16] is that $\tau \longmapsto [\theta_\tau]$ is an isomorphism of $\mathbb{Q}_p$ with $H^1(R, V')$. If ψ is the additive character of $\mathbb{Q}_p$ fixed earlier, then $\psi \circ \theta_\tau$ is the corresponding element of $Z^1(R, V^*)$.

We can now determine the multiplier μ_τ corresponding to the θ_τ by the isomorphism of Theorem 1 Let

$$r = ((u, \eta), (v, W)), \quad r' = ((u', \eta'), (v', W')).$$

Then,

$$\mu_\tau(r, r') = \psi\Big(\theta_\tau((v, W)^{-1})(u', \eta')\Big).$$

But,

$$\theta_\tau((v, W)^{-1}) = \theta_\tau(-W^{-1}v, W^{-1}) = (-2\tau W^{-1}v, -\tau(v, v))$$

so that

$$\theta_\tau((v, W)^{-1})(u', \eta') = -2\tau(W^{-1}v, u') - \tau\eta'(v, v) = -2\tau(v, Wu') - \tau\eta'(v, v).$$

Hence,

$$\mu_\tau(r, r') = \psi\Big(-2\tau(v, Wu') - \tau\eta'(v, v)\Big).$$

In view of Theorem 1, we have the isomorphism:

$$H^2(G) \approx H^2(R) \times H^1(R, V').$$

Now R itself is a semidirect product $V_0 \rtimes R_0$ but now R_0 is semisimple. As R_0 acts irreducibly on V_0 with a *symmetric* non-degenerate invariant bilinear form, we see as before that 0 is the only invariant *skew symmetric* invariant bilinear form. Hence, all multipliers of R are trivial when restricted to V. Thus, by Theorem 1 we have

$$H^2(R) \approx H^2(R_0) \times H^1(R_0, V_0').$$

But R_0 is connected semisimple and so, by Theorem 3 of Section 6 of [16] we have $H^1(R_0, V_0') = 0$. Hence, $H^2(R) \approx H^2(R_0)$. In other words, every multiplier of R is equivalent to a lift to R of a multiplier of R_0.

These remarks allow us to give a complete explicit description of $H^2(G)$. Let n_0 be a multiplier for R_0. We lift n_0 to the multiplier n of G by the composition of the maps $G \longrightarrow R, \qquad R \longrightarrow R_0$. We then define the multiplier $m_{n_0,\tau} = n\mu_\tau$ of G. Thus,

$$m_{n_0,\tau}(r, r') = n_0(W, W')\psi\Big(-2\tau(v, Wu') - \tau\eta'(v, v)\Big).$$

We now describe the Galilean particles. First we fix $\tau \neq 0$. The affine action corresponding to the cocyle θ_τ is given by $(v, W) : (\xi, t) \mapsto (W\xi + 2\tau v, t - (W\xi, v) - \tau(v, v))$.

The function $M : (\xi, t) \mapsto (\xi, \xi) + 4\tau t$ maps V to k and is easily verified to be invariant under the affine action. Hence, the level sets of M are invariant under the affine action. Since $M((0, a/4\tau)) = a$, we see that M maps onto k.

Fix $a \in \mathbb{Q}_p$ and consider the level set $M[a] = \big\{(\xi, t) \bigm| M(\xi, t) = a\big\}$. The element $(0, a/4\tau) \in M[a]$; if $(\xi, t) \in M[a]$ then the element $(\xi/2\tau, I)$ of R sends $(0, a/4\tau)$ to (ξ, t) by the affine action, as is easily verified. Hence, $M[a]$ is a single orbit. The orbits are thus all closed and so, by Effros's theorem, the affine action is regular. One can see this also explicitly by observing that the set $\big\{(0, b) \bigm| \big(b \in \mathbb{Q}_p\big)\big\}$ meets each affine orbit in exactly one point. The stabilizer in R of $(0, a/4\tau)$ is R_0.

For a given orbit, the corresponding $m_{n_0,\tau}$—representations are parameterized by the n_0-representations of R_0. However, as we shall now show, these representations are *projectively the same for different a.*

