

# Notes: Algebraic Groups

## SS 20, ETH Z

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# 1 Introduction

Let  $k$  be an algebraically closed field.

**Definition 1.** For  $I \subseteq k[X] := k[X_1, \dots, X_n]$ , we define its **vanishing set** by

$$V(I) := \{p \in k^n \mid \forall f \in I : f(p) = 0\}.$$

A set  $S \subset k^n$  is called **algebraic**, if

$$S = V(I)$$

for some  $I \subseteq k[X]$ .

**Example 1.** The group  $\mathrm{GL}_n(k)$  is not an algebraic subset of  $k^{n \times n}$ . But, we can identify it with an algebraic subset of  $(k^{n \times n})^2$  by

$$\mathrm{GL}_n(k) \cong \{(x, y) \in k^{n \times n} \mid xy = 1_n\} = V(X \cdot Y - 1_n).$$

**Definition 2.** Let  $\iota : \mathrm{GL}_n(k) \hookrightarrow k^{n \times n^2}$  be the injection

$$A \mapsto (A, A^{-1}).$$

A **linear algebraic group** over  $k$  is a subgroup  $U \subseteq \mathrm{GL}_n(k)$  s.t.  $\iota(k)$  is an algebraic subset of  $k^{2n^2}$ .

I.e., a linear algebraic group is a matrix-group which can be defined by polynomials over the entries of a matrix and its inverse.

**Example 2.** The following groups are linear algebraic groups:

1. The multiplicative group  $\mathcal{G}_m(k) := k^\times = k \setminus \{0\} = \mathrm{GL}_1(k)$ .
2. The general linear group  $\mathrm{GL}_n(k)$ .
3. The special linear group

$$\mathrm{SL}_n(k) := \{A \in \mathrm{GL}_n(k) \mid \det(A) = 1\}.$$

4. The orthogonal group

$$\mathcal{O}_n(k) := \{A \in \mathrm{GL}_n(k) \mid A^T \cdot A = 1\}.$$

5. The special orthogonal group

$$\mathrm{SO}_n(k) := \mathcal{O}_n(k) \cap \mathrm{SL}_n(k).$$

6. The upper triangle-matrix group

$$\left\{ \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ & \ddots & \vdots \\ & & a_{n,n} \end{pmatrix} \mid a_{i,j} \in k \right\} \cap \mathrm{GL}_n(k).$$

7. The normed upper triangle-matrix group

$$\left\{ \begin{pmatrix} 1 & \cdots & a_{1,n} \\ & \ddots & \vdots \\ & & 1 \end{pmatrix} \mid a_{i,j} \in k \right\} \cap \mathrm{GL}_n(k).$$

8. The group of  $n$ -th roots of unity

$$\mu_n(k) := \{x \in k \mid x^n = 1\}.$$

9. The additive group  $(k, +)$  is not a subgroup of  $\mathrm{GL}_n(k)$ , but it can be identified with the linear algebraic group

$$\left\{ \begin{pmatrix} 1 & a \\ & 1 \end{pmatrix} \mid a \in k \right\} \subset \mathrm{GL}_2(k)$$

10. For  $k = \mathbb{C}$ , the unit sphere and the unitary groups are NOT linear algebraic groups.

## 2 Algebraic Groups and Hopf Algebras

**Definition 3.** A **morphism**  $f : X \rightarrow Y$  of algebraic sets  $X \subset k^m, Y \subset k^n$  is a map which is coordinatewise described by polynomials.

**Definition 4.** An **algebraic group** is an algebraic set  $G \subset k^n$  together with a fixed element  $e \in G$  and morphisms  $m : G \times G \rightarrow G, i : G \rightarrow G$  s.t.  $(G, m, i, e)$  is a group.

A **morphism of algebraic groups** is a morphism of algebraic sets that is also a group homomorphism.

**Definition 5.** Let  $V \subset k^n$  be any subset. Then, we define the vanishing ideal of  $V$  by

$$I(V) := \{f \in k[x] \mid f(V) = 0\}.$$

**Definition 6.** For a commutative ring  $R$  we define the **radical** of an ideal  $I \subseteq R$  by

$$\sqrt{I} := \{r \in R \mid r^m \in I \text{ for some } m \in \mathbb{N}_0\}.$$

$R$  is called **reduced**, if  $\sqrt{0} = 0$ .

**Lemma 1** (Zariskis Lemma). *Let  $L \supseteq k$  be fields. If  $L$  is finitely generated as a  $k$ -algebra, then the extension  $L \supseteq k$  is finite, i.e.,  $L$  is a finitely-generated  $k$ -vector space.*

**Theorem 1** (Hilberts Nullstellensatz). *For any ideal  $I \subseteq k[x]$ , we have*

$$I(V(I)) = \sqrt{I}.$$

*Proof.* It is easy to see that

$$I \subset \sqrt{I} \subset I(V(I)).$$

Now, let  $f \in I(V(I))$  and assume – for the sake of contradiction – that  $f \notin \sqrt{I}$ . Since  $\sqrt{I}$  is the intersection of its upper prime ideals, there is a prime ideal  $p \supset I$ , s.t.  $f \notin p$ . Now, define the zero divisor-free ring

$$R := (k[x]/p)[f^{-1}].$$

And let  $\phi : k[x] \rightarrow R$  be the corresponding ring homomorphism.

Let  $m \subseteq R$  be a maximal ideal in  $R$ . Then,  $R/m$  is a field, which contains  $k$  and is finitely generated as  $k$ -algebra. According to Zariski's lemma,  $R/m$  is a finite (ergo algebraic) extension of  $k$ . Since  $k$  is algebraically closed, we have  $R/m = k$ . Let  $\pi_m : R \rightarrow k$  be the corresponding ring homomorphism.

Now, for  $x_1, \dots, x_n$ , set

$$t_i := \pi_m(\phi(x_i)).$$

Then,  $t = (t_1, \dots, t_n) \in k^n$ . We now have

1.  $t \in V(I)$ : For each  $g \in I$ , we have  $\phi(g) = 0$ . On the other hand

$$g(t) = g(\pi_m \circ \phi(x)) = \pi_m \circ \phi(g) = 0.$$

2.  $f(t) \neq 0$ :  $\phi(f)$  is invertible in  $R$ , therefore  $\phi(f) \neq 0$  and  $\phi(f) \notin m$ . Ergo

$$f(t) = \pi_m \circ \phi(f) \neq 0.$$

Ergo, there is a point  $t \in V(I)$  s.t.  $f(t) \neq 0$ . This yields a contradiction, since we assumed  $f \in I(V(I))$ .  $\square$

**Definition 7.** For an algebraic set  $X \subset k^n$ , we define its **coordinate ring** by

$$k[X] := k[x_1, \dots, x_n]/I(X).$$

**Lemma 2.** For a morphism  $f : X \rightarrow Y$  of algebraic sets define the following homomorphism of  $k$ -algebras.

$$\begin{aligned} f^* : k[Y] &\longrightarrow k[X] \\ p &\longmapsto p \circ f. \end{aligned}$$

We have a contravariant functor  $_*$  from the categories of algebraic sets over  $k$  to the category of  $k$ -algebras:

$$\begin{aligned} X &\longmapsto k[X] \\ \text{Hom}(X, Y) &\longmapsto \text{Hom}_k(k[Y], k[X]) \\ f &\longmapsto f^*. \end{aligned}$$

**Lemma 3.** We have

$$k[X \times Y] \cong k[X] \otimes k[Y].$$

*Proof.*

$$k[X] \otimes k[Y] = k[x]/I(X) \otimes_k k[y]/I(Y) = k[x, y]/I(X) \otimes k[y] + k[x] \otimes I(Y).$$

But

$$V(I(X) \otimes k[y] + k[x] \otimes I(Y)) = V(I(X) \otimes k[y]) \cap V(k[x] \otimes I(Y)) = X \times Y.$$

$\square$

**Theorem 2.** *Every finitely generated reduced  $k$ -algebra  $A$  is isomorphic to some  $k[X]$  for some algebraic  $X$ .*

