University of California
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# Structure of Elementary Particles in Non-Archimedean Spacetime

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Mathematics

by

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The dissertation of Jukka Tapio Virtanen is approved.

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*To my parents for instilling in me a love of science and mathematics, and to my wife, Katy, for her enduring love and support*

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# Vita

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Abstract of the Dissertation

# Structure of Elementary Particles in Non-Archimedean Spacetime

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We show that a generalized Poincaré group defined over a field of characteristic $\neq 2$ can be imbedded into the conformal group as a stabilizer of a null vector. We also show that the spacetime $W$ is imbedded as a Zariski dense open subset of a smooth projective variety. In the case of a local field, this is a compactification of spacetime.

We present a treatment of the Mackey machine for projective representations and discuss an analogue of the theorem of Mackey for projective representations and introduce a new affine action. This theorem is used to give the particle classification over $p$-adic Minkowski space. Lastly, we prove that the massive particles and the so called eventually massive particles of the Poincaré group do not have conformal symmetry.

# CHAPTER 1

# Introduction

*In fact, I would not be too surprised if discrete mod p mathematics and the p-adic numbers would eventually be of use in the building of models for very small phenomena.* -Raoul Bott, 1975 [1]

*Now it seems that the empirical notions on which the metric determinations of Space are based, the concept of a solid body and light ray, lose their validity in the infinitely small; it is therefore quite definitely conceivable that the metric relations of Space in the infinitely small do not conform to the hypothesis of geometry; and in fact, one ought to assume this as soon as it permits a simpler way of explaining phenomena.* -Bernhard Riemann, Inaugural lecture, 1854 [2]

## 1.1 The Dirac mode in theoretical physics

Relativistic quantum mechanics was quite a new field in 1928 when Dirac wrote his famous equation describing the relativistic electron. He had sought to replace the Klein-Gordon (K-G) equation, which was the only known equation that described relativistic particles ([3] p. 39). His objection to the K-G equation was both on practical and aesthetic grounds. Using the K-G equation led to difficulties when trying to define probability densities and currents. Dirac also felt that the wave equation should be of the first order in both time and space, since time should be treated on an equal footing with the space coordinates. In part it was this search for mathematical beauty that eventually led Dirac to his equation.

Dirac's equation had a disturbing side effect. It allowed for two possible kinds of states: One state of positive energy (and negative charge), that described the electron, and another state of negative energy (and positive charge), which Dirac initially thought should be ruled out as unphysical. He later deviced the so called *hole theory*, which postulated that when a particle is created out of a vacuum, a corresponding hole of negative energy is also created. He thought initially that this mysterious particle with positive charge had to be the proton; however, the proton is about 1800 times heavier than the electron. It was Hermann Weyl who first suggested that the other solution should correspond to a positively charged particle whose mass had to be the same as that of the electron, due to the symmetry of the Dirac equation. This particle, when it was eventually discovered, was named the *positron*.

In 1932, Anderson discovered in his cloud chambers a single track of the positron and promptly published his results, although it appears that Blackett had discovered three tracks but was looking for more evidence. Anderson was awarded the Nobel prize for his discovery. Blackett did eventually receive the Nobel prize as well for his investigation of cosmic rays using cloud chambers. When Dirac was later asked why he had not taken the leap of calling this positively charged particle a new particle, he half jokingly replied: "Pure cowardice!" He was ultimately vindicated, however, as he received the Nobel prize in 1933.

This story of the positron is one of the early instances of what is nowadays called the *Dirac mode*. The Dirac mode can be thought of as a mathematical insight to a physical discovery. Dirac believed that if a mathematical idea was beautiful and simple, then it was quite likely that it described a real physical phenomenon. We quote the master himself:

"The steady progress of physics requires for its theoretical formulation a mathematics that gets continually more advanced. This is only natural and to be expected... The theoretical worker in the future will therefore have to proceed in a more indirect way. The most powerful method of advance that can be suggested at the present is to employ all the resources of pure mathematical formalism that forms the existing basis of theoretical physics, and *after* each success in this direction, to try to interpret the new mathematical features in terms of physical entities." [5].

This methodology may seem somewhat contrary to the one usually encounters. Rather than try to interpret experimental results and try to create a mathematical theory that describes the results, one instead follows an aesthetically pleasing idea that perhaps comes from a thought experiment or as a generalization of an existing idea. But one must remember that there is a method to this "madness". To paraphrase Nambu: The Dirac mode is to invent a new mathematical framework first and then try to find its relevance in the real world, with the expectation that a mathematically beautiful idea must have been adopted by God [6]. One may think of the Dirac mode as an Ockham's razor of theoretical physics.

Indeed, theoretical physicists have embraced this idea in recent years. Most notably, string theory and supersymmetry are examples of the Dirac mode at work. At the time of writing this thesis, no experimental confirmations of these theories exist. Yet, the mathematics involved is captivating and continually attracts theoretical physicists and mathematicians. This thesis will be in the spirit of the Dirac mode. We start with an interesting, plausible, and beautiful mathematical idea, and we let the mathematics guide us. We would hope that the results we obtain describe real physical phenomena, but even if this does not turn out to be the case, as mathematicians, the beauty of the mathematics itself will suffice to satisfy us.

## 1.2 Non-Archimedean Spacetime

### 1.2.1 A historical perspective

Physicists have long been aware of the problems that one encounters while trying to extrapolate from the macroscopic level to the quantum level. Even Coulomb's law predicts that there should be infinite force between point-like charged particles as the distance between them approaches zero. The advent of quantum mechanics helped explain many of the mysteries of the microscopic world, but it created problems of its own. It has proven to be exceedingly difficult to construct convergent theories of elementary particles and their interactions. String theory attempts to bypass these difficulties by assuming that the fundamental objects themselves are strings. In fact, much of modern particle physics is now based on the assumption that the geometry of space is non-conventional at ultra small distances and times.

For centuries it had been taken for granted that space itself is an affine space over $\mathbb{R}$. Eventually, Einstein pointed out that spacetime, not space, is the invariant object and that it cannot be assumed that it is an affine space, but it must be that spacetime is a smooth manifold at least. Eventually, some mathematicians and physicists started to explore the possibility that spacetime could be defined over other fields, and a new viewpoint started to slowly emerge. In the 1930's Weyl tried to solve some problems in quantum mechanics where the underlying space was defined over finite rings $\mathbb{Z}_p = \mathbb{Z}/pZ$ [7]. Later on Ulam [8], Beltrametti [9] and Beltrametti and Cassinelli[10] investigated quantum theory over $p$-adic and finite fields. Beltrametti suggested that one ought to consider representations in fields of characteristic $p$ to model spin structures. Nambu extended these ideas to field theory [6] (p 371). However, it was Volovich [13] who suggested that the limitations in the very nature of measurements force us to look at *non-Archimedean* models of spacetime. Since the Volovich hypothesis, there has been increasing interest in non-Archimedean quantum mechanics. Those who have worked on this area include Arafyeva, Freund, Grossman, Manin, Varadarajan, Vladimirov, Volovich, Witten and others. For a more substantial summary of the history of these ideas we refer to [14].

## 1.3 Non-archimedean models for spacetime

In 1987, I V. Volovich published a ground breaking paper in which he made a case for a non-Archimedean structure of spacetime at the Planck scale.

In quantum mechanics, the Heisenberg uncertainty principle $\Delta p \Delta q \geq \hbar$ asserts that there is a limit to the extent to which one may know the momentum and the location of a particle simultaneously. In addition to this, however, there is an absolute limit to the size of the region where one may make a measurement. When both general relativity and quantum theory are relevant, the scale that emerges from these considerations is the Planck scale.

### Planck scale

When one wants to probe smaller and smaller regions of spacetime, more energy must be put into the region. In order to locate an elementary particle, the energy needed is greater than Planck mass, $m_{pl} = (\hbar c G^{-1})^{\frac{1}{2}}$. In such a case, the gravitational energy will have an event horizon at $r = 2Gm_{pl}c^{-2} = 2l_{pl}$, where $l_{pl}$ is the Planck length. This means that in order to observe a region smaller than the Planck length, one would need to put in an amount of energy that would create a miniature black hole, and subsequently no information could be obtained. This principle, therefore, determines that a smallest possible measurable quantity of length and time exists, namely Planck length and Planck time [15]. Planck length is of the order of $10^{-33}$ cm, and Planck time, which is the time that a photon takes to travel the Planck length, is of the order of $10^{-43}$ seconds.

**The Volovich hypothesis**

Our usual notions of space and time are built from axioms of geometry, which started with Euclid but eventually were formalized by Hilbert. Of particular interest to us is the Archimedean axiom, which states that given a line segment of any length and a shorter line segment, successive additions of the short line segment along the long line segment will eventually surpass the long line segment. This axiom is, at its core, a statement about measurability. One is comparing two different scales. However, let us assume that one of the lengths is a sub-Planck scale length and that the other is a macroscopic length. We now see that this axiom breaks down. Instead, Volovich suggested that one ought to build a new theory of geometry of spacetime that is non-Archimedean. The underlying field in the ordinary theory of spacetime is the real number field. To build a new theory that has non-Archimedean features, the underlying field itself, at the outset, should be non-Archimedean [13]. The simplest non-Archimedean fields are the fields $\mathbb{Q}_p$ of *p-adic* numbers. They are constructed by completing the rational numbers with respect to the $p$-adic absolute value. (See the appendix A for a brief discussion of the $p$-adic number fields and their properties.)

## 1.4 Some preliminaries and a summary of thesis

In this thesis we shall adopt the Volovich point of view of $p$-adic spacetime and examine some of its consequences. In particular, we will define the symmetry groups (Poincaré, Galilean, conformal etc.) over $\mathbb{Q}_p$ and investigate their structure. We now discuss some preliminary concepts needed to introduce the main ideas and results of this thesis.

### 1.4.1 Poincaré group and the conformal group

#### Poincaré group

Classically, *Minkowski space* is the 4-dimensional vector space $\mathbb{R}^4$ over the real numbers with a quadratic form $q$ given by

$$q(x) = x_0^2 - x_1^2 - x_2^2 - x_3^2.$$

However, one can generalize this considerably. We may think of an *$n$-Minkowski space*. Let us start with a vector space $V$ of dimension $n$ over $\mathbb{R}$ with a non-singular quadratic form $q$ of signature $(1, n-1)$; this means that there is an orthogonal basis $(e_\mu)_{0\leq\mu\leq n-1}$, such that $q(e_0) = 1$ and $q(e_\mu) = -1$ for $\mu \geq 1$. We denote such an $n$-Minkowski space by $\mathbb{R}^{1,n-1}$.

More generally, one may consider a real-valued quadratic vector space $V$ with signature $(m, n)$. That is, in a suitable basis, the quadratic form of $V$ is

$$q(x) = x_0^2 + ... + x_{m-1}^2 - y_0^2 - ... - y_{n-1}^2$$

Such a quadratic vector space is denoted by $\mathbb{R}^{m,n}$ [16]. This can be generalized even more by considering an arbitrary finite-dimensional vector space $V$ over a field $k$ of characteristic $\neq 2$. A pair $(V, Q)$, where $Q$ is a quadratic form on $V$, is called a *quadratic vector space*, which we often write as $V$, suppressing mention of $Q$. We will call such a quadratic vector space *spacetime* if $Q$ is nonsingular.

By the usual convention, the group of invertible linear transformations of $\mathbb{R}^{1,n-1}$ that leaves $q$ invariant is known as the *Lorentz group*. Let $V$ be a quadratic vector space over $k$; we will call the subgroup $SO(Q)$ of $SL(V)$ preserving $Q$ the Lorentz group, $L_V$. We shall often write $SO(V)$ for $SO(Q)$ and refer to its elements as *rotations*. Over $\mathbb{R}$, $O(V)$ has four connected components corresponding to $det = \pm 1$ and $a_{00} \geq\leq 0$; here $a_{00}$ is the 00-entry of the matrix of the element of $SO(V)$. Over an arbitrary field $k$, it only makes sense to speak of $det = \pm 1$. The connected component of $O(V)$ in the Zariski topology over the algebraic closure is defined by $det = 1$. Over a general field we do not have a notion of signature; rather, we use Witt classes as a substitute (Refer to Chapter 2 for discussion of Witt classes).

We may now think of the group $P_V = \{(t, R) | t \in V \ R \in L_V\}$. $P_V$ acts on spacetime by a translation and a Lorentz rotation. Given $(t, R) \in P_V$ where $t \in V$ is a translation and $R \in SO(V)$ is a rotation, the action on $v \in V$ of $(t, R)$ is given by $(t, R)v \mapsto t + Rv$. If we consider the composition of these maps, we get $(t, R)(t', R')v \mapsto (t, R)(t' + R'v) \mapsto t + R(t' + R'v) = t + Rt' + RR'v = (t + Rt', RR')v$.

Thus, the multiplication law for the group $P_V$ is given by

$$(t, r)(t', r') = (t + rt', rr').$$

We summarize this in the following definition.

**Definition 1.** *The Poincaré group $P_V$ of a quadratic vector space $V$ is defined as*

$$P_V = V \rtimes L_V.$$

We note that $SO(V)$ is a linear algebraic group defined by the equation $r^t F r = F$, where $F$ is the matrix of the quadratic form on $V$ relative to a basis of $V$. Hence, the defining equations of $SO(V)$ are over the field $k$, showing that $SO(V)$ is a linear algebraic group defined over $k$. We shall think of $P_V$, $V$, $L_V$ as linear algebraic groups defined over $k$. We shall also use this notation to denote the groups of $k$-points of these algebraic groups; it will be clear from the context to which object is being referred.

We shall often call $P_V$ the *inhomogeneous group* and denote it by $ISO(V)$. The properties of the Poincaré group are discussed in Chapter 2.

### The conformal group

Conformal maps are maps between pseudo-Riemannian or Riemannian manifolds that take the metric of one manifold to a multiple of the metric of the other, where the multiplying function may vary from point to point.

**Examples:**

(a) Let $\mathbb{R}^{m,n}$ be a quadratic vector space over the real numbers with a quadratic form of signature $(m, n)$. We define the action on $\mathbb{R}^{m,n}$ as dilation $x \mapsto cx$, where $c > 0$.

(b) Complex analytic maps $f$ from a domain $D$ in the complex plane to another domain $D'$ where $df$ is never 0, are conformal.

[3] page 95.

Weyl was the first person to notice that Maxwell's equations are invariant under conformal transformations. The conformal group of Minkowski space is the largest group that preserves the forward light cone. This is why it arises naturally as a candidate for a symmetry group in radiation problems.

In euclidean space $R^n$, one can define inversion as

$$x \mapsto x' = \frac{x}{\|x\|^2}.$$

Then the metric changes by

$$ds'^2 = \frac{1}{r^4} ds^2.$$

This metric is undefined at the origin, but after a one-point compactification of $R^n$ into $S^n$ using the stereographic projection and defining $\infty$ to be the image of the origin, one obtains a conformal map of $S^n$. It is typical that conformal maps are globally defined only after a suitable compactification of the space in question. Compactification of spacetime and the extension of the Poincaré group to the conformal were first considered by Felix Klein [3] p. 95. More recently the conformal group has been considered by Penrose [3] p. 95.

One usually compactifies $R^{m,n}$ to a projective cone in $\mathbb{R}^{m+1,n+1}$ in such a way that $SO(m+1, n+1)$ acts conformally and transitively on this cone. This is why $SO(m+1, n+1)$ is called a conformal group in this setting. It is possible to show that $SO(m+1, n+1)$ contains the Poincaré group, $ISO(m, n)$, as a subgroup. We shall discuss these ideas over an arbitrary field $k$ of ch $\neq 2$. We shall discuss the conformal group and the imbedding of the Poincaré group in it in Chapter 2.

### 1.4.2 Projective completion of spacetime and the conformal metric

Given a quadratic vector space $W$ over field $k$ of characteristic $\neq$ to 2, one can then ask:

(a) Is it possible to imbed $W$ as a Zariski open dense subspace $A$ of some smooth projective variety $[\Omega]$?

(b) Can we imbed $P_W$ as a subgroup $H$ of $SO(V)$, where $W$ is Witt equivalent to $V$ and $dim(V) = dim(W) + 2$, which leaves $A$ invariant? Furthermore, are all such imbeddings conjugate over $SO(V)$?

(c) Is the action of $SO(V)$ on $[\Omega]$ transitive?

(d) Is the action of $H$ on $A$ isomorphic to the action of $P_W$ on $W$?

We will show in Chapter 2 that this is indeed the case. This process is called *projective completion of spacetime.* If $k$ is a locally compact non-discrete field, the projective completion becomes compact and one speaks of *compactification of spacetime.* Unfortunately, we will see that the metric of $W$ does not extend to $[\Omega]$; rather at each point $[x]$ of $[\Omega]$, we have a family of metrics differing by scalar multiples that contains the metric of $W$ on $A$. While the group $SO(V)$ preserves $[\Omega]$, it preserves the quadratic form at each $[x]$ only up to a multiplicative constant. Thus we say that $[\Omega]$ has *a conformal structure*; and as $SO(V)$ keeps this structure invariant, we shall call $SO(V)$, in this context, the *conformal group.* Compactification of spacetime is well known for an arbitrary signature when the underlying field defining spacetime is real. Our explicit results for spacetime over an arbitrary field of ch $\neq 2$ are new.

### 1.4.3 Symmetries of a quantum system

#### Principles of Quantum Mechanics

To specify a quantum mechanical system one must describe the states of the system, the observables, and the various probabilities associated with measurements. Our starting point will be a complex separable Hilbert space $\mathcal{H}$. The states of our system are points of the projective space $\mathbf{P}(\mathcal{H})$ [3] p. 10. Another way of stating this is to say that the states are given by unit vectors in $\mathcal{H}$, where two vectors $\psi$ and $\psi'$ define the same state if and only if $\psi = c\psi'$ for some complex number $c$ of unit modulus, called a *phase factor*. We shall denote vectors in $\mathcal{H}$ by $\psi$ and the corresponding states by $[\psi]$. What physicists call *the superposition principle* is nothing more than the statement that the points of the projective space $\mathbf{P}(\mathcal{H})$ of $\mathcal{H}$ correspond to the states of the quantum system. A state $\psi_3 \in \mathcal{H}$ is a superposition of states $\psi_1$ and $\psi_2$ if $[\psi_3]$ lies on the line in $\mathbf{P}(\mathcal{H})$ joining $[\psi_1]$ and $[\psi_2]$. [3] p. 17.

The observables of a quantum system correspond to self-adjoint operators of $\mathcal{H}$. Suppose $A$ is the operator of an observable on $\mathcal{H}$ having discrete eigenvalues $a_1, a_2, ...$, with corresponding orthonormal eigenvectors $\psi_1, \psi_2, ....$ Then one can ask the question: given a state $\psi$, what is the probability that measurement of $A$ upon the state $\psi$ will yield the value $a_i$ for some fixed $i$? This probability is given by

$$Prob_{\psi}(A = a_i) = |(\psi, \psi_i)|^2.$$

One cannot, in general, hope to have a discrete spectrum of eigenvalues; for example, position and momentum operators have continuous spectra. Hence, a more sophisticated spectral theory of operators is needed. If $A$ is an arbitrary self-adjoint operator, one can associate to $A$ a *spectral measure* $P^A$, which is a projection-valued measure and which replaces the notion of eigenspaces. We have

$$A = \int_{-\infty}^{+\infty} \lambda dP^A(\lambda).$$

When one measures $A$ in the state $[\psi]$, the probability of finding a value of $A$ in some set $E \subset \mathbb{R}$ is given by

$$Prob_\psi(A \in E) = \parallel P_E^A \psi \parallel^2 = (P_A^E \psi, \psi).$$

If $\psi$ and $\psi'$ are unit vectors, we define the quantity

$$|(\psi, \psi')|^2$$

as the *transition probability.* This is the probability that when the system is in state $\psi$ and a measurement is made to determine if the state is $\psi'$, the measurement finds the system in the state $\psi'$. In $\mathbf{P}(\mathcal{H})$, the states are given by points $[\psi]$. We can define the transition probability in $\mathbf{P}(\mathcal{H})$ to be:

$$p([\psi], [\phi]) = |(\psi, \phi)|^2, \quad \psi, \ \phi \in \mathcal{H}.$$

Note that this is well defined. [3] p. 17.

### Symmetry of a quantum system

Our ultimate goal is to describe particles of a quantum mechanical system, but first we need to define the concept of a symmetry in a quantum system.

**Definition 2.** *A symmetry of a quantum system with associated Hilbert space $\mathcal{H}$ is a bijection $\mathbf{P}(\mathcal{H}) \to \mathbf{P}(\mathcal{H})$ that preserves $p([\psi], [\phi])$ for any two $[\psi]$, $[\phi] \in \mathbf{P}(\mathcal{H})$.*

If $s$ is a symmetry of $\mathcal{H}$, then

$$s : \mathbf{P}(\mathcal{H}) \rightarrow \mathbf{P}(\mathcal{H}), \quad p(s[\psi], s[\phi]) = p([\psi], [\phi]).$$

Clearly, for unitary (resp. anti-unitary) operators $U$ (resp. $U'$) of $\mathcal{H}$ we have:

$$|(U\psi, U\phi)|^2 = |(\psi, \phi)|^2 \qquad |(U'\psi, U'\phi)|^2 = |(\phi, \psi)|^2.$$

Hence, both unitary and anti-unitary operators preserve the transition probability and so preserve $p$. If $U$ is a unitary or anti-unitary operator, we say that the symmetry $s$ is *induced* by $U$ if $[U\psi] = s \cdot [\psi]$ for all $\psi \in \mathcal{H}$ [3] p. 17.

**Theorem 1.4.1.** ***(Wigner)*** *Every symmetry is induced by a unitary or antiunitary operator on $\mathcal{H}$ which is determined uniquely up to a phase factor. The symmetries form a group and the unitary ones form a normal subgroup of index $\leq 2$. [3] p. 17.*

We note that if $G$ is a group that acts as a group of symmetries and $G$ is generated by squares, then every element of $G$ acts as a unitary symmetry. This is simply because a square of every symmetry is unitary. Given a Lie group $G$, the connected component of $G$ is generated by elements of the form $expX = (exp\frac{X}{2})^2$. So every element of the connected component of $G$ acts as a unitary symmetry. If now $G$ is a connected Lie group, and $\lambda : g \rightarrow \lambda(g)$ is a homomorphism of $G$ into the group of symmetries of $\mathcal{H}$, then for each $g$ there is a unitary operator $L(g)$ of $\mathcal{H}$ such that $\lambda(g)$ is induced by $L(g)$. Thus, one can obtain a map $L : g \rightarrow L(g)$. But this cannot in general be expected to be a unitary representation of $G$ in $\mathcal{H}$ because $L(g)$ is unique only up to a phase factor. In order for $L$ to be a representation of $G$, it must be a continuous homomorphism of $G$ into $\mathcal{U}(\mathcal{H})$, the unitary group of $\mathcal{H}$. Continuity is with respect to the strong operator topology of $\mathcal{H}$. The continuity is satisfied when $G$ and $\mathcal{H}$ are separable and the maps $\phi, \psi \mapsto (L(g)\phi, \psi)$ are Borel [3] p. 18.

Even though $L$ is not typically a unitary representation of $G$ in $\mathcal{H}$, we have that

$$L(g)L(h) = m(g,h)L(gh) \text{ where } |m(g,h)| = 1.$$

The function $m(g,h)$ is known as a *multiplier*. Typically, $L(g)$ is not uniquely determined by $\lambda(g)$; however, the image $L^{\sim}(g)$ in the projective unitary group $\mathcal{U}(\mathcal{H})/\mathbb{C}^{\times}1$ is well defined. We note that as soon as all the maps involved are Borel, the continuity of the map $L^{\sim}$ and the action of $G$ is guaranteed. So even though we do not have a unitary representation of $G$, we do have a *projective unitary representation* (PUR) of $G$. [3] p. 18. Projective representations are of great importance to us, and we shall devote later sections to their systematic study.

### 1.4.4 Particle classification given an arbitrary symmetry group $G$

Let $G$ be a group that acts on spacetime. We have noted that the symmetries preserve transition probabilities and therefore leave laws of physics invariant. If we are to have a quantum system compatible with the action of $G$, we must have a projective unitary representation of the symmetry group $G$ on the Hilbert space $\mathcal{H}$ of the system. We note that if $G$ is the Poincaré group, compatibility with $G$ is the statement that the description of the system obeys the principles of special relativity [3] p. 26.

It is a natural postulate that elementary particles are described by *projective unitary irreducible representations* (PUIRs). [3] p. 27. If a representation acts reducibly, then the Hilbert space can be written as a direct sum of two orthogonal subspaces each compatible with the action of $G$. The system can then be regarded as a direct sum of two relativistically invariant subsystems and so is no longer elementary. However, not all irreducible projective unitaries define elementary particles. Some of them are not physically realizable. Any process that weeds out some projective unitaries as unphysical is called *a selection rule* [4] p. 5. For example the Poincaré group has irreducible representations that correspond to particles having an imaginary mass [3] p. 35. Such representations are ruled out as unphysical.

We shall define the *real Poincaré group* to be $P = \mathbb{R}^4 \rtimes SL(2,C)_{\mathbb{R}}$, where $SL(2,C)_{\mathbb{R}}$, the complex group $SL(2,\mathbb{C})$ viewed as a real Lie group, is viewed as the universal covering group of $SO(1,3)^0$ acting on $R^4$ through a covering map. We make this definition, to distinguish it from Poincaré groups over other fields, which will be discussed later.

A classification of elementary particles was carried out by E. Wigner in his famous 1939 paper [12]. Wigner proved that there was a one to one correspondence between the irreducible projective unitary representations of the real Poincaré group and ordinary irreducible representations of its two-fold cover. He then found these irreducible representations of the two-fold cover and determined those of positive energy, thus completing the classification of elementary particles corresponding to the real Poincaré group [3] p. 26.

### 1.4.5 Projective representations

#### Extension of PUIRs of Poincaré into conformal group the classical description

Originally, the real Poincaré group was shown to be imbedded into the conformal group [16]. One may ask whether any of the PUIRs of the real Poincaré group extend to be PUIRs of the conformal group. Another way of stating this question is: Which relativistic particles have conformal symmetry? Classically, only massless particles have conformal symmetry. Intuitively this makes sense since the conformal group will "dilate" the mass of any particle defined by the real Poincaré group. As a result, the only particles that can stay invariant under the conformal group are the massless ones. This question, in the classical setting, was completely settled by Angelopoulous [16]. In this thesis we have extended the imbedding of the Poincaré group into the conformal group for arbitrary fields of characteristic $\neq 2$, in particular for all $\mathbb{Q}_p$. One may thus ask the above extendability question in the $p$-adic setting.