To see this we observe first that the projection map $(\xi, t) \longmapsto \xi$ is a bijection of the level set $M[a]$ onto V_0'; in fact, the point

$$\left(\xi, \frac{a - (\xi, \xi)}{4\tau}\right)$$

is the unique point of $M[a]$ above ξ. The affine action on $M[a]$ corresponds to the action $(v, W), \xi \longmapsto W\xi + 2\tau v$. We shall therefore identify $M[a]$ with V_0' and the affine action by the above action. Hence, Lebesgue measure λ is invariant. *We note that the parameter a has disappeared in the action.* Hence, by Theorem 4, the action of R in the representation corresponding to the cocycle $m_{n_0,\tau}$ takes place on $L^2(V_0', \mathcal{K}, \lambda)$, where $\mathcal{K}$ is the Hilbert space for the n_0-representation of R_0 and is independent of a. Furthermore, by the same theorem, the translation action by (u, η) is just multiplication by $\psi((u, \xi) + t\eta)$ on $M[a]$ which reduces to multiplication by

$$\psi\left((u, \xi) + \frac{\eta(a - (\xi, \xi))}{4\tau}\right).$$

on $L^2(V_0', \mathcal{K}, \lambda)$. We now notice that the factor $\psi\left(\frac{\eta a}{4\tau}\right)$ *is independent of the variable* ξ and so it is a phase factor. It can therefore be pulled out and the remaining part is independent of a. Hence, projectively the entire representation can be written in a form that is independent of the parameter a. This proves that the representations with different a are projectively equivalent and describe the same particle.

The relevant parameters are thus $\tau(\neq 0)$ and the projective representations μ of R_0. We interpret τ as the *Schrödinger mass* and μ as the *spin*.

We still have to consider the case $\tau = 0$ when the multiplier is the lift to G of a multiplier n_0 for R_0 via the maps $G \longrightarrow R, \qquad R \longrightarrow R_0$. The affine action is now the ordinary action $(v, W), (\xi, t) \longmapsto (W\xi, t - (W\xi, v)$. The function $N : (\xi, t) \longmapsto (\xi, \xi)$ is clearly invariant and maps onto $\mathbb{Q}_p$. We claim that the level sets of $N[a]$, where N takes the values a are orbits. The subset when $t = 0$ is clearly an orbit for R_0. If $(\xi, t) \in N[a]$, select $v \in V_0$ such that $(\xi, v) = -t$; then the element $(v, I) \in R$ takes $(\xi, 0)$ to (ξ, t). There is obviously an invariant measure on $N[a]$, namely the measure $d\sigma_a \times dt$, where $d\sigma_a$ is the "surface" measure on the subset in V_0' where (ξ, ξ) takes the value a (the "sphere.") The spectrum is thus contained in a subvariety of V_0'. Over $\mathbb{R}$ this leads to unphysical relations between momenta [13]. Over $\mathbb{Q}_p$ there is no such argument but the representations do not seem to represent particles.

5. THE CONFORMAL GROUP AND CONFORMAL SPACE TIME

5.1. Imbedding of the Poincaré group in the conformal group

Theorem 6. *Let k be a field of ch $\neq 2$. Suppose W and V are two Witt equivalent quadratic vector spaces over k with $dim(V) = dim(W) + 2$ and let $p \in V$ be a null vector. Denote by H_p the stabilizer of p in $SO(V)$. Then there exists an isomorphism of algebraic groups $h : P_W \xrightarrow{\sim} H_p$ over k.*

Proof. Fix a null vector $q \in V$ such that (p, q) is a hyperbolic pair in V and let $W_p = \langle p, q\rangle^{\perp}$. Then $V = W_p \oplus \langle p, q\rangle$ and $W_p \simeq W$. For brevity we write W for W_p.

Let h be in H_p. We want to write h in an explicit block matrix form with respect to $V = \langle p\rangle \oplus \langle q\rangle \oplus W$. Let $R \in \mathrm{End}(W)$ be defined by $ht \equiv Rt \mod \langle p, q\rangle$ for $t \in W$. A calculation shows $hp = p$, $hq = -\frac{(t,t)}{2}p + q + t, hw = -(t, Rw)p + Rw$ for $w \in W$. Let $e(t, R) \in \mathrm{Hom}(W, \langle p\rangle)$ be the map $e(t, R) : w \mapsto -(t, Rw)p$. Then one can write the matrix of h as

$$h = h(t, R) = \begin{pmatrix} 1 & -\frac{(t,t)}{2} & e(t, R) \\ 0 & 1 & 0 \\ 0 & t & R \end{pmatrix}.$$

Since $1 = \det(h) = \det(R)$ and $(w, w) = (hw, hw) = (Rw, Rw)$, it follows that $R \in \mathrm{SO}(W)$.

We note that h is completely determined by t and R. Moreover, for any $t \in W$, $R \in \mathrm{SO}(W)$, $h = h(t, R)$ as defined above makes sense and has the following properties:

(1) $hp = p$, hq is a null vector, and $(hq, p) = 1$.

(2) $hw \perp p, hw \perp hq, (hw, hw) = (w, w)$.

These properties are sufficient to ensure that h preserves the form on V. From the formula for h we see that $\det(h) = 1$ and so $h \in \mathrm{SO}(V)$. Since $hp = p$ we see finally that $h \in H_p$.

