Ex 7·74

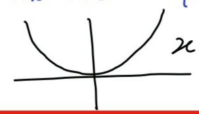
Show by Jensen's inequality that $E(X^2) \geq E(X)^2$.

Jensen's Inequality Let X be a random variable taking values in the (possibly finite) interval (a, b) such that $E(X)$ exists. Let $g:(a,b) \to \mathbb{R}$ be a convex function s.t. $E|g(X)| < \infty$. Then:

$$E(g(X)) \geq g(E(X)).$$

Let $g(x) = x^2$. To use Jensen's inequality, x^2 must be convex.

Convex function A continuous function whose value at the midpoint of every interval of its domain is $\leq$ the arithmetic mean of the end points!

e.g.  x^2.

A necessary + sufficient ($\Leftrightarrow$) condition for convexity is:

$$\frac{d^2 g(x)}{dx^2} \geq 0 \quad \forall\ x \in [a, b].$$

$g'(x) = 2x, \quad g''(x) = 2 \geq 0$! so x^2 is convex :)

Assuming $E(X)$ exists, and $E|g(X)| < \infty$, then by Jensen's Inequality, we have:

$$E(g(X)) \geq g(E(X))$$

$$\Rightarrow \quad E(X^2) \geq E(X)^2 \quad , \text{ as required.}$$

Ex 7.75 Harmonic mean (η) is defined:

$$\frac{1}{\eta} = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{x_i}, \qquad x_i \in \mathbb{R}.$$

Show that η is no greater than the geometric mean of x_i.

geometric mean: $\left(\prod_{i=1}^{n} x_i\right)^{1/n}$.

we need to show: $\rightarrow$ $\eta = \dfrac{1}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}} \leq \left(\prod_{i=1}^{n} x_i\right)^{1/n}$

OR we need to show:

$\hookrightarrow$ $$-\log\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}\right) \leq \log\left(\prod_{i=1}^{n} x_i\right)^{1/n} = \frac{1}{n}\sum_{i=1}^{n}\log x_i$$

OR

$\hookrightarrow$ $$\underbrace{\log \sum_{i=1}^{n} \frac{1}{x_i}\cdot\frac{1}{n}}_{} \geqslant \underbrace{-\sum_{i=1}^{n} \log(x_i)\cdot\frac{1}{n}}_{}. \quad (*)$$

looks very much like an expected value $\log E(X)$ for rand. var. X with $P(X = 1/x_i) = 1/n$

looks very much like an expected value $E(\log X)$ for a rand. var. $P(X = 1/x_i) = 1/n$. (since $-\log(x_i) = \log\left(\frac{1}{x_i}\right)$).

So, to prove $(*)$ maybe we can use Jensen's Inequality!

OK! we want to prove:

$$\log \sum_{i=1}^{n} \frac{1}{x_i}\cdot\frac{1}{n} \geqslant \sum_{i=1}^{n} \log\left(\frac{1}{x_i}\right)\cdot\frac{1}{n}.$$

By J.I., $E(-\log X) \geqslant -\log E(X)$ $\quad (\dagger)$

since $-\log x$ concave $\forall x > 0$.

let X be a rand. var. with $P(X = 1/x_i) = 1/n$, then:

$$(\dagger) \Rightarrow \quad \sum_{i=1}^{n} -\log\left(\frac{1}{x_i}\right)\frac{1}{n} \geqslant -\log \sum_{i=1}^{n} \frac{1}{x_i}\cdot\frac{1}{n} \; = -\log\left(\frac{1}{\eta}\right) = \log \eta .$$

But! $\displaystyle\sum_{i=1}^{n} -\log\left(\frac{1}{x_i}\right)\frac{1}{n} = \frac{1}{n}\sum_{i=1}^{n} \log(x_i) = \frac{1}{n}\log \prod_{i=1}^{n} x_i = \log\left(\left(\prod_{i=1}^{n} x_i\right)^{1/n}\right).$

So! $\displaystyle\log\left(\left(\prod_{i=1}^{n} x_i\right)^{1/n}\right) \geqslant \log \eta .$

$$\Rightarrow \quad \left(\prod_{i=1}^{n} x_i\right)^{1/n} \geqslant \eta ,$$ as was to be proved!

Example 7.79

If X has the exponential distribution with parameter λ, then:

$$\phi_X(t) = \int_0^\infty e^{itx} \lambda e^{-\lambda x}\,dx = \frac{\lambda}{\lambda - it} \quad , t \in \mathbb{R}.$$

Characteristic function of a random variable X is defined to be:

$$\boxed{\phi_X(t) = E\left(e^{itX}\right) \qquad \forall t \in \mathbb{R}.}$$

In the example, we are dealing with the exponential distribution given by: (p. 64)

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 - e^{-\lambda x} & \text{if } x > 0 \end{cases} \qquad \text{(distribution)}.$$

(p. 66)

$$\Rightarrow f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \lambda e^{\lambda x} & \text{if } x > 0 \end{cases} \qquad \text{(density function)}$$

$$\phi_X(t) = E(e^{itX}) = E(g(X)) \qquad g(x) = e^{itx}.$$

$$= \int_{-\infty}^{\infty} e^{itx} f(x)\,dx = \underline{\underline{\int_0^{\infty} e^{itx} \lambda e^{-\lambda x}\,dx.}}$$

lets evaluate this integral!!...

$$\lambda \int_0^{\infty} e^{itx} \cdot e^{-\lambda x}\,dx = \lambda \int_0^{\infty} e^{x(it-\lambda)}\,dx.$$

$$= \lambda \left[\frac{1}{it-\lambda} e^{x(it-\lambda)} \right]_0^{\infty}$$

$$= \lambda \lim_{x \to \infty} \frac{1}{it-\lambda} e^{x(it-\lambda)} - \lambda \lim_{x \to 0} \frac{1}{it-\lambda} \cancel{e^{x(it-\lambda)}}^{\,1}$$

$$= \frac{\lambda}{it-\lambda} \lim_{x \to \infty} \frac{e^{ixt}}{e^{x\lambda}} + \frac{\lambda}{\lambda - it}$$

$\underline{\text{But!}}$ $$\lim_{x \to \infty} \left| \frac{e^{ixt}}{e^{x\lambda}} \right| \leq \lim_{x \to \infty} \frac{1}{|e^{x\lambda}|} = \lim_{x \to \infty} \frac{1}{e^{x\lambda}} = 0.$$

$$\Rightarrow \phi_X(t) = 0 + \frac{\lambda}{\lambda - it} = \frac{\lambda}{\lambda - it}, \text{ as req'd.}$$

<u>Note!</u> As with all integrals you can use whatever tools you want to evaluate them

This approach seems the easiest to me.

But! you could split the integral into real and imaginary parts, or use the residue Calculus. (Complex Analysis).