We have seen that understanding PURs is the key to classifying particles of a quantum system with respect to a symmetry group $G$. In Chapter 3 we outline the necessary background of PURs. In Chapter 4 we specialize the discussion to projective unitary representations of semidirect product groups. In Chapter 5 we further specialize the results to the case where the groups are defined over $\mathbb{Q}_p$. Our main results are discussed in Chapter 6 in which we determine some necessary conditions for a PUIR of a $p$-adic Poincaré group to extend to the conformal group. We shall show in particular that neither massive particles extend nor the so-called eventually massive particles.

### 1.4.6 Further problems of interest

Chapter 7 describes future research questions and ideas raised by this thesis. The main concern of this thesis is that of extendability of PUIRs. We would like to complete this study by proving the converse of Theorem 6.5.2; namely, we want to prove that massless particles, and only massless particles, have full conformal symmetry. Specifically, we want to look at which PUIRs of the Poincaré group extend to be PUIRs of the conformal group. In the future we would also like to expand the ideas of this thesis to the $p$-adic Galilean group. It is also reasonable to ask whether the theory of the groups defined over $p$-adic numbers discussed in this thesis may be extended to groups defined over other interesting fields such as fields of characteristic $p$.

# CHAPTER 2

# Poincaré and conformal groups over a field of characteristic $\neq 2$

## 2.1 Introduction

In this chapter we discuss the algebraic theory of the Poincaré and conformal groups over fields of characteristic $\neq 2$. Throughout the chapter $k$ will denote a fixed field of characteristic $\neq 2$. In Section 2.2, the theory of quadratic vector spaces and in particular Witt rings is outlined. These are important concepts that are used throughout the rest of this thesis. We state the main theorems of this chapter in Section 2.3. We prove the imbedding Theorem 2.3.1 in Section 2.4. We show how to compactify spacetime and prove Theorem 2.3.2 in Section 2.5. Section 2.6 is dedicated to proving the conjugacy of the imbeddings of the Poincaré groups inside the conformal group, namely Theorem 2.3.3. Finally, in Section 2.7 we define the partial conformal group and investigate some of its properties.

## 2.2 Quadratic forms

We begin by recalling some standard facts about quadratic vector spaces. Let $k$ be a field of characteristic $\neq 2$. We are interested in classifying quadratic forms over $k$.

**Definition 3.** *A quadratic vector space over $k$, is a pair $(V, Q)$ where $V$ is a finite-dimensional vector space over $k$ and $Q$ is a quadratic form on $V$, namely $Q(x) = B(x, x)$ where $B$ is a symmetric bilinear form on $V$.*

We shall by abuse of language refer to $V$ as a quadratic vector space. Since the characteristic of $k \neq 2$, having a quadratic form $Q$ is equivalent to having a symmetric bilinear form $B$. This is because we may define one from the other using the formula $B(x, y) = \frac{1}{2}(Q(x+y) - Q(x) - Q(y))$. A quadratic vector space $V$ is *nondegenerate* iff $B$ is nondegenerate. If $W$ is a subspace of $V$, $(W, Q|_W)$ is a quadratic subspace of $(V, Q)$. The subspace $W$ is nondegenerate, if $Q|_W$ is nondegenerate. This happens iff $W \cap W^{\perp} = 0$ or equivalentlly, $V = W \oplus W^{\perp}$. If $(V, Q)$ is a quadratic vector space, we often omit mentioning $Q$ and write $(v, w) := B(v, w)$ and $v^2 := Q(v)$.

**Definition 4.** *The radical of $V$ is the subspace of all $v \in V$ such that $(v, w) = 0$ for all $w \in V$.*

If $R$ is the radical and $W$ is complementary to $R$ then $(W, Q|_W)$ is nondegenerate and we can find an orthogonal basis $(w_i)$ for $W$ such that $(w_i, w_j) = a_i \delta_{ij}$ for suitable $a_i \in k, \;\; a_i \neq 0$.

**Definition 5.** *Given a quadratic vector space $V$, a vector $v \in V$ is said to be a null vector if $v^2 = 0$.*

**Definition 6.** *A subspace $W$ of a quadratic vector space $V$ is a null space if $v^2 = 0$ for all $v \in W$; $W$ is said to be isotropic if it has (nonzero) null vectors.*

We note that the above definition implies that if $W$ is a null space then $(w, w') = 0$ for all $w, w' \in W$. In fact if $w, w' \in W$, $w^2 = w'^2 = (w + w')^2 = 0$ and $0 = (w + w')^2 = w^2 + w'^2 + 2(w, w') = 2(w, w')$. Hence $(w, w') = 0$.

**Definition 7.** *A quadratic vector space $V$ of dimension $2n$ is called a hyperbolic space if there is an orthogonal basis $(e_i, f_i)_{1 \leq i \leq n}$ such that $e_i^2 = f_i^2 = 0$, $(e_i, f_j) = \delta_{ij}$ for all $i, j$. In particular, it is nondegenerate.*

**Definition 8.** *A quadratic vector space is called a hyperbolic plane if it is a 2-dimensional hyperbolic space.*

**Definition 9.** *We shall say that a pair of null spaces $E, F$ is hyperbolic if $E \cap F = 0$ and $E \oplus F$ is nondegenerate.*

This is equivalent to saying that $\dim(E) = \dim(F)$, and there are bases $(e_i)$ of $E$ and $(f_j)$ of $F$ such that $e_i^2 = 0$, $f_j^2 = 0$ and $(e_i, f_j) = \delta_{ij}$. Indeed, the non-degeneracy of $E \oplus F$ is equivalent to requiring that $Q$ is nondegenerate when restricted to $E \oplus F$.

**Definition 10.** *Let $V$ be a quadratic vector space. If for all $v \in V$, $v^2 = 0 \Leftrightarrow v = 0$ we say $V$ is anisotropic or definite.*

**Lemma 2.2.1.** *If $\dim(V) = 2$, $V$ is hyperbolic iff there is a basis of $V$, $u, v$, such that $u^2 = -v^2 = a \neq 0$, $(u, v) = 0$.*

Null vectors can be enlarged to hyperbolic spaces. We have, in fact, the following lemma.

**Lemma 2.2.2.** *Let $V$ be a nondegenerate quadratic vector space and let $v \neq 0$ be a null vector. Then we can find a null vector $w$ such that $(v, w) = 1$ and the subspace spanned by $v, w$ is hyperbolic. If $W$ is a subspace of $V$ with radical $E$ and $W = E \oplus U$, we can find a null space $F \subset V$ such that $E, F$ is a hyperbolic pair and $E \oplus F \perp U$. In particular, if $E$ is a null space, we can find a null space $F$ such that $E, F$ is a hyperbolic pair.*

See [17] p. 590.

**Definition 11.** *A map $J : V \to V'$ between quadratic vector spaces which is a linear isomorphism such that $(x, y)_V = (J \cdot x, J \cdot y)_{V'}$, is called an isometry.*

We have the following key result.

**Theorem 2.2.1.** ***(Witt)*** *Let $V$ be a nondegenerate quadratic vector space and $W$ and $W'$ be subspaces of $V$. Let $g' : W \to W'$ be an isometry. Then there is an isometry $g : V \to V$ extending $g'$.*

For the proof, see [17] p. 591.

**Corollary 2.2.1.** *Let $V$ and $V'$ be isometric quadratic vector spaces and let $W$ and $W'$ be subspaces of $V$ and $V'$ respectively. Let $g' : W \to W'$ be an isometry. Then $g'$ may be extended to an isometry $g : V \to V'$.*

**Theorem 2.2.2.** *If $V$ is a nondegenerate quadratic vector space, then maximal null spaces (resp. hyperbolic spaces) exist. Any null space of $V$ (resp. hyperbolic space) can be enlarged to a maximal one. Any two maximal null spaces of $V$ (resp. hyperbolic spaces) are conjugate under the group of isometries of $V$. Finally, every maximal null space may be enlarged to a maximal hyperbolic space.*

The proof of this theorem is a trivial consequence of above theorems and results which may be found in [17] p. 590-594.

Let $V$ be a quadratic space and $W$ its radical. We can then write $V = W \oplus U$, where $U$ is nondegenerate. The quadratic structure on $V$ naturally induces one on $V/W$. Let $\pi : V \to V/W$ be the canonical map. We define the induced quadratic form $\tilde{Q}$ on $V/W$ by $\tilde{Q}(v') = Q(v)$, where $v' \in V/W$, $v \in V$, and $\pi(v) = v'$. It is easy to see that this quadratic form $\tilde{Q}$ is well defined. We also see that $U$ and $V/W$ are isometric. This implies that $U$ is unique up to isometry: if $U'$ is also such that $V = W \oplus U'$, then $U$ and $U'$ are isometric since they are both isometric to $V/W$.

If $H$ is a maximal hyperbolic subspace of $U$ and $A$ the orthogonal complement of $H$ in $U$, then $A$ is anisotropic and we have the decomposition

$$V = W \oplus H \oplus A.$$

Let

$$V = W \oplus H' \oplus A'$$

be another such decomposition of $V$. Now $U = H \oplus A$ is isometric to $U' = H' \oplus A'$ by the above discussion. Also, $H$ and $H'$ are isometric since they are both maximal hyperbolic subspaces of $U$ and $U'$ respectively and hence by corollary to Theorem 2.2.1, there is an isometry of $U$ with $U'$ that takes $H$ to $H'$. Hence $A \simeq U/H$ and $A' \simeq U'/H'$ are isometric.

**Definition 12.** *Given two quadratic vector spaces $V = H \oplus A$, $V' = H' \oplus A'$, where $H$ and $H'$ are maximum hyperbolic subspaces and $A$ and $A'$ are anisotropic subspaces, we say that $V$ and $V'$ are Witt equivalent if $A$ and $A'$ are isometric.*

It is clear from the above discussion that Witt equivalence is an equivalence relation. We write $\simeq$ for Witt equivalence. Every Witt equivalence class contains a unique anisotropic isomorphism class and Witt equivalence between anisotropic quadratic spaces, of the same dimension, is the same as isomorphism.

Let $W(k)$ be the set of Witt equivalence classes. If $V = V_1 \oplus V_2$, it is clear that varying the $V_i$ in their Witt classes does not change the Witt class of $V$. The same is true of $V' = V_1 \otimes V_2$, since the tensor product of hyperbolic spaces is hyperbolic. Thus the set $W(k)$ of Witt classes has two binary operations, $\oplus$ (addition) and $\otimes$ (multiplication), both commutative, and multiplication, obeying distributive laws with respect to addition. The 0 class is the class of hyperbolic spaces and the 1 is the class of the one dimensional vector space with basis $e$ and $Q(e) = 1$. We thus have the following.

**Theorem 2.2.3.** *$W(k)$ is a ring.*

**Definition 13.** *The ring $W(k)$ is called the Witt ring of the field $k$.*

The Witt ring is an invariant of $k$. If $V$ is nondegenerate and we choose an orthogonal basis $(e_i)$ of $V$ such that $e_i^2 = a_i \neq 0$, we write $(a_1, a_2, ..., a_n)$ for the class of $V$. We note that $(e_i, e_j) = a_i\delta_{ij}$ and we may adjust $e_i$ in such a way as to pick $a_i$ to be any member of the coset $a_i k^{2^\times}$.

**Example:**

If $k = \mathbb{R}$, any quadratic space which is nondegenerate is of signature $(m, n)$; $(m, n)$ determines the isomorphism class while $s = m - n$ determines the Witt class. Thus the Witt ring is $\mathbb{Z}$.

**Witt ring of a finite field and classification of quadratic forms over finite fields**

Let $F$ be a finite field of characteristic $\neq 2$. Suppose $|F| = q$.

In a finite field the map $x \mapsto x^2$ is a bijective homomorphism from $F^\times$ onto $F^{\times^2}$ with kernel $\{\pm 1\}$. So it is an isomorphism and $|\frac{F^\times}{F^{\times^2}}| = 2$. We write $\varepsilon$ for the representative of the nontrivial coset. It is well known that $-1$ is not a square in $F$ if and only if $q \equiv 3 \mod 4$. If $-1$ is not a square, we take the representative to be $-1$; otherwise, we will take the representative to be an arbitrary $\varepsilon$.

**Lemma 2.2.3.** *Every element of $F$ is a sum of two squares.*

*Proof.* Let $x \in F$. Note that $|F^2| = \frac{1}{2}(q-1)+1$ while the set $x - F^2$ has order $\frac{1}{2}(q+1)$. By counting elements the two sets must have a non-empty intersection. $\square$

**Lemma 2.2.4.** *An anisotropic quadratic vector space $V$ over $F$ has dimension $\leq 2$, hence of dimension $0$ or $2$.*

*Proof.* Let $\dim(V) = 3$. First, let us consider the case where $q \equiv 3 \bmod 4$. If $V$ corresponds to $(a_1, a_2, a_3) \neq (1, 1, 1)$, then $V$ clearly has null vectors by Lemma 2.2.3. Let us consider the case $(1, 1, 1)$. If $x \in F^\times$, by Lemma 2.2.3 we may write $-x^2 = y^2 + z^2$ for some $y, z \in F$. Hence $(1, 1, 1)$ contains a null vector. Now consider the case $q \equiv 1 \mod 4$. Then $V$ has a class $(a, b, c)$ where $a, b, c \in \{1, \varepsilon\}$. Since we may multiply by $\varepsilon$ and permute the $a, b, c$, the only cases we need to consider are $(1, 1, 1)$ and $(1, 1, \varepsilon)$. The case $(1, 1, 1)$ is exactly the same as above. For $(1, 1, \varepsilon)$, note that we may write $-\varepsilon = x^2 + y^2$ and hence we have a null vector in $(1, 1, \varepsilon)$. $\square$

Let us denote the unit element of the Witt ring by (1).

**Lemma 2.2.5.** *Let $q \equiv 3$ mod $4$ then in the Witt ring of $F$ $(1, 1, 1) = -(1)$ and $(1, 1) \neq 0$.*

*Proof.* By Lemma 2.2.4 we know that $(1, 1, 1)$ has null vectors, so $(1, 1, 1) \simeq (\pm 1)$. If $(1, 1, 1) = (1)$, then $(1, 1) + (1) = (1)$, which means that $(1, 1)$ is hyperbolic. So $ae_1 + be_2$ is a null vector where $e_1^2 = e_2^2 = 1$, $(e_1, e_2) = 0$. Hence $a^2 + b^2 = 0$, which implies $a^2 = -b^2$ which contradicts the fact that $-1$ is not a square. Hence $(1, 1) \neq (0)$. So we must have $(1, 1, 1) = (-1)$. $\square$

**Theorem 2.2.4.** *If $q \equiv 3$ mod $4$ then the Witt ring of $F$ is isomorphic to the ring $\mathbb{Z}/4\mathbb{Z}$. The generator is $(1)$. The elements are $(0)$, $(1)$, $(1, 1) = (1)^2$, and $(1, 1, 1) = (1)^3 = -(1)$.*

*Proof.* Since $(1,1,1) = (-1)$ and hence $(1,1) = (-1,-1)$, the possible classes are $(0),(1),(-1),(1,1)$. This is clearly the additive group $\mathbb{Z}/4\mathbb{Z}$. For the ring structure we need only to check that $(1,1)\times(1,1,1) = (1,1)$. But $(1,1)\times(-(1)) = -(1,1) = (1,1)$. $\square$

**Theorem 2.2.5.** *If $q \equiv 1 \mod 4$ then the Witt ring of $F$ is the group ring of $F^\times/F^{\times 2}$ with values in $\mathbb{Z}/2\mathbb{Z}$. If $\varepsilon$ is a representative of the nontrivial coset in $F^\times/F^{\times 2}$, then the elements of $W(F)$ are $(0)$, $(1)$, $(\varepsilon)$, $(1,\varepsilon)$.*

*Proof.* We note that $(1) = (-1)$, hence $(1,1) = (0)$. The possible classes are $(0),(1),(\varepsilon),(1,\varepsilon)$. It is now a straightforward verification that $W(F)$ is the group ring over $F^\times/F^{\times 2} \simeq \mathbb{Z}/2\mathbb{Z}$ with values in $\mathbb{Z}/2\mathbb{Z}$. The isomorphism is given by $(0) \mapsto 0, (1) \mapsto \delta_0, (\varepsilon) \mapsto \delta_1, (1,\varepsilon) \mapsto 1$, where $\delta_x$ is a function that is 1 at $x$ and 0 everywhere else, and 1 is the constant function equal to 1. $\square$

The above results are all contained in the following description of the quadratic forms of a finite field.

**Theorem 2.2.6.** *The isomorphism classes of nondegenerate quadratic forms of a vector space of dimension $D$ over a finite field $F$ of characteristic $\neq 2$ have the following form:*

*$|F| \equiv 3 \mod 4$, $D$ even:*

*Corresponding to the class* $(0)$

$$x_1^2 + x_2^2 + ... + x_{D/2}^2 - y_1^2 - y_2^2 - ... - y_{D/2}^2$$

*Corresponding to the class* $(1,1)$

$$x_1^2 + ...x_{D/2+1}^2 - y_1^2 - ... - y_{D/2-1}^2$$

*$|F| \equiv 3 \mod 4$, $D$ odd:*

*Corresponding to the class* $(1)$

$$x_1^2 + x_2^2 + ... + x_{(D+1)/2}^2 - y_1^2 - y_2^2 - ... - y_{(D-1)/2}^2$$

*Corresponding to the class* $(-1) = (1,1,1)$

$$x_1^2 + x_2^2 + ... + x_{(D-1)/2}^2 - y_1^2 - y_2^2 - ... - y_{(D+1)/2}^2$$

*$|F| \equiv 1 \mod 4$,, $D$ even:*

*Corresponding to the class* $(0)$

$$x_1^2 + x_2^2 + ... + x_{D/2}^2 - y_1^2 - y_2^2 - ... - y_{D/2}^2$$

*Corresponding to the class* $(1, \varepsilon)$

$$x_1^2 + x_2^2 + ... + x_{D-1}^2 - \varepsilon y^2$$

*$|F| \equiv 1 \mod 4$, $D$ odd:*

*Corresponding to the class* $(1)$

$$x_1^2 + x_2^2 + ... + x_{(D+1)/2}^2 - y_1^2 - y_2^2 - ... - y_{(D-1)/2}^2$$

*Corresponding to the class* $(\varepsilon)$

$$x_1^2 + x_2^2 + ... + x_{(D-1)/2}^2 - y_1^2 - y_2^2 - ... - y_{(D-1)/2} + \varepsilon y_{(D+1)/2}^2$$

## 2.3 Statements of main theorems

Before we state the main results of this chapter, we first prove a lemma to clarify the setup.

Consider a pair of quadratic vector spaces $(W, V)$ that are related in the following manner:

(a) $\dim(V) = \dim(W) + 2$

(b) $V$ is Witt equivalent to $W$.

Note that $V \simeq W \oplus H$, where $H$ is a hyperbolic plane. In particular, $V$ is necessarily isotropic.

**Lemma 2.3.1.** *Given a quadratic vector space $W$, there exists a quadratic vector space $V$ unique up to an isometry such that $W$ and $V$ are Witt equivalent with $dim(V) = dim(W) + 2$. Conversely, given a quadratic vector space $V$ that is isotropic, there exists a quadratic vector space $W$, uniquely determined up to isometry, such that $W$ and $V$ are Witt equivalent, with $dim(V) = dim(W) + 2$.*

*Proof.* If $W$ is a quadratic vector space, define $V = W \ \oplus \langle p, q\rangle$, where $(p, q)$ is a hyperbolic pair and $W = \langle p, q\rangle^{\perp}$. Then it is clear that $W$ and $V$ are Witt equivalent with $\dim(V) = \dim(W) + 2$. It is clear from our discussion of Witt classes that $V$ is uniquely defined up to an isomorphism of quadratic vector spaces.

Let $V$ be an isotropic quadratic vector space. Choose a null vector $p \in V$. Then $V = W_p \oplus \langle p, q\rangle$, where $q$ and $p$ span a hyperbolic plane. It is clear that $W_p$ and $V$ are Witt equivalent with $\dim(V) = \dim(W_p)+2$, and that $\langle p\rangle^{\perp} = kp \oplus W_p$, so that $W_p \simeq W = \langle p\rangle^{\perp}/kp$, and also that the natural map gives an isomorphism of quadratic vector spaces, $W_p \simeq W$. It is also clear from our discussion of Witt equivalence that $W$ is uniquely determined up to an isometry.

$\square$

**Imbedding of the Poincaré group into the conformal group and the conjugacy of such imbeddings**

The first major theorem of this chapter states that given a Poincaré group $P_W$, we can imbed it as a stabilizer of a null vector in $SO(V)$ of a quadratic vector space $V$, such that $W$ is Witt equivalent to $V$ with $\dim(V) = \dim(W) + 2$.

**Theorem 2.3.1.** *Suppose $W$ and $V$ are two Witt equivalent quadratic vector spaces with dim$(V)$ = dim$(W) + 2$ and let $p \in V$ be a null vector. Denote by $H_p$ the stabilizer of $p$ in $SO(V)$. Then there exists an isomorphism of algebraic groups*

$$f : P_W \xrightarrow{\sim} H_p$$

*over $k$.*

**Compactification of space time and the conformal metric**

Given a quadratic vector space $W$, one can ask:

(a) Is it possible to imbed $W$ as a Zariski dense subspace $A$ of some smooth projective variety $[\Omega]$?

(b) Can we imbed $P_W$ as a subgroup $H$ of some group $G$ which acts transitively on $[\Omega]$, and further, that the action of $H$ on $A$ is isomorphic to the action of $P_W$ on $W$?

We will show that this is indeed the case. When the underlying field is local, this process is called *compactification of spacetime*. Unfortunately, we will see that the metric of $W$ does not extend to $[\Omega]$; rather, at each point $[x]$ of $[\Omega]$, we have a family of metrics differing by a scalar multiple, that contains the metric of $W$ on $A$. While the group $G$ preserves $[\Omega]$, it preserves the quadratic form at each $[x]$ only up to a multiplicative constant. Thus we say that $[\Omega]$ has *a conformal structure*; and as $G$ keeps this structure invariant we call $G$ *the conformal group.*

These ideas are summarized in the following theorem.

**Theorem 2.3.2.** *Given two Witt equivalent quadratic vector spaces $W$ and $V$ over $k$ with $dim(V) = dim(W) + 2$ then:*

*(a) There exists an imbedding of $W$ as a Zariski dense open subset $A$ of a smooth projective variety $[\Omega]$, and a $k$-imbedding of $P_W$ as a subgroup $H \subset SO(V)$.*

*(b) The action on $H$ on $A$ is isomorphic to the action of $P_W$ on $W$.*

*(c) The action of $SO(V)$ on $[\Omega]$ is conformal.*

### Conjugacy of imbeddings

The following theorem is a converse of sorts to Theorem 2.3.1. It states that the subgroups of $SO(V)$ which are isomorphic to a Poincaré group $P_W$ arise only as stabilizers of null vectors. Furthermore, $SO(V)$ acts by conjugacy transitively on the set of all Poincaré groups inside $SO(V)$.

**Theorem 2.3.3.** *Let $W$ and $V$ be two quadratic vector spaces with $W$ Witt equivalent to $V$ with $dim(V) = dim(W) + 2$ and $dim(V) = D$. If there is an imbedding $f : P_W \hookrightarrow SO(V)$ of algebraic groups over $k$, then for $D \geq 5$,*

*(a) $f(P_W) = H_p$, where $H_p$ is a stabilizer of some null vector $p \in SO(V)$.*

*(b) All such imbeddings $f$ are conjugate under $SO(V)(k)$.*

We note that this theorem can be proved using the theory of parabolic subgroups, but our proof is elementary and direct.

The goal of this chapter is to prove theorems 2.3.1, 2.3.2 and 2.3.3. For $k = \mathbb{R}$ the results are known [16].

We begin with the proof of Theorem 2.3.1.

## 2.4 Proof of Theorem 2.3.1

**Isomorphism of $P_W$ with $H_p$ when $p$ is a null vector**

Let $W$ and $V$ be two quadratic vector spaces with $W$ Witt equivalent to $V$ so that $\dim(V) = \dim(W) + 2$. Let $p$ be a null vector in $V$. Then $\langle p \rangle^{\perp}/kp$ is isometric to $W$. Fix a null vector $q \in V$ such that $(p, q)$ is a hyperbolic pair in $V$, and let $W_p = \langle p, q \rangle^{\perp}$. Then $V = W_p \oplus \langle p, q \rangle$ and $W_p \simeq W$. Thus to imbed $P_W$ in $SO(V)$ as a stabilizer of a null vector, it suffices to show that $P_{W_p}$ is isomorphic to $H_p \subset SO(V)$, the stabilizer of $p$. The construction of this imbedding will show that it is defined over $k$.

*Proof.* Let $h$ be in $H_p$. We want to write $h$ in an explicit block matrix form with respect to $V = \langle p \rangle \oplus \langle q \rangle \oplus W_p$. We begin by investigating how $h$ acts on $q$ and $w \in W$. We write $hq = ap + bq + t$, where $t \in W_p$. Then $(hq, hp) = 1$, so $(ap + bq + t, p) = b = 1$. So we get that $b = 1$.

We also have that $(hq, hq) = (ap + q + t, ap + q + t) = 2a + (t, t) = 0$ so that $a = -\frac{(t,t)}{2}$.

Let $R$ be $h$ followed by the projection $V \to W_p$ mod $\langle p, q \rangle$. Then for $w \in W_p$, $hw = cp + dq + Rw$. But $0 = (w, p) = (w, h^{-1}p) = (hw, p) = d$, so $d = 0$. We also have that $0 = (q,w) = (hq,hw) = (-\frac{(t,t)}{2}p + q + t, cp + Rw) = c + (t, Rw)$, so $c = -(t, Rw)$.

So we have the following relations:

$$hp = p$$

$$hq = -\frac{(t,t)}{2}p + q + t$$

$$hw = -(t, Rw)p + Rw.$$

Therefore one can write the matrix of $h$ as

$$h(t,R) = \begin{pmatrix} 1 & -\frac{(t,t)}{2} & e(t,R) \\ 0 & 1 & 0 \\ 0 & t & R \end{pmatrix}.$$

Here $e(t,R)$ is the map $e(t,R) : w \mapsto -(t, Rw)$. We note that $h$ is completely determined by $t$ and $R$. The relation $(w,w) = (hw,hw)$ means that

$$(w,w) = (-(t,Rw)p + Rw, -(t,Rw)p + Rw) = (Rw, Rw).$$

It follows that $R$ is in $O(W_p)$, but if we calculate the determinant of $h(t,R)$, we see that $R$ is in fact in $SO(W_p)$. Moreover, any element $h(t,R)$ with $t \in W_p,\ R \in SO(W_p)$ is in $H_p$. Indeed, $h(t,R)$ is defined so that $h_p = p,\ \ (hq,hq) = 0,\ \ (hq,hp) = 1,\ \ (hq,hw) = 0\ \ \forall w \in W_p,\ \ (hw,hw) = (w,w)\ \ \forall w \in W_p$, thus showing that $h \in SO(V)$, and hence in $H_p$.