It is now trivial to verify that h is a homomorphism from P_W to H_p, i.e.,

$$h(t, R) \cdot h(t', R') = h(t + Rt', RR').$$

We omit the calculation. Thus h is a morphism of algebraic groups $P_W \longrightarrow H_p$ which is defined over the ground field k and is bijective. The inverse map is a morphism of algebraic varieties because it can be seen as the restriction to H_p of the map from a closed subvariety of $\mathrm{GL}(V)$ to $W \rtimes \mathrm{GL}(W)$ defined by

$$\begin{pmatrix} 1 & b & c \\ 0 & 1 & 0 \\ g & t & R \end{pmatrix} \longmapsto (t, R).$$

We thus see that we have an isomorphism of algebraic groups from P_W to H_p, defined over k. $\square$

5.2. Conformal compactification of space time

Let W, V be as above. Let $G = \mathrm{SO}(V)$. We shall now construct a smooth irreducible projective variety $[\Omega]$ such that

(a) There is a k-imbedding of W as a Zariski open subspace A_W of $[\Omega]$.

(b) The group G acts transitively on $[\Omega]$ and there is a k-isomorphism of P_W with a subgroup G_W of G which leaves A_W invariant.

(c) The action of G_W on A_W is isomorphic (via the imbedding) to the action of P_W on W.

The metric of W does not extend to $[\Omega]$; rather at each point $[x]$ of $[\Omega]$ we have a family of metrics differing by scalar multiples that contains the metric of W on A_W. The group G preserves this family of metrics. Thus, we say that $[\Omega]$ has *a conformal structure*; and as G keeps this structure invariant, we call G *the conformal group*. We refer to $([\Omega], G)$ as the *conformal compactification* of (W, P_W). When k is a *local field*, $[\Omega]$ (or rather, the set of its k-points) is compact, thus justifying our terminology. These ideas are summarized in the following theorem.

Theorem 7. *Given two Witt equivalent quadratic vector spaces W and V over k with $dim(V) = dim(W) + 2$ there exists a conformal compactification of (W, P_W).*

We prove this theorem in a series of lemmas.

Definition 33. Let V, W be as above. We define $\Omega = M_0 = \{p \in V \mid p \neq 0, (p, p) = 0\}$.

There is a basis of V for which the quadratic form becomes: $Q(x) = a_0x_0^2 + a_1x_1^2 + ... + a_{n+1}x_{n-1}^2, \;\; a_i \neq 0$, where $n = \dim(V)$. Thus the equation defining Ω i $a_0x_0^2 + ... + a_{n-1}x_{n-1}^2 = 0$. This homogeneous polynomial defines a smooth irreducible quadric cone $[\Omega]$ of dimension $n - 2$ in the projective space $\mathbb{P}(V)$. Let $P(x \longmapsto [x])$ be the map from $V \setminus \{0\}$ to $\mathbb{P}(V)$. Then $[\Omega]$ is the image under P of Ω in $\mathbb{P}(V)$, and is stable under the action of $\mathrm{SO}(V)$. The tangent space at $x \in \Omega$ is $V_x = \{v \in V \mid (x, v) = 0\}$, and for $[x] \in [\Omega]$, the tangent space at $[x]$ is $[\Omega]_{[x]}$ and is defined as the image of the tangent map dP_x of V_x.

Lemma 4. $[\Omega]$ *has a natural G-invariant conformal structure.*

Proof. We note that tangent map $dP_x : V_x \to [\Omega]_{[x]}$ is surjective because P is submersive. Hence, the kernel of dP_x is one dimensional. We know that P is constant on the line kx so dP_x vanishes on kx. Thus the kernel of dP_x is the line kx. Hence, the quadratic form Q on V induces a quadratic form $\tilde{Q}$ on $[\Omega]_x$. We note that if we use the map $dP_{\lambda x} : V_{\lambda x} \to [\Omega]_{[x]}$ to define the induced quadratic form $\tilde{Q}'$ then $\tilde{Q}' = \lambda^2\tilde{Q}$. Furthermore, if we have $g \in \mathrm{SO}(V)$ and $x' = \lambda x$, then the set of metrics at $[\Omega]_{[x]}$ induced from V_x goes over to the set of metrics induced from $V_{\lambda x}$. Thus, $[\Omega]$ has a conformal structure defined by these induced metrics. The definition of the conformal structure makes it clear that it is G-invariant. □