*Proof.* Choose some  $\pi : k[x_1, \dots, x_n] \twoheadrightarrow A$  and set  $X := V(\ker \pi)$ . Then  $\ker \pi = I(X)$ , since  $\pi$ 's kernel is radical since  $A$  is reduced.  $\square$

**Corollary 1.** *The contravariant functor  $_* : \mathcal{C}_{\text{algSets}} \rightarrow \mathcal{C}_{k\text{-alg.s}}$  gives an antiequivalence of categories.*

**Lemma 4.** *An algebraic set  $X$  is isomorphic to some algebraic subset of  $Y$  iff there is an epimorphism  $k[Y] \twoheadrightarrow k[X]$ .*

**Lemma 5.** *Let  $G \subset k^n$  be an algebraic group. Then, we have maps*

$$\begin{aligned} m : G \times G &\longrightarrow G \\ i : G &\longrightarrow G \\ e : * &\longrightarrow G. \end{aligned}$$

*They induce dual maps in the category of  $k$ -algebras:*

$$\begin{aligned} \Delta &:= m^* : k[G] \longrightarrow k[G] \otimes_k k[G] \\ \iota &:= i^* : k[G] \longrightarrow k[G] \\ \varepsilon &:= e^* : k[G] \longrightarrow k \end{aligned}$$

**Definition 8.** A **Hopf-algebra** over  $k$  is a (reduced?!)  $k$ -algebra together with maps  $\Delta, \varepsilon, \iota$  as above s.t. the following holds:

$$\begin{aligned} (\Delta \otimes \text{Id})\Delta &= (\text{Id} \otimes \Delta)\Delta \\ s^* \circ (\iota \otimes \text{Id})\Delta &= s^* \circ (\text{Id} \otimes \iota)\Delta = \varepsilon \\ (\varepsilon \otimes \text{Id})\Delta &= (\text{Id} \otimes \varepsilon)\Delta = \text{Id} \end{aligned}$$

where  $s : G \rightarrow G \times G, g \mapsto (g, g)$  is the diagonal map.

A morphism of Hopf-algebras is a homomorphism of  $k$ -algebra  $F : A \rightarrow B$  s.t.

$$\Delta \circ F = (F \otimes F) \circ \Delta.$$

**Theorem 3.** *The contravariant functor  $_*$  gives an anti-equivalence of the categories of algebraic groups and the categories of finitely generated Hopf-algebras over  $k$ .*



**Example 3.** 1. Let  $G = \mathcal{G}_a = (k, +)$ . Then,  $k[G] = k[x]$ , since  $I(x) = 0$ . Then, we have

$$\begin{aligned}\Delta(x) &= x \otimes 1 + 1 \otimes x \\ \iota(x) &= -x \\ \varepsilon(x) &= 0.\end{aligned}$$

2. Let  $G = \mathcal{G}_m = \{(a, a^{-1}) \mid a \neq 0\} \cong k^\times$ . Then,  $k[G] = k[x, y]/(xy - 1) = k[x, x^{-1}]$ . Then, we have

$$\begin{aligned}\Delta(x) &= x \otimes x \\ \iota(x) &= x^{-1} \\ \varepsilon(x) &= 1.\end{aligned}$$

3. Let  $G = \mathrm{GL}_n(k)$ . Then,  $k[G] = k[x, y]/(xy - 1_n) = k[x_{i,j}, \frac{1}{\det}]$ . Then, we have

$$\begin{aligned}\Delta(x_{i,j}) &= \sum_k x_{i,k} \otimes x_{k,j} \\ \Delta\left(\frac{1}{\det(x)}\right) &= \frac{1}{\det(x)} \otimes \frac{1}{\det(x)} \\ \iota(x_{i,j}) &= (x^{-1})_{i,j} \\ \varepsilon(x_{i,j}) &= \delta_{i,j}.\end{aligned}$$

## 2.1 An Aside on the General Group

Let  $G = \mathrm{GL}_n(k) = \{(x, y) \mid xy = \mathrm{Id}_n\}$ . Since we have

$$x^{-1} = \frac{1}{\det(x)} \cdot \mathrm{adj}(x)$$

where the adjoint  $\mathrm{adj}(x)$  can be expressed by polynomials in the entries of  $x$ , we have isomorphisms

$$\begin{aligned}k[x, y]/(xy - 1_n) &\longrightarrow k[x, 1/\det(x)] = k[x, t]/(\det(x) \cdot t = 1) \\ (x, y) &\longmapsto (x, \det(y))\end{aligned}$$

and

$$\begin{aligned}k[x, 1/\det(x)] &\longrightarrow k[x, y]/(xy - 1_n) \\ (x, t) &\longmapsto (x, t \cdot \mathrm{adj}(x)).\end{aligned}$$

**Lemma 6.**

$$k[\mathbf{GL}_n(k)] \cong k[x_{i,j}, \frac{1}{\det(x)}].$$

**Lemma 7.** *Let  $V$  be a finite-dimensional  $k$ -vector space. If we choose a basis for  $V$ , we get an isomorphism  $\mathbf{GL}(V) \cong \mathbf{GL}_n(k) \subset k^{n \times n}$ . Hence,  $\mathbf{GL}(V)$  is an algebraic group whose structure is up to a unique isomorphism independent of the choice of basis.*

### 3 Actions

*Remark 1.* Let  $G \curvearrowright M$  be a group action of algebraic sets, then the morphism

$$G \times M \longrightarrow M$$

yields an homomorphism

$$\Delta : k[M] \rightarrow k[G] \otimes k[M].$$

This turns  $k[M]$  to a **comodule** of the Hopf-Algebra  $k[G]$ .

**Definition 9.** Let  $V$  be vector space and  $G$  an algebraic group. A morphism  $r_V : G \rightarrow \mathrm{GL}(V)$  of groups is called **representation** of  $G$ , if there is a linear map

$$\Delta : V \rightarrow V \otimes_k k[G](= \mathrm{Hom}_{alg}(G, V))$$

s.t. we have for each  $v \in V$  and  $g \in G$

$$r_V(g) \cdot v = \sum_i v_i \cdot f_i(g)$$

where  $\Delta v = \sum_i v_i \otimes f_i$ .

That is,  $V$  is a comodule for  $k[G]$ .

A map  $\phi : V \rightarrow W$  is called **equivariant** for two representations  $r_V, r_W$  of  $G$ , if

$$\phi(r_V(g)v) = r_W(g)\phi(v)$$

for all  $g, v$ .

**Example 4.** Let  $G = \mathrm{GL}_n(k)$ ,  $V = k^n$  and  $r_V$  be the canonical representation. For an orthonormal basis  $(b_i)_{i=1, \dots, n}$ , we for example can set

$$\Delta v = \sum_{i=1}^n b_i \otimes f_i$$

where

$$f_i(A) := b_i^T A v.$$

Then, we have

$$r_V(A) \cdot v = A \cdot v = \sum_{i=1}^n b_i \cdot b_i^T A v = \Delta(v)(A).$$

**Example 5.** Let  $M$  be a right  $G$ -set. Then,  $G$  also acts on  $k[M]$ , therefore we have a map

$$\rho : G \rightarrow \mathrm{GL}(k[M])$$

by, for  $v \in k[M]$ ,

$$(\rho(g)v)(m) := v(m.g).$$

Further, we have an algebra morphism

$$\Delta : k[M] \rightarrow k[M] \otimes k[G] = k[M \times G]$$

with

$$(\Delta v)(m, g) = v(m.g).$$

With  $\Delta v = \sum_i v_i \otimes f_i$

$$\rho(g)v(m) = v(mg) = \Delta v(m, g) = \sum_{i=1} f_i(g)v_i(m).$$

Ergo,  $\rho$  is a representation of  $G$ .

When  $M = G$  with action given by the right translation, then  $\rho : G \rightarrow \mathrm{GL}(k[G])$  is called the **right regular representation** of  $G$ .