We now verify that we have a homomorphism from $P_{W_p}$ to $H_p$, i.e.,

$$h(t,R) \cdot h(t',R') = h(t + Rt', RR').$$

We have, for $h(t,R)\cdot h(t',R')$ the expansion

$$\begin{pmatrix} 1 & -\frac{(t,t)}{2} & e(t,R) \\ 0 & 1 & 0 \\ 0 & t & R \end{pmatrix} \begin{pmatrix} 1 & -\frac{(t',t')}{2} & e(t',R') \\ 0 & 1 & 0 \\ 0 & t' & R' \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -\frac{(t,t)}{2} - \frac{(t,t)}{2} + e(t,R)\cdot t' & e(t',R') + e(t,R)\cdot R' \\ 0 & 1 & 0 \\ 0 & t+Rt' & RR' \end{pmatrix}$$

The matrix of $h(t+Rt', RR')$ is

$$\begin{pmatrix} 1 & -\frac{(t+Rt',t+Rt')}{2} & e(t+Rt',RR') \\ 0 & 1 & 0 \\ 0 & t+Rt' & RR' \end{pmatrix}$$

One now has to verify that

(a) $-\frac{(t+Rt',t+Rt')}{2} = -\frac{(t',t')}{2} - \frac{(t,t)}{2} + e(t,R)\cdot t'$

(b) $e(t+Rt',RR') = e(t',R') + e(t,R)\cdot R'$.

**Verification of** $(a)$**:**

We have that

$$-\frac{(t+Rt',t+Rt')}{2} = -\frac{(t,t)}{2} - \frac{(Rt',Rt')}{2} - (Rt',t) = -\frac{(t,t)}{2} - \frac{(t',t')}{2} - (Rt',t).$$

So it remains to show that

$$-(Rt',t) = e(t,R)\cdot t'.$$

This is clear since

$$e(t, R) \cdot t' = -(t, Rt') = -(Rt', t).$$

**Verification of** $(b)$**:**

We want to show that

$$e(t + Rt', RR') \cdot w = e(t', R') \cdot w + e(t, R) \cdot R'w.$$

We have

$$e(t', R') \cdot w + e(t, R) \cdot R'w = -(t', R'w) - (t, RR'w)$$

$$= -(Rt', RR'w) - (t, RR'w) = -(t + Rt', RR'w) = e(t + Rt', RR')w.$$

So our assertions are proven and thus we see that $h$ is a homomorphism.

Next we verify that $h$ is injective. Now $h$ is the identity iff $t = 0$ and $R$ is the identity on $W_p$. The mapping $h$ has already been observed to be surjective. We thus see that we have an isomorphism $h$, from $P_{W_p}$ to $H_p$.

The explicit form of $h$ makes it clear that it is a morphism of algebraic groups defined over $k$. To see that it is an isomorphism we note two things: $H_p$ is a closed subgroup of $GL(V)$ defined over $k$ and $f^{-1}$ is the restriction to $H_p$ of the polynomial which maps

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \mapsto (h, i) \in P_W$$

of $GL(V)$ into $P_W$. $\square$

This completes the proof of 2.3.1.

## 2.5 Proof of Theorem 2.3.2

As $SO(V)$ acts on $V$, there is one invariant, namely the quadratic form. Thus the orbits of $SO(V)$ are contained in the level sets of $Q$. More specifically, we have the following lemma:

**Lemma 2.5.1.** *Let $k$ be an arbitrary field of characteristic $\neq 2$, $V$ a quadratic vector space over $k$. If dim$(V) \geq 3$, then $SO(V)$ acts transitively on all sets of the form $M_a = \{x \in V|Q(x) = a \ \ a \neq 0\}$, and on the set $\Omega = \{x \in V|Q(x) = 0$ and $x \neq 0\}$*

*Proof.* **Case 1:** Consider first the sets $M_a = \{x \in V|Q(x) = a\}$ where $a \neq 0$. If $Q(p) = Q(p') = a$, then consider the subspaces $F = kp$ and $F' = kp'$. Let us take any linear isomorphism $F \to F'$ mapping $p$ to $p'$. Then this map is an isometry of $F$ to $F'$. Now Witt's theorem (Theorem 2.2.1) gives a linear isomorphism $\sigma : V \xrightarrow{\sim} V$ preserving the quadratic form. Hence $\sigma$ is in $O(V)$. If $\sigma$ has determinant $-1$, we want to show that there is $\sigma' \in SO(V)$ which takes $p$ to $p'$. Let $K_p$ be the stabilizer of $p$ in $O(V)$. If we take any element of $\tau \in K_p$ with determinant $-1$ and put $\sigma' = \sigma\tau$, then $\sigma'$ has the required property. Let us show that such a $\tau$ exists. Since $Q(p) \neq 0$, we can write $V = kp \ \oplus \langle p\rangle^{\perp}$. We take $\tau$ to be the identity on $kp$ and $\tau$ restricted to $\langle p\rangle^{\perp}$ to be some element of $O(\langle p\rangle^{\perp})$ having negative determinant. This is possible as soon as $\dim(\langle p\rangle^{\perp}) \geq 1$ or $\dim(V) \geq 2$. Then $\sigma' = \sigma \cdot \tau$ is the required isometry.

**Case 2:** Consider now $\Omega = \{x \in V|Q(x) = 0 \text{ and } x \neq 0\}$. Let $Q(p) = Q(p') = 0$. We proceed exactly as in case 1) until we get a $\sigma$ which is in $O(V)$. If $\sigma$ has determinant $-1$, we want to show that there is $\sigma' \in SO(V)$ which takes $p$ to $p'$. As before, if we can find $\tau \in K_p$ such that $det(\tau) = -1$, we can take $\sigma' = \sigma \cdot \tau$. We find a $q$ such that $(q, q) = 0$ and $(q, p) = 1$. Then we take $\tau$ to be identity on $\langle p, q\rangle$ and on $\langle p, q\rangle^{\perp}$ we take $\tau$ to be some element of $O(\langle p, q\rangle^{\perp})$ having a negative determinant. We are using here the fact that $\dim(V) \geq 3$ so that $\dim(\langle p, q\rangle^{\perp}) \geq 1$. Then $\sigma' = \sigma \cdot \tau$ is the required isometry. $\square$

**Definition 14.** *Let $V$ be a quadratic vector space of dimension $n$. We define the light cone of $V$ as*

$$\Omega = \{p \in V | p \neq 0, (p, p) = 0\}.$$

The light cone is simply the collection of all nonzero null vectors of $V$. There is a basis of $V$ for which the quadratic form becomes:

$$Q(x) = a_0 x_0^2 + a_1 x_1^2 + ... + a_{n-1} x_{n-1}^2, \quad a_i \neq 0$$

Thus the equation defining $\Omega$ is

$$a_0 x_0^2 + ... + a_{n-1} x_{n-1}^2 = 0.$$

This is a homogeneous polynomial and it defines a quadric cone in the projective space $\mathbb{P}(V)$. The cone is stable under the action of $SO(V)$ by Lemma 2.5.1. Let $P$ be the map from $V$ to $\mathbb{P}(V)$. We write $[\Omega]$ for the image of $\Omega$ in $\mathbb{P}(V)$ and $[x]$ for the image of $x \in V$.

**Lemma 2.5.2.** *Let $f(x_0, ..., x_{n-1})$ be a homogeneous polynomial of degree prime to the characteristic of $k$ defined in affine space $k^n$ with the property that $\nabla f(x) \neq 0$ when $x \neq 0$ and $f(x) = 0$. Let $U_i$ be the subset of $\mathbb{P}(k^n)$ where $x_i \neq 0$ and define $f_i(x)$ on $U_i$ by $f_i(x) = f(x_0, ...x_{i-1}, 1, x_{i+1}, ...x_{n-1})$ for all $[x_0 : ... : x_{i-1} : 1 : x_{i-1} : ... : x_{n-1}] \in U_i$. Then on each $U_i$, $\nabla f_i(x) \neq 0$ when $x \neq 0$ and $f_i(x) = 0$.*

*Proof.*

$$\nabla f = (\frac{\partial f}{\partial x_0}, \frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_{n-1}}).$$

We look at the open sets $U_i = \{[x_0 : x_1 : ... : x_{i-1} : 1 : x_{i+1} : ... : x_{n-1}]\}$. On $U_i$

$$\nabla f_i = (\frac{\partial f_i}{\partial x_0}, \frac{\partial f_i}{\partial x_1}, ..., \widehat{\frac{\partial f_i}{\partial x_i}}, ..., \frac{\partial f_i}{\partial x_{n-1}}).$$

Assume that for some $x \in U_i$, $x = [x_0 : ..., x_{i-1} : 1 : x_{i+1} : ... : x_{n-1}]$ we have that $\nabla f_i(x) = 0$ and $f_i(x) = 0$. Then $f(x_0, ..., x_{i-1}, 1, x_{i+1}, ..., x_{n-1}) = 0$. Since $\nabla f_i(x) = 0$, we have $\frac{\partial f}{\partial x_j} = 0$ for all $j \neq i$, Euler's identity says:

$$x_0 \frac{\partial f}{\partial x_0} + x_1 \frac{\partial f}{\partial x_1} + ... + x_{n-1} \frac{\partial f}{\partial x_{n-1}} = deg(f) f(x).$$

So that, at $x$

$$\frac{\partial f}{\partial x_i} = deg(f) f(x_0, ..., x_{i-1}, 1, x_{i+1}, ..., x_{n-1}) = 0.$$

Hence $\frac{\partial f}{\partial x_i} = 0$ also. So $\nabla f(x) = 0$ and $f(x) = 0$ when

$$x = (x_0, ..., x_{i-1}, 1, x_{i+1}, ..., x_{n-1})$$

. This contradicts our assumption. □

**Corollary 2.5.1.** $[\Omega]$ *is a smooth projective variety.*

*Proof.* $\Omega$ is defined by $Q(x) = a_0 x_0^2 + ... + a_{n-1} x_{n-1}^2 = 0$ and $x \neq 0$. We have

$$\nabla Q(x) = 2(a_0 x_0 \frac{\partial}{\partial x_0} + ... + a_{n-1} x_{n-1} \frac{\partial}{\partial x_{n-1}}).$$

Hence $\nabla Q(x) = 0$ iff $x_0 = 0, ..., x_{n-1} = 0$, but by assumption we exclude the origin. We note that $[\Omega]$ is a union of open subsets of the form $U_i \cap [\Omega]$, where $U_{x_i} = \{[x_0 : ... : x_{n-1}] | x_i \neq 0\}$ It now follows from Lemma 2.5.2 that on $U_i \subset [\Omega]$, $[\Omega]$ is smooth by the implicit function theorem. Since smoothness is a local condition, it follows that $[\Omega]$ is a smooth projective variety. $\square$

Since $\dim(V) = n$, $\mathbb{P}(V)$ has dimension $n-1$, and since $Q = 0$ defines an $n-2$ dimensional surface in $\mathbb{P}(V)$, $\dim([\Omega]) = n-2$. The tangent space at $x \in \Omega$ is $V_x = \{v \in V | (x,v) = 0\}$, and for $[x] \in [\Omega]$, the tangent space at $[x]$ is $[\Omega]_{[x]}$ and is defined as the image of the tangent map $dP_x$ of $V_x$.

**Lemma 2.5.3.** $[\Omega]$ *has a conformal structure.*

*Proof.* In general, if one has a quadratic vector space $V$ with a quadratic form $Q$ and $W$ a quotient space of $V$, one may induce a quadratic form on $W$ provided that the kernel of the projection is contained in the radical of $Q$.

We now demonstrate this for $V = V_x$ and $W = [\Omega]_{[x]}$. The tangent map $dP_x : V_x \to [\Omega]_{[x]}$ is surjective because $P$ is submersive. Thus the kernel of $dP_x$ is one dimensional. We know that $P$ is constant on the line $kx$ so $dP_x$ vanishes on $kx$. Hence the kernel of $dP_x$ must actually be the line $kx$.

We now define a quadratic form $\tilde{Q}(t)$ for $[\Omega]_{[x]}$. Given $w \in [\Omega]_{[x]}$ we define $\tilde{Q}(w) = Q(v)$ where $v$ is any $v \in V_x$ such that $dP_x(v) = w$.

We now show that $\tilde{Q}$ is well defined. Let us take $v$ and $v'$ such that $dP_x : v \mapsto w$ and $dP_x : v' \mapsto w$ then $v' = v + ax$. Then $Q(v+ax) = Q(v) + Q(ax) + 2aB(x,v)$, but $v$ is perpendicular to $x$ so $B(x,v) = 0$. Also, $x$ is null, so $Q(x) = 0$, and hence $Q(w + ax) = Q(w)$, and $\tilde{Q}$ is well defined.

Consider the following set of maps. We have the map $a_\lambda : V \to V$ such that $v \mapsto \lambda v$. We also have the map $P : V \to \mathbb{P}(V)$. Then the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "a_\lambda"] \arrow[d, "P"] & V \arrow[dl, "P"] \\ \mathbb{P}(V) & \end{tikzcd}$$

Thus diagram of tangent maps also commutes:

$$\begin{tikzcd} V_x \arrow[r, "a_\lambda"] \arrow[d, "dP_x"] & V_{\lambda x} \arrow[dl, "dP_{\lambda x}"] \\ \mathbb{P}(V)_{[x]} & \end{tikzcd}$$

Now $a_\lambda$ takes $x \in V_x$ to $\lambda x \in V_{\lambda x}$. If $dP_x$ takes $x$ to $w \in [\Omega]_{[x]}$, then $dP_{\lambda x}$ takes $\lambda x$ to $w$ also. The quadratic form $Q$ is the same on both $V_x$ and $V_{\lambda x}$. Hence the induced quadratic form at $w \in [\Omega]_{[x]}$ using $x \in V_x$ is $Q(x)$, whereas the induced quadratic form at $w \in [\Omega]_{[\lambda x]} = [\Omega]_{[x]}$ using $\lambda x \in V_{\lambda x}$ is given by $Q(\lambda x) = \lambda^2 Q(x)$. Hence there is no well defined metric at each point $[x]$ of $[\Omega]$; the induced metric depends on which point we choose on the line projecting down to $[x]$ and which tangent map we use. We have a family of metrics differing at each point of $[\Omega]$ by a scalar multiple. Furthermore, if we have $g \in SO(V)$ such that $gx' = \lambda x$, then under the action of $g$ the set of metrics at $[\Omega]_{[x]}$ induced from $V_x$ goes over to the set of metrics induced from $V_{\lambda x}$. Thus $[\Omega]$ has a conformal structure. $\square$

This proves Theorem 2.3.2 *c*).

**Definition and structure of $A_{[p]}$**

We define $A_{[p]} := \{[a] \in [\Omega] | (p, a) \neq 0\}$. We also introduce $C_p$ as the set of null vectors of $V_p$ and $C_{[p]}$ the image of $C_p$ in the projective space. Then we may write $A_{[p]} = [\Omega] \backslash C_{[p]}$. This means that if $[a] \in A_{[p]}$ and we write $a = \alpha p + \beta q + w$, where $w \in W$, then, as $(p, a) \neq 0$, we must have $\beta \neq 0$. We also have $(a, a) = 0$. Since $(a, a) = 0$, we have that

$$(\alpha p + \beta q + w, \alpha p + \beta q + w) = 2\alpha\beta(p, q) + (w, w) = 0.$$

So $2\alpha\beta = -(w,w)$. Since we are only interested in the image of $a$ in the projective space we may take $\beta$ to be 1. We then have that $\alpha = \frac{-(w,w)}{2}$. Then $[a]$ is given by $[\frac{-(w,w)}{2} : 1 : w]$, so $[a]$ is entirely determined by $w$. We thus have bijection $W \simeq A_{[p]}$, $J : w \mapsto [\frac{-(w,w)}{2} : 1 : w]$.

**Lemma 2.5.4.** *$A_{[p]}$ is a Zariski open dense subset of $[\Omega]$.*

*Proof.* It is clear that $A_{[p]}$ is a Zariski open subset of $[\Omega]$; it is dense since $[\Omega]$ is irreducible. $\square$

This proves Theorem 2.3.2 $a$).

**Action of $H_p$ on $A_{[p]}$**

**Lemma 2.5.5.** *The map $J$ intertwines the actions of $H_p$ on $A_{[p]}$ and $P_{W_p}$ on $W_p$.*

*Proof.* We have to show that the action of $(t,R) \in P_{W_p}$ on $W$ goes over to the action of $h(t,R) \in H_p$ on $A_{[p]}$.

Let $(t,R)$ be in $P_W$. We look at the action of $h(t,R)$ on $J(w)$.

$$\begin{pmatrix} 1 & -\frac{(t,t)}{2} & e(t,R) \\ 0 & 1 & 0 \\ 0 & t & R \end{pmatrix} \begin{pmatrix} -\frac{(w,w)}{2} \\ 1 \\ w \end{pmatrix} = \begin{pmatrix} -\frac{(w,w)}{2} - \frac{(t,t)}{2} - (t,R)w \\ 1 \\ t + Rw \end{pmatrix}$$

Now, $-\frac{(w,w)}{2} - \frac{(t,t)}{2} - (t,Rw) = -\frac{(t+Rw,t+Rw)}{2}$. So in $A_{[p]}$, $[-\frac{(w,w)}{2} : 1 : w] = J(w)$ goes to $[-\frac{(t+Rw,t+Rw)}{2} : 1 : t+Rw] = J(t+rW)$. $(t,R) \in P_W$ maps $w$ to $t+Rw$. So we see that action of $P_W$ on $W$, under $J$, goes over to the action of $H_p$ on $A_{[p]}$. $\square$

This completes the proof of Theorem 2.3.2 *b*). All claims of Theorem 2.3.2 have now been proven.

**Lemma 2.5.6.** *If $k = \mathbb{Q}_p$ then $[\Omega]$ is compact.*

*Proof.* Since $[\Omega]$ is a closed subset of $\mathbb{P}(\mathbb{Q}_p^{n+1})$, it suffices to prove that $\mathbb{P}(\mathbb{Q}_p^{n+1})$ is compact. This is a well known fact, but we will prove it here for completeness. We first show that the unit ball $B_{n+1} = \{x \in \mathbb{Q}_p^{n+1} | \ |x| \leq 1\}$ is compact. A $p$-adic expansion of $x \in \mathbb{Q}_p$ is defined by $x = \sum_{i=k}^{\infty} a_i p^i$. Here $k$ may be negative and $a_i \in F = \{0, ..., p-1\}$. Hence the map of $x \in \mathbb{Q}_p$ to its $p$-adic extension defines a map from $\mathbb{Q}_p$ to the countable product $F^\infty = F \times F \times ....$ This infinite product is compact by Tychonoff's theorem. We may define a metric on $F^\infty$ in a standard way. If $a, b \in F^\infty$ then $a = (a_0, a_1, ...)$ and $b = (b_0, b_1 ...)$ then $|a - b|$ is defined to be $p^{-n}$ where $n$ is the first coordinate where $a$ and $b$ differ.

Let $x, y \in B_1$ and the image of $x$ in $F^\infty$ be $(a_0, a_1, ...)$ and the image of $y$ be $(b_0, b_1, ...)$. If $|x - y|_p \leq p^{-n}$ then $a$ and $b$ agree up to the $n$th coordinates. Hence in $F^\infty$ $|a - b| = p^{-n}$. So we have that $B_1$ is homeomorphic to $F^\infty$.

We define the norm on $|| \cdot ||_V$ on $V = \mathbb{Q}_p^{n+1}$ in the following way. If $x = (x_0, ..., x_n), y = (y_0, ..., y_n) \in V$ then $||x - y||_V = max(||x_i - y_i||_p)$. This sup norm topology is equivalent to the product topology when the product is finite. Thus the statement $x = (x_0, ..., x_n) \in B_{n+1}$ is equivalent to that each $x_i \in B_1$. Hence $B_{n+1}$ can be thought of as a product of $n + 1$ copies of $B_1$. Hence it is compact.

We need to show that given $x = [a_0, ..., a_n] \in \mathbb{P}(\mathbb{Q}_p^{n+1})$ there exists a $c \in \mathbb{Q}_p$ such that $c(a_0, a_1, ..., a_n) \in B$. It is enough, for any given $a \in \mathbb{Q}_p$, to find a $c$ such that $0 \leq ca \leq 1$. If $|a|_p = p^m$, take $c = 1/p^{m+1}$. This proves that $\mathbb{P}(V)$ is compact. $\square$

Lemma 2.5.6 shows that if the underlying field is $\mathbb{Q}_p$, then the projective imbedding becomes the compactification of spacetime.

## 2.6 Proof of Theorem 2.3.3

We now aim to prove that given $P_W \hookrightarrow SO(V)$ with $W$ Witt equivalent to $V$ and $\dim(V) = \dim(W)+2$, the image of $P_W$ is the stabilizer of a null vector in $V$. Our first step is to show that $P_W$ stabilizes a vector in $V$. Since $P_W = W \rtimes SO(W)$, we begin by showing that $W$ has a non-zero fixed vector. For what follows we shall write $L$ for $SO(W)$.

### $V$ has a subspace on which $W$ acts trivially

Let $\bar{k}$ be an algebraic closure of $k$ and $\bar{V} = \bar{k} \otimes_k V$, $\bar{W} = \bar{k} \otimes_k W$. We can then work in the framework of algebraic groups. We thus have a closed algebraic subgroup $\bar{P}_W \simeq \bar{W} \rtimes SO(\bar{W})$, $P_W \subset SO(\bar{V})$. We want to prove that $P_W = \bar{P}_W(k)$ fixes a null vector $p \in V$. For what follows, we shall assume that we have a $k$-imbedding of $P_W$ into $SO(V)$. We may then consider $V$ as a $k$-representation of $P_W$.

**Lemma 2.6.1.** *In any $W$-stable subspace $V_1 \subset V$, $W$ fixes a vector in $V_1$.*

*Proof.* We give the argument for $V = V_1$ for notational simplicity, but the argument is general. First assume that $k$ itself is algebraically closed. Then, as $W$ is abelian, it is solvable. By Lie's theorem there is a basis of $V$ in which matrices of $W$ are upper triangular. Let $\phi$ be a representation of $W$ in $V$ which takes each $w \in W$ to an upper triangular matrix. Let $\alpha_i$ be the function which takes $w \in W$ to the $i$th diagonal entry of the matrix $\phi(w)$. It is easy to verify that $\alpha_i$ is a homomorphism from $k^n$ to $k^\times$. This is also a morphism of algebraic groups. Since each entry of $\phi(w)$ is given by a polynomial, we have that $\alpha_i$ is a polynomial. Since $\alpha_i$ must be a constant, it is 1. So all diagonal entries of $\phi(w)$ are 1. Hence the last vector in the basis is fixed by $W$.

Now we want to consider the case where $k$ is not necessarily algebraically closed. We still have the above basis over $\bar{k}$. Thus each $w \in W$ satisfies the equation $(w - I)^d = 0$ over $\bar{k}$, and so over $k$ this equation also holds. Now $w = I + x$ where $x$ is nilpotent. There exists a nonzero vector $v$ and an integer $r \geq 1$ for which $x^r v = 0$ but $x^{r-1}v \neq 0$. Let us denote $x^{r-1}v$ by $v'$. Now $wv' = (I + x)v' = v'$. Thus $v'$ is a nonzero vector fixed by $w$.

Next, we want to show that there exists a nonzero vector fixed by all of $W$. We proceed by induction dimension of $V$. If $V$ is one dimensional, then since all $w \in W$ are unipotent, they are all the identity matrix and thus fix a vector. Assume that the result holds for all lower dimensional spaces. If all $w \in W$ are identity matrices, we are done. So assume there is one $w_0$ which is not the identity. This $w_0$ has a tangent subspace of $V$ on which it is the identity. All the other $w \in W$ commute with $w_0$, so they preserve this subspace. This subspace is of smaller dimension than $V$ and so by induction there is a vector fixed by all of $W$.

□

Let $U$ be a $P_W$-subspace of $V$ of minimal dimension $r$ on which $P_W$ acts irreducibly. Let $U_1$ be the maximal subspace of $U$ on which $W$ acts trivially. $U_1$ is $L$-stable, hence $P_W$-stable, so $U_1 = U$ by the minimality of $U$. Thus $W$ acts trivially on $U$. Since $U$ is $P_W$-stable, so is $U^\perp$. Hence $\dim(U^\perp) \geq \dim(U)$, so $D - r \geq r$, hence $r \leq \frac{D}{2}$.

We now consider the restriction of the quadratic form to $U$. The form need not be nondegenerate. Let $U_1$ be the radical of $U$. $U_1$ is stable under $P_W$, so by irreducibility of $U$, either $U_1 = 0$ or $U_1 = U$. Observe that $\dim(P_W) = \frac{(D-2)(D-3)}{2} + D - 2 = \frac{(D-1)(D-2)}{2}$.

**Lemma 2.6.2.** *$U_1 = U$, i.e., $U$ is isotropic.*

*Proof.* Assume that $U_1 = 0$. In this case the quadratic form is nondegenerate, so $V = U \oplus U^{\perp}$. Both $U$ and $U^{\perp}$ are $P_W$-stable, and thus $P_W \subset O(U) \times O(U^{\perp})$. But since $P_W$ is a connected group, it must be mapped to the connected component of $O(U) \times O(U)^{\perp}$, so really we have $P_W \subset SO(U) \times SO(U^{\perp})$. Hence $\dim(P_W) \leq \dim(SO(U)) + \dim(SO(U^{\perp}))$. We now have

$$\frac{(D-1)(D-2)}{2} \leq \frac{r(r-1)}{2} + \frac{(D-r)(D-r-1)}{2}$$

This gives, after a simple calculation,

$$D(r-1) \leq r^2 - 1 = (r+1)(r-1).$$

So either $r = 1$ or $D \leq r + 1$. Note that we already know that $r \leq \frac{D}{2}$. So if $D \leq r + 1$, then we must have that $D \leq 2$. We are assuming $D \geq 5$, hence $r = 1$. So $L$ must be trivial on $U$, since $L$ is semisimple and there is no nontrivial homomorphism into any abelian group. Thus $P_W$ must fix a basis vector $u$ of $U$. Now $V = \langle u \rangle \oplus \langle u \rangle^{\perp}$, and the stabilizer of $u \simeq SO(\langle u \rangle^{\perp})$, hence is of dimension $\frac{(D-1)(D-2)}{2}$ which is also $\dim(P_W)$. So $P_W \simeq SO(\langle u \rangle^{\perp})$, but this is impossible since $SO(\langle u \rangle^{\perp})$ is semisimple while $P_W$ has a normal nontrivial connected abelian subgroup. This rules out the possibility $U_1 = 0$. Hence $U_1 = U$. $\square$

Thus $P_W$ imbeds into the normalizer in $SO(V)$ of $U$. Let $r = \dim(U) \geq 1$. We shall investigate the structure of this normalizer, which we denote by $Q$.