We write $V = W \oplus \langle p, q\rangle$, where the sum is orthogonal, and $\langle p, q\rangle$ is hyperbolic with $(p, p) = (q, q) = 0, \quad (p, q) = 1$. We define $A_{[p]} = \{[a] \in [\Omega] \mid (p, a) \neq 0\}$ and we introduce C_p as the set of null vectors of V_p. Thus, $C_p = V_p \cap \Omega$. Write $C_{[p]}$ for the image of C_p in $[\Omega]$. Then we have $A_{[p]} = [\Omega]\backslash C_{[p]}$ since V_p is defined by the equation $(p, v) = 0$. Let $[a] \in A_{[p]}$, we write $a = \alpha p + \beta q + w$, where $w \in W$, then, as $(p, a) \neq 0$, we must have $\beta \neq 0$. A quick calculation shows that $\alpha = \frac{-(w,w)}{2}$. Since we are only interested in the image of a in the projective space, we may take β to be 1. Then $[a]$ is given by $[\frac{-(w,w)}{2} : 1 : w]$ so $[a]$ is entirely determined by w. We thus have the bijection

$$J : W \simeq A_{[p]} \qquad J : w \mapsto \left[\frac{-(w, w)}{2}p + q + w\right].$$

Lemma 5. *$A_{[p]}$ is a Zariski open dense subset of $[\Omega]$.*

Proof. It is clear that $A_{[p]}$ is a Zariski open subset of $[\Omega]$; it is dense since $[\Omega]$ is irreducible. □

Lemma 6. *Let H_p be the subgroup of $\mathrm{SO}(V)$ that fixes p. Then H_p leaves invariant the image $A_{[p]}$ of W under J. Moreover, the map J intertwines the actions of (t, R) on W and $h(t, R)$ on $A_{[p]}$ (see theorem 6).*

Proof. Notice first that if $h \in H_p$, then h stabilizes V_p. Therefore, C_p, hence $A_{[p]}$. Let (t, R) be in P_W. □

All claims of Theorem 7 have now been proven.

Lemma 7. *If $k = \mathbb{Q}_p$ then $[\Omega]$ is compact.*

Proof. Since $[\Omega]$ is a closed subset of $\mathbb{P}(\mathbb{Q}_p^{n+1})$, and $\mathbb{P}(\mathbb{Q}_p^{n+1})$ is compact, $[\Omega]$ is compact. □

Lemma 7 shows that if the underlying field is $\mathbb{Q}_p$, then the projective imbedding becomes the compactification of spacetime.

5.3. Conjugacy of imbeddings

The following theorem is a converse of sorts to Theorem 6. It states that the subgroups of $\mathrm{SO}(V)$ that are isomorphic to a Poincaré group P_W arise only as stabilizers of null vectors. $\mathrm{SO}(V)$ acts by conjugacy transitively on the set of all Poincaré groups inside $\mathrm{SO}(V)$.

Theorem 8. *Let W and V be two quadratic vector spaces with W Witt equivalent to V with $dim(V) = dim(W) + 2$. If there is an imbedding $f : P_W \hookrightarrow SO(V)$ of algebraic groups over k, then for $\dim(W) \geq 3$,*

(a) $f(P_W) = H_p$, where H_p is a stabilizer of some null vector $p \in SO(V)$.

(b) All such imbeddings f are conjugate under $SO(V)(k)$.

Theorem 8 is of great theoretical interest. But its proof is long and technically involved. Since neither the theorem nor its proof itself are used in the rest of the paper, we do not give it here; instead we refer the reader to the Arxiv version of this article [23].

5.4. Partial conformal group

In this section we introduce a subgroup $\tilde{P}_V$ of $\mathrm{SO}(V)$, the stabilizer of the line kp. It is an easy computation that $h \in \mathrm{SO}(V)$ stabilizes the line kp if and only if h has the form

$$h = \begin{pmatrix} c & -c\frac{(t,t)}{2} & ce(t,R) \\ 0 & \frac{1}{c} & 0 \\ 0 & t & R \end{pmatrix}.$$

Here $c \in k^{\times}$, $t \in W$, $R \in \mathrm{SO}(W)$ and $e(t,R) \in \mathrm{Hom}(W, \langle p \rangle)$. We write $h = h(c,t,R)$. We denote the set of all such matrices h as $\tilde{P}_V = \{h(c,t,R) \mid c \in k^{\times},\ t \in W,\ R \in \mathrm{SO}(W)\}$. Let us denote by $\tilde{c}$ the matrix

$$\tilde{c} = \begin{pmatrix} c & 0 & 0 \\ 0 & \frac{1}{c} & 0 \\ 0 & 0 & I \end{pmatrix} \quad (c \in k^{\times}).$$

Given $h(c,t,R) \in \tilde{P}_V$, then $h(c,t,R) = \tilde{c}h(t,R)$, where $h(t,R) \in P_V$.

The following is immediate.

Lemma 8. $\tilde{P}_V = \{\tilde{c}h(t,R) \mid c \in k^{\times}\ ,\ t \in W\ ,\ R \in SO(W)\}$.