**Lemma 8.** *Let  $G$  be an algebraic group and  $V$  a finite-dimensional  $k$ -vector space. Then  $\rho : G \rightarrow \mathrm{GL}(V)$  is morphism of algebraic groups iff it is a representation.*

**Definition 10.** Let  $G$  be an algebraic group and  $V$  a representation of  $G$ . A subspace  $W \subset V$  is called **invariant** or **subrepresentation**, if we have  $W.G = W$ .

**Lemma 9.** *The following are equivalent:*

1.  $W$  is invariant.
2.  $\Delta(W) \subseteq W \otimes k[G]$ .

**Lemma 10.** *Any representation  $V$  is a filtered union of its finite-dim. subrepresentations:*

1. Each  $v \in V$  is contained in some fin.-dim. subrep.
2. Any two finite-dim. subrep. are contained in some bigger fin.-dim. subrep.

**Theorem 4.** *Every algebraic group  $G$  is isomorphic to a linear algebraic group.*

*Proof.* Let  $\rho : G \rightarrow \mathrm{GL}(k[G])$  be the right regular representation.  $k[G]$  is a finitely-generated  $k$ -algebra. Then, there is a finite-dim. subrepresentation  $V \subseteq k[G]$  s.t.  $V$  generates  $k[G]$  as  $k$ -algebra. Then

$$\phi : G \longrightarrow \mathrm{GL}(V)$$

is morphism of algebraic groups.

Consider the dual map

$$\phi^* : k[\mathrm{GL}(V)] \rightarrow k[G].$$

We need to show that  $\phi^*$  is surjective. It is enough to show that  $V \subset \mathrm{Im} \phi^*$ . Define

$$\begin{aligned} l : V \subset k[G] &\longrightarrow k \\ f &\longmapsto f(e). \end{aligned}$$

Let  $f \in V$  and set  $a(g) := l(g \cdot f)$  for  $g \in \mathrm{GL}(V)$ . Then  $a \in k[\mathrm{GL}(V)]$  is regular. Further,

$$\phi^*(a)(g) = a(\rho(g)) = l(\rho(g)f) = f(eg) = f(g).$$

Therefore,  $f = \phi^*(a) \in \mathrm{Im}(\phi^*)$ . Since  $V$  generates  $k[G]$ , the surjectivity of  $\phi^*$  follows.  $\square$

**Theorem 5.** *Let  $H$  be an algebraic subgroup of an algebraic group  $G$ . There is a finite-dim. representation  $V$  of  $G$  and a line  $L \subset V$  s.t.  $H$  is the stabilizer in  $G$  of  $L$ , i.e.*

$$H = \{g \in G \mid L.g = L\}.$$

*Proof.* Let  $V$  be like in the previous proof. Consider

$$I \hookrightarrow k[G] \twoheadrightarrow k[H].$$

We can now set  $L' := V \cap I$ . We then have for  $g \in G$ .

$$L'.g \subseteq I \iff g \in H.$$

Now, in general  $L'$  is not of dimension one. Set  $d = \dim(L')$  and consider the one-dimensional subspace  $L := \Lambda^d(L') \subseteq \Lambda^d(V)$ .  $G$  acts on  $\Lambda^d(V)$  in the natural way.

It is clear, that  $H$  stabilizes  $L$ . For the other direction, let  $g \notin H$  and let  $e_1, \dots, e_n$  be a basis of  $V$  s.t.  $L' = \langle e_1, \dots, e_d \rangle$ . Then,

$$L = \langle e_1 \wedge \dots \wedge e_d \rangle$$

and, since  $g$  does not stabilize  $L'$ , w.l.o.g. we can assume  $e_1.g = e_{d+1}$ . Then, we have  $g(e_1 \wedge \dots \wedge e_d) = g(e_1) \wedge \dots \wedge g(e_d) =: v$ . Now,  $v$  cannot be zero and it cannot lie in  $L$  because  $e_1.g = e_{d+1}$ . Therefore,  $g \notin H$  does not stabilize  $L$ .  $\square$

**Theorem 6.** *Let  $H$  be a normal algebraic subgroup of an algebraic group  $G$ . Then, there is a finite-dimensional  $\rho : G \rightarrow \mathbf{GL}(V)$  s.t.  $H = \ker(\rho)$ .*

*Proof.* Let  $V, L$  and  $\phi : G \rightarrow \mathbf{GL}(V)$  be like in the preceding theorem. Set

$$V_H := \{v \in V \mid H.v \subset \langle v \rangle\}.$$

Then,  $V_H$  is  $G$ -invariant, since

$$h.(g.v) = (hg).v = (gh').v = g.(h'v) = g.(\kappa \cdot v) = \kappa \cdot g.v$$

for all  $g \in G, h \in H, v \in V_H$  and fitting  $h' \in H, \kappa \in k^\times$ . W.l.o.g. we have  $V = V_H$ .  $V$  is not trivial, because  $L \subset V$ .

Let  $\chi$  range through all homomorphism  $H \rightarrow k^\times$ , then we have

$$V = \bigotimes_{\chi} V_{\chi}$$

where

$$V_{\chi} = \{v \in V \mid h.v = \chi(h) \cdot v\}.$$

Then each  $g \in G$  permutes those eigenspaces by

$$g.V_{\chi} = V_{\chi(g^{-1} \cdot g)}.$$

Now, let  $W := \bigoplus_{\chi} \text{End}(V_{\chi}) \subset \text{End}(V)$ . For  $g \in G$  and  $\lambda \in \text{End}(V)$ , define

$$\begin{aligned} \tilde{\gamma} : G &\longrightarrow \mathbf{GL}(\text{End}(V)) \\ g &\longmapsto \tilde{\gamma}(g) : [\lambda \mapsto \phi(g) \circ \lambda \circ \phi(g)^{-1}]. \end{aligned}$$

The action  $\tilde{\gamma}(g)$  stabilizes  $W$ , since each  $\phi(g)$  just permutes the  $V_{\chi}$  and  $\phi(g)^{-1}$  permutes them back. Therefore, we have a subrepresentation

$$\gamma : G \rightarrow \mathbf{GL}(W).$$

We now have to show

$$\ker(\gamma) = H.$$

Since elements of  $H$  don't permute  $V_{\chi}$ , we have  $\gamma(H) = \text{Id}$ .

One the other side, let  $g \in G$  with  $\gamma(g) = \text{Id}$ . Then, we can choose the projection  $\pi : V \twoheadrightarrow L$  in  $W$  and get

$$\phi(g) \circ \pi = \pi \circ \phi(g).$$

Therefore,  $g$  leaves each  $L$  invariant. But now, we have  $g \in H$ . □

## 4 Connected Components

**Lemma 11.** *Let  $I_1, I_2, I_\lambda \subset k[x]$  be ideals, then*

$$\begin{aligned} V(I_1 \cap I_2) &= V(I_1) \cup V(I_2) \\ V\left(\bigcup_{\lambda} I_{\lambda}\right) &= \bigcap_{\lambda} V(I_{\lambda}). \end{aligned}$$

**Definition 11.** A topological space  $X$  is called **connected**, if any of the following equivalent condition holds:

- There is no pair of non-empty closed subsets  $Z_1, Z_2 \subseteq X$ , s.t.  $X = Z_1 \dot{\cup} Z_2$ .
- There is no pair of non-empty open closed subsets  $U_1, U_2 \subseteq X$ , s.t.  $X = U_1 \dot{\cup} U_2$ .

**Definition 12.** A topological space  $X$  is called **irreducibel**, if any of the following equivalent condition holds:

- There is no pair of proper closed subsets  $Z_1, Z_2 \subsetneq X$ , s.t.  $X = Z_1 \cup Z_2$ .
- For each pair  $U_1, U_2 \subseteq X$  of non-empty open subsets we have  $U_1 \cap U_2 \neq \emptyset$ .
- Each nonempty open subset of  $X$  is dense.

**Example 6.**  $V(xy)$  is connected but not irreducible.

**Recall:** Last time we introduced the **Zariski-Topology** on  $X$ .

There, algebraic sets equal closed sets.