**Lemma 2.6.3.** *We have $Q \simeq N \rtimes S$, where* *(a) $N$ is unipotent of dimension $r(D-2r)+\frac{1}{2}r(r-1)$.*

*(b) $S \simeq GL(r)$.*

*Proof.* As we described in Section 2.2, we can find a null subspace $U'$ of dimension $r$ of $V$, such that for a suitable basis $(u_i)$ of $U$ and $(u_i')$ of $U'$, we have $(u_i, u_j') = \delta_{ij}$. Let $R = (U \oplus U')^{\perp}$. Then $V = U \oplus U' \oplus R$. It is now possible to find a basis for $R$ such that the matrix of the quadratic form is

$$M = \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & \Lambda \end{pmatrix},$$

where $\Lambda$ is a nonsingular diagonal matrix with entries in $k$. The stabilizer $Q$ of $U$ in $SO(V)$ is the subgroup of all matrices in $SO(V)$ of the form

$$X = \begin{pmatrix} A & B_1 & C_1 \\ 0 & B_2 & C_2 \\ 0 & B_3 & C_3 \end{pmatrix}$$

which satisfy

$$X^T M X = M, \text{ and } det(X) = 1.$$

Now

$$X^T M X = \begin{pmatrix} 0 & A^T B_2 & A^T C_2 \\ B_2^T A & B_1^T B_2 + B_2^T B_1 + B_3^T \Lambda B_3 & B_1^T C_2 + B_2^T C_1 + B_3^T \Lambda C_3 \\ C_2^T A & C_1^T B_2 + C_2^T B_1 + C_3^T \Lambda B_3 & C_1^T C_2 + C_2^T C_1 + C_3^T \Lambda C_3 \end{pmatrix}.$$

Hence the required conditions are:

$$A^T B_2 = I \Rightarrow B_2 = (A^T)^{-1};$$

$$C_2 = 0;$$

$$B_1^T B_2 + B_2^T B_1 + B_3^T \Lambda B_3 = 0;$$

$$B_2^T C_1 + B_3^T \Lambda C_3 = 0 \Rightarrow C_1 = -A B_3^T \Lambda C_3;$$

$$C_3^T \Lambda C_3 = \Lambda, \ \ det(C_3) = 1.$$

We use these relations to write $X$ as

$$X = \begin{pmatrix} A & B_1 & -A B_3^T \Lambda C_3 \\ 0 & (A^T)^{-1} & 0 \\ 0 & B_3 & C_3 \end{pmatrix},$$

where $C_3 \in SO(R)$ and we have the relation

$$B_1^T (A^T)^{-1} + A^{-1} B_1 + B_3^T \Lambda B_3 = 0. \tag{2.1}$$

If $\beta_1 = B_1 A^T$, $\beta = B_3 A^T$, we can write this relation as

$$\beta_1 + \beta_1^T + \beta^T \Lambda \beta = 0 \tag{2.2}$$

We can write $X = YZ$ with

$$Y = \begin{pmatrix} I & \beta_1 & -\beta^T \Lambda \\ 0 & I & 0 \\ 0 & \beta & I \end{pmatrix}, \quad Z = \begin{pmatrix} A & 0 & 0 \\ 0 & (A^T)^{-1} & 0 \\ 0 & 0 & C_3 \end{pmatrix}.$$

Notice that $Y, Z \in Q$, since they both satisfy (2.1). Let $S$ be the subgroup of $Q$ of all matrices of the form of $Z$. The map $Z \mapsto (A, C_3)$ is a homomorphism of $Q$ onto $GL(r) \times SO(R)$. Let $N$ be its kernel. Then $N$ consists of the matrices of the form $Y$ with the relation

$$\beta_1 + \beta_1^T + \beta^T \Lambda \beta = 0.$$

We shall now give a simpler description of $N$. The condition

$$\beta_1 + \beta_1^T + \beta^T \Lambda \beta = 0$$

means that $\frac{1}{2}(\beta_1 + \beta_1^T) = -\frac{1}{2}\beta^T \Lambda \beta$. Therefore, $\beta$ determines the symmetric part of $\beta_1$. Let $\sigma = \frac{1}{2}(\beta_1 - \beta_1^T)$, the skew symmetric part of $\beta_1$; then $\sigma$ can be arbitrary. Thus, $N$ is the group of matrices

$$\eta(\beta, \sigma) = \begin{pmatrix} I & -\frac{1}{2}\beta^T \Lambda \beta + \sigma & -\beta^T \Lambda \\ 0 & I & 0 \\ 0 & \beta & I \end{pmatrix},$$

where $\beta \in M(D - 2r \times r)$, $\sigma \in Skew(r \times r)$. The multiplication law in $N$ is given by

$$\eta(\beta, \sigma)\eta(\beta', \sigma') = \eta(\beta + \beta', \sigma + \sigma' + \frac{1}{2}(\beta'^T \Lambda \beta - \beta^T \Lambda \beta')).$$

We can verify that $N$ is normalized by $S$:

$$\begin{pmatrix} A^{-1} & 0 & 0 \\ 0 & A^T & 0 \\ 0 & 0 & C_3^{-1} \end{pmatrix} \begin{pmatrix} I & \beta_1 & -\beta^T \Lambda \\ 0 & I & 0 \\ 0 & \beta & I \end{pmatrix} \begin{pmatrix} A & 0 & 0 \\ 0 & (A^T)^{-1} & 0 \\ 0 & 0 & C_3 \end{pmatrix}$$

$$= \begin{pmatrix} I & A^{-1}\beta_1(A^T)^{-1} & -A^{-1}\beta^T\Lambda C_3 \\ 0 & I & 0 \\ 0 & C_3^{-1}\beta(A^T)^{-1} & I \end{pmatrix}.$$

Note that if we take $\beta' = C_3^{-1}\beta(A^T)^{-1}$ and $\beta_1' = A^{-1}\beta_1(A^T)^{-1}$, then

$$\beta_1' + (\beta_1')^T + (\beta')^T\Lambda\beta' = A^{-1}(\beta_1 + \beta_1^T + \beta^T((C_3^T)^{-1}\Lambda C_3^{-1})\beta)(A^T)^{-1}$$

$$= A^{-1}(\beta_1 + \beta_1^T + \beta^T\Lambda\beta)(A^T)^{-1}$$

$$= 0,$$

since $(C_3^T)^{-1}\Lambda C_3^{-1} = \Lambda$. If we go by the previous description of the homomorphism of $Q$ to $GL(r) \times SO(D-2r)$, we should have the semidirect product decomposition:

$$Q = N \rtimes S.$$

From the formula for $\eta(\beta, \sigma)$, we see that it is unipotent; in fact, $\eta(\beta, \sigma) = I$ on $U$, $\simeq \begin{pmatrix} I & 0 \\ \beta & I \end{pmatrix}$ mod $U$, so that it is unipotent. Hence $N$ is a unipotent group. Thus $N$ is the unipotent radical of $Q$. This finishes the proof of Lemma 2.6.3. $\square$

We shall need the following technical lemma to prove the proposition which follows it. Denote by $\bar{L}$ the group $SO(\bar{W})$.

**Lemma 2.6.4.** *Suppose $\bar{L}$ acts faithfully on $\bar{k}^3$. Then $\bar{L}$ acts irreducibly on $\bar{k}^3$.*

*Proof.* $\bar{L}$ cannot act trivially, since it is faithful. Suppose it is not irreducible.

Then $\bar{k}^3$ has a submodule of dimension 1 or 2. If $\dim(M) = 1$ we have

$$\bar{L} \subset \left\{ \begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\},$$

so $\bar{L}$ imbeds into $SL(2) \rtimes \bar{k}^2$. The left side has dimension 6, but the right side has dimension 5. This is impossible. If $\dim(M) = 2$, then

$$\bar{L} \subset \left\{ \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 1 \end{pmatrix} \right\},$$

and again we would have an imbedding of $L$ into $SL(2) \rtimes \bar{k}^2$, which is impossible.

□

**Proposition 2.6.1.** *$U$ has dimension 1 and is spanned by a null vector $p$. In particular, $P_W$ fixes $p$.*

*Proof.* We have shown that $P_W$ can be embedded inside $Q$. We identify $P_W$ with its image in $Q$. We then have the inclusion $L \subset Q$. We consider the canonical projection $\pi$ mapping $Q$ to $Q/N$. Then the kernel of $\pi|_L$ is $L \cap N$. Since $N$ is a unipotent normal subgroup of $Q$, $L \cap N$ is a unipotent normal subgroup of $L$. But $L$ is semisimple, so it does not have any nontrivial normal connected unipotent subgroups. Thus, $L \cap N$ must be finite. This means that $\dim(L/L \cap N) = \dim(L)$ and that $\dim(L) \leq \dim(S)$. Hence we have

$$\frac{(D-2)(D-3)}{2} \leq r^2 + \frac{(D-2r)(D-2r-1)}{2}.$$

We note that since $P_W$ is its own commutator, it must be contained in the commutator subgroup of $N \rtimes S$. Hence $L$ is mapped into $S'$, the commutator subgroup of $S$. Therefore, we may write:

$$\frac{(D-2)(D-3)}{2} < r^2 + \frac{(D-2r)(D-2r-1)}{2}.$$

We now get

$$2D(r-1) < 3r^2 + r - 3,$$

so

$$2D(r-1) \leq 3r^2 + r - 4 = (3r+4)(r-1).$$

If $r = 1$, $L$ must be trivial on $U$, so $P_W$ fixes a basis vector of $U$ which is a null vector, and the proposition is proved. We assume $r \geq 2$ in the rest of the argument and show that this leads to a contradiction. If $r \geq 2$, then,

$$2D \leq 3r + 4.$$

Since $r \leq \frac{D}{2}$,

$$2D \leq 3r + 4 \leq 3\frac{D}{2} + 4$$

$$D \leq 8.$$

So this cannot happen if $D > 8$. Hence we need only consider the possibilities $D = 5, 6, 7, 8$.

**Case $D = 8$:**

$2D \leq 3r + 4 \leq 3\frac{D}{2} + 4 \Rightarrow 16 \leq 3r + 4 \leq 16 \Rightarrow r = 4$. Note that $r = \frac{D}{2}$. So

$V = U \oplus U'$, $R = 0$. Hence $N$ consists of matrices of the form

$$Y = \begin{pmatrix} I & \sigma \\ 0 & I \end{pmatrix},$$

where $\sigma \in Skew(4 \times 4)$ (the reduction to $SL(4)$ from $GL(4)$ is as before) and $S \simeq GL(4)$, where $ISO(6) \hookrightarrow N \rtimes S$. We also note that $N \simeq k^6$ so that

$$ISO(6) \hookrightarrow k^6 \rtimes SL(4).$$

The dimension of the left side is equal to the dimension of the right side, so the above map must be an isomorphism. The semisimple part of $ISO(6)$, $SO(6)$, must be sent to the semisimple part of $k^6 \rtimes SL(4)$; we thus get $SO(6) \simeq SL(4)$. But this is impossible, since $SL(4)$ is the two fold cover of $SO(6)$. So this case is ruled out.

**Case $D = 7$:**

If $D = 7$ then,

$$2D \leq 3r + 4 \leq 3\frac{D}{2} + 4 \Rightarrow 10 \leq 3r \leq \frac{21}{2}.$$

No integer exists with these properties.

**Case $D = 6$:**

$2D \leq 3r + 4 \leq 3\frac{D}{2} + 4 \Rightarrow 8 \leq 3r \leq 9 \Rightarrow r = 3$. So again, $R = 0$, so that $P_W = ISO(4)$. Clearly, $N \simeq \bar{k}^3$ and $S \simeq GL(3)$. We have $P_W = ISO(4) \hookrightarrow N \rtimes S$, and we know that $SO(4)$ is its own commutator, so we must in fact, have $ISO(4) \hookrightarrow \bar{k}^3 \rtimes SL(3)$. Let $\bar{P}$ be the image of $P_W$ in $SL(3)$. Since the $SO(4)$ part must map onto a subgroup of dimension 6, it follows that $\dim(\bar{P})$ has to be $6, 7$, or $8$. If $\dim(\bar{P}) = 8$ (resp. $6$), then $\bar{P} = SL(3)$ (resp. $\bar{P}$ is the image of $SO(4)$). In either case, $\bar{P}$ is semisimple, and has the image of $\bar{k}^4$ as a normal connected unipotent subgroup. Hence the image of $\bar{k}^4$ must be trivial, which means $\bar{k}^4 \hookrightarrow \bar{k}^3$ which is impossible.

Assume $\dim(\bar{P}) = 7$. Let $\bar{T}$ (resp. $\bar{L}$) be the image of $\bar{k}^4$ (resp. $SO(4)$). Then $\bar{T} \simeq \bar{k}$, and $\bar{L}$ normalizes $\bar{T}$, so that $\bar{L}$ acts trivially on $\bar{T}$, $\bar{P} = \bar{T} \times \bar{L}$. $\bar{L}$ acts faithfully on $\bar{k}^3$. From Lemma 2.6.4 we know that $\bar{L}$ acts irreducibly on $\bar{k}^3$. Hence $\bar{T}$ acts as a scalar, which must be 1 since $\bar{T} \simeq \bar{k}$. This is impossible since $\bar{T}$ must act faithfully.

**Case** $D = 5$**:**

$2D \leq 3r + 4 \leq 3\frac{D}{2} + 4 \Rightarrow 6 \leq 3r \leq \frac{15}{2} \Rightarrow r = 2$. We have $ISO(3) = k^3 \rtimes SO(3)$. Moreover,

$$\eta(\beta, \sigma) = \begin{pmatrix} I & -\frac{1}{2}\beta^T\beta + \sigma & -\beta^T \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix},$$

where $\sigma \in Skew(r \times r) = Skew(2 \times 2)$, $\beta \in M(D - 2r \times r) = M(1 \times 2)$. This shows that $\dim(N) = 3$. Moreover, $S = GL(2)$. But $L$ is mapped into the commutator subgroup of $S$, hence into $SL(2)$. But $\dim(ISO(3)) = \dim(N \rtimes SL(2))$. So we have an isomorphism $ISO(3) \simeq N \rtimes SL(2)$. so $SO(3) \simeq SL(2)$. But $SO(3)$ is a two-fold cover of $SL(2)$, and this is a contradiction. $\square$

**Corollary 2.6.1.** *$P_W$ is the stabilizer of $p$ which is a basis for $U$.*

*Proof.* We know that $P_W$ is a subgroup of the stabilizer of $p$, but the stabilizer of the null vector $p$ is isomorphic to a Poincaré group $P$ which has same dimension as $P_W$, and both are connected, so $P_W$ is the stabilizer of $p$. $\square$

**Completion of the proof of 2.3.3**

**Theorem 2.6.1.** *All subgroups of $SO(V)$ isomorphic to the Poincaré group $P_W$ of $SO(V)$ are conjugate.*

*Proof.* All null vectors $p$ are conjugate over $SO(V)(k)$ by Lemma 2.5.1. Hence their stabilizers are conjugate. By the theorems 2.3.1 and 2.3.3 $a$), a subgroup of $SO(V)$ is a stabilizer of a null vector iff it is isomorphic to the Poincaré group $P_W$. So all Poincaré subgroups inside $SO(V)$ are conjugate. □

## 2.7 Partial conformal group

In this section we introduce a subgroup $\tilde{P}_V$ of $SO(V)$, the stabilizer of the line $kp$. It is an easy computation that $h \in SO(V)$ stabilizes the line $kp$ iff $h$ has the form

$$h = \begin{pmatrix} c & -c\frac{(t,t)}{2} & ce(t,R) \\ 0 & \frac{1}{c} & 0 \\ 0 & t & R \end{pmatrix}$$

Here $c \in k$, $t \in W_p$, $R \in SO(W)$ and $e(t,R) \in \mathrm{Hom}(W_p, \langle p \rangle)$. We write $h = h(c,t,R)$. We shall denote the set of all such matrices $h$ as

$$\tilde{P}_V = \{h(c,t,R) | c \in k,\ t \in W,\ R \in SO(W)\}.$$

Let us denote by $\tilde{c}$ the group of matrices of the form

$$\tilde{c} = \begin{pmatrix} c & 0 & 0 \\ 0 & \frac{1}{c} & 0 \\ 0 & 0 & I \end{pmatrix} \quad (c \in k).$$

Note that $\tilde{c} \simeq k^\times$. Given $h(c,t,R) \in \tilde{P}_V$ then $h(c,t,R) = \tilde{c}h(t,R)$, where $h(t,R) \in P_V$.

The following is immediate.

**Lemma 2.7.1.** $\tilde{P}_V = \{\tilde{c}h(t,R) | c \in k \ , \ t \in W \ , \ R \in SO(W)\}$.

*(a)* $\tilde{P}_V \simeq V \rtimes (SO(V) \times k^\times)$.

*(b) Multiplication is given by:* $\tilde{c}h(t,R)\tilde{c'}h(t',R') = \widetilde{cc'}h(\frac{1}{c'}t + Rt', RR')$

*(c) The conjugation action of* $\tilde{c}$ *on a translation part is to dilate it by a factor of* $c$*. That is* $\tilde{c}h(t,R)\widetilde{c^{-1}} = h(ct,R)$*. Note that* $\tilde{c}$ *commutes with the* $R$ *action.*

**Lemma 2.7.2.** $\tilde{P}$ *is the subgroup of* $SO(V)$*, which leaves* $A_{[p]}$ *invariant.*

*Proof.* Let $g$ be any element of $SO(V)$ which leaves $A_{[p]}$ invariant. We want to first show that $g \cdot p = \alpha p + w$ where $w \in W$, which is equivalent to showing that $gp \in V_p$. If $g$ preserves $A_{[p]}$, then $g$ also preserves the compliment of $A_{[p]}$, which is $V_p \cap C$. Now $p \in V_p \cap C$ so that $gp \in V_p \cap C$. Thus $gp \in V_p$, as was to be shown. Since $g \cdot p = \alpha p + w$ then $(g \cdot p, g \cdot p) = (\alpha p + w, \alpha p + w) = (w,w) = 0$, which means that $w$ is a null vector. We want to show $w = 0$.

Assume $w$ is nonzero. So $gp = \alpha p + w$. We first pick $w' \in W$ so that $(w,w') = -\alpha$. This is possible, because as $w$ is null, $\langle w \rangle \subset \langle w^\perp \rangle \nsubseteq W$, so that we can find a $w' \in W$ such that $(w,w') \neq 0$ and as soon as $(w,w')$ is nonzero since then we can adjust $w'$ so that $(w, w') = -\alpha$. Having chosen a $w'$, we define $a' = -\frac{(w',w')}{2}p + q + w'$, $a = g^{-1}a'$ then $(p, a') = 1$, and $(a', a') = -(w', w') + (w', w') = 0$. Thus $[a'] \in A_{[p]}$. Let $a = g^{-1}a'$. Since $g$ preserves $A_{[p]}$ we have that $[a] \in A_{[p]}$. So we must have $(p, a) \neq 0$. But $(p, a) = (gp, ga) = (\alpha p + w, a') = (\alpha p + w, -\frac{(w',w')}{2}p + q + w') = \alpha + (w, w) = 0$, a contradiction. Hence $w = 0$, showing $gp = \alpha p$.

$\square$

**Definition 15.** *We will call $\tilde{P}$ the partial conformal group.*

This is a reasonable definition since $\tilde{P}$ is the subgroup stabilizing $A_{[p]}$. We introduce the notation $\tilde{L} = \tilde{c} \cdot L$.

# CHAPTER 3

# Background for the theory of PURs theory

## 3.1 Introduction

This chapter lays the foundation for the rest of this thesis. It contains no new results, but instead we establish terminology and notation that will be used later. For completeness, we also include some well known results that will be used in the remaining chapters. In Section 3.2, we discuss several concepts relating to locally compact second countable (lcsc) topological spaces and groups. In Section 3.2.1, we discuss the Mackey machine of lcsc groups as well as a theorem of Effros. In Section 3.3, we recall some standard terminology of cocycles. In Section 3.4, we discuss the theory of projective unitary representations of locally compact groups.

## 3.2 Locally compact second countable topological spaces

### Borel structure terminology

We recall some standard Borel space terminology that will be used throughout this thesis. A *Borel structure* on a space $X$ is a $\sigma$-algebra $\mathcal{B}$, of subsets of $X$. We refer to the pair $(X, \mathcal{B})$ as a *Borel space.* If $(X, \mathcal{B})$ and $(Y, \mathcal{C})$ are Borel spaces, then a function $f : X \to Y$ is said to be *Borel* if $f^{-1}[\mathcal{C}] \subset \mathcal{B}$. A *measure* on a Borel space is any non-negative function that is countably additive on its Borel structure, $+\infty$ being an allowed value. A set $X$ is said to be of *$\sigma$-finite measure* if $X$ can be written as a countable union of elements of $\mathcal{B}$, each element having finite measure. A Borel space with a measure $m$ is called a *measure space* and is denoted by $(X, \mathcal{B}, m)$. We single out an important subclass of Borel spaces: if $T$ is a complete separable metric space and $X$ a Borel subset of $T$, then $X$ is called a *standard Borel space*, and we say that $X$ has a *standard Borel structure*. Suppose that a group $G$ is also a Borel space. Then $G$ is a *Borel group* if the map $x, y \mapsto xy^{-1}$ of $G \times G$ into $G$ is Borel. A Borel group is called *standard* if the underlying Borel structure is standard. If the Borel group in question is a lcsc group, then its Borel structure is standard.

**Measure theory on lcsc spaces**

We recall standard notions of measure theory on lcsc spaces. Let $X$ be lcsc Hausdorf topological space. By a *measure on* $X$ we mean a measure that gives finite measure to all compact subsets of $X$. The measure $m$ is *finite* if $m(X) < \infty$, and $m$ is a *probability measure* if $m(X) = 1$. Two measures $m$ and $n$ are equivalent ($m \equiv n$) if they have the same null sets. The relation $\equiv$ is an equivalence relation and partitions the set of all measures on $X$ into equivalence classes called *measure classes on* $X$. A measure $\nu$ is *absolutely continuous with respect to a measure* $\mu$ if $E \in \mathcal{B}$ and $\mu(E) = 0$ implies $\nu(E) = 0$.

Let $G$ be a group with a Borel action on $X$, i.e. the map $G \times X \to X$ is a Borel map. A measure $m$ on $X$ is called *quasi-invariant under* $G$ if for any $E \subset X$ for which $m(E) = 0$, $m_g(E) = m(Eg^{-1}) = 0 \;\; \forall g \in G$.

If $m$ is quasi-invariant, then $m_g$ is absolutely continuous with respect to $m$ and vice versa.

### Homogeneous spaces and the Haar measure

Let $G$ be a lcsc group. We consider the action of $G$ on itself. A measure $m_l$ is *left invariant* if $m_l(gE) = m_l(E)$ for all Borel sets $E$. It is well known that there exists an essentially unique measure with this property. By essentially unique we mean that if $m_l'$ is another measure with this property, then $m_l = c \cdot m_l'$, where $c$ is some positive constant. $m_l$ is called a *left Haar measure*. Similarly there exists an essentially unique right invariant measure $m_r$, called the *right Haar measure*. Left invariant measures need not be right invariant and vice versa, but in fact $m_l \equiv m_r$, and it is well known that there exists a unique quasi-invariant measure class on $G$ with respect to left and right translations. This is the class containing $m_l$ and $m_r$.

The following theorem is well known.

**Theorem 3.2.1.** *Let $G$ be a lcsc group and $H$ a closed subgroup of $G$. Let $X = G/H$ be the corresponding homogeneous space. Then there exists a unique quasi-invariant measure class on $X$ under the left $G$-action. Let $\pi$ be the canonical map from $G$ to $X$. If $P$ is a probability measure and $P_X(E) = P(\pi^{-1}E)$, then $P_X$ belongs to this class. The null sets of the class are the ones for which $\pi^{-1}(E)$ has measure $0$ on $G$.*

See [19] p. 172 for a proof.

### 3.2.1 The Mackey Machine for lcsc groups

We are interested in the following setup. Let $G = W \rtimes L$, where $W$ and $L$ are lcsc groups and $W$ is abelian. We want to describe the representations of $G$. This is done via the theory of Mackey and is known in the literature as the *Mackey machine*. Denote the character group of $W$ by $W^*$ and let $\chi \in W^*$ be a character of $W$. $G$ acts on $W^*$ by $l[\chi(t)] = \chi(l^{-1}tl)$ for $l \in L$ and $t \in W$. Note that $W$ acts trivially on $W^*$. We say that the action of $L$ on $W^*$ is *regular* if there is a Borel set $E$ in $W^*$ that meets every orbit exactly once. The semidirect product $G = W \rtimes L$ is called *regular* if the action of $L$ on $W^*$ is regular. Denote the stabilizer of $\chi$ in $G$ by $G_\chi$. Then $G_\chi = W \rtimes L_\chi$, where $L_\chi$ is the stabilizer of $\chi$ in $L$. The group $L_\chi$ is known as *the little group*. We may extend $\chi$ to be character of $G_\chi$ by setting $\chi(tl) = \chi(t)$. We verify that this does define a character:

$$\chi(tlt'l') = \chi(tlt'l^{-1}ll') = \chi(thll').$$

Here $h$ is in $W$

$$\chi(thll') = \chi(th) = \chi(t)\chi(h).$$

We have used the definition of $\chi$ extended to $G_\chi$, and on the right hand side we use the fact that $t$ and $h$ are both in $W$ and $\chi$ is a character of $W$.

$$\chi(t)\chi(h) = \chi(t)(l^{-1}[\chi])(t').$$

But $l$ is in $G_\chi$, so $l^{-1}[\chi] = \chi$, and we have

$$\chi(tlt'l') = \chi(tt').$$

Thus we have proven that $\chi$ extended in this manner to $G_\chi$ has the homomorphism property and is a character of $G_\chi$. We denote this character by $\chi$ as well. If $\sigma$ is an irreducible representation of $L_\chi \simeq G_\chi/W$, then we may extend $\sigma$ to be a representation of $G_\chi$ by setting $\sigma(tl) = \sigma(l)$ for $t \in T$ and $l \in L_\chi$. We now put $\chi$ and $\sigma$ together to get a representation of $G_\chi$ by taking the tensor product

$\chi \otimes \sigma$, which is the representation $(t, x) \longmapsto \chi(t)\sigma(x)$ of $G_\chi$. Let

$$\Theta_{\chi,\sigma} = Ind_{G_\chi}^{G} \chi \otimes \sigma.$$

Let $\chi$ and $\chi'$ be two characters that are in the same orbit, with little groups $L_\chi$ and $L_{\chi'}$ respectively. Then the representations of the little groups are equivalent. To see this, note that since $\chi$ and $\chi'$ are in the same orbit, there exists $x \in G$ such that $\chi' = x[\chi]$. Then the stabilizer of $\chi'$ is $L_{\chi'} = L_\chi x^{-1}$. Hence the stabilizers are conjugate, so $L_{\chi'} \simeq L_\chi$, and for every representation $\sigma$ of $L_\chi$, there is an equivalent representation $\sigma'(g) = \sigma(x^{-1}gx)$ of $L_{\chi'}$.