(a) $\tilde{P}_V \simeq V \rtimes (SO(V) \times k^{\times})$.

(b) *Multiplication is given by:* $\tilde{c}h(t,R)\tilde{c'}h(t',R') = \widetilde{cc'}h(\frac{1}{c'}t + Rt', RR')$.

(c) *The conjugation action of* $\tilde{c}$ *on the translation part is to dilate it by a factor of* c. *That is* $\tilde{c}h(t,R)\widetilde{c^{-1}} = h(ct,R)$. *Note that* $\tilde{c}$ *commutes with the* R *action.*

Lemma 9. $\tilde{P}_V$ *is the largest subgroup of* $SO(V)$ *that leaves* $A_{[p]}$ *invariant.*

Proof. We note that it is easier to work with $A_p := \{a \in \Omega \mid (p,a) \neq 0\}$. Let g be any element of $\mathrm{SO}(V)$ that leaves A_p invariant. Then g leaves $A_{[p]}$ invariant as well. We want to first show that $gp = \alpha p + w$ where $w \in W$, which is equivalent to showing that $gp \in V_p$. If g preserves A_p, then g also preserves the compliment of A_p, which is $V_p \cap C_p$. Now $p \in V_p \cap C_p$ so that $gp \in V_p \cap C_p$.

If g preserves A_p, it also preserves $A_p \backslash C_p = V_p \cap C_p$. We must show that $g \cdot \langle p \rangle = \langle p \rangle$. If $\langle p \rangle$ is the only null line in V_p, then $g \cdot \langle p \rangle = \langle p \rangle$ trivially. So assume that $V_p = \langle p \rangle + W$ has other null lines. Now $W \cap C_p$ is stable under $\mathrm{SO}(W)$ and $\mathrm{SO}(W)$ acts irreducibly on W, so $W \cap C_p$ spans W. We have that $g(W) \subset \mathrm{Span}(g(W \cap C_p)) \subset \mathrm{Span}(g(V_p \cap C_p)) \subset \mathrm{Span}(V_p \cap C \subset V_p)$. Hence, $g(W) \subset V_p$. On the other hand, as $p \in V_p \cap C_p$, $g \cdot p \in V_p$. So $g(V_p) \subset V_p$ and $g(V_p^{\perp}) \subset V_p^{\perp}$. Hence, $g\langle p \rangle = \langle p \rangle$. $\square$

Definition 34. We will call $\tilde{P}_V$ the partial conformal group.

This is a reasonable definition since $\tilde{P}_V$ is the subgroup stabilizing $A_{[p]}$.

6. EXTENDABILITY OF PUIRS OF THE POINCARÉ GROUP TO THE PUIRS OF THE CONFORMAL GROUP

As we discussed in Section 5.5.1, if V_1 and V_0 are two quadratic vector spaces with V_1 Witt equivalent to V_0 with $dim(V_0) = dim(V_1) + 2$, then the Poincaré group P_{V_1}, can be imbedded as a subgroup of the conformal group $\mathrm{SO}(V_0)$, and furthermore, that any two such imbeddings are conjugate over $\mathrm{SO}(V_0)$. A natural question that one may ask is the following: are there PUIRs of the Poincaré group that extend to be PUIRs of the conformal group? PUIRs that do extend to the conformal group are said to have conformal symmetry. Classically, only massless particles (photons) have conformal symmetry and the corresponding PUIRs of the real Poincaré group extend to PUIRs of the real conformal group [20]. We would like to explore this question in the p-adic setting. Our ultimate goal is to establish some necessary conditions for this extension to be possible.

Definition 35. Let V_1 and V_0 be two Witt equivalent quadratic vector spaces over $\mathbb{Q}_p$ with $dim(V_0) = dim(V_1) + 2$. When a PUIR U of P_{V_1} can be extended to be a PUIR V of $\mathrm{SO}(V_0)$ we say that the particle corresponding to U has *conformal symmetry*.

Definition 36. When a PUIR of U of P_V can be extended to be a PUIR $\tilde{U}$ of the group $\tilde{P}_V$, we say that the particle corresponding to U has *partial conformal symmetry*.

We make the following trivial, but important observation: If a particle does not have partial conformal symmetry, then it does not have conformal symmetry. In the next section we aim to establish some necessary conditions for a particle to have partial conformal symmetry.

6.1. Extensions of m-representations of semidirect products

Let A, L and M be lcsc groups with A being abelian and L being a closed subgroup of M. Suppose M acts on A so that we may form the semidirect products $G = A \rtimes L$, $H = A \rtimes M$. We assume: a) all multipliers of G and H are trivial on A; b) that $H^1(L, A^*) = 0$, $H^1(M, A^*) = 0$; c) 1 is the only character of A fixed by L; and d) the actions of M and L on A^* are regular.