We called a set  $X$  **irreducible** iff each open subset lies dense in  $X$ .

**Lemma 12.** *For an algebraic set  $X$ , the following are equivalent:*

- (1)  $X$  is irreducible.
- (2)  $k[X] = k[x_1, \dots, x_n]/I(X)$  is an (integral) domain.
- (3)  $I(X)$  is a prime ideal.

The proof of (2)  $\iff$  (3) is a basic algebraic result.

**Lemma 13.** *An open base for the Zariski-Topology on an algebraic set  $X$  is given by sets:*

$$D(f) := \{p \in X \mid f(p) \neq 0\}$$

for each  $f \in k[X]$ . We call the  $D(f)$  **basic open sets**.

*Proof.* Suppose  $U \subseteq X$  is nonempty and open. Set

$$Z := X \setminus U$$

then  $Z$  is closed. Thus

$$Z = \{x \in X \mid f(x) = 0 \ \forall f \in I\} = V(I)$$

for some ideal  $I \subseteq k[X]$ . Let  $p \in U$ , then there is an  $f \in I$  s.t.

$$f(p) \neq 0.$$

Also,  $D(f) \cap Z = \emptyset$ , thus  $p \in D(f) \subseteq U$ . □

*Proof: Lemma 1.* It is clear that (2) is equivalent to (3).

(1) is equivalent to

$$\forall \text{ nonempty, open } U_1, U_2 \subset X : U_1 \cap U_2 \neq \emptyset$$

$$\stackrel{\text{Lemma 2}}{\iff} \forall \text{ nonempty, basic open } D(f_1), D(f_2) \subset X : D(f_1) \cap D(f_2) \neq \emptyset$$

Since  $D(f_1) \cap D(f_2) = D(f_1 f_2)$ , this is equivalent to the statement

$$\forall f_1, f_2 \in k[X] : f_1, f_2 \neq 0 \implies f_1 f_2 \neq 0.$$

Which states that  $k[X]$  is a domain. □



**Lemma 14.** *Let  $X$  be an algebraic set. We have bijections*

$$\{\text{closed subsets } Z \subseteq X\} \leftrightarrow \{\text{radical ideals } I \subset k[X]\}$$

*and*

$$\{\text{irreducible, closed subsets } Z \subseteq X\} \leftrightarrow \{\text{prime ideals } I \subset k[X]\}$$

*and*

$$\{\text{points of } X\} \leftrightarrow \{\text{maximum ideals } I \subset k[X]\}.$$

**Lemma 15** (Primary Decompositions, Atiyah, Macdonald Ch. 4). *For an ideal  $I$  we call  $P \supseteq I$  a **minimal prime** of  $I$ , if  $P$  is a prime ideal and we have for each prime ideal  $Q$ :*

$$P \supseteq Q \supseteq I \implies P = Q.$$

*Any radical ideal  $I$  of  $k[x_1, \dots, x_n]$  has only finitely many **minimal** primes  $P_1, \dots, P_r$ . In particular,*

$$I = \bigcap_{i=1}^r P_i$$

*and for each  $i$*

$$P_i \not\supseteq \bigcap_{j:j \neq i} P_j.$$

**Definition 13.** An **(irreducible) component**  $Z$  of  $X$  is a maximal irreducible closed subset, i.e., an irreducible closed  $Z \subseteq X$  s.t. there does not exist an irreducible closed  $Y \subset X$  s.t.  $Y \supsetneq Z$ .

Then, we have the bijection

$$\{\text{irreducible components of } X\} \leftrightarrow \{\text{minimal primes of } I(X)\}.$$

**Lemma 16.** *Any algebraic set  $X$  has finitely many irreducible components  $Z_1, \dots, Z_r$ . We have*

$$X = Z_1 \cup \dots \cup Z_r$$

*and for each  $i$*

$$Z_i \not\subset \bigcup_{j:j \neq i} Z_j.$$

**Example 7.** 1. Let  $X = V(x \cdot y) \subset k^2$ . Then  $X = Z_1 \cup Z_2$  where  $Z_1 = V(x)$ ,  $Z_2 = V(y)$ .

$X$  is connected, but not irreducible ( $D(x)$  does not lie dense in  $X$ ).

2. Let  $X$  be a **finite** algebraic set. It is easy to check that every subset of  $X$  is closed:

$$\{p\} = V(x_1 - p_1, \dots, x_n - p_n)$$

for each  $p \in X$ . Further

$$X = \{p_1\} \cup \dots \cup \{p_r\}.$$

Moreover: Any function  $f : X \rightarrow k$  is regular (i.e. given by polynomials).

**Lemma 17.** *We call an element  $e \in k[X]$  **idempotent** iff  $e^2 = e$ .*

*Let  $X$  be an algebraic set. Then*

$$\begin{aligned} X \text{ connected} &\iff \text{the only idempotents } e \in k[X] \text{ are } 0 \text{ and } 1 \\ &\iff k[X] \not\cong A \times B \text{ for any } k\text{-algebras } A, B. \end{aligned}$$

**Lemma 18.** *Morphisms of algebraic sets are continuous.*

*Proof.* Let  $\phi : X \rightarrow Y$  be a morphism. It suffices to show that for all closed  $Z \subset Y$  that  $\phi^{-1}(Z) \subset X$  is closed.

But, if

$$Z = V_Y(S) := \{q \in Y \mid f(q) = 0 \ \forall f \in S\}$$

for some ideal  $S \subset k[Y]$ , then

$$\phi^{-1}(Z) = V_X(\phi^*(S)) = \{q \in X \mid f(\phi(q)) = 0 \ \forall f \in S\}.$$

□

**Lemma 19.** *Isomorphisms of algebraic sets are homeomorphisms. In particular, any isomorphism of algebraic sets  $\phi : X \rightarrow X$  permutes the irreducible components  $Z_1, \dots, Z_r$  of  $X$ :*

$$\forall i \exists j : \phi(Z_i) = Z_j.$$

**Theorem 7.** *Let  $G$  be an algebraic group.*

- (i) *There is a unique irreducible component  $G^0$  of  $G$  with  $e \in G^0$ .*
- (ii) *Every irreducible component  $Z$  of  $G$  is a coset  $gG^0$  of  $G$  for some  $g \in Z$ .*
- (iii)  *$G^0$  is a normal algebraic subgroup of  $G$ .*
- (iv)  *$G^0$  is of finite index, i.e.*

$$[G : G^0] = \#(G/G^0) < \infty.$$

(v) *The irreducible components are also the connected components.*

*Proof.* Let  $G = Z_1 \cup \dots \cup Z_r$  be the decomposition into components. We may assume that  $e \in Z_1$ .

Recall that  $Z_1 \not\subset \bigcup_{j \geq 2} Z_j$ . Then, there is an  $x \in Z_1 \setminus \bigcup_{j \geq 2} Z_j$ . Thus, for all algebraic set isomorphisms  $\phi : G \rightarrow G$ , we have by some previous lemma that  $\phi(x)$  is likewise contained in some unique component of  $G$ . For example, we may take  $\phi$  to be

$$\begin{aligned}\phi_g : G &\rightarrow G \\ y &\mapsto gy\end{aligned}$$

for any  $g \in G$ . Then, for all  $g \in G$ , the element  $gx = \phi_g(x)$  is contained in only one component of  $G$ . Ergo, each  $g \in G$  is contained in exactly one component.

(i) Take  $g = e$ .

(iii)  $G^0$  is an algebraic subset, by construction. Denote by  $m : G \times G \rightarrow G$  and  $i : G \rightarrow G$  the continuous multiplication and inversion map on  $G$ . **Why is  $G^0$  a subgroup?** We need to show

$$\begin{aligned}m(G^0 \times G^0) &\subseteq G^0. \\ i(G^0) &\subseteq G^0.\end{aligned}$$

We know that  $i(G^0)$  is some component of  $G$ , since  $i$  is an isomorphism. But it contains the identity  $e$ , since  $e^{-1} = e$ . Therefore,  $i(G^0) = G^0$ .