**Theorem 3.2.2. (Mackey)** *The $\Theta_{\chi,\sigma}$ are UIR's of $G$. Moreover, $\Theta_{\chi,\sigma} \simeq \Theta_{\chi',\sigma'}$ if and only if $\chi, \chi'$ are in the same orbit and $\sigma, \sigma'$ are equivalent representations of the little groups.*

**Theorem 3.2.3. (Mackey)** *If $G$ is a regular semidirect product, the $\Theta_{\chi,\sigma}$ are precisely all the UIR's of $G$.*

The following theorem of Effros is an invaluable tool in determining when a given action is regular [18].

**Theorem 3.2.4. (Effros)** *Let $X$ be a lcsc Hausdorf space and $L$ a lcsc group acting continuously on $X$. Then the following are equivalent:*

*(a) All $L$-orbits in $X$ are locally closed.*

*(b) There exists a Borel set $E \subset X$ that meets each $L$-orbit exactly once.*

*(c) If $Y = L \backslash X$ is equipped with the quotient topology, the Borel structure of $Y$ consists of all sets $F \subset Y$ whose preimages in $X$ are Borel, and the space $Y$ with this Borel structure is standard.*

(d) *Every invariant measure on $X$ that is ergodic for $L$ is supported on an orbit and is the unique (up to a normalizing scalar) invariant measure on this orbit.*

## 3.3 Cocycles

Let $G$ be a lcsc group and $X$ a standard Borel $G$-space. Let $\mathcal{L}$ be an invariant measure class on $X$. Let $M$ be a standard Borel group.

**Definition 16.** *Let $f : G \times X \to M$ be a Borel function. Then $f$ is a $(G, X, M)$-cocycle relative to $\mathcal{L}$ if the following hold:*

(a) *$f(e, x) = 1$ for almost all $x \in X$.*

(b) *$f(g_1 g_2, x) = f(g_1, g_2[x])f(g_2, x)$ for almost all $(g_1, g_2, x) \in G \times G \times X$.*

Note that the null sets in $(a)$ are taken with respect to the measure class $\mathcal{L}$ on $X$ and in $(b)$ with respect to the product measure on $G \times G \times X$.

When there is no ambiguity concerning the group $M$ or the measure class $\mathcal{L}$, we refer to $f$ as $(G, X)$ cocycle.

**Definition 17.** *Suppose $f_1$ and $f_2$ are two cocycles relative to some measure class $\mathcal{L}$. If there exists a Borel map $b : X \to M$ such that $f_2(g, x) = b(g[x])f_1(g, x)b(x)^{-1}$ for almost all $(g, x) \in G \times X$, then the cocyles are said to be cohomologous and we write $f_1 \sim f_2$.*

**Definition 18.** *If $f$ is a $(G, X, M)$-cocycle relative to $\mathcal{L}$, then $f$ is called a strict $(G, X, M)$-cocycle relative to $\mathcal{L}$ if it satisfies the stronger conditions:*

(a) *$f(e, x) = 1$ for all $x \in X$.*

*(b)* $f(g_1g_2, x) = f(g_1, g_2[x])f(g_2, x)$ *for all* $(g_1, g_2, x) \in G \times G \times X$.

Notion of being *strictly cohomologous* is defined in the obvious way [19] p. 174-175.

**Theorem 3.3.1.** *If $f$ is any $(G, X)$-cocycle, there exists a strict $(G, X)$-cocycle $f'$ such that*

$$f(g, x) = f'(g, x)$$

*for almost all $(g, x) \in G \times X$; $f'$ is determined uniquely up to strict cohomology.*

For a proof, see [19] p. 179.

## 3.4 Projective representations

As discussed in Chapter 1, projective representations (PR) are the key to classifying quantum systems corresponding to a given symmetry group. We begin with a study of multipliers.

### Multipliers

Let $G$ be a lcsc group and $T$ the group of complex numbers of unit modulus.

**Definition 19.** *A multiplier for $G$ is a Borel map $m : G \times G \to T$ such that*

*(a)* $m(x, yz)m(y, z) = m(xy, z)m(x, y)$ *for all* $x, y \in G$.

*(b)* $m(x, e) = m(e, x) = 1$ *for all* $x \in G$.

The set of multipliers forms a commutative group under pointwise multiplication. We shall write $Z(G)$ for this group.

**Definition 20.** *If $a$ is a Borel map of $G$ to $T$ and $m$ and $m'$ are two multipliers such that*

$$m'(x, y) = m(x, y)\frac{a(x)a(y)}{a(xy)},$$

*then the two multipliers related as above are called equivalent and we write $m \simeq m'$.*

If a multiplier $m$ is equivalent to 1, then $m(x, y) = \frac{a(xy)}{a(x)a(y)}$, and we say that $m$ is a *trivial multiplier.*

The subgroup of trivial multipliers is denoted by $B(G)$. We now define the multiplier group of $G$ as $H^2(G) = Z(G)/B(G)$.

### 3.4.1 $m$-representations

Let $G$ be a lcsc group and $\mathcal{H}$ a Hilbert space.

**Definition 21.** *An m-representation of $G$ is a pair $(m, U)$ such that $m$ is a multiplier and $U$ is a Borel map of $G$ into the group of unitary operators of $\mathcal{H}$ such that $U(1) = 1$ and $U(x)U(y) = m(x, y)U(xy)$.*

**Definition 22.** *A Borel mapping $U$ of $G$ into unitary group $\mathcal{U}$ of $\mathcal{H}$ is a projective representation if there exists a multiplier $m$ for $G$ such that $U$ is an m-representation.*

If $a$ is a Borel map of $G$ to $T$, and $U'(g) = a(g)U(g)$, then $U'$ is an $m'$ representation with a multiplier $m'$, which is related to $m$ in the following way

$$m'(x, y) = m(x, y)\frac{a(x)a(y)}{a(xy)},$$

If a multiplier $m$ is trivial for some $U$, then $aU(g) = a(g)U(g)$ is an ordinary unitary representation of $G$. We say that the $m$-representation is unitarizable when the multiplier is trivial.

Later we shall discuss the fact that for any multiplier $m$ of $G$ there exist an irreducible $m$-representation of $G$.

### Multipliers and group extensions

Given a lcsc group $G$ we are interested in finding its PURs. Unfortunately PURs are difficult to work with. It turns out, however, that for any given projective representation $U$ of $G$ corresponding to a multiplier $m$ of $G$, there exists a lcsc group $G_m$ which is a group extension of $G$. This is established in Theorem 3.4.1. Theorem 3.4.2 then shows that when lifted to $G_m$ the projective representation $U$ with a multiplier $m$ becomes unitarizable. Furthermore, the irreducible projective representations (PUIRs) with a multiplier $m$ are in one to one correspondence with certain ordinary UIRs of $G_m$, the correspondence preserving irreducibility.

Let $G$ and $C$ be lcsc groups and $C$ a closed normal subgroup of the center of $G$.

**Definition 23.** *A central extension of $G$ by $C$ is a lcsc group $G^{\sim}$ such that $G = G^{\sim}/C$ and $C$ is in the center of $G^{\sim}$.*

We have the following exact sequence

$$0 \to C \to G^{\sim} \to G \to 0.$$

One can show that given any central extension $G^{\sim}$ of $G$, one can associate to $G^{\sim}$ a multiplier $m$ for $G$. This is done by showing that with $C = T$, $G^{\sim} \simeq G \times T$

(Borel isomorphism) and that there exists a multiplier $m : G \times G \to T$ such that on $G^{\sim}$ the product structure is given by

$$(x_1, t_1)(x_2, t_2) = (x_1 x_2, m(x_1 x_2) t_1 t_2).$$

Conversely, given a multiplier $m$ for $G$, one would like to associate to it a central extension $G_m$. One might try to associate to $m$ the group $G_m = G \times T$ with the product structure

$$(x_1, t_1)(x_2, t_2) = (x_1 x_2, m(x_1 x_2) t_1 t_2).$$

Indeed, this is possible provided that $m$ is continuous. The main difficulty here is that when the multiplier $m$ is not continuous on $G \times G$, the product topology on $G_m$ does not make it into a lcsc group. However, the Mackey-Weil topology on $G_m$ will convert $G_m$ into a lcsc group. The key theorem is due to Mackey.

**Theorem 3.4.1.** *Let $m$ be a multiplier, $G$ a lcsc group and $T$ the group of complex numbers of modulus 1. If we define, for $(x_1, t_1), (x_2, t_2) \in G \times T$,*

$$(x_1, t_1)(x_2, t_2) = (x_1 x_2, m(x_1 x_2) t_1 t_2),$$

*then $G \times T$ becomes a standard Borel group under this product, and there exists a unique lcsc topology (Mackey-Weil topology) for $G \times T$ such that*

*(a) $G \times T$ becomes a lcsc group, denoted by $G_m = G \times_m T$, under this topology.*

*(b) The Borel structure of this topology coincides with the product structure on $G \times T$.*

*$G_m = G \times T$ is a central extension of $G$ by $T$. Every central extension of $G$ is equivalent to some $G_m$. The extensions corresponding to multipliers $m_1$ and $m_2$ are equivalent iff $m_1$ and $m_2$ are similar.*

For proof see [19] p. 253.

The following theorem shows that there is a one to one correspondence between $m$-representations of $G$ and certain ordinary representations of $G_m$.

**Theorem 3.4.2.** *Let $m$ be any multiplier for $G$ and let $U$ be an $m$-representation of $G$ in a separable Hilbert space $\mathcal{H}$. Then $V_m : (g,t) \mapsto tU(g)$ is an ordinary representation of $G_m$ in $\mathcal{H}$. Conversely, if $V_m$ is a representation of $G_m$ in $\mathcal{H}$ such that $V_m(e,t) = t1$ for all $t \in T$, then $U(g) := V_m(g,1)$ is an $m$-representation of $G$ and $V_m(g,t) = tU(g)$. Finally, the correspondence $U \leftrightarrow V_m$ preserves irreducibility.*

For the proof, see [19], p. 259.

The following proposition and its corollary can be found in [19], p. 259. They show that as long as a group has a multiplier $m$, it has $m$-representations and these $m$-representations are built out of irreducible $m$-representations.

**Proposition 3.4.1.** *Let $m$ be any multiplier for $G$. Then there exists an irreducible $m$-representation of $G$.*

**Corollary 3.4.1.** *If $G$ is compact and $m$ is a multiplier for $G$, then there exists finite-dimensional $m$-representations of $G$. The irreducible $m$-representations of $G$ are finite-dimensional and every $m$-representation of $G$ is a direct sum of irreducible $m$-representations.*

We shall abbreviate central extension as ce.

#### Universal central extensions

We have seen that given a lcsc group $G$ and an $m$-representation, we can find an extension $G_m$ where the $m$-representation becomes unitarizable. A natural question is that of the existence of a universal central extention $E$, where all $m$-representations of $G$ become unitarizable.

**Definition 24.** *A ce $E$ of a lcsc group $G$ is said to be a universal central extension (uce) if every ce of $G$ is an image of $E$ by a unique map.*

**Lemma 3.4.1.** *Let $G$ be a lcsc group and $E$ a ce of $G$. Then $E$ is a uce iff*

*(a) (E,E) dense in E.*

*(b) $H^2(E) = 0$.*

Note that for $G$ to have a uce it is necessary that $(G, G)$ be dense in G.

The following lemma is from [20]

**Lemma 3.4.2.** *Let $G$ be a lcsc group and $E$ a ce of $G$ with $(E, E)$ dense in $E$. If all PURs of $G$ become unitarizable when lifted to $E$, then $E$ is a uce of $G$.*

#### D. Wigner's theorem

We have seen that in order to assure that the ces were in the lcsc category, we needed to appeal to the Mackey-Weil topology. However, if the lcsc group $G$ in question is *totally disconnected* (td), this is not necessary. The following theorem of David Wigner shows that, in the td setting, any multiplier is equivalent to a continuous multiplier, and hence the product topology on $G_m$ makes it a lcsc group.

**Theorem 3.4.3.** ***(David Wigner)*** *Let $G$ be a t.d. group.*

*(a) Any multiplier for $G$ is equivalent to a continuous one.*

*(b) If $m_1 \simeq m_2$, where $m_i \in Z(G)$ are continuous, the map $a : G \to T$ is a Borel map with $a(1) = 1$, and $m_2(g, h) = m_1(g, h)a(gh)a(g)a(h)^{-1}$, then $a$ is continuous.*

*(c) Let $H_c^2(G)$ be the group of continuous multipliers. If $m \in H_c^2(G)$ is a continuous multiplier for $G$, and $U$ is an $m$-representation, then $U$ is already continuous. In particular, the natural map $H_c^2(G) \to H^2(G)$ is an isomorphism.*

To prove his theorem, D. Wigner used a special case of the following selection theorem of E. Michael.

**Theorem 3.4.4.** ***(E. Michael)*** *Let $X$ and $Y$ be second countable Hausdorff spaces, $Y$ metrizable as a complete metric space and $X$ totally disconnected. If $F : Y \to X$ is an open continuous surjective mapping, then there exists a continuous mapping $G : X \to Y$ such that $F \circ G(x) = Id_X$.*

For the proof, see [21]. We note that the groups in which we are interested are linear algebraic groups defined over $\mathbb{Q}_p$; hence their groups of $\mathbb{Q}_p$-points are lcsc and td. When this is the case, we may assume that our multipliers are continuous and ce exist, without appealing to the Mackey-Weil topology.

# CHAPTER 4

# Projective representations of semidirect products

## 4.1 Introduction

We begin in Section 4.2 with the discussion of PUIRs of lcsc groups, which are semidirect products of lcsc groups. In particular, we focus on a recent paper of V. S. Varadarajan [20] which contains many results that will be of use to us. In Section 4.3, we discuss the theory of $m$-systems of imprimitivity. We then use this material to establish the Mackey machine for projective representations of semidirect products in Section 4.4. Our results are variants of Mackey's results, but the appearance of the affine actions defined by cocycles is new.

## 4.2 PUIRs of semidirect product groups

### 4.2.1 Multipliers for semidirect products

The theory of representations of semidirect products was first studied by E. Wigner in the case of the Poincaré group, but was developed in depth by Mackey [22]. We follow Mackey's work in the following sections.

To understand representations of semidirect product groups, we need to study their multipliers. Our setup is the following. Let $A$ be an abelian lcsc group and

$G$ a lcsc group, $H = A \rtimes G$. We start by discussing multipliers for abelian groups.

### Multipliers for abelian groups

Let $A$ be a lcsc Abelian group.

**Definition 25.** *A bicharacter of $A$ is a continuous map $b$ of $A \times A$ into $T$ which is a character in each argument.*

We shall soon show that bicharacters of $A$ are closely linked to the multipliers of $A$.

One can show that any function $f : A \times A \to T$ that is a Borel homomorphism in each argument is a bicharacter [23].

**Remark:** Any bicharacter is a continuous multiplier.

**Definition 26.** *A bicharacter is called alternating or antisymmetric if it satisfies*

$$b(x, y)b(y, x) = 1, \quad b(x, x) = 1.$$

Let $m$ be any multiplier of $A$, not necessarily continuous. Set

$$m^{\sim}(x, y) = \frac{m(x, y)}{m(y, x)}.$$

One can now verify that $m^{\sim}$ is in a Borel homomorphsim in each argument, hence a bicharacter, which is clearly alternating. For details for this, and for Lemma 4.2.1, see [23].

Let $\Lambda^2(A)$ be the group of alternating bicharacters for $A$. Then the mapping $m \mapsto m^{\sim}$ is a homomorphism of $Z(A)$ into $\Lambda^2(A)$.

**Definition 27.** *Let $A$ be a lcsc abelian group and $p$ a prime number. If the* *mapping $a \mapsto pa$ is an isomorphism of $A$ with itself, then we say that $A$ is $p$-regular.*

**Lemma 4.2.1.** ***(Varadarajan, Digernes)*** *The map $m \mapsto m^{\sim}$ is a homomorphism whose kernel is the group of trivial multipliers, hence it induces an injection of $H^2(A)$ into $\Lambda^2(A)$. Furthermore, if $A$ is 2-regular this map is an isomorphism.*

We note that $\mathbb{Q}_p$ is 2-regular for all $p$. In later chapters, we consider semidirect products $H = A \rtimes G$ where $A$ is a finite-dimensional abelian vector space over $\mathbb{Q}_p$, and $G$ is a linear algebraic group defined over $\mathbb{Q}_p$. Hence we will be able to identify the multipliers of $A$ with the $\Lambda^2(A)$. In particular, we will be able to apply Lemma 4.2.2 to $H$.

The following lemma can be found in [20].

**Lemma 4.2.2.** ***(Varadarajan)*** *Let $A$ be 2-regular. If 1 is the only element of $\Lambda^2(A)$ invariant under $G$, then, for any multiplier $m$ for $H$, the restriction of $m$ to $A \times A$ is trivial, and so $m \simeq m'$ where $m' = 1$ on $A \times A$. If $m$ is continuous, then $m'$ can be chosen to be continuous as well.*

### Mackey's theorem on multipliers of semidirect products

We now return to studying multipliers of semidirect products. Let $H = A \rtimes G$ where $A$ and $G$ are lcsc groups and $A$ is abelian. Let $A^*$ be the character group of $A$. We first investigate the multipliers of $H$ that are trivial when restricted to $A$. We begin by defining a 1-cocycle for $G$ with coefficients in $A^*$ as a Borel map $f(G \to A^*)$ such that

$$f(gg') = f(g) + g[f(g')] \quad (g, g' \in G).$$

This is equivalent to saying that $g \mapsto (f(g), g)$ is a Borel homomorphism of $G$ into the semidirect product $A^* \rtimes G$. The multiplicative property of this map is clear from its definition. When $f$ is a Borel homomorphism and has a multiplicative property, it is automatically continuous ([19] p. 181). So $f$ is a homomorphism. Thus, all 1-cocycles are continuous. We denote the abelian group of continuous 1-cocycles by $Z^1(G, A^*)$. The coboundaries are the cocycles of the form $g \mapsto g[a]-a$ for some $a \in A^*$. The coboundaries form a subgroup $B^1(G, A^*)$ of $Z^1(G, A^*)$. We now form the cohomology group $H^1(G, A^*) = Z^1(G, A^*)/B^1(G, A^*)$.

Denote by $M_A(H)$ (resp. $M_{AG}(H)$) the group of multipliers on $H$ that are trivial when restricted to $A \times A$ (resp. $A \times A$ and $G \times G$). Let $H^2_A(H)$ (resp. $H^2_{AG}(H)$) denote its image in $H^2(H)$. Let $M'_A(H)$ be the group of multipliers $m$ for $H$ with $m|_{A\times A} = m|_{A\times G} = 1$. The following theorem describes the multipliers of $H$. Full details can be found in [20].

**Theorem 4.2.1.** ***(Mackey)*** *Any element in $M_A(H)$ is equivalent to one in $M'_A(H)$. If $m \in M'_A(H)$ and $m_0 = m|_{G\times G}$ and $\theta_m(g^{-1})(a') = m(g, a')$, then $m \mapsto (m_0, \theta_m)$ is an isomorphism $M'_A(H) \simeq Z^2(G) \times Z^1(G, A^*)$ which is well-defined in cohomology and gives the isomorphisms $H^2_A(H) \simeq H^2(G) \times H^1(G, A^*)$ and $H^2_{AG}(H) \simeq H^1(G, A^*)$.*

**Corollary 4.2.1.** *If $m_0 \in Z^2(G)$ and $\theta \in Z^1(G, A^*)$ are given, then one may define $m \in M'_A(H)$ by $m(ag, a'g') = m_0(g, g')\theta(g^{-1})(a')$. If $m_0 = 1$, then $m$ is a continuous multiplier and $m(ag, a'g') = \theta(g^{-1})(a')$.*

**Definition 28.** *A multiplier $m$ for $H$ is said to be standard if $m|_{A\times A} = m|_{A\times G} = m|_{G\times A} = 1$.*

**Lemma 4.2.3.** *A multiplier for $H$ is standard iff it is the lift to $H$ of a multiplier for $G$ via $H \to H/A \simeq G$.*

*Proof.* The lift to $H$ of a multiplier for $G$ is certainly standard. Conversely, let

$m$ be a standard multiplier for $H$. We must show that $m(a_1x_1, a_2x_2) = m(x_1, x_2)$ for $a_1, a_2 \in A$ and $x_1, x_2 \in G$. We have

$$m(a_1, a_2x_2) = m(a_1, a_2x_2)m(a_2, x_2) = m(a_1a_2, x_2)m(a_1, a_2) = 1$$

$$m(a_1x_2, a_2) = m(a_1x_1, a_2)m(a_1, x_1) = m(a_1, x_1a_2)m(x_1, a_2) =$$

$$= m(a_1, x_1[a_2]x_1)m(x_1, a_2) = 1,$$

where we have used the previous formula. Hence,

$$m(a_1x_1, a_2x_2) = m(a_1x_1, a_2x_2)m(a_2, x_2) = m(a_1x_1a_2, x_2)m(a_1x_1, a_2)$$

$$= m(a_1x_1[a_2]x_1, x_2) = m(a'x_1, x_2) \quad (a' = a_1x_1[a_2])$$

$$= m(a'x_1, x_2)m(a', x_1) = m(a', x_1x_2)m(x_1, x_2) = m(x_1, x_2).$$

□

**Corollary 4.2.2.** *If $H^1(G, A^*) = 0$ then every multiplier of $H$ is equivalent to the lift to $H$ of a multiplier for $G$.*

## 4.3 $m$-Systems of imprimitivity

Mackey described a method of studying PURs based on systems of imprimitivity. Such an approach has also been tried by others, such as D. R. Grigore [25], although he considers only PURs of the real Poincaré group and the Galilean group over $\mathbb{R}^{1,2}$. Our method, while not entirely new, is a modification of this approach. In our method, the multiplier group is studied from the cocycle point of view and has as a new aspect, the so-called affine action. We are also able to formulate Mackey's theorem in the context of $m$-representations, which utilizes this affine action. Mackey himself never explicitly stated his theorem for $m$-representations, so this theorem is new as well.

For the remainder of the chapter we assume the following setup. Let $G$ be a lcsc group. Let $X$ be a $G$-space that is also a standard Borel space. Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{U}$ be the group of unitary transformations of $\mathcal{H}$.

**Definition 29.** *An m-system of imprimitivity is a pair $(U, P)$, where $P(E \to P_E)$, is a projection-valued measure on the class of Borel subsets of $X$, the projections being defined in $\mathcal{H}$, and $U$ is an $m$-representation of $G$ in $\mathcal{H}$ such that*

$$U(g)P(E)U(g)^{-1} = P(g[E]) \ \ \forall\, g \in G \text{ and all Borel } E \subset X.$$

We note that this is simply a variant of the notion of ordinary systems of imprimitivity, where the unitary representations have been replaced by $m$-representations. We say that the pair $(U, P)$ is *based on* $X$.

We need the following standard result for ordinary systems of imprimitivity cf [19] p. 215. Let $G$ be a lcsc group and $X$ be a standard Borel $G$-space. We choose an invariant measure class $\mathcal{L}$ on $X$ and let $\alpha$ be a measure in this class. We let $\mathcal{H} = L^2(X)$ and $\mathcal{U}$ to be the unitary group on $\mathcal{H}$.

**Theorem 4.3.1.** *Let $\phi$ be a $(G, X, \mathcal{U})$-cocycle. For each $g \in G$ let $\rho_g = d\alpha/d\alpha^{(g^{-1})}$. Then there exists a unique system of imprimitivity $(U, P)$ such that for all Borel sets $E \subset X$,*

$$P_E f = \chi_E f,$$

*and for almost all $g$,*

$$(U_g f)(x) = \{\rho_g(g^{-1}[x])\}^{\frac{1}{2}} \phi(g, g^{-1}[x]) f(g^{-1}[x])$$

*for each $f \in \mathcal{H}$ for almost all $x$. Moreover the equivalence class of $(U, P)$ depends only on $\mathcal{L}$ and on the cohomology class of $\phi$.*

**Definition 30.** *When we have the setup of Theorem 4.3.1, we say that the system of imprimitivity $(U, P)$ is associated to $\phi$ and $\alpha$.*

For what follows, we take $X$ to be a transitive $G$-space. We fix some $x_0 \in X$ and let $G_0$ be the stabilizer of $x_0$ in $G$, so that $X \simeq G/G_0$. We also fix a multiplier $m$ for $G$ and let $m_0 = m|_{G_0 \times G_0}$.

Our goal is to prove the following theorem:

**Theorem 4.3.2.** *There is a natural one-to-one correspondence between $m_0$-representations $\mu$ of $G_0$ and $m$-systems of imprimitivity $(U, P)$ of $G$ based on $X$. Under this correspondence, we have a ring isomorphism of the commuting ring of $\mu$ with that of $(U, P)$.*

This theorem ties together $m$-systems of imprimitivity of $G$ and $m_0$-representations of $G_0$. We begin our proof by establishing some lemmas.