Because of the assumptions that $H^1(G, A^*) = 0$, and $H^1(H, A^*) = 0$, and that the actions of M and L are regular, irreducible m-representations U of G (resp. H) correspond to pairs (χ, u) where $\chi \in A^*$ and u is an irreducible m_χ-representation of the stabilizer G_χ (resp. H_χ) of χ in G (resp. H).

We will need the following technical result.

Lemma 10. *Let U be an m-representation of G where m is standard. Let V_1 be an m_1-representation of H extending U. Then we can find a standard multiplier m' for H such that $m'|_{G\times G} = m$ and U has an extension V to H as an m'-representation with $V(ah) = F(ah)V_1(ah)$ $(ah \in H)$ for some Borel function $F : H \to T$ with $F = 1$ on G.*

Proof. From Mackey's work (see [22]) we know that $V : ah \mapsto m_1(a, h)V_1(ah)$ is an m'-representation of H with $m'|_{A\times A} = 1$ and $m'|_{A\times H} = 1$. Clearly, V extends U, $m' \simeq m_1$ and $m'|_{G\times G} = m$. As $H^1(M, A^*) = 0$, we have $m'(ah, a'h') = m'_0(h, h')f(h[a'])/f(a')$ where m'_0 is a multiplier for M and $f \in A^*$. Since $m'|_{G\times G} = m$, $f(g[a'])f(a')^{-1} = 1$ $\forall g \in L$, $a' \in A$. Hence, $f = 1$ by the assumption that 1 is the only character fixed by L. Thus m' is already standard. □

We will need the following technical lemma:

Lemma 11. *Let δ_i $(i = 1, 2)$ be two strict m-cocycles for G such that for each $g \in G$, $\delta_1(g, x) = \delta_2(g, x)$ for almost all $x \in X$. Let ν_i be the m-representations of G_0 defined by δ_i $(i = 1, 2)$. Then $\nu_1 \simeq \nu_2$.*

We refer the reader to [23] for the proof.

The following is a key lemma that will be utilized often to prove the impossibility of the extension of both massive and eventually massive particles.

Lemma 12. *Let U be an irreducible m-representation of G for a standard multiplier m for G. Let U correspond to the L-orbit of $\chi \in A^*$ and an irreducible m_χ-representation u of the stabilizer L_χ of χ in L, m_χ being $m|_{L_\chi \times L_\chi}$. Then the following are equivalent:*

(1) *U extends to a projective unitary representation V_1 of H.*

(2)(a) *$M[\chi] \setminus L[\chi]$ is a null set in $M[\chi]$.*

(b) *There is a standard multiplier m' for H with $m'|_{G\times G} = m$.*

(c) *u extends to a m'_χ-representation of M_χ.*

In this case, there is an m'-representation V of H such that V belongs to the same equivalence class as V_1 with:

(I) *$V|_G = U$.*

(II) *V corresponds to χ and v where v is an m'_χ-representation of M_χ.*

(III) *$v|_{L_\chi} = u$.*

Proof. (1) ⇒ (2): We may assume U extends to an m'-representation V of H belonging to the same equivalence class as V_1 where m' is standard and $m'|_{G\times G} = m$. Clearly V is irreducible. Hence, the spectrum of V lives on an M-orbit in A^*. But as V and U have the same restriction to A, the spectrum of V must meet $L[\chi]$ so that we may assume it to be $M[\chi]$. But then $M[\chi] \setminus L[\chi]$ must be null. This proves (2)(a) and (2)(c).

We may now write V in the form:

$$(V(ah)f)(\zeta) = \langle a, \zeta\rangle \rho_h(h^{-1}\zeta)^{\frac{1}{2}} C(h, h^{-1}\zeta) f(h^{-1}\zeta), \quad (\zeta \in M[\chi],\ h \in M)$$

where C is a strict m'-cocyle that defines the m'_χ-representation v. On the other hand, U is given by:

$$(U(ag)f)(\zeta) = \langle a, \zeta\rangle \rho_g(g^{-1}\zeta)^{\frac{1}{2}} D(g, g^{-1}\zeta) f(h^{-1}\zeta) \quad (\zeta \in L[\chi],\ g \in L)$$

where D is a strict m-cocyle defining the m-representation u. Since $V|_G = U$, it follows that $D(g,\nu) = C(g,\nu)$ for each g for almost all $\nu \in M[\chi]$. Hence, by Lemma 11, u is equivalent to the restriction of v to L_χ. If $u(g) = rv(g)r^{-1}$ $(g \in L_\chi)$, where r is a unitary representation in the space of v, it is clear that u extends to rvr^{-1}. This proves (2)(b).