If  $g \in G$ , then  $gG^0$  is some component of  $G$ . Suppose  $g \in G^0$ . Then  $gG^0 \cap G^0 \supseteq \{g\}$ , therefore  $gG^0 = G^0$ . Ergo,  $G^0$  is closed under multiplication.

**Why is  $G^0$  a normal?** If  $g \in G$ , then  $gG^0g^{-1}$  is a component that contains  $e$ , therefore  $G^0 = gG^0g^{-1}$ .

(Alternative proof that  $m(G^0 \times G^0) = G^0$ : Consider

- any continuous image of an irreducible set is irreducible.
- the closure of any irreducible set is irreducible.

Ergo  $\overline{m(G^0 \times G^0)}$  is a closed irreducible set containing  $e$ . Ergo,  $\overline{m(G^0 \times G^0)} = G^0$ .

(ii) Let  $Z \subset G$  be a component. Let  $g \in Z$ . Then  $g \in (gG^0 \cap Z)$ , so  $gG^0 = Z$ .

- (iv) This follows from some previous lemma.
- (v) This is left as a topological exercise. It is true whenever the irreducible components do not intersect.

□

It now follows:

$$\{\text{finite algebraic groups}\} \longleftrightarrow \{\text{finite groups}\}$$

where the above arrow is an equivalence of categories.

**Example 8.** • Let  $G = \{g_1, \dots, g_r\}$  be a finite algebraic group. Then,

$$G^0 = \{e\}.$$

- Without proofs:

$$G \in \{\mathrm{GL}_n(k), \mathrm{SO}_n(k), \mathrm{SL}_n(k)\} \implies G^0 = G.$$

Further,

$$G = \mathrm{O}_n(k) \implies G^0 = \mathrm{SO}_n(k).$$

And if  $-1 = 1$  i.e.  $\mathrm{char} k = 2$ , then  $[G : G^0] = 1$ . Otherwise  $[G : G^0] = 2$ .

## 5 Jordan Decomposition

As usual,  $k = \bar{k}$  is an algebraically closed field.

**Definition 14.** Let  $V$  be a finite-dimensional vector space.

An element  $x \in \text{End}(V)$  is **semisimple**, if it is diagonalizable, i.e. it has a basis of eigenvectors, or equivalently, if the minimal polynomial of  $x$  is square-free.

Then, there is a decomposition  $V = \bigoplus_{i=1}^r V_i$  and distinct elements  $\lambda_1, \dots, \lambda_n \in k$  s.t.

$$x|_{V_i} = \lambda_i.$$

If  $\dim(V_i) = n_i$ , then

$$\text{characteristic polynomial of } x = \prod_{i=1}^r (T_i - \lambda_i)^{n_i} \in k[T]$$

and

$$\text{minimal polynomial of } x = \prod_{i=1}^r (T_i - \lambda_i) \in k[T].$$

(Where the minimal polynomial of  $x$  is defined as the least degree monic polynomial  $m \in k[T]$  s.t.  $m(x) = 0$ .)

*Remark 2.* Let  $m(T) \in k[T]$  be the minimal polynomial of  $x \in k^{n \times n}$ .

The theorem of Cayley and Hamilton states that we have for each  $p \in k[T]$ :

$$p(x) = 0 \implies m|p.$$

**Definition 15.**  $x \in \text{End}(V)$  is **nilpotent** if  $x^n = 0$  for some  $n$ .

$x$  is **unipotent**, if  $x - 1$  is nilpotent.

**Lemma 20.**  $x$  is nilpotent iff the characteristic polynomial of  $x$  is  $T^{\dim(V)}$ . (Use Cayley-Hamilton for one of the directions).

**Lemma 21.** If  $x$  is semisimple and nilpotent, then  $x = 0$ .

If  $x$  is semisimple and unipotent, then  $x = 1$ .

**Lemma 22.** If  $x, y$  are commuting elements, that are semisimple resp. unipotent resp. nilpotent, then so is  $xy$ .

*Proof.* It is easy to see, that this is true for nilpotent  $x, y$ .

Now, let  $x, y$  be unipotent and commuting. Then, we have

$$xy - 1 = (x + 1)(y - 1) + (x - y).$$

Since  $x, y$  commute,  $(x+1)(y-1)$  must be nilpotent.  $(x-y)$  must be nilpotent because the sum of commuting nilpotent elements must be nilpotent. Because everything commutes, also  $xy - 1$  as the sum of two commuting, nilpotent elements must be nilpotent.

Now, let  $A, B \in k^{n \times n}$  be two diagonalizable and commuting matrices. Let  $\lambda_1, \dots, \lambda_r$  be different eigenvalues of  $A$  and let  $E_i$  be the corresponding eigenspaces. We then have

$$A \cdot (BE_i) = BAE_i = \lambda_i \cdot BE_i.$$

Ergo, each  $E_i$  is invariant under  $B$ . Since  $B|_{E_i}$  stays diagonalizable, we can simply choose a basis of eigenvectors  $b_1, \dots, b_n \in \bigcup_i E_i$  of  $B$ . Since each  $b_i$  lies in a  $E_j$ , those vectors are also eigenvectors for  $A$ . Therefore,  $b_1, \dots, b_n$  is basis of eigenvectors for both matrices.  $\square$

**Theorem 8** (Goal). *For all algebraic groups  $G$  and for all  $g \in G$ , there exist unique group elements  $g_s, g_u \in G$  s.t.*

$$g = g_s g_u = g_u g_s$$

*and for all finite-dimensional representations  $\rho : G \rightarrow GL(V)$ ,  $\rho(g_s)$  is semisimple and  $\rho(g_u)$  is unipotent.*

**Example 9.** If  $g = \begin{pmatrix} \lambda & 1 & 0 \\ & \lambda & 1 \\ & & \lambda \end{pmatrix} \in G = GL_3(k)$ , then  $g_s = \begin{pmatrix} \lambda & 0 & 0 \\ & \lambda & 0 \\ & & \lambda \end{pmatrix}$ ,  $g_u = \begin{pmatrix} 1 & \lambda^{-1} & 0 \\ & 1 & \lambda^{-1} \\ & & 1 \end{pmatrix}$ .

**Theorem 9** (Goal Theorem). *Let  $G$  an algebraic group. For all  $g \in G$  there is exactly one pair  $g_s, g_u \in G$  s.t.*

$$g = g_s g_u = g_u g_s$$

*and for all finite-dimensional representations  $r : G \rightarrow GL_n(V)$ , the element  $r(g_s)$  resp.  $r(g_u)$  is semisimple resp. unipotent.*

Last time, we saw:

**Lemma 23.** • *If  $g, h$  are commuting and semisimple resp. commuting and unipotent then so is  $gh$ .*

• *If  $g$  is semisimple and unipotent, then  $g = 1$ .*

**Proposition 1.** *Let  $V$  be a finite-dimensional vector space and  $g \in GL(V)$ . There exist unique elements  $g_s, g_u \in GL(V)$  s.t.*

$$g = g_s g_u = g_u g_s$$

*and  $g_s$  is semisimple and  $g_u$  is unipotent.*

*Moreover,  $g_s, g_u \in k[g] = \{\sum_{i=1}^m a_i g^i \mid a_i \in k\} \subseteq \text{End}(V)$ .*

*Proof. Existence* (Sketch): Say

$$g = \begin{pmatrix} \lambda & 1 & 0 \\ & \lambda & 1 \\ & & \lambda \end{pmatrix}$$

then take

$$g_s = \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{pmatrix}, \quad g_u = \begin{pmatrix} 1 & \lambda^{-1} & 0 \\ & 1 & \lambda^{-1} \\ & & 1 \end{pmatrix}.$$

For  $\lambda \in k$ , define the **generalized  $\lambda$ -eigenspace** of  $g$  by

$$V_\lambda := \{v \in V \mid \exists n \in \mathbb{N}_0 : (g - \lambda)^n v = 0\}.$$

Then

$$V = \bigoplus_{\lambda \in k} V_\lambda.$$

Here  $V_\lambda$  = sum of domains of all Jordan blocks with  $\lambda$ s on the diagonal. (It follows from the Jordan Decomposition for matrices that such a decomposition exist.)