**Lemma 4.3.1.** *Let $G$ be a lcsc group acting transitively on a lcsc space $X$. Fix $x_0 \in X$ and let $G_0$ be the stabilizer of $x_0$ in $G$. Suppose that $C \subset G_0$ is a closed central subgroup of $G$ and $\chi$ is a character of $C$. Let $\gamma$ be a strict $(G, X)$-cocycle with values in the unitary group $\mathcal{U}$ of Hilbert space $\mathcal{K}$, and let $\nu$ be the map $G_0 \to \mathcal{U}$ defined by $\nu(g) = \gamma(g, x_0) \quad g \in G_0$. Then $C$ acts trivially on $X$ and the following are equivalent:*

*(a) $\nu(c) = \chi(c)$.*

*(b) $\gamma(c, x) = \chi(c)$ for each $c \in C$ for almost all $x \in X$.*

*(c) $\gamma(c, x) = \chi(c)$ for each $c \in C$ and for all $x \in X$.*

*Proof.* We first show that $C$ acts trivially on $X$. Since $G$ acts transitively on $X$, any element of $x \in X$ may be written as $g[x_0]$ for some $g \in G$. Let $c \in C$; since $c$ commutes with $g$, $c[g[x_0]] = g[c[x_0]]$. By definition, $c$ fixes $x_0$, hence,

$$c[g[x_0]] = g[x_0].$$

This shows that $c$ fixes all of $X$.

We first show that $(a)$ implies $(c)$. Let $c \in C$. Now, any $x \in X$ may be written as $x = g[x_0]$ for some $g \in G$. Hence

$$\gamma(c, g[x_0]) = \gamma(cg, x_0)\gamma(g, x_0)^{-1}.$$

But $cg = gc$ so $\gamma(cg, x_0) = \gamma(gc, x_0)$, giving

$$\gamma(gc, x_0) = \gamma(g, c[x_0])\gamma(c, x_0) = \gamma(g, x_0)\gamma(c, x_0).$$

Therefore,

$$\gamma(c, g[x_0]) = \gamma(g, x_0)\gamma(c, x_0)\gamma(g, x_0)^{-1}$$

$$= \gamma(g, x_0)\chi(c)\gamma(g, x_0)^{-1} = \chi(c).$$

Since $(c)$ implies $(b)$ trivially, we need only prove $(b)$ implies $(a)$. For each $c \in C$ and almost all $x \in X$, $\gamma(c, x) = \chi(c)$. Again, we write the arbitrary $x$ as $g[x_0]$ for some $g \in G$. For all $x$ $\gamma(c, g[x_0]) = \gamma(g, x_0)\gamma(c, x_0)\gamma(g, x_0)^{-1}$. By assumption, $\gamma(c, x) = \chi(c)$ for almost all $x \in X$. Hence, for almost all $g$ we have

$$\gamma(c, g[x_0]) = \chi(c) = \gamma(g, x_0)\gamma(c, x_0)\gamma(g, x_0)^{-1}.$$

In particular, this holds for some $g_1$, and we get

$$\gamma(c, g_1[x_0]) = \gamma(g_1, x_0)\gamma(c, x_0)\gamma(g_1, x_0)^{-1} = \chi(c)$$

$$\gamma(c, x_0) = \gamma(g_1, x_0)^{-1}\chi(c)\gamma(g_1, x_0) = \chi(c)$$

Hence $\gamma(c, g[x_0]) = \gamma(g, x_0)\gamma(c, x_0)\gamma(g, x_0)^{-1} = \chi(c)$ for all $g$, and $\gamma(c, x) = \chi(c)$ for all $x$.

□

Let $E = E_m$ be the ce of $G$ defined by $m$, and let $E_0$ be the preimage of $G_0$ in $E$ under the map $E \to G$. From previous discussions we know that $E = G \times T$ and $E_0 = G_0 \times T$. These are Borel groups. The multiplication in $E$ is given by $(g, t)(g', t') = (gg', tt'm(g, g'))$, so, by restriction, the multiplication in $E_0$ is given by $(g, t)(g', t') = (gg', tt'm_0(g, g'))$, where $m_0 = m|_{G_0 \times G_0}$. Hence $E_0$ is Borel isomorphic to $E_{m_0}$, the ce of $G_0$. This identification is canonical because $E_0$ inherits a lcsc topology from $E$. We note that $E$ acts on $X$ through $G$, so $E_0$ may be viewed as the stabilizer of $x_0$ in $E$.

Let $X' = G[x_0]$. Let $\nu$ be a representation of $G_0$. Then there exists a strict $(G, X', \mathcal{U})$-cocycle $\phi_0$ that defines the representation $\nu$ at $x_0$. That is, $\nu(g) = \phi_0(g, x_0)$. We may extend $\phi_0$ to a $(G, X, \mathcal{U})$-cocycle $\phi$ by, for example, setting $\phi(g, x) = 1$ for $x \notin X'$. Now, by Theorem, 4.3.1 we may associate to $\phi$ a system of imprimitivity $(U, P)$ for $G$, where $P$ is a projection-valued measure living on $X'$. Such systems are said to be *induced by the representation* $\nu$ *of* $G_0$. The following theorem, which may be found in [19] p. 232, states that this is a canonical one to one correspondance.

**Theorem 4.3.3.** *Let $X$ be a standard Borel $G$ space and $(U, P)$ be a transitive* *system of imprimitivity based on $X$. There is a one to one correspondence between $(U, P)$ equivalence classes of systems of imprimitivity with $P$ living on $X'$ and equivalence classes of representations $\nu$ of the stabilizer $G_0$ of $x_0$. Each system of imprimitivity may be obtained by inducing from some representation $\nu$ of $G_0$.*

**Lemma 4.3.2.** *In the correspondence between systems of imprimitivity $(V, Q)$ for $E$ based on $X$ and URs $\nu$ of $E_0$, the systems with $V(t) = tI$ $(t \in T)$ correspond to URs $\nu$ of $E_0$ with $\nu(t) = tI$.*

*Proof.* Let $\nu$ and $V$ correspond. Then $\nu$ gives rise to a strict $(E_0, X)$ cocycle $\gamma$ such that $\gamma(g, x_0) = \nu(g)$. The representation $V$ acts on the Hilbert space $\mathcal{H} = L^2(X : \mathcal{K} : p)$ of $\mathcal{K}$-valued functions on $X$ and $p$ is a quasi-invariant probability measure on $X$. The action of $V$ is given by

$$(V(h)f)(x) = \rho_h(h^{-1}[x])^{\frac{1}{2}}\gamma(h, h^{-1}[x])f(h^{-1}[x]),$$

where $\rho_h = dp/dp^{(h^{-1})}$.

Suppose now that $\nu(t) = tI$ for $t \in T$. Since $T$ is central in $E$, it acts trivially on $X$ by Lemma 4.3.1. We also note that $\rho_t = 1$. By Lemma 4.3.1, $\gamma(t, x) = t$ for all $x \in X$. Hence $V(t) = tI$. Conversely, suppose that $V(t) = tI$ for all $t \in T$. Then for each $t \in T$, $\gamma(t, t^{-1}x) = t$ for almost all $x$, so that $\gamma(t, x) = t$ for almost all $x$. By Lemma 4.3.1 we have that $\nu(t) = t$ for all $t \in T$. □

We are now ready to prove Theorem 4.3.2.

*Proof.* If $(U, P)$ is an $m$-system of imprimitivity for $G$ and we define $V$ on $E_m$ by $V(x, t) = tU(x)$, then $V$ is an ordinary representation by Lemma 3.4.2 and $(V, P)$ is thus an ordinary system of imprimitivity for $E_m$ based on $X$. Lemma 4.3.2 now says that there is a bijection between these $(V, P)$ systems of imprimitivity and UR's $\nu$ of $E_0$ such that $\nu(1,t) = tI$. Since $E_0 \simeq E_{m_0}$, Theorem 3.4.2 gives a bijection between $\nu$-representations of $E_0$ for which $\nu(1,t) = tI$ and $m_0$-representations $\mu(x) = \nu(x,1)$ of $G_0$. Hence there exists a bijection between $(U,P)$-$m$-systems of imprimitivity for $G$ and $m_0$-representations of $G_0$.

$\square$

**Definition 31.** *A strict $m$-cocycle for $(G,X)$ is a Borel map $\delta : G \times X \to \mathcal{U}$ such that*

$$m(g_1,g_2)\delta(g_1g_2,x) = \delta(g_1,g_2[x])\delta(g_2,x) \quad \forall g_i \in G, \ \ x \in X.$$

Two such cocycles $\delta_i$ $(i = 1,2)$ are cohomologous $(\simeq)$ if there exists a Borel function $\phi : X \to \mathcal{U}$ such that $\delta_2(g,x) = \phi(g[x])\delta_1(g,x)\phi(x)^{-1}$ for all $g \in G, \ \ x \in X$.

The following lemma shows that certain strict $(G,X)-m-$ cocyles are related to certain strict $(E,X)$-cocyles. This is analogous to our earlier results where we found that certain $m$-representations of $G$ are related to the certain ordinary representations in the ce.

**Lemma 4.3.3.** *There is a natural bijection between strict $(E,X)$-cocycles $\gamma$ such that $\gamma((1,t),x) = tI \ \ \forall t \in T, \ x \in X$ and strict $(G,X)$-$m$-cocyles $\delta$ given by $\delta(g,x) = \gamma((g,1),x)$, $\gamma((g,t),x) = t\delta(g,x)$. This bijection respects equivalences, and induces a bijection of the respective cohomology sets.*

*Proof.* We have,

$$\gamma((g_1,t_1)(g_2,t_2),x) = \gamma((g_1g_2,t_1t_2m(g_1g_2)),x)$$

$$= t_1t_2m(g_1,g_2)\delta(g_1g_2,x) = t_1t_2\delta(g_1,g_2[x])\delta(g_2[x])$$

$$= \gamma((g_1,t_1),(g_2,t_2)[x])\gamma((g_2,t_2),x).$$

Also,

$$\delta(g_1g_2,x) = \gamma((g_1g_2,1),x) = \gamma(g_1,1)(g_2,m(g_1,g_2)^{-1},x)$$

$$= \gamma((g_1,1),g_2[x])\gamma((g_2,m(g_1,g_2)),x)$$

$$\delta(g_1,g_2[x])m(g_1,g_2)^{-1}\delta(g_2,x).$$

That the correspondence respects equivalence is clear, since $X$ is the same for both, and the condition $\gamma((1,t),x) = tI \;\forall\; t \in T,\; x \in X$ is unchanged under equivalence. □

The following lemma shows how to build from an $m_0$-representation of $G_0$ a corresponding $m$-system $(U,P)$ of $G$.

**Lemma 4.3.4.** *Given an $m_0$-representation $\mu$ of $G_0$ there is a strict $m$-cocycle $\delta$ with values in $\mathcal{U}$ such that $\delta(g,x_0) = \mu(g)$. In the corresponding $m$-system $(U,P)$, the action of $U$ is given as follows: $U$ acts on $L^2(X,\mathcal{K},p)$ with*

$$(U(g)f)(x) = \rho_g(g^{-1}[x])^{\frac{1}{2}}\delta(g,g^{-1}[x])f(g^{-1}[x]).$$

*The $\rho$ factors drop out if $p$ is invariant.*

*Proof.* Define $\nu(x,t) = t\mu(x)$ for $(x,t) \in E_0$. Then $\nu$ is a UR of $E_0$. This is a consequence of the fact that $E_0 \simeq E_{m_0}$ and Lemma 3.4.2. Next we build a strict $(E,X)$-cocyle $\gamma$ with $\gamma((g,t),x_0) = \nu((g,t)),\;\; (g,t) \in E_0$. By Lemma 4.3.3 such strict $(E,X)$-cocycles are in bijection with strict $(G,X)$-$m$-cocyles $\delta$ given by $\delta(g,x) = \gamma((g,1),x)$ and $\gamma((g,t),x) = t\delta(g,x)$. We now construct a $(U,P)$ system of imprimitivity for $G$. A trivial modification of proof of Theorem 4.3.1 shows that if $\delta$ is a $(G,X,\mathcal{U})$-cocycle, then there exists a unique $m$-system of imprimitivity $(U, P)$ such that for all Borel sets $E \subset X$,

$$P_E f = \chi_E f,$$

and for almost all $g$,

$$(U_g f)(x) = \{\rho_g(g^{-1}[x])\}^{\frac{1}{2}} \delta(g, g^{-1}[x]) f(g^{-1}[x])$$

for each $f \in \mathcal{H}$ for almost all $x$.

$\square$

The following lemma will be needed later.

**Lemma 4.3.5.** *Let $\delta_i$ $(i = 1, 2)$ be two strict $m$-cocycles for $G$ such that for each $g \in G$, $\delta_1(g, x) = \delta_2(g, x)$ for almost all $x \in X$. Let $\nu_i$ be the $m$-representations of $G_0$ defined by $\delta_i$ $(i = 1, 2)$. Then $\nu_1 \simeq \nu_2$.*

*Proof.* Let $\gamma_i$ be the strict $(E, X)$ cocycle defined by $\gamma_i((g, t), x) = t\delta_i(g, x)$. Then for each $(g, t) \in E$, $\gamma_1((g, t), x) = \gamma_2((g, t), x)$ for almost all $x \in X$. Let $\mu_i((g, t)) = \gamma_i((g, t), x_0)$, $g \in G_0$. Then by [19] p. 178 lemma 5.25, $\mu_1 \simeq \mu_2$. Since $\nu_i(g) = \mu_i(g, 1))$, $\nu_1 \simeq \nu_2$.

$\square$

## 4.4 The Mackey machine for projective representations of semidirect products

We now turn our attention to the Mackey treatment of lcsc groups with a semidirect product structure. Most of the following discussion is implicit in Mackey's work (for example [26]), but some aspects are new and make the theory more transparent when we specialize our assumptions to the groups in which we are interested. In this section, we consider a group $H = A \rtimes G$, where $G$ and $A$ are lcsc groups and $A$ is abelian. We concern ourselves only with multipliers of $H$ that are trivial when restricted to $A \times A$. We recall that these multipliers are completely described by Theorem 4.2.1.

The following lemma shows that there is a close connection between the cocyles $Z^1(G, A^*)$ and actions of $G$ on $A^*$. This lemma introduces a new idea into the existing theory.

**Lemma 4.4.1.** *Let $\phi : G \to A^*$ be a continuous map with $\phi(1) = 0$. Define $g\{\chi\} = g[\chi] + \phi(g)$, for $g \in G, \chi \in A^*$. Then $g : \chi \mapsto g\{\chi\}$ defines an action of $G$ on $A^*$ iff $\phi \in Z^1(G, A^*)$.*

*Proof.* If $g\{\cdot\}$ is to be an action on $A^*$, then $g_2\{g_1\{\chi\}\} = g_2g_1\{\chi\}$ for all $\chi \in A^*$. Now

$$g_2\{g_1\{\chi\}\} = g_2[g_1[\chi] + \phi(g_1)] + \phi(g_2) = g_2[g_1[\chi]] + g_2[\phi(g_1)] + \phi(g_2).$$

On the other hand

$$g_2g_1\{\chi\} = g_2g_1[\chi] + \phi(g_2g_1).$$

Equating the two, we see that the condition on $\phi$ is

$$\phi(g_2g_1) = g_2[\phi(g_1)] + \phi(g_2),$$

that is, $\phi \in Z^1(G, A^*)$.

□

We remark that this action depends on the choice of the cocycle $\phi \in Z^1(G, A^*)$.

So we write it as $g_\phi\{\chi\}$.

But if $\phi' \in Z^1(G, A^*)$ defines the same element as $\phi$ in $H^1(G, A^*)$, then $\phi'(g) = \phi(g) + g[\chi_0] - \chi_0$ for some $\chi_0 \in A^*$. So,

$$g_{\phi'}\{\chi\} = g[\chi] + \phi'(g) = g[\chi] + g[\chi_0] - \chi_0 = g[\chi + \chi_0] - \chi_0.$$

Let $\tau : \chi \mapsto \chi + \chi_0$ be the translation by $\chi_0$ in $A^*$. Then

$$g_{\phi'}\{\chi\} = \tau^{-1} g_\phi(\tau(\chi))$$

or

$$g_{\phi'} = \tau^{-1} g_\phi \tau.$$

So the actions defined by $\phi$ and $\phi'$ are equivalent in this strong sense.

**Definition 32.** *The action $g_\phi : \chi \mapsto g\{\chi\}$ is called the affine action of $G$ on $A^*$ determined by $\phi$.*

We will need the following technical lemma later.

**Lemma 4.4.2.** *Let $Y$ be a standard Borel space and $t : Y \to Y$ a Borel isomorphism defined by $t : y \mapsto t[y]$. Let $Q$ be a projection valued measure on $Y$. Then for all bounded Borel functions $f$ on $Y$,*

$$\int_Y f(t^{-1}[y])dQ(y) = \int_Y f(y)dQ_t(y)$$

*where $Q_t$ is a projection valued measure given by $Q_t(E) = Q(t[E])$.*

*Proof.* For any Borel set $E \subset Y$ and characteristic function $1_E$, we have $1_E(t^{-1}[y]) = 1_{t[E]}(y)$. Hence the assertion is clear for characteristic functions. Moreover, any bounded Borel function is a uniform limit of simple functions (which are finite sums of characteristic functions). Hence the lemma is proven. $\square$

The following theorem now shows how the $m$-representations of $H$ correspond to $m_0$-systems of imprimitivity based on $A^*$, where the action of $G$ on $A^*$ is given by the affine action.

**Theorem 4.4.1.** *Fix $\theta \in Z^1(G, A^*)$ and $m \in M'_A(H)$, $m \simeq (m_0, \theta)$. There is a natural bijection between $m$-representations $V$ of $H = A \rtimes G$ and $m_0$-systems of imprimitivity $(U, P)$ on $A^*$ for the affine action $g_\phi : \chi \mapsto g\{\chi\} = g_\theta\{\chi\}$ defined by $\theta$. The bijection is given by*

$$V(ag) = U(a)U(g), \quad U(a) = \int_{A^*} \langle a, \chi \rangle dP(\chi).$$

*Proof.* Consider the multiplier $m$ for $H$ defined by

$$m(ag, a'g') = m_0(g, g')\theta(g^{-1})(a')$$

where $m_0$ is a multiplier for $G$ and $\theta$ is a cocycle in $Z^1(G, A^*)$. Then

$$\begin{aligned} U(g)U(a)U(g)^{-1} &= \frac{m(g,a)}{m_0(g,g^{-1})} U(ga)U(g)^{-1} \\ &= \frac{m(ga,g^{-1})}{m_0(g,g^{-1})} m(g,a)U(g[a]). \end{aligned}$$

Now, $m(ga, g^{-1}) = m_0(g, g^{-1})$, where $m(g, a) = \theta(g^{-1})(a)$. Hence,

$$U(g)U(a)U(g)^{-1} = \theta(g^{-1})(a)U(g[a]).$$

Now $U$ is an ordinary representation on $A$, since $m|_{A\times A} = 1$. This means that there exists a unique pvm $P$ on $A^*$ such that

$$U(a) = \int_{A^*} \langle a, \chi \rangle dP(\chi) \quad (a \in A).$$

Thus

$$U(g) \left( \int_{A^*} \langle a, \chi \rangle dP(\chi) \right) U(g)^{-1} = \theta(g^{-1})(a) \int_{A^*} \langle g[a], \chi \rangle dP(\chi)$$

$$= \int_{A^*} \langle a, g^{-1}[\chi] + \theta(g^{-1}) \rangle dP(\chi)$$

$$= \int_{A^*} \langle a, g^{-1}\{\chi\} \rangle dP(\chi)$$

$$= \int_{A^*} \langle a, \chi \rangle dQ_g(\chi).$$

Here the last step is by Lemma 4.4.2, $Q_g$ being the pvm defined by $Q_g(E) = P(g\{E\})$. So

$$U(g)P(E)U(g)^{-1} = P(g\{E\}).$$

We have shown that for the action of $G$ on $A^*$ by $g : \chi \mapsto g\{\chi\}$, $(U, P)$ is an $m_0$-system of imprimitivity. Conversely, suppose $(U, P)$ is an $m_0$-system of imprimitivity for this action. Then, by retracing the steps in the above calculation with $U(a) = \int_{A^*} \langle a, \chi \rangle dP(\chi)$, we find

$$U(g)U(a)U(g)^{-1} = \theta(g^{-1})(a)U(g[a]).$$

If we define $V(ag) = U(a)U(g)$, then $V$ becomes an $m$-representation where $m(ag, a'g') = m_0(g, g')\theta(g^{-1})(a')$. □

We need the following definition.

**Definition 33.** *If $U$ is a UR of $A$ and $E \mapsto P(E)$ is its associated pvm, we say that Spec$(U) \subset F$ if $P(F) = I$. Here $F$ is some Borel set in $A^*$. We extend this terminology to any PUR of $H = A \rtimes G$ that is a UR on $A$.*

From the work in Section 4.3, we now obtain the basic theorem of irreducible $m$-representations of $H$. This is a variant of a theorem that in the ordinary system of imprimitivity context is known as Mackey's theorem. However, this theorem in the context of $m$-representations is formulated in terms of the affine action, and is new.

**Theorem 4.4.2.** *Fix $\chi \in A^*$, $m \simeq (m_0, \theta)$. If the affine action is regular, then there is a natural bijection between irreducible $m$-representations $V$ of $H = A \rtimes G$ with Spec$(V) \subset G\{\chi\}$ (the orbit of $\chi$ under the affine action) and irreducible $m_0$-representations of $G_\chi$, the stabilizer of $\chi$ in $G$ for the affine action. Let $X = G\{\chi\}$ and let $P$ be a $\sigma$-finite quasi-invariant measure for the action of $G$. Then for any irreducible $m_0$-representation $\mu$ of $G_\chi$ in the Hilbert space $\mathcal{K}$, the corresponding $m$-representation $V$ acts on $L^2(X, \mathcal{K}, P)$ and has the following form:*

$$(V(ag)f)(q) = \langle a, q\rangle \rho_g(g^{-1}\{q\})^{\frac{1}{2}})\delta(g, g^{-1}\{q\})f(g^{-1}\{q\})$$

*where $\delta$ is any strict $m_0$-cocyle for $(G, X)$ with values in $\mathcal{U}$, the unitary group of $\mathcal{K}$, such that $\delta(g, q) = \mu(g),\ g \in G_\chi$.*

We note that the $\rho_g$-factors drop out if $P$ is an invariant measure.

**Corollary 4.4.1.** *Suppose $H^1(G, A^*) = 0$. Then we can take $\theta(g) = 1$ so that $m(ag, a'g') = m_0(g, g')$. In this case, the affine action reduces to the ordinary action and Theorem 4.4.2 becomes the ordinary Mackey theorem for $m$-representations.*

# CHAPTER 5

# Elementary Particles Over $\mathbb{Q}_p$

## 5.1 Introduction

In this chapter, we discuss our results concerning the classification of the particles of the $p$-adic Poincaré group. We first specialize the results established in Chapter 4 to the $p$-adic setting. In Section 5.2, we state some results of V. Varadarajan [20] on multipliers of $p$-adic vector spaces. Our discussion in Section 5.3 on cohomology group $H^1(G, U)$ for $p$-adic Lie groups also follows [20] closely. In the Section 5.4, we classify the particles of the $p$-adic Poincaré group. These results on the particle classification are new.

## 5.2 Multipliers for $p$-adic vector spaces

In our setup so far, $H = A \rtimes G$, where $A$ and $G$ are lcsc groups and $A$ is abelian. We now take $A$ to be an $n$-dimensional vector space $V$ over $\mathbb{Q}_p$.

Let $C_n(V)$ be the abelian group of $n$-characters of $V$. The $n$-characters are continuous maps $V \to T$ that are characters in each argument. The topology is given by uniform convergence on compact sets. We fix a nontrivial additive character $\psi : \mathbb{Q}_p \to T$. (See Appendix A.3 on how $\psi$ is chosen.) Let $M_n$ be the vector space of $n$-linear maps from $V$ into $\mathbb{Q}_p$.

The following proposition and its corollary can be found in [20].

**Proposition 5.2.1.** *The map $\beta \mapsto \psi(\beta)$ is a topological additive isomorphism of $M_n$ with $C_n$.*

**Corollary 5.2.1.** *Let $b$ be a skew symmetric bilinear form on $V \times V$. Then $\psi(b)$ is a multiplier for $V$ and the map $F : b \mapsto [\psi(b)]$ is an isomorphism of skew-symmetric bilinear forms with $H^2(V)$.*

## 5.3 $H^1(G,U)$ for $p$-adic Lie groups $G$

We now take a closer look at the cohomology group $H^1(G,U)$. This group is important for Theorem 4.2.1. In this section, we state results from [20]. We take $G$ to be a $p$-adic Lie group and $U$ a finite-dimensional vector space over $\mathbb{Q}_p$ on which $G$ acts linearly. By the discussion in Section 4.2, a map $f : G \to U$ is a 1-cocycle iff $\gamma_f : g \mapsto (f(g), g)$ is a homomorphism of topological groups from $G$ to $U \rtimes G$. When the underlying structure is that of a Lie group, $f$ is analytic [24]. Hence, all the cocycles are analytic and differentiable. It is useful to look at the Lie algebra structure of $G$. Later on, we will be able to determine when $H^1(G,U)$ is trivial by looking at the corresponding Lie algebra cohomology. Let $\mathfrak{g}$ be the Lie algebra of $G$ and let $\mathfrak{u}$ be the Lie algebra of $U$. Then $\mathfrak{u} \oplus' \mathfrak{g}$ is the Lie algebra of $U \rtimes G$.

**Lemma 5.3.1.** *The bracket structure is given by*

$$[(u,g),(u',g')] = (g[u'] - g'[u], [g,g']).$$

Let $\partial\gamma_f$ be the differential of $\gamma_f$. Then $\partial\gamma_f$ is a Lie algebra morphism of $\mathfrak{g}$ to $\mathfrak{u} \oplus' \mathfrak{g}$. $\partial\gamma_f$ is of the form $\partial\gamma_f = (\partial f(X), X)$ for $X \in \mathfrak{g}$. This equation defines $\partial f$. Now $\partial_{\gamma_f}([X,Y)] = [\partial\gamma_f(X), \partial\gamma_f(Y)]$ $(X, Y \in \mathfrak{g})$ thus $\partial f([X,Y]) = X \cdot \partial f(Y) - Y \cdot \partial f(X)$. This defines a 1-cocycle for $\mathfrak{g}$ with values in $\mathfrak{u}$. The coboundaries will be cocycles of the form $\partial f'$ where $f'(g) = g[u] - u$.

**Lemma 5.3.2.** *The map* $f \mapsto \partial f$ *induces a well defined map* $\partial : H^1(G,U) \to H^1(\mathfrak{g},\mathfrak{u})$.