(2) ⇒ (1): Extend u to an m'_χ-representation of M_χ and build a strict $(M, M[\chi])$-cocyle C for the multiplier m' for M that defines the m'_χ-representation at χ. The restriction of C to L is a strict cocycle for $m_1 = m'|_{L\times L}$. The m'-representation of H corresponding to (χ, m') restricts on G to the m_1-representation defined by (χ, m_1), and hence is equivalent to U. So U extends to a PUR of H.

The above proof also establishes (I),(II) and (III). □

6.2. Impossibility of partial conformal symmetry for massive particles

We now show that massive particles do not posses partial conformal symmetry. We begin with some important lemmas.

Lemma 13. *The orbit of a massive point under $SO(V) \times \mathbb{Q}_p^\times$ is open in V.*

Proof. Let $x \in V$ be such that $Q(x) = a \neq 0$; then if $g \in \mathrm{SO}(V) \times \mathbb{Q}_p^\times$ and $g[x] = tx$, then $Q(g[x]) = at^2$. Thus, the orbit of $Q(x)$ under $\tilde{P}$ is $a(\mathbb{Q}_p^\times)^2$. Hence, the orbit of x is $Q^{-1}(a(\mathbb{Q}_p^\times)^2)$. Since Q is a continuous function, the orbit of x will be open in V if we can show that $a(\mathbb{Q}_p^\times)^2$ is open in $\mathbb{Q}_p^\times$. We note that it suffices to prove that $(\mathbb{Q}_p^\times)^2$ is open in $\mathbb{Q}_p$. This is an easy verification and we omit it here. □

Lemma 14. *Let $p \in V$ be a massive point, then the quasi-invariant measure class on the orbit $SO(V) \times \mathbb{Q}_p^\times \cdot p$, is the Lebesgue (Haar) measure class.*

Proof. By Lemma 13 the orbit $\mathrm{SO}(V) \times \mathbb{Q}_p^\times[p] = \omega_p$ is open in V. Let $E \subset \omega_p$ be a set of Haar measure 0 in V. Since ω_p is open in V, the Haar measure is also defined on ω_p. Let μ be the Haar measure. Since $\mathrm{SO}(V) \times \mathbb{Q}_p^\times$ acts linearily for any $(g,c) \in \mathrm{SO}(V) \times \mathbb{Q}_p^\times$, we have that $\mu((g,c)\cdot E) = |\det((g,c))|_p \mu(E)$. Hence, $\mu((g,c)\cdot E) = 0$. Thus, the measure class on ω_p is the Haar measure class, and it is quasi-invariant under $\mathrm{SO}(V) \times \mathbb{Q}_p^\times$. $\square$

Corollary 14. *Both massive and massless orbits under $SO(V)$ have Haar measure 0 in V.*

Proof. Let us now take $x \in V$ such that $x \neq 0$ and $Q(x) = a$. Then $f(x) = Q(x) - a$ is an analytic function and defines a subset $Q_a = \{x \in V \mid f(x) = 0\}$ of V. We want to show that Q_a has measure 0 in V. We may assume that there is a basis of V, (e_i), such that $(e_i, e_j) = a_i\delta_{ij}$ and that $Q(x) = \sum a_i x_i^2$. Since $x = 0$ is not in Q_a we see that Q has a nonzero gradient on all of Q_a. It follows that Q_a has Haar measure 0 in V. $\square$

Theorem 9. *A massive PUIR of P_V does not have extension to $\tilde{P}_V$.*

Proof. Let p correspond to a massive orbit. We have that $P_V = V \rtimes \mathrm{SO}(V)$ and $\tilde{P}_V = V \rtimes (\mathrm{SO}(V) \times \mathbb{Q}_p^\times)$. Let us denote $\mathrm{SO}(V)$ by L and $\mathrm{SO}(V) \times \mathbb{Q}_p^\times$ by M. If a massive PUIR of P_V were to have an extension to $\tilde{P}_V$, then by Lemma 12 $M[p] \setminus L[p]$ would be a null set. However, by Corollary 14 $L[p]$ has Lebesgue (Haar) measure 0, so $M[p]$ would have to have Lebesgue measure 0. But by Lemma 13, the orbit $M[p]$ is open and so has nonzero Lebesgue measure. Hence, the massive representations cannot extend even to the partial conformal group and therefore cannot extend to the full conformal group. $\square$

6.3. Impossibility of partial conformal symmetry for eventually massive particles

Since the massive particles do not have partial conformal symmetry, we now turn our attention to massless particles.