Let's define  $g_s \in \text{GL}(V)$  by

$$g_s|_{V_\lambda} = \lambda \cdot \text{Id}.$$

Note that  $gV_\lambda \subset V_\lambda$ , hence  $g$  commutes with  $g_s$ , hence  $g, g_s$  commutes with  $g_u := gg_s^{-1}$ . Then,  $g = g_s g_u = g_u g_s$ .

Write  $\det(T - g) = \prod_\lambda (T - \lambda)^{n(\lambda)}$ ,  $n(\lambda) = \dim(V_\lambda)$ . Since the polynomials  $T - \lambda$  for  $\lambda \in k$  are coprime, the chinese remainder theorem implies that there is a  $Q \in k[T]$  s.t.

$$Q \equiv \lambda \pmod{(T - \lambda)^{n(\lambda)}}$$

for each  $\lambda \in k$ .

We claim that

$$Q(g) = g_s.$$

Indeed, since  $gV_\lambda \subseteq V_\lambda$ , we have

$$Q(g)V_\lambda \subseteq V_\lambda.$$

So, it suffices to show for all  $v \in V_\lambda$

$$Q(g)v = g_s v = \lambda v.$$

Note that, by Cayley-Hamilton,

$$V_\lambda = \{v \in V \mid (g - \lambda)^{n(\lambda)} v = 0.\}$$

Write

$$Q = \lambda + R \cdot (T - \lambda)^{n(\lambda)}$$

for some  $R \in k[T]$ . Since  $(g - \lambda)^{n(\lambda)} v = 0$ , deduce that  $Q(g)v = \lambda v$ , as required.

If  $P \equiv \lambda^{-1} \pmod{(T - \lambda)^{n(\lambda)}}$ , then  $P(g) = g_s^{-1}$ .

Therefore,

$$g_u = g \cdot P(g)$$

for  $T \cdot P(T) \in k[T]$ .

**Uniqueness:** Suppose given some other decomposition

$$g = g'_s g'_u = g'_u g'_s$$

with  $g'_s$  semisimple and  $g'_u$  unipotent. Then  $g'_s$  commutes with  $g'_s$  and  $g'_u$ , hence with  $g$ , hence also with any element in  $k[g]$ . Ergo,  $g'_s$  commutes with  $g_s$  and  $g_u$ . Similarly,  $g'_u$  commutes with  $g_s$  and  $g_u$ .



Consider

$$h := g'_s g_s^{-1} = g'_s g'_u (g'_u)^{-1} g_s^{-1} = g(g'_u)^{-1} g_s^{-1} = g_u (g'_u)^{-1}.$$

Then  $h = g'_s g_s^{-1}$  is a product of semisimple elements and  $h = g_u (g'_u)^{-1}$  is a product of unipotent elements. By proceeding lemmas,  $h$  is semisimple and unipotent, ergo trivial. It follows  $g'_s = g_s$  and  $g'_u = g_u$ .  $\square$

**Corollary 2.** *Let  $g \in GL(V)$ , let  $W \subset V$  be any  $g$ -invariant subspace, i.e.  $gW \subseteq W$ .*

*Then,  $W$  is  $g_s$ -invariant and  $g_u$ -invariant.*

*Proof.* This is clear, since  $g_s$  and  $g_u$  are algebraically generated by  $g$  over  $g$ .  $\square$

**Lemma 24.** *Let  $\phi : V \rightarrow W$  be a linear map between finite-dimensional vector spaces.*

*Let  $\alpha \in GL(W)$  and  $\beta \in GL(W)$  s.t.*

$$\begin{array}{ccc} V & \xrightarrow{\alpha} & V \\ \downarrow \phi & & \downarrow \phi \\ W & \xrightarrow{\beta} & W, \end{array}$$

*i.e.  $\phi \circ \alpha = \beta \circ \phi$ .*

*Then,*

$$\begin{aligned} \phi \circ \alpha_s &= \beta_s \circ \phi, \\ \phi \circ \alpha_u &= \beta_u \circ \phi. \end{aligned}$$

*Proof.* Write  $V = \bigoplus_{\lambda \in k} V_\lambda$ ,  $W = \bigoplus_{\lambda \in k} W_\lambda$  where  $V_\lambda$  are the generalized  $\alpha$ -eigenspaces and  $W_\lambda$  are the generalized  $\beta$ -eigenspaces.

We claim that

$$\phi(V_\lambda) \subset W_\lambda.$$

Indeed, let  $v \in V_\lambda$ , then

$$(\beta - \lambda)^n \phi(v) = \phi((\alpha - \lambda)^n v) = 0.$$

Since  $(\alpha - \lambda)^n v = 0$ , the claim follows.

Since,  $\alpha_s|_{V_\lambda} = \lambda \text{Id}$  and  $\beta_s|_{W_\lambda} = \lambda \text{Id}$ , deduce that

$$\phi \circ \alpha_s = \beta_s \circ \phi.$$

Indeed, both sides are given on  $V_\lambda$  by  $\lambda \cdot \phi$ . Thus

$$\begin{aligned}\phi \circ \alpha_u &= \phi \circ \alpha \alpha_s^{-1} \\ &= \beta \beta_s^{-1} \circ \phi \\ &= \beta_u \circ \phi.\end{aligned}$$

□

**Lemma 25.** *Let  $\alpha \in GL(V)$ ,  $\beta \in GL(W)$ . Then the **tensor**  $\alpha \otimes \beta \in GL(V \otimes W)$  is defined by*

$$(\alpha \otimes \beta)(u \otimes v) = \alpha(u) \otimes \beta(v).$$

Then, we have

$$\begin{aligned}(\alpha \otimes \beta)_s &\stackrel{(1)}{=} \alpha_s \otimes \beta_s \\ (\alpha \otimes \beta)_u &\stackrel{(2)}{=} \alpha_u \otimes \beta_u.\end{aligned}$$

*Proof.* It suffices to prove (1), since

$$\begin{aligned}(\alpha \otimes \beta)_u &= (\alpha \otimes \beta) \circ (\alpha \otimes \beta)_s^{-1} \\ &\stackrel{(1)}{=} (\alpha \otimes \beta) \circ (\alpha_s \otimes \beta_s)^{-1} \\ &= \alpha \alpha_s^{-1} \otimes \beta \beta_s^{-1} \\ &= \alpha_u^{-1} \otimes \beta_u^{-1}\end{aligned}$$

For (1), consider

$$\begin{aligned}V &= \bigoplus_{\lambda \in k} V_\lambda, \\ W &= \bigoplus_{\lambda \in k} W_\lambda.\end{aligned}$$

It follows

$$V \otimes W = \bigoplus_{\lambda, \mu \in k} V_\lambda \otimes W_\mu.$$

Now,

$$\alpha_s \otimes \beta_s|_{V_\lambda \otimes W_\mu} = \lambda \mu \cdot \text{Id}.$$

Ergo,  $\alpha_s \otimes \beta_s$  is semisimple. By Proposition, we reduce to checking that  $\alpha_u \otimes \beta_u$  is unipotent. Indeed,

$$\alpha_u \otimes \beta_u - 1 = (\alpha_u - 1) \otimes (\beta_u - 1) + 1 \otimes (\beta_u - 1) + (\alpha_u - 1) \otimes 1$$

is nilpotent (You can also check that  $\alpha_u \otimes \beta_u = (\alpha_u \otimes 1) \circ (1 \otimes \beta_u)$  is unipotent.) □

**Example 10.** Let  $1 \in GL(V)$ . Then  $1_s = 1$  and  $1_u = 1$ .

**Summary** : Let  $G$  be an algebraic group. Let  $r_V : G \rightarrow \mathrm{GL}(V)$  be a finite-dimensional representation. Also, fix  $g \in G$ .