For any cocycle $f$ we write $[f]$ for its cohomology class. We note that $H^1(G,U)$ is a $\mathbb{Q}_p$-vector space and $\partial$ is a $\mathbb{Q}_p$ linear map.

**Theorem 5.3.1.**

*(a) The kernel* $\Gamma$ *of the map* $\partial : H^1(G,U) \to H^1(\mathfrak{g},\mathfrak{u})$ *is the space of all cohomology classes that contain a cocycle vanishing on a compact open subgroup* $K$ *of* $G$.

*(b) If for any* $0 \neq u \in U$ *the stabilizer* $G_u$ *of* $u$ *in* $G$ *is either all of* $G$ *or has measure* $0$*, then* $\Gamma \simeq Hom_0(G, U^G)$*, where* $Hom_0(G, U^G)$ *is the space of morphisms* $G \to U^G$ *that are* $0$ *on small compact open subgroups of* $G$ *and* $U^G$ *is the space of vectors in* $U$ *fixed by* $G$.

**Lemma 5.3.3.** *Let* $V$ *be a finite-dimensional* $p$*-adic vector space and* $R$ *a* $p$*-adic Lie group acting on* $V$*. Let* $G = V \rtimes R$*. Suppose that* $(R,R)$ *is an open subgroup of finite index in* $R$*. Then* $Hom_0(G,\mathbb{Q}_p) = 0$*. Let* $V'$ *be the dual of* $V$*. If* $V'^R = 0$*, then* $G/(G,G) \simeq R/(R,R)$.

**Corollary 5.3.1.** *Under the conditions of* $(b)$ *of Theorem 5.3.1,* $\partial : H^1(G,U) \to H^1(\mathfrak{g},\mathfrak{u})$ *is injective if either* $U^G = 0$ *or* $Hom_0(G,\mathbb{Q}_p) = 0$*, in particular if* $G$ *is compact.*

Let $V$ be a finite-dimensional vector space over $\mathbb{Q}_p$. Let be $G$ the group of $\mathbb{Q}_p$-rational points of a closed algebraic subgroup $G \subset GL(\bar{V})$ defined over $\mathbb{Q}_p$. Here $\bar{V} = \bar{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} V$, where $\bar{\mathbb{Q}}_p$ is the algebraic closure of $\mathbb{Q}_p$. Then $G = G(\mathbb{Q}_p)$ is a $p$-adic Lie group and the theory discussed above applies.

**Lemma 5.3.4.** *Let $V$, $G$ be as above. Then condition $(b)$ of Theorem 5.3.1 is satisfied.*

**Lemma 5.3.5.** *Suppose that $G$ is semisimple. Then $H^1(G, V') = 0$.*

*Proof.* Since $\mathfrak{g}$ is semisimple, $H^1(\mathfrak{g}, V') = 0$. So, we only need to verify that $\partial$ is injective. By Theorem 5.3.1 and Lemma 5.3.4, we need to only consider the case when $V'^G \neq 0$. It is known that $(G, G)$ is an open subgroup of finite index in $G$, so the condition of Lemma 5.3.3 is satisfied.

□

## 5.4 Particle classification for the $p$-adic Poincaré group

We can now apply the theory that we have developed in chapters 2-4 and in the first half of this chapter to discuss the PUIRs of the $p$-adic Poincaré group. For the remainder of this chapter we shall work over the field $\mathbb{Q}_p$. All the groups described will be algebraic groups defined over $\mathbb{Q}_p$, so that the groups of their $\mathbb{Q}_p$-points are $p$-adic Lie groups and, in particular, lcsc. By the Poincaré group we mean the group $P = V \rtimes SO(V)$, where $V$ is a finite-dimensional quadratic vector space over $\mathbb{Q}_p$. In Chapter 1 we described that elementary particles correspond to certain PUIRs, of the Poincaré group. Our goal for this chapter is to describe these PUIRs and thus classify the elementary particles associated to the $p$-adic Poincaré group.

We want to establish that the representations of the Poincaré group are indeed described by Theorem 4.4.2. To do this, we must establish that $P$ satisfies the criteria of Theorem 4.4.2.

Our setup requires $H = A \rtimes G$, with $A$ and $G$ being lcsc groups and $A$ being abelian. Furthermore, the only alternating bicharacter of $A$ invariant under $G$ is the trivial one. We first show that this holds true for the Poincaré group.

**Lemma 5.4.1.** *The only bicharacter of $V$ invariant under $SO(V)$ is the trivial one.*

*Proof.* Let $\beta$ be the nondegenerate bilinear form corresponding to the quadratic form $Q$ of $V$. Then $\beta$ induces an isomorphism $\tilde{\beta} : V \to V'$. Now $V$ and $V'$ are irreducible $SO(V)$ modules. One may check this directly by going to the Lie algebra. Hence Schur's lemma applies and any bilinear form $\alpha$ which is invariant under $SO(V)$ is a scalar multiple of $\beta$. In particular, if $\alpha$ is a skew-symmetric bilinear form invariant under $SO(V)$ it must be 0, since $\beta$ is symmetric. □

**Theorem 5.4.1.** *The cohomology $H^1(SO(V), V')$ is trivial.*

*Proof.* $SO(V)$ is semisimple so Lemma 5.3.5 applies. □

**Corollary 5.4.1.** *Every multiplier of $P$ is equivalent to the lift of a multiplier for $L = SO(V)$.*

*Proof.* Corollary 4.2.2 applies since $H'(SO(V), V') = 0$. □

We note that the Theorem 4.4.2 requires that we know that the action of $SO(V)$ on $V'$ is regular. Since the quadratic form on $V$ is nondegenerate, we have that $V \simeq V'$. Furthermore, this isomorphism respects both the action of $GL(V)$ on $V$ and the induced action on $V'$. Hence the action of $SO(V)$ on $V'$ is isomorphic to the action of $SO(V)$ on $V$. We recall from Chapter 4 that the orbits of the action of $SO(V)$ in $V$ are:

(a) The sets $M_a = \{p \in V' | (p, p) = a \neq 0\}$.

(b) The set $M_0 = \{p \in V' | (p,p) = 0,\ p \neq 0\}$.

(c) The set $\{0\}$.

**Definition 34.** *We will call the orbits of type* $(a)$ *massive and the orbits of type* $(b)$ *massless and* $(c)$ *trivial-massless.*

Mackey's theorem also requires that the action on $V'$ is regular. The lemma below tells us that this condition is also satisfied by the Poincaré group.

**Lemma 5.4.2.** *The action of* $SO(V)$ *on* $V$ *is regular.*

*Proof.* $V$ is divided into massive, massless and trivial-massless orbits. By the theorem of Effros (3.2.4), it suffices to show that each of these sets is open in its closure. Let us look at the massless orbit first. We have shown that if $x \neq 0$ is a null vector, then the orbit of $x$ under $SO(V)$ consists of all the nonzero null vectors. $M_0$ is open in its closure $M_0 \cup \{0\}$. The massive orbit $M_a = \{v \in V | (v,v) = a \neq 0\}$ is defined as the variety $\{v \in V | Q(v) - a = 0\}$. It is closed, so it is trivially open in its closure. Likewise, the trivially-massless orbit is open in its closure. □

We now see that Theorem 4.4.2 applies to the Poincaré group, and we summarize our results in the following theorem, which completely describes the particles of the $p$-adic Poincaré group.

**Theorem 5.4.2.** *Let* $P_V = V \rtimes SO(V)$ *be the* $p$*-adic Poincaré group. Fix* $p \in V'$ *and let* $m_0$ *be a multiplier of* $SO(V)$ *and* $m$ *be its lift to* $P_V$*. Then there is a natural bijection between irreducible* $m$*-representations of* $P_V = V \rtimes SO(V)$ *with* $Spec(V) \subset SO(V)[p]$*, the orbit of* $p$ *under the ordinary action of* $SO(V)$ *and irreducible* $m_0$*-representations of* $SO(V)_p$*, the stabilizer of* $p$ *in* $SO(V)$*. Every*

*PUIR of $P_V$, up to unitary equivalence, is obtained by this procedure. Let $X = SO(V)[p]$, and $\lambda$ be a $\sigma$-finite quasi-invariant measure for the action of $SO(V)$. Then, for any irreducible $m_0$-representation $\mu$ of $SO(V)_p$ in the Hilbert space $\mathcal{K}$, the corresponding $m$-representation $U$ acts on $L^2(X, \mathcal{K}, \lambda)$ and has the following form:*

$$(U(ag)f)(q) = \langle a, q\rangle \rho_g(g^{-1}\{q\})^{\frac{1}{2}})\delta(g, g^{-1}\{q\})f(g^{-1}\{q\}),$$

*where $\delta$ is any strict $m_0$-cocyle for $(SO(V)), X)$ with values in $\mathcal{U}$, the unitary group of $\mathcal{K}$, such that $\delta(g, q) = \mu(g),\ g \in SO(V)_p$.*

Thus, to determine the PUIRs of the Poincaré group, one must determine the multipliers of $L = SO(V)$, and for each given multiplier $m$, determine the irreducible $m$-representations. The PUIRs then correspond to $\nu \in V'$ and $m_\nu$-representations of the stabilizer $L_\nu$ of $\nu$ in $L = SO(V)$, $m_\nu$ being $m|_{L_\nu \times L_\nu}$.

We now define some terminology, that will be needed in Chapter 6.

**Definition 35.** *A vector $p \in V'$ is called a massless momentum vector if $p \neq 0$ and $(p, p) = 0$.*

**Definition 36.** *A vector $p \in V'$ is called a* massive momentum vector *if $(p, p) \neq 0$.*

We refer to an orbit of a massless momentum vector as a massless orbit, and an orbit of a massive momentum vector as a massive orbit. We now define massless and massive particles.

**Definition 37.** *A PUIR of the Poincaré group is called an elementary particle. A particle is called massless if it corresponds to a null momentum vector $p$. A particle is called trivially massless if $p = 0$; otherwise it is called massive.*

# CHAPTER 6

# Conformal Symmetries for $p$-adic Particles

## 6.1 Introduction

In Section 6.2, we establish some definitions and state the main goal of this chapter. In Section 6.3, we prove a key lemma that is used repeatedly to establish results about the non-extendability of massive and eventually massive particles. In Section 6.4, we prove that massive particles do not have partial conformal symmetry. Finally, in Section 6.5, we establish that eventually massive particles do not have partial conformal symmetry.

### 6.1.1 Some motivation for the approach

The particle classification over $\mathbb{Q}_p$ for the $p$-adic Poincare group is similar to, but more subtle than that over $\mathbb{R}$. Over $\mathbb{R}$, one goes to the 2-fold cover of $V \rtimes SO(V)$, the uce, and classifies the irreducible representations of $V \rtimes SL(2, \mathbb{C})_{\mathbb{R}}$. This cannot be done over $\mathbb{Q}_p$ because the 2-fold cover map $V \rtimes Spin(V) \rightarrow V \rtimes SO(V)$ is not surjective, thus one loses information by going to the uce. So not all projective representations may be recovered. Hence, one has to work directly with the multipliers and projective representations.

## 6.2 Statement of the problem

We recall from Chapter 4 that if $V_1$ and $V_0$ are two quadratic vector spaces, with $V_1$ Witt equivalent to $V_0$, and $dim(V_0) = dim(V_1)+2$, then the Poincaré group $P_{V_1}$ can be imbedded as a subgroup of the conformal group $SO(V_0)$, and furthermore, that any two such imbeddings are conjugate over $SO(V_0)$. A natural question is the following: are there PUIRs of the Poincaré group that extend to be PUIRs of the conformal group? PUIRs that do extend to the conformal group are said to have conformal symmetry. Classically, only massless particles (photons) have conformal symmetry and the corresponding PUIRs of the real Poincaré group extend to PUIRs of the real conformal group [16]. We would like to explore this question in the $p$-adic setting. Our ultimate goal is to establish some necessary conditions for this extension to be possible.

**Definition 38.** *Let $V_1$ and $V_0$ be two Witt equivalent quadratic vector spaces over $\mathbb{Q}_p$ with $dim(V_0) = dim(V_1) + 2$. When a PUIR $U$ of $P_{V_1}$ can be extended to be a PUIR $V$ of $SO(V_0)$, we say that the particle corresponding to $U$ has conformal symmetry.*

Recall that in Chapter 4 we defined the partial conformal group $\tilde{P} = V \rtimes (SO(V) \times \mathbb{Q}_p^\times)$, where $c \in \mathbb{Q}_p^\times$ commutes with $SO(V)$ and acts on $V$, as multiplication by $c$.

**Definition 39.** *When a PUIR of $U$ of $P$ can be extended to be a PUIR $V$ of the group $\tilde{P}$, we say that the particle corresponding to $U$ has partial conformal symmetry.*

We make the following trivial, but important, observation: if a particle has no partial conformal symmetry, then it does not have conformal symmetry.

In the next section, we establish some necessary conditions for a particle to have partial conformal symmetry.

## 6.3 Extensions of $m$-representations of semidirect products

Let $A, L$ and $M$ be lcsc groups with $A$ being abelian and $L$ being a closed subgroup of $M$. Suppose that $M$ acts on $A$ so that we may form the semidirect products $G = A \rtimes L$ and $H = A \rtimes M$. We assume: $a$) All multipliers of $G$ and $H$ are trivial on $A$; $b$) $H^1(L, A^*) = 0$, and $H^1(M, A^*) = 0$; $c$) 1 is the only element of $A^*$ fixed by $L$; and $d$) The actions of $M$ and $L$ on $A^*$ are regular.

Because of the assumptions that $H^1(G, A^*) = 0$, $H^1(H, A^*) = 0$, and the actions of $M$ and $L$ are regular, the theory in Chapter 4 tells us that the irreducible $m$-representations $U$ of $G$ (resp. $H$) correspond to pairs $(\chi, u)$ where $\chi \in A^*$ and $u$ is an irreducible $m_\chi$-representation of the stabilizer $G_\chi$ (resp. $H_\chi$) of $\chi$ in $G$ (resp. $H$).

We need the following technical result.

**Lemma 6.3.1.** *Let $U$ be an $m$-representation of $G$, where $m$ is standard. Let $V_1$ be an $m_1$-representation of $H$ extending $U$. In this case, we can find a standard multiplier $m'$ for $H$ such that $m'|_{G\times G} = m$ and $U$ has an extension $V$ to $H$ as an $m'$-representation with $V(ah) = F(ah)V_1(ah)$ $(ah \in H)$ for some Borel function $F : H \to T$ with $F = 1$ on $G$.*

*Proof.* From the proof of Theorem 4.2.1 (see [20]), we know that $V : ah \mapsto m_1(a, h)V_1(ah)$ is an $m'$-representation of $H$ with $m'|_{A\times A} = 1$ and $m'|_{A\times H} = 1$. Clearly, $V$ extends $U$, $m' \simeq m_1$ and $m'|_{G\times G} = m$. As $H_1(M, A^*) = 0$, we have

$m'(ah, a'h') = m'_0(h, h')f(h[a'])/f(a')$, where $m'_0$ is a multiplier for $M$ and $f \in A^*$. Since $m'|_{G\times G} = m$, $f(g[a'])f(a')^{-1} = 1 \ \ \forall g \in L, \ a' \in A$. Hence $f = 1$, by the assumption that 1 is the only character fixed by $L$. Thus $m'$ is already standard.

$\square$

The following is a key lemma of the chapter that will be used multiple times to prove the impossibility of the extension of both massive and eventually massive particles.

**Lemma 6.3.2.** *Let $U$ be an irreducible $m$-representation of $G$ for a standard multiplier $m$ for $G$. Let $U$ correspond to the $L$-orbit of $\chi \in A^*$ and an irreducible $m_\chi$-representation $u$ of the stabilizer $L_\chi$ of $\chi$ in $L$, $m_\chi$ being $m|_{L_\chi \times L_\chi}$. The following are equivalent:*

***(1)*** *$U$ extends to an irreducible projective unitary representation $V_1$ of $H$.*

***(2)*** *(a) $M[\chi] \setminus L[\chi]$ is a null set in $M[\chi]$.*

*(b) There is a standard multiplier $m'$ for $H$ with $m'|_{G\times G} = m$.*

*(c) $u$ extends to a $m'_\chi$-representation of $M_\chi$.*

*In this case, there is an $m'$-representation $V$ of $H$ such that $V$ belongs to the same equivalence class, as $V_1$ with:*

*(a) $V|_G = U$.*

*(b) $V$ corresponds to $\chi$ and $v$ where $v$ is an $m'_\chi$-representation of $M_\chi$.*

*(c) $v|_{L_\chi} = u$.*

*Proof.* (1) $\Rightarrow$ (2): We may assume $U$ extends to an $m'$-representation $V$ of $H$ belonging to the same equivalence class as $V_1$ where $m'$ is standard and $m'|_{G\times G} =$

$m$. Clearly $V$ is irreducible. Hence, the spectrum of $V$ lives on an $M$-orbit in $A^*$. But as $V$ and $U$ have the same restriction to $A$, the spectrum of $V$ must meet $L[\chi]$ so that we may assume it to be $M[\chi]$. But then $M[\chi] \setminus L[\chi]$ must be null. This proves (2)($a$).

We may now write $V$ in the form:

$$(V(ah)f)(\zeta) = \rho_h(h^{-1}\zeta)^{\frac{1}{2}}\langle a, \zeta\rangle C(h, h^{-1}\zeta)f(h^{-1}\zeta), \quad (\zeta \in M[\chi],\ h \in M),$$

where $C$ is a strict $m'$-cocyle that defines the $m'_\chi$-representation $v$. On the other hand, $U$ is given by:

$$(U(ag)f)(\zeta) = \rho_g(g^{-1}\zeta)^{\frac{1}{2}}\langle a, \zeta\rangle D(g, g^{-1}\zeta)f(h^{-1}\zeta) \quad (\zeta \in L[\chi],\ g \in L),$$

where $D$ is a strict $m$-cocyle defining the $m$-representation $u$. Since $V|_G = U$, it follows that $D(g, \nu) = C(g, \nu)$ for each $g$, for almost all $\nu \in M[\chi]$. By Lemma 4.3.5, $u$ is equivalent to the restriction of $v$ to $L_\chi$. If $u(g) = rv(g)r^{-1}$ $(g \in L_\chi)$, where $r$ is a unitary representation in the space of $v$, it is clear that $u$ extends to $rvr^{-1}$. This proves (2) ($b$).

(2) $\Rightarrow$ (1): Extend $u$ to an $m'_\chi$-representation of $M_\chi$ and build a strict $(M, M[\chi])$-cocyle $C$ for the multiplier $m'$ for $M$ that defines the $m'_\chi$-representation at $\chi$. The restriction of $C$ to $L$ is a strict cocycle for $m_1 = m'|_{L\times L}$. The $m'$-representation of $H$ corresponding to $(\chi, m')$ restricts on $G$ to the $m_1$-representation defined by $(\chi, m_1)$ and hence is equivalent to $U$. So, $U$ extends to a PUR of $H$.

□

## 6.4 Impossibility of partial conformal symmetry for massive particles

In this section, we show that massive particles do not possess partial conformal symmetry. We begin with some important lemmas.

**Lemma 6.4.1.** *The orbit of a massive point under $SO(V) \times \mathbb{Q}_p^\times$ is open in $V$.*

*Proof.* Let $x \in V$ be such that $Q(x) = a \neq 0$; then if $g \in \mathbb{Q}_p^\times$ and $g[x] = tx$ then $Q(g[x]) = at^2$. Thus, the orbit of $Q(x)$ under $\tilde{P}$ is $a(\mathbb{Q}_p^\times)^2$. Hence the orbit of $x$ is $Q^{-1}(a(\mathbb{Q}_p^\times)^2)$. Since $Q$ is a continuous function, the orbit of $x$ will be open in $V$ if we can show that $a(\mathbb{Q}_p^\times)^2$ is open in $\mathbb{Q}_p^\times$. We note that it suffices to prove that $(\mathbb{Q}_p^\times)^2$ is open in $\mathbb{Q}_p$. If we find a neighborhood $N$ of identity in $\mathbb{Q}_p^\times$ that is open such that $N^2$ is open, then given any $y \in N^2$, $y[N]$ is an open neighborhood of $y$. First, we show that such a neighborhood exists. We claim that $(1+x)^{\frac{1}{2}} \in \mathbb{Q}_p^2$ for $|x|_p << 1$. We consider the binomial expansion of $(1+x)^{\frac{1}{2}}$:

$$(1+x)^{\frac{1}{2}} = 1 + \sum_{n=0}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot \ldots \cdot (2n+1)}{2^n \cdot n!} x^n.$$

We want to show the above expansion converges in the $p$-adic absolute value for all $x$ with $|x|_p << 1$. Since, $|a+b|_p = \max(|a|_p, |b|_p)$, it suffices to show that the $n$th term tends to 0 for $|x|_p << 1$. Since $|x^n|_p$ tends to zero exponentially already, we need only check that the other terms in the sum do not tend to infinity faster. Now, $|1 \cdot 3 \cdot \ldots \cdot (2n+1)|_p \leq 1$, since $1 \cdot 3 \cdot \ldots \cdot (2n+1) \in \mathbb{Z}$. Furthermore $|2^{-n}|_p = (|2^{-1}|_p)^n$ and so it is enough to show that $|n!|_p^{-1} = c^n$ for some $c > 0$. Thus, the only term that remains is $|n!|_p^{-1}$.

All the terms in $n!$, that are not divisible by $p$, can be thrown out since they have $p$-adic norm 1. So we need to look at the product $p \cdot 2p \cdot \ldots \cdot \frac{n}{p} \cdot p$. Now, we ask again what is the largest power of $p$ dividing $p \cdot 2p \cdot ... \cdot \frac{n}{p} \cdot p = p^{\frac{n}{p}} \cdot 1 \cdot 2... \cdot \frac{n}{p}$. So, the largest power of $p$ dividing $n!$ is

$$\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + ... + \left[\frac{n}{p^m}\right].$$

Hence,

$$|n!|_p^{-1} = p^{\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+...+\left[\frac{n}{p^m}\right]} \leq p^{\frac{n}{p}+\frac{n}{p^2}+...+\frac{n}{p^m}} = p^{\frac{n}{p-1}} = (p^{\frac{1}{p}-1})^n.$$

□

**Lemma 6.4.2.** *Let $p \in V$ be a massive point. Then the quasi-invariant measure class on the orbit $SO(V) \times \mathbb{Q}_p^{\times} \cdot p$, is the Lebesgue (Haar) measure class.*

*Proof.* By Lemma 6.4.1, the orbit $SO(V) \times \mathbb{Q}_p^{\times} \cdot p = \omega_p$ is open in $V$. Let $E \subset \omega_p$ be a set of Haar measure 0 in $V$. Since $\omega_p$ is open in $V$, the Haar measure is also defined on $\omega_p$. We will now show that the Haar measure is quasi-invariant on $\omega_p$.

Let $\mu$ be the Haar measure. Since $SO(V) \times \mathbb{Q}_p^{\times}$ acts linearly for any $(g, c) \in SO(V) \times \mathbb{Q}_p^{\times}$, we have that $\mu((g, c) \cdot E) = |\det((g, c))|_p$. Hence, $\mu((g, c) \cdot E) = 0$. Thus, the measure class on $\omega_p$ is the Haar measure class and it is quasi-invariant under $SO(V) \times \mathbb{Q}_p^{\times}$. □

**Lemma 6.4.3.** *Let $f : k^n \to k$ be a nonzero analytic function and $N = \{x \in k^n | f(x) = 0\}$. If the gradient of $f$ is nowhere vanishing on $N$, then $N$ has Haar measure 0.*

*Proof.* Note that to show that $N$ has Haar measure 0, it is enough to show that given any $p \in N$, then for some open set $U$ of $k^n$ containing $p$, $U \cap N$ has Haar measure 0 in $k^n$. Since $N$ may be covered with countably many of these neighborhoods of measure 0, it will then follow that $N$ has measure 0. By assumption, at any point $p$ of $N$ the gradient of $f$ is nonzero on some neighborhood $U \cap N$ of $p$. We may assume that $\frac{\partial f}{\partial x_1} \neq 0$ at $p$ by a permutation of coordinates if necessary. The map $g(x) = (f(x), x_2, ..., x_n)$ is a diffeomorphism near $p$. The Jacobian matrix is given by

$$\begin{pmatrix} \frac{\partial f}{\partial x_1} & \cdots \\ 0 & I \end{pmatrix},$$

which has nonzero determinant near $p$ by assumption. Let $E$ be a neighborhood around $p$. Then $\int_E d^n x = \int_{g(E)} |\det J| d^n y$, where $y = (f(x), y_2, ...y_n)$. Because $f(x) = 0$, $g$ maps $E$ into a subspace $E' = \{y_1 = 0\}$ of $k^n$. Each line perpendicular to $E'$ intersects it at a single point. Fubini's theorem can now be used to show that $E'$ has measure 0 in $k^n$. $\square$

**Corollary 6.4.1.** *Both massive and massless orbits under $SO(V)$ have Haar measure 0 in $V$.*

*Proof.* Let us now take $x \in V$ such that $x \neq 0$ and $Q(x) = a$. Then $f(x) = Q(x) - a$ is an analytic function and defines a subset $Q_a = \{x \in V | f(x) = 0\}$ of $V$. We want to show that $Q_a$ has measure 0 in $V$. We may assume that there is a basis of $V$, $(e_i)$, such that $(e_i, e_j) = a_i \delta_{ij}$ and that $Q(x) = \sum a_i x_i^2$. Since $x = 0$ is not in $Q_a$ we see that $Q$ has a nonzero gradient on all of $Q_a$. All the requirements of Lemma 6.4.3 are satisfied and thus $Q_a$ has Haar measure 0 in $V$. $\square$

We can now prove the first main result of this chapter.

**Theorem 6.4.1.** *A massive PUIR of $P_V$ does not have an extension to $\tilde{P}_V$*

*Proof.* Let $p$ correspond to a massive orbit. We have that $P_V = V \rtimes SO(V)$ and $\tilde{P}_V = V \rtimes (SO(V) \times \mathbb{Q}_p^\times)$. Let us denote $SO(V)$ by $L$ and $SO(V) \times \mathbb{Q}_p^\times$ by $M$. If a massive PUIR of $P_V$ were to have an extension to $\tilde{P}_V$, then by

Lemma 6.3.2, $M[p] \setminus L[p]$ would be a null set. However, by Corollary 6.4.1, $L[p]$ has Lebesgue (Haar) measure 0, so $M[p]$ would have to have Lebesgue measure 0. But, by Lemma 6.4.1, the orbit $M[p]$ is open and so has nonzero Lebesgue measure. Hence the massive representations cannot extend even to the partial conformal group and therefore cannot extend to the full conformal group.