Let V be a quadratic vector space and p a nontrivial null vector in V. Let $P_V = V \rtimes \mathrm{SO}(V)$ and let $\tilde{P}_V = V \rtimes (\mathrm{SO}(V) \times k^\times)$. As discussed in Section 5.5.4 the action of $c \in k^\times$ on $v \in V$ is $c : v \mapsto cv$, and $k^\times$ commutes with $\mathrm{SO}(V)$. We know from Theorem 6 that the stabilizer of p in $\mathrm{SO}(V)$ is isomorphic to $P_{V_1} = V_1 \rtimes \mathrm{SO}(V_1)$ where V_1 is a vector space Witt equivalent to V with $dim(V) = dim(V_1) + 2$. We now claim the following:

Proposition 1. *Let $\tilde{P}_{Vp}$ be the stabilizer of p in $SO(V) \times k^\times$. Then $\tilde{P}_{Vp}$ is isomorphic to $\tilde{P}_{V_1}$.*

Proof. Let $(g,c) \in \mathrm{SO}(V) \times k^\times$. Then (g,c) acts on p by $(g,c) : p \mapsto cg[p]$. Hence, (g,c) fixes p if and only if $g[p] = \frac{1}{c}p$. In other words, g stabilizes the line kp. As proven in Section 5.5.4 the stabilizer of the line is $\tilde{P}_{V_1}$. $\square$

Lemma 15. *We have $H^1(SO(V) \times \mathbb{Q}_p^\times, V') = 0$. Moreover, the action of $(g,c) \in SO(V) \times \mathbb{Q}_p^\times$ on $\lambda \in V'$ is by $(g,c) : \lambda \mapsto cg \cdot \lambda$.*

Proof. Let $F \in H^1(\mathrm{SO}(V) \times \mathbb{Q}_p^\times, V')$. We set $L = \mathrm{SO}(V)$ and write elements of $\mathrm{SO}(V) \times \mathbb{Q}_p^\times$ as (l,c). Then $F((l,1))$ is a trivial cocycle for all $l \in L$, since $H^1(\mathrm{SO}(V), V') = 0$. Thus, we can find a $\lambda \in V'$ such that $F((l,1)) = l[\lambda] - \lambda \ \forall\, l \in \mathrm{SO}(V)$. If $F_1((l,c)) = F((l,c)) - ((l,c))[\lambda] - \lambda$, then $F_1 \simeq F$ and F_1 is 0 on L. So we may assume that F is 0 on L to begin with. We may identify $(l,1)$ with L and $(1,c)$ with $c \in \mathbb{Q}_p^\times$ and we can then write $(l,c) = lc$. We now use the fact that $lc = cl$ to write: $F((l,c)) = F(lc) = F(l) + l[F(c)] = F(c) + c[F(l)]$. Since $F(l) = 0$, we get $F(c) = l[F(c)]$. However, L does not fix any nontrivial vector in V' so we must have $F(c) = 0$ and so $F = 0$. $\square$

Lemma 16. *Suppose p is a null vector in V^* and U is an irreducible m-representation of P_V corresponding to p and an irreducible m_0-representation u of $SO(V)_p = L_p$. Suppose that U has an extension to $\tilde{P}$ as a projective unitary representation. Then, identifying $SO(V)_p$ with $P_{V_1} = V_1 \rtimes SO(V_1)$, u is a massless PUIR of P_{V_1} that has partial conformal symmetry.*

Proof. By Lemma 12 2) c), u extends to be a representation of $\tilde{P}_{V_1} = V_1 \rtimes (\mathrm{SO}(V_1) \times \mathbb{Q}_p)$. Now by Theorem 9, u must be massless. □

Lemma 16 shows that if one has a PUIR U_0 of a Poincaré group P_{V_0} that extends to a PUIR of the conformal group $\tilde{P}_{V_0}$, then U_0 corresponds to a PUIR U_1 of a stabilizer P_{V_1} of a massless $p_0 \in V_0^*$. We note that $P_{V_1} = V_1 \rtimes \mathrm{SO}(V_1)$ where V_1 is a quadratic vector space Witt equivalent to V_0 with $dim(V_0) = dim(V_1) + 2$. So P_{V_1} is itself a Poincaré group. Now in turn, U_1 has partial conformal symmetry and will correspond to a PUIR U_2 of the stablizer P_{V_2} of some $p_1 \in V_1$. So it is clear that this process can be repeated until one reaches a stage R where V_R is anisotropic. At the anisotropic stage, the only massless character in V_R^* is the trivial one. One may also end the process by picking the trivial null vector at any stage. We thus have a chain of Poincaré groups $P_{V_0}, P_{V_1}, \ldots$ and corresponding massless representations $U_0, U_1, \ldots$. From our discussion we have the following theorem:

Theorem 10. *If U is massless and has partial conformal symmetry, all the U_ν are massless.*

We say that U is *eventually massive* if some U_ν is massive.

Theorem 11. *Eventually massive particles do not have partial conformal symmetry.*

Both theorems 10 and 11 are immediate from Theorem 9 and Lemma 16.

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