Let  $\lambda_V := r_V(g)_s$  (or  $r_V(g)_u$ ).

We get a family of operators  $\lambda_V \in \mathrm{End}(V)$  with the following properties:

- (i) if  $V = k$  and  $r_V(g') = 1$  for all  $g' \in G$ , then  $\lambda_V = 1$ .
- (ii) for any two representations in  $V$  and  $W$ , we have

$$\lambda_{V \otimes W} = \lambda_V \otimes \lambda_W.$$

- (iii) for all  $G$ -equivariant  $\phi : V \rightarrow W$  we have

$$\phi \circ \lambda_V = \lambda_W \circ \phi.$$

**Theorem 10.** *Let  $G$  be an algebraic group. Let  $\lambda_V \in \mathrm{End}(V)$  (i.e.  $V = (r_V, V)$  is a finite-dim. representation of  $G$ ) be a family of operations satisfying (i), (ii), (iii).*

*Then, there is exactly one  $g \in G$  s.t.  $\lambda_V = r_V(g)$  for all  $V$ .*

Note, that this theorem implies our goal theorem.

Applying the theorem to  $\lambda_V = r_V(g)_s$  implies

$$\exists_1 g_s \in G : r_V(g_s) = r_V(g)_s$$

and

$$\exists_1 g_u \in G : r_V(g_u) = r_V(g)_u.$$

*Proof of Goal Theorem.* There exist unique  $g_s, g_u \in G$  s.t.

$$g \stackrel{(*)}{=} g_u g_s = g_s g_u,$$

Then,  $r_V(g) = r_V(g_s)r_V(g_u) = r_V(g_u)r_V(g_s)$ .

Since  $r_V(g_u)$  is unipotent and  $r_V(g_s)$  is semisimple, it follows  $r_V(g_u) = r_V(g)_u$  and  $r_V(g_s) = r_V(g)_s$ .

To deduce  $(*)$ , take any  $r_V : G \hookrightarrow \mathrm{GL}(V)$ . We know for each  $V$

$$r_V(g) = r_V(g_s)r_V(g_u) = r_V(g_u)r_V(g_s). \quad \square$$

*Proof of Theorem 10.* We first extend the assignment

$$V \mapsto \lambda_V$$

to all<sup>1</sup> representations of  $G$ .

Say  $V = \bigcup_j W_j$  where each  $W_j$  is a finite-dimensional  $G$ -invariant subspace. Try to define  $\lambda_V \in \text{End}(V)$  by

$$\lambda_V|_{W_j} := \lambda_{W_j}.$$

For this to be well-defined, we need to show for each  $i, j$

$$\lambda_{W_i}|_{W_i \cap W_j} \stackrel{(*)}{=} \lambda_{W_j}|_{W_i \cap W_j}.$$

**Proof of (\*):** Apply assumption (iii) to the  $G$ -equivariant linear maps

$$\begin{aligned} W_i \cap W_j &\xrightarrow{\phi} W_i, \\ W_i \cap W_j &\xrightarrow{\phi'} W_j. \end{aligned}$$

Then,

$$\begin{aligned} \lambda_{W_i}|_{W_i \cap W_j} &= \lambda_{W_i} \circ \phi \\ &\stackrel{(iii)}{=} \phi \circ \lambda_{W_i \cap W_j} \\ &= \phi' \circ \lambda_{W_i \cap W_j} \end{aligned}$$

and

$$\lambda_{W_j}|_{W_i \cap W_j} = \lambda_{W_j} \circ \phi' = \phi' \circ \lambda_{W_i \cap W_j}.$$

Recall here that any finite-dimensional  $G$ -invariant  $W \subset V$  is a representation. ...

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<sup>1</sup>Not necessarily finite-dimensional, but may be written as a filtered union of finite-dimensional  $G$ -invariant subspaces of  $W$ .

Let  $G$  be an algebraic group.

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from  
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**Easy Exercise** : If  $V_1, V_2$  are representations  $r_1, r_2$  of  $G$ , then  $V_1 \otimes V_2$  is also a representation with

$$r = r_1 \otimes r_2 : G \rightarrow \mathbf{GL}(V_1 \otimes V_2)$$

given by

$$r(g)(v_1 \otimes v_2) = (r_1(g)v_1) \otimes (r_2(g)v_2).$$

*Proof.* Given  $\Delta_j : V_j \rightarrow V_j \otimes k[G]$ , define

$$\Delta : V_1 \otimes V_2 \longrightarrow V_1 \otimes V_2 \otimes k[G]$$

by: if

$$\Delta_1 u = \sum_i u_i \otimes f_i, \quad \Delta_2 v = \sum_j v_j \otimes h_j,$$

then

$$\Delta(u \otimes v) = \sum_i \sum_j u_i \otimes v_j \otimes f_i h_j.$$

Set  $A := k[G]$ , then

$$r_A := \text{right regular representation with } r_A(g)f(x) = f(xg).$$

The map

$$\begin{aligned} A \otimes A &\xrightarrow{m} A \\ f_1 \otimes f_2 &\longmapsto f_1 f_2 \end{aligned}$$

defines a morphism of representations

$$(A, r_A) \otimes (A, r_A) \rightarrow (A, r_A).$$

Indeed,

$$\begin{aligned} m((r_A \otimes r_A)(g)(f_1 \otimes f_2))(x) &= f_1(xg)f_2(xg), \\ &= f_1 f_2(xg) = r_A(g)(m(f_1 \otimes f_2))(x), \end{aligned}$$

since  $f_1(\_g) \otimes f_2(\_g) = (r_A \otimes r_A)(g)(f_1 \otimes f_2)$ .

Ergo  $m \circ (r_A \otimes r_A)(g) = r_A(g) \circ m$ . □

Recall: We wanted to prove the following theorem

**Theorem 11.** *Let  $\lambda_V \in \text{End}(V)$  be given s.t. for all finite-dim. rep.s  $V$  of  $G$  s.t.:*

$$(i) \lambda_k = 1$$

$$(ii) \lambda_{V \otimes W} = \lambda_V \otimes \lambda_W$$

(iii) *for all morphisms of rep.s  $\phi : V \rightarrow W$  we have*

$$\phi \circ \lambda_V = \lambda_W \circ \phi.$$

*Then, there is exactly one  $g \in G$  s.t.  $\lambda_V = r_V(g)$  for all  $V$ .*

*Proof.* Last time, we saw that any such family  $V \mapsto \lambda_V$  extends to **all** rep.s  $V$  of  $G$ .

Let's note also that, if  $(V_0, r_0)$  is any representation of  $G$  with trivial action, i.e.  $r(g) = 1$  for all  $g$ , then  $\lambda_{V_0} = 1$ . Indeed, let  $v \in V_0$ . We must check that  $\lambda_{V_0} v = v$ . Since the action is trivial, any subspace of  $V_0$  is  $G$ -invariant.

Consider the map

$$\begin{aligned} \phi : k &\longrightarrow V_0 \\ \alpha &\longmapsto \alpha v \end{aligned}$$

where  $v = \phi(1)$ . Then,  $\phi$  is a morphism of rep.s because the action is trivial.

Thus,

$$\lambda_V v = (\lambda_V \circ \phi)(1) \stackrel{(iii)}{=} (\phi \circ \lambda_k)(1) \stackrel{(i)}{=} \phi(1) = v.$$

Consider  $\lambda_A \in \text{End}(A)$ . Then,

$$\lambda_{A \otimes A} = \lambda_A \otimes \lambda_A.$$

It is an easy exercise to see that  $m : (A, r_A) \otimes (A, r_A) \rightarrow (A, r_A)$  is a morphism of rep.s.

By (iii) it follows,  $m \circ (\lambda_A \otimes \lambda_A) = \lambda_A \circ m$ , i.e.

$$\lambda_A(f_1 f_2) = \lambda_A(f_1) \lambda_A(f_2)$$

for all  $f_1, f_2 \in A$ . Thus,  $\lambda_A$  is an algebra morphism (check, using the morphism  $k \hookrightarrow A$ , that  $\lambda_A(1) = 1$ ).