$\square$

## 6.5 Impossibility of partial conformal symmetry for eventually massive particles

Since the massive particles cannot be extended to the conformal group, we now turn our attention to massless particles.

Let $V$ be a quadratic vector space and $p$ a nontrivial null vector in $V$. Let $P_V = V \rtimes SO(V)$ and let $\tilde{P}_V = V \rtimes (SO(V) \times k^\times)$. As discussed in Section 2.7, the action of $c \in k^\times$ on $v \in V$ is $c : v \mapsto cv$, and $k^\times$ commutes with $SO(V)$. We know from Theorem 2.3.3 that the stabilizer of $p$ in $SO(V)$ is isomorphic to $P_{V_1} = V_1 \rtimes SO(V_1)$, where $V_1$ is a vector space Witt equivalent to $V$ with $dim(V) = dim(V_1) + 2$. We now claim the following:

**Proposition 6.5.1.** *Let $\tilde{P}_{Vp}$ be the stabilizer of $p$ in $SO(V) \times k^\times$. Then $\tilde{P}_{Vp}$ is isomorphic to $\tilde{P}_{V_1}$.*

*Proof.* Let $(g, c) \in SO(V) \times k^\times$. Then $(g, c)$ acts on $p$ by $(g, c) : p \mapsto cg[p]$. Hence $(g, c)$ fixes $p$ iff $g[p] = \frac{1}{c}p$. In other words, $g$ stabilizes the line $kp$. As proved in Section 2.7, the stabilizer of the line is $\tilde{P}_{V_1}$. $\square$

**Lemma 6.5.1.** *$H^1(SO(V) \times \mathbb{Q}_p^\times, V') = 0$. Action of $(g, c) \in SO(V) \times \mathbb{Q}_p^\times$ on $\lambda \in V'$ is by $(g, c) : \lambda \mapsto cg \cdot \lambda$.*

*Proof.* Let $F \in H^1(SO(V) \times \mathbb{Q}_p^\times, V')$. We set $L = SO(V)$ and write elements of $SO(V) \times \mathbb{Q}_p^\times$ as $(l, c)$. Then $F((l, 1))$ is a trivial cocycle for all $l \in L$ since $H^1(SO(V), V') = 0$. Thus, we can find $\lambda \in V'$ such that $F((l, 1)) = l \cdot \lambda - \lambda \ \forall\, l \in SO(V)$. If $F_1((l, c)) = F((l, c)) - ((l, c)) \cdot \lambda - \lambda$, then $F_1 \simeq F$ and $F_1$ is 0 on $L$. So we may assume that $F$ is 0 on $L$ to begin with. We may identify $(l, 1)$ with $L$ and $(1, c)$ with $c \in \mathbb{Q}_p^\times$, and we can then write $(l, c) = lc$. We now use the fact that $lc = cl$ to write:

$$F((l, c)) = F(lc) = F(l) + l[F(c)] = F(c) + c[F(L)].$$

Since $F(l) = 0$, we have $F(c) = l[F(c)]$. However, $L$ does not fix any nontrivial vector in $V'$, so we must have $F(c) = 0$ therefore $F = 0$. $\square$

**Lemma 6.5.2.** *Suppose $p$ is a null vector in $V'$ and $U$ is an irreducible m-representation of $P$ corresponding to $p$ and an irreducible m-representation $u$ of $SO(V)_p = L_p$. Suppose that $U$ has an extension to $\tilde{P}$ as a projective unitary representation. Then, identifying $SO(V)_p$ with $P_{V_1} = V_1 \rtimes SO(V_1)$, $u$ is a massless PUIR of $P_{V_1}$ that has partial conformal symmetry.*

*Proof.* By Lemma 6.3.2 2)$c$), $u$ extends to be a representation of $\tilde{P}_{V_1} = V_1 \rtimes (SO(V_1) \times \mathbb{Q}_p)$. Now, by Theorem 6.4.1, $u$ must be massless. $\square$

Lemma 6.5.2 shows that if one has a PUIR $U_0$ of a Poincaré group $P_{V_0}$ that extends to a PUIR of the partial conformal group $\tilde{P}_{V_0}$, then $U_0$ corresponds to a PUIR $U_1$ of a stabilizer $P_{V_1}$ of a massless $p_0 \in V_0'$. We note that $P_{V_1} = V_1 \rtimes SO(V_1)$, where $V_1$ is a quadratic vector space Witt that is equivalent to $V_0$ with $dim(V_0) = dim(V_1) + 2$. So $P_{V_1}$ is itself a Poincaré group. Now, $U_1$ has partial conformal symmetry and will correspond to a PUIR $U_2$ of the stablizer $P_{V_2}$ of some $p_1 \in V_1$. So it is clear that this process can be repeated until one reaches a stage $R$ where

$V_R$ is anisotropic. At the anisotropic stage, the only massless momentum vector in $V_R'$ is the trivial one. One may also end the process by picking the trivial null vector at any stage. We thus have a chain of Poincaré groups $P_{V_0}, P_{V_1}, \ldots$ and corresponding massless representations $U_0, U_1, \ldots$. From our discussion, we have the following theorem:

**Theorem 6.5.1.** *If $U$ is massless and has partial conformal symmetry, all the $U_\nu$ are massless.*

We say that $U$ is *eventually massive* if some $U_\nu$ is massive.

**Theorem 6.5.2.** *Eventually massive particles do not have conformal symmetry.*

Theorems 6.5.1 and 6.5.2 are immediate consequences of Theorem 6.4.1 and Lemma 6.5.2.

# CHAPTER 7

# Further Problems of Interest

## 7.1 Classification of the particles of the $p$-adic Poincaré group with full conformal symmetry

In this thesis, we showed that the massive and eventually massive particles do not have full conformal symmetry. We suspect that the massless particles may extend to have full conformal symmetry and we would like to find the sufficient condition for a particle to have full conformal symmetry. Classically, the projective representations of the real Poincaré group that extend to the real conformal group are the massless ones. If we can show that this holds for the $p$-adic Poincaré group and the $p$-adic conformal group, then it can be seen that this extendability is a more intrinsic property of the linear algebraic groups involved rather than the fields over which they are defined.

## 7.2 Galilean group

### 7.2.1 Galilean group over $\mathbb{R}$

Classically, the Galilean group is the group of translations, rotations, and boosts of spacetime consistent with Newtonian mechanics. Let $x = (x_1, x_2, x_3)$ denote the space coordinates and $t$ the time coordinate. We define spacetime as $V =$

$V_0 \oplus V_1$, and we write for $w \in V$, $w = (x,t)$. A Galilean transformation $g : (x,t) \mapsto (x',t')$ is defined by

$$g : x \mapsto Wx + tv + u$$

and

$$g : t \mapsto t + \nu.$$

Here, $W \in SO(3)$, $u$ and $v$ are vectors in 3-space, and $\nu$ is a real number. In this transformation, $u$ is a spacial translation, $\nu$ is a time translation and $v$ is a boost. We may think of $v$ as a velocity vector and $W$ a rotation in 3-space. The set of all such transformations form the Galilean group.

The Galilean group may also be defined for an arbitrary finite-dimensional vector space as a semidirect product $V \rtimes R$ of the group $V$ of all translations in spacetime and the group $R = V_0 \rtimes R_0$. Here, $R_0 = SO(V_0)$. The subgroup $R_0$ is not semisimple. This creates some subtle differences between the theory involving the Lorentz group and the theory involving the Galilean group [19] p. 283-284.

### 7.2.2 Galilean group over $\mathbb{Q}_p$

One may define an analogue of the Galilean group over $\mathbb{Q}_p$. Let $V$ be a finite-dimensional vector space over $\mathbb{Q}_p$ such that $V = V_0 \oplus V_1$, where $V_0$ is an isotropic vector space and $V_1$ has dimension 1. The Galilean group is now defined as $G = V \rtimes R$, where $R = V_0 \rtimes SO(V_0)$. Technically, one should think of this as a pseudo-Galilean group, since in the real case $V_0$ is anisotropic. As before, the action of $((u,\nu),(v,W)) \in G$ on $V$ is given by

$$((u,\nu),(v,W)) : (x,t) \mapsto (Wx + tv + u, t + \nu).$$

Let $(\cdot,\cdot)$ be a bilinear form on $V_0$. Given a pair $(\zeta, t) \in V$, we define a bilinear form $\langle(\zeta,t),\cdot\rangle$ on $V$ by $\langle(\zeta,t),(u,\nu)\rangle = (\zeta,u) + t\nu$. We identify the algebraic dual $V'$ with set of all such pairs $(\zeta,t)$. The action of $R$ on $V$ is given by:

$$(v,W) : (u,\nu) \mapsto (Wu + \nu v, \nu).$$

The action of $R$ on $V'$ is given by:

$$(v,W) : (\zeta,t) \mapsto (W\zeta, t - (W\zeta, v)).$$

We denote $\mathrm{Lie}(G) = \mathfrak{g}$. We further denote $\mathrm{Lie}(R) = \mathfrak{r}$ and $\mathrm{Lie}(R_0) = \mathfrak{r}_0$. Then $\mathfrak{r} = V_0 \oplus \mathfrak{r}_0$. The actions of $\mathfrak{r}$ on the Lie algebras are given by:

$$(v,W) : (u,\nu) \mapsto (Wu + \nu v, 0) \quad (\text{On } V)$$

$$(v,W) : (\zeta,t) \mapsto (W\zeta, -(\zeta,v)) \quad (\text{On } V).$$

### 7.2.3 Particle classification of the $p$-adic Galilean group

The study of particles of the real Galilean group corresponds to the study of particles of ordinary non-relativistic quantum mechanics. A natural question that arises is that of classifying particles of the $p$-adic Galilean group. This classification is given by the Theorem 4.4.2. It is noteworthy that in the presence of a nontrivial affine action, the theorem differs from the usual Mackey theorem.

### 7.2.4 $H^1(G, V')$ of the Galilean group

V. Varadarajan proved in [20] that the cohomology group $H^1$ is nontrivial for the $p$-adic Galilean group. Therefore, the affine action will play a role in the particle classification. For completeness we give the argument here.

Our goal is to compute the cohomology $H^1(R, V')$. As shown by Lemma 5.3.2, $\partial$ is a well defined mapping of $H^1(R, V')$ into $H^1(\mathfrak{r}, V')$. We begin by computing $H^1(\mathfrak{r}, V')$.

Let $\lambda : \mathfrak{r} \to V'$ be a cocycle. Since $\mathfrak{r}_0$ is semisimple, $H^1(\mathfrak{r}_0, V') = 0$. Thus, $\lambda$ vanishes on $\mathfrak{r}_0$. This implies that there exists some $w \in V'$ such that for all $Z \in \mathfrak{r}_0$, $\lambda(Z) = Zw$. By subtracting from $\lambda$ the coboundary $X \mapsto Xw$, we may assume that $\lambda$ vanishes on $\mathfrak{r}_0$. We write $\lambda(v) = (Lv, f(v))$, where $L : V_0 \to V_0$ and $f : V_0 \to V_1$. The map $(\zeta, t) \mapsto \zeta$ is an $\mathfrak{r}$-module homomorphism of $V'$ to $V_0'$. The module action in $V'$ is given by

$$(v, W) : (\zeta, t) \mapsto (W\zeta, -(\zeta, v)),$$

and the corresponding action on $V_0'$ is given by

$$(V, W) : \zeta \mapsto W\zeta.$$

**Claim:** $(v, Z) \mapsto Lv$ is a cocycle. This is clear, since $\lambda$ is a cocycle, and as said above, the mapping to $V_0'$ is an $\mathfrak{r}$ module homomorphism. This gives that $LZv = ZLv$ for all $Z \in \mathfrak{r}_0$ and $v \in V_0$. So $L$ has to be a constant on $V_0$. Write $L = cI$. The cocycle condition for $\lambda$ gives $\lambda(Zv) = (LZv, f(Zv)$. Hence,

$$\lambda(Z \cdot v) = \lambda((0, Z) \cdot (0, I)) = \lambda(Zv, 0) = \lambda(Z) + Z \cdot \lambda(v) = Z\lambda(v).$$

Hence, we have that $(LZv, f(Zv)) = Z \cdot \lambda(v) = Z \cdot (Lv, f(v)) = (0, Z) \cdot (cv, f(v))$. By the Lie algebra action on $V'$, this is $(cZv, -(cv, 0)) = (cZv, 0)$. Hence $f(Zv) = 0$ for all $Z \in \mathfrak{r}_0$. The $Zv$ span $V_0$ so $f$ must be identically 0. Hence, $\lambda(v, Z) = (cv, 0)$. One may now verify that if $c$ is taken to be 1, the resulting map is a cocycle. We write $\lambda_0(v, Z) = (v, 0)$. We claim that this map is not a coboundary. If it were, then there would exist a $(\zeta_0, t_0) \in V'$ such that for all $(v, Z) \in \mathfrak{r}_0$

$$\lambda_0((v, Z)) = (v, Z) \cdot (\zeta_0, t_0) = (v, 0).$$

But $(v, Z) \cdot (\zeta_0, t_0) = (Z\zeta_0, -(\zeta_0, v)) = (v, 0)$. Hence, $(Z\zeta_0, (\zeta_0, v)) = (v, 0)$ for all $v \in V_0$ This is not possible unless $\zeta_0 = 0$. But then $v$ must be 0. This is a contradiction and so $\lambda_0$ cannot be a coboundary. Thus, $\lambda_0$ is the basis for $H^1(\mathfrak{r}, V')$.

The next step is to find $H^1(R, V')$. We use the fact that $\partial$ is a homomorphism of $H^1(R, V')$ to $H^1(\mathfrak{r}, V')$. We first find a cocycle $\theta$ in $H^1(R, V')$ that maps to $\lambda$, and then we show that the kernel of $\partial$ is trivial.

We begin by investigating the cocycle $\theta$. We write $\theta : (v, W) \mapsto (v, f(v, W))$. We now use the cocycle condition $\theta((v, W) \cdot (v', W')) = \theta(v + Wv', WW') = (v, f(v, W)) + (v, W) \cdot (v', f(v', W'))$. Using the group action defined earlier we get

$$(v, f(v, W)) + (v, W) \cdot (v', f(v', W')) = (v, f(v, W)) + (Wv', f(v', W')) - (Wv', v)).$$

On the other hand, $\theta(v + Wv', WW') = (v + Wv', f(v + Wv', WW'))$. From the two above equations we see that $f(v + Wv', WW') = f(v, W) + f(v', W') -$

$(Wv', v)$. Let us take $W = W' = I$. Then

$$f(v + v', I) = f(v, I) + f(v', I) - (v', v).$$

A solution to this is $f(v, I) = -\frac{1}{2}(v, v)$. We also see from the above that $f(0, WW') = f(0, W) + f(0, W')$. So $f$ is a homomorphism from $R$ to $V'$. This implies that $f(0, W) = 0$. We have that $(v, W) = (v, I) \cdot (0, W)$, so using the cocycle condition we get

$$\theta((v, I) \cdot (0, W)) = \theta(v, W) = (v, f(v, I)) + (v, I) \cdot (0, f(0, W))$$

$$= (v, f(v, I)) + (0, -(0, v)) = (v, f(v, I)).$$

This implies that $f(v, W) = f(v, I) = -\frac{1}{2}(v, v)$. Set $\theta(v, W) = (v, -\frac{1}{2}(v, v))$. It is easy to verify that $\partial\theta = \lambda_0$. The coycle $\theta$ is not a coboundary since $\lambda_0$ is not a coboundary.

Since $\lambda_0$ spans $H^1(\mathfrak{r}, V')$, it only remains to show that the kernel of $\partial$ is trivial. It will then follow that $H^1(R, V')$ is one dimensional. By Theorem 5.3.1 and part *b*) of Lemma 5.3.1, this kernel is isomorphic to $\mathrm{Hom}_0(R^{\sim}, \mathbb{Q}_p)$, since $V'^R$ has dimension 1. Since $(R^{\sim}, R^{\sim}) = R^{\sim}$, Lemma 5.3.3 implies that $\mathrm{Hom}_0(R^{\sim}, \mathbb{Q}_p) = 0$.

### 7.2.5 Regular action and orbits

Theorem 4.4.2 requires us to find the little group $G_\chi$ that is a stabilizer of $\chi \in A^*$ under the affine action. One must also show that the affine action on $A^*$ is regular. One possible way to proceed is as in the proof of 5.4.2. One may be able to show that the action is regular using the theorem of Effros.

### 7.2.6 Multipliers of the $p$-adic Galilean group

To classify the particles of the $p$-adic Galilean group, one must also find the $m_0$-representations of the groups $G_\chi$. In [20], Varadarajan found the multipliers for the $p$-adic Galilean group. This will help in showing that the action is regular and also it will give some of the multipliers of the little groups by restriction. However, work remains to uncover all the multipliers of the little groups that are needed for the full particle classification.

## 7.3 Problems over finite fields

Another natural question is that of extending the ideas introduced in this thesis to the algebraic groups defined over other fields. For instance one might consider the Poincaré group defined over a finite field. The notion of particle classification is well defined as is that of extendability of the representations of the Poincaré group to the conformal group. In fact, we have made some progress in this matter.

# APPENDIX A

# Non-Archimedean number systems

## A.1 Introduction

In this appendix, we briefly discuss some basic concepts of non-Archimedean number systems. Specifically, we introduce the field of $p$-adic numbers. Details of the following discussion may be found in [27] p. 85-87, as well as in [14].

### A.1.1 Global fields

By a global field $k$ we mean either a finite extension of $\mathbb{Q}$ or a finite extension of the field $\mathbf{F}(T)$ of rational functions in the variable $T$ with coefficients in a finite field $\mathbf{F}$.

### A.1.2 Valuations

**Definition 40.** *Let $k$ be a global field. An absolute value on $k$ is a map $|\cdot| : k \to \mathbb{R}_{\geq 0}$ such that*

*(a)* $|x| = 0 \Leftrightarrow x = 0$

*(b)* $|xy| = |x||y|$ *for all* $x, y \in k$.

*(c) There is a positive constant $C$ such that $|x + y| \leq C\, max\{|x|, |y|\}$ for all $x, y \in k$.*

**Definition 41.** *If an absolute value satisfies the triangle inequality*

$$|x + y| \leq |x| + |y|,$$

*then the absolute value is called a valuation on $k$.*

When a valuation is dependent on some parameter $v$, we often write $|\cdot|_v$ for this valuation on $k$. There are two important classes of valuations: if the constant $C$ in the definition 40 $c$) can be taken to be 1, then the valuation is called *non-Archimedean*; if the valuation is not equivalent to a non-Archimedean evaluation it is called *Archimedean.*

We denote by $k_v$ the completion of $k$ with respect to $|\cdot|_v$.

### A.1.3 $\mathfrak{p}$-adic valuations

Let $R$ be a Dedekind ring and let $\mathfrak{p}$ be a nonzero prime ideal of $R$. For any nonzero $x \in R$, we may form the following factorization of the ideal $Rx$,

$$Rx = \prod_{\mathfrak{p} \text{ prime}} \mathfrak{p}^{v_{\mathfrak{p}}(x)}.$$

Here, $v_{\mathfrak{p}}(x)$ is the power of $\mathfrak{p}$ appearing in the factorization. Denote by $K$ the quotient field of $R$. If $x \in K$, then $Rx$ is a fractional ideal, and again we have a factorization of $Rx$ as above. Hence $v_{\mathfrak{p}}$ is defined for every nonzero element of $K$. We now localize at the prime $\mathfrak{p}$. We write $\mathfrak{p}$ for the ideal $\mathfrak{p}R_{\mathfrak{p}}$. Now, any $y \in K$ may be written

$$y = u\pi^{v_{\mathfrak{p}}(y)}$$

where $u$ is a unit and $\pi$ is the generator of $\mathfrak{p}$. It is easy to verify that $v_{\mathfrak{p}}$ satisfies the following:

(a) $v_{\mathfrak{p}}(y)$ is an integer for all $y \in K, \;\; y \neq 0$

(b) $v_{\mathfrak{p}}(xy) = v_{\mathfrak{p}}(x) + v_{\mathfrak{p}}(y)$

(c) $v_{\mathfrak{p}}(x+u) \geq \min \{v_{\mathfrak{p}}(x), v_{\mathfrak{p}}(y)\}$.

We also adopt the convention that $v_{\mathfrak{p}}(0) = +\infty$. A function $v_{\mathfrak{p}}$ satisfying the above conditions is called *an exponential valuation of* $K$. An exponential valuation yields an ordinary valuation by setting

$$|x|_v = c^{v(x)} \quad \text{for } c \text{ real and } 0 < c < 1.$$

One may show that $|\cdot|_v$ so defined, satisfies

$$|x+y|_v \leq \max \{|x|_v, |y|_v\}.$$

Hence, $|x|_v$ is a non-Archimedean valuation. When $K$ is an algebraic number field and $\mathfrak{p}$ is a prime ideal in the ring of algebraic integers of $K$, then the valuation $|x|_{v_{\mathfrak{p}}}$ is called the $\mathfrak{p}$-adic valuation on $K$. The most important example for us is when $K = \mathbb{Q}$ and $\mathfrak{p}$ is the ideal generated by some prime number $p$.

### A.1.4 Classification of local fields

A local field is a completion of a global field with respect to a valuation. Equivalently, a local field is a locally compact nondiscrete field.

Any field completed with respect to an Archimedian valuation is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$. Such a field is called an*Archimedean field.*

The $p$-adic numbers are constructed by starting with $\mathbb{Q}$ and a prime $p$. Then $\mathbb{Q}$ is completed with respect to the valuation $|\cdot|_p$.

Ostrowski's theorem shows that up to an equivalance there is only one non-Archimedean completion of $\mathbb{Q}$.

**Theorem A.1.1.** ***(Ostrowski)*** *Any nontrivial absolute value on $\mathbb{Q}$ is equivalent to either to the usual real absolute value or to the $p$-adic absolute value.*

## A.2 Some important properties of $p$-adic numbers

### A.2.1 $p$-adic expansion

Any $p$-adic number $r \in \mathbb{Q}_p$ can be expanded as a

$$r = \sum_{i=k}^{\infty} a_i p^i$$

where the $a_i$ are in the set $\{0, 1, .., p-1\}$. The $p$-adic numbers for which $a_i = 0$ for $i < 0$ are called the $p$-adic integers.

## A.3 Nontrivial additive character of $\mathbb{Q}_p$

Let $x$ in $\mathbb{Q}_p$. We define an additive character $\psi$ for $\mathbb{Q}_p$ in the following manner. We have a unique $p$-adic expansion of $x = \sum_{k=-r}^{\infty} a_k p^k$. We write $\sigma(x)$ for the part of the sum involving negative exponents. In other words, $x = \sigma(x) + x'$, where $x'$ is a $p$-adic integer and $\sigma(x) \in \mathbb{Z}[\frac{1}{p}]$. We now define the character to be $\psi(x) = e^{2\pi i \sigma(x)}$. It is trivial to verify that $\psi$ is an additive character and $\psi : \mathbb{Q}_p \to \mathbb{C}^{\times}$. Also $\psi : \mathbb{Z}_p \to \{1\}$.

# APPENDIX B

## Identification of topological and algebraic duals

The following discussion may be found in [28] p. 38-40. Let $G$ be a locally compact commutative group and $\chi$ a character of $G$. That is, $\chi$ is a continuous representation of $G$ into the unit modulus elements of $\mathbb{C}$. Let $G^*$ be the group of characters of $G$. One may topologize $G$ by assigning to it the topology of uniform convergence on compact sets of $G$. This makes $G^*$ into a locally compact commutative group. We call this the *topological dual* of $G$. For any $g \in G$, $g^* : \chi \mapsto \chi(g)$ is a character of $G^*$. The mapping $g \mapsto g^*$ is an isomorphism of $G$ with $(G^*)^*$.

Let $V$ be a finite-dimensional vector space over $k$. Then, by choosing a basis for $V$, we have $V \simeq k^{(dim(V))}$. $V$ is a locally compact group under addition and $V^*$ is defined in the obvious way. We may regard $V^*$ as a vector space over $k$ by the rule

$$a \cdot \chi(v) = \chi(av) \text{ for } v \in V, a \in k.$$

On the other hand, we can consider $V' = \mathrm{Hom}_k(V, k)$, the algebraic dual of $V$. We denote the value of linear form $v' \in V'$ on $v$ by $\langle v, v' \rangle_V$. We note that $V'$ has a structure of a $k$ vector space. If $\Psi$ is any character of the additive group of $k$, then for every $v' \in V'$, there is an element $v^*$ of the topological dual $V^*$ such that $v^*(v) = \Psi(\langle v, v' \rangle)$ for all $v \in V$.

**Theorem B.0.1.** *Let $k$ be a non-discrete locally compact field and $V$ be a vector* *space of finite dimension $n$ over $k$. Let $\Psi$ be a non trivial character of the additive group of $k$. Then the topological dual of $V^*$ is a vector space of dimension $n$ over $k$ and the formula*

$$v^*(v) = \Psi(\langle v, v' \rangle) \textit{ for all } v \in V$$

*defines a bijective mapping $v' \mapsto v^*$ of the algebraic dual $V'$ of $V$ onto $V^*$. This mapping is an isomorphism for the structures $V'$ and $V^*$ as vector spaces over $k$, and homeomorphism with respect to the topologies.*

Next, we show that the action of $GL(V)$ on $V^*$ goes over to the $GL(V)$ action on $V'$. We have the isomorphism

$$v' \mapsto \Psi(\langle v, v' \rangle) = \chi_{v'}(v)$$

mapping $V'$ to $V^*$. Let $g \in GL(V)$, then action of $g$ on $\chi$ is $(g \cdot \chi)(v) = \chi(g^{-1}v)$ on $V^*$. Action of $g$ on $V'$ is $\langle v, gv' \rangle = \langle g^{-1}v, v' \rangle$. Now, using the isomorphism, we see that $gv'$ corresponds to

$$\Psi(\langle v, gv' \rangle) = \Psi(\langle g^{-1}v, v' \rangle) = \chi(g^{-1}v) = (g \cdot \chi)(v).$$

Hence, the action of $GL(V)$ on $V^*$ goes over to the $GL(V)$ action on $V'$. Thus the action of $SO(V)$ on $V^*$ is the same as its action on $V$. From now on we will think of $V^*$ as $V$ and the characters of $V$ are points of $V$. We thus have the same quadratic form induced on $V^*$. For $SO(V)$ the characters are split to level sets according to the quadratic form.

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