Thus,  $\lambda_A = \phi^*$  for some unique morphism  $\phi$  of algebraic sets  $\phi : G \rightarrow G$ .

We claim that  $\phi$  commutes with left multiplication i.e.

$$\phi(hx) = h\phi(x)$$

for all  $h, x \in G$ . Indeed, let's consider the map

$$\begin{aligned} A &\longrightarrow A \\ f &\longmapsto f(h \cdot \_). \end{aligned}$$

This induces a morphism

$$(A, r_A) \xrightarrow{\psi} (A, r_A).$$

By (ii),  $\psi \circ \lambda_A = \lambda_A \circ \psi$ .

Since  $\lambda_A = \phi^*$ , this implies the claim.

Now, set  $g := \phi(e)$ . Then for all  $h \in G$ ,

$$\phi(h) = \phi(h e) = h g.$$

Thus,  $\lambda_A = \phi^* = r_A(g)$ .

(It remains to show that

$$\lambda_V = r_V(g)$$

for each finite-dim. rep.  $V$ .)

Let  $V = (V, r)$  be any rep. This induces a map

$$\Delta : V \longrightarrow V \otimes A.$$

If  $\Delta v = \sum v_i \otimes f_i$ , then

$$h v = \sum f_i(h_i) \otimes v_i.$$

Let

$$\begin{aligned} \varepsilon : V \otimes A &\longrightarrow V \\ v \otimes f &\longmapsto f(1)v. \end{aligned}$$

It follows  $\varepsilon \circ \Delta : V \rightarrow V$  is the identity map.

Let  $(V_0, r_0)$  be the representation of  $G$  with  $V_0 := V$  and  $r_0$  the trivial action. Then,  $\Delta : V \rightarrow V_0 \otimes A$  is a morphism of representations.

(Indeed, if  $\Delta v = \sum v_i \otimes f_i$ , then

$$\Delta(r(h)v) \stackrel{?}{=} (r_0(h) \otimes r_A(h))\Delta v$$

since

$$\begin{aligned}
\Delta v &= \sum v_i \otimes f_i \\
\iff xv &= \sum f_i(x_i)v_i \quad \forall x \in G \\
\iff xhv &= \sum f_i(xh)v_i \quad \forall x, h \in G.
\end{aligned}$$

Since  $r(h)v = hv$ , it follows

$$\Delta(hv) = \sum v_i \otimes f_i(\cdot h) \implies (?)$$

We want to show

$$\lambda_V = r_V(g).$$

We have

$$\begin{aligned}
\Delta \circ \lambda_V &\stackrel{(iii)}{=} \lambda_{V_0 \otimes A} \circ \Delta \\
&\stackrel{(ii)}{=} \lambda_{V_0} \otimes \lambda_A \\
&= 1 \otimes \lambda_A = 1 \otimes r_A(g).
\end{aligned}$$

This implies

$$\Delta \circ \lambda_V = (1 \otimes r_A(g)) \circ \Delta$$

but also

$$\Delta \circ r_V(g) = (1 \otimes r_A(g)) \circ \Delta.$$

Because of the injectivity of  $\Delta$  it now follows

$$\lambda_V = r_V(g). \quad \square$$

**Corollary 3.** *Let  $\phi : G \rightarrow H$  be any morphism of algebraic groups. Then, for all  $g \in G$*

$$\begin{aligned}
\phi(g)_s &= \phi(g_s) \\
\phi(g)_u &= \phi(g_u).
\end{aligned}$$

*Proof.* Let  $V$  be any **faithful** representation of  $H$ , i.e.  $r_V : H \rightarrow \mathbf{GL}(V)$  is injective, (for a finite-dim.  $V$ ).



Then,  $r_V \circ \phi$  is a rep. of  $G$ . To prove (i), it suffices to show

$$r_V(\phi(g)_s) = r_V(\phi(g_s))$$

since  $H$  operates faithfully on  $V$ .

We know that

$$r_V(\phi(g)_s) = r_V(\phi(g))_s$$

(characterizing property of  $h_s$  for  $h \in H$ ). On the other hand,

$$r_V(\phi(g_s)) = (r_V \circ \phi)(g_s) = r_V(\phi(g))_s.$$

Therefore, claim (i) follows. (ii) works analogously.  $\square$

**Definition 16.** Let  $g \in G$  where  $G$  is an algebraic group. We call  $g$  **semisimple**, if  $g = g_s$ .

We call  $g$  **unipotent**, if  $g = g_u$ .

**Lemma 26.** For  $g \in G$ , the following are equivalent:

- (i)  $g$  is semisimple.
- (ii)  $r_V(g)$  is semisimple for all finite-dim. rep.  $V$ .
- (iii)  $r_V(g)$  is semisimple for at least one faithful f.d. rep.  $V$  of  $G$ .

We get an analogous lemma for unipotent group elements.

*Proof.* We have

$$\begin{aligned}
(i) &\iff g = g_s \\
&\stackrel{\text{Def. of } g_s \text{ by goal thm.}}{\iff} r_V(g) = r_V(g)_s \forall \text{ f.d. } V \\
&\iff r_V(g) \text{ is semisimple} \\
&\iff (ii) \implies (iii).
\end{aligned}$$

On the other hand,

$$(iii) \implies \exists \text{ faithful f.d. } V \text{ s.t. } r_V(g) = r_V(g)_s = r_V(g_s) \implies g = g_s. \square$$

## 6 Non-Commutative Algebra

**Definition 17.** A ring  $R$  (for now) is unital, associative but not necessarily commutative.

**Example 11.** The ring of matrices over some field or ring.

**Definition 18.** A **left ideal**  $I \subset R$  is a subset that is an abelian subgroup of  $(R, +)$  s.t.  $ra \in I$  for all  $r \in R, a \in I$ .

A **right ideal**  $I \subset R$  is a subset that is an abelian subgroup with

$$IR \subset I.$$

A two-sided ideal  $I$  is a subset that is a left and a right ideal of  $R$ .

It is easy to check that for any homomorphism of rings  $\phi : R \rightarrow S$ ,  $\text{Kern}\phi$  is a two-sided ideal. Also, if  $J \subset R$  is any two-sided ideal, then there exists a unique ring structure on  $R/J$  s.t. the projection  $R \rightarrow R/J$  is a ring homomorphism.

**Definition 19.** A **left module**  $M$  for  $R$  is an abelian group equipped with a ring homomorphism

$$R \xrightarrow{\alpha} \text{End}(M)$$

where  $\text{End}(M)$  acts on the left of  $M$ . We write

$$rm := \alpha(r)m.$$

We have

$$(r_1 r_2)(m) = r_1(r_2(m)).$$

If  $R$  acts on  $M$  by the left, we write

$$R \curvearrowright M.$$

**Example 12.**  $M_n(k) \curvearrowright k^n$  where  $k^n$  is the space of column vectors.

If  $k^n$  denotes the space of row vectors, we have  $k^n \curvearrowleft M_n(k)$ .

**Definition 20.** A **(left) submodule**  $N \subset M$  is an algebraic subgroup s.t.

$$RN \subset N.$$

It follows that  $N$  itself is a left module.

**Definition 21.** A (left) module  $M$  of  $R$  is **simple** (or irreducible) if it has exactly the two submodules:  $0 = \{0\}$  and  $M$ .

**Definition 22.** A ring  $R$  is a **division ring** (aka **skew field**) if it satisfies any of the following equivalent requirements:

- (i)  $R^\times = R \setminus \{0\}$  where<sup>2</sup>  $R^\times = \{r \in R \mid \exists a, b \in R : ar = rb = 1.\}$
- (ii)  $R$  has no nontrivial left or right ideals.

**Definition 23.** If  $R \curvearrowright M$ , then we can define

$$\text{End}_R(M) := \{\phi \in \text{End}(M) \mid \phi(rm) = r\phi(m) \forall r \in R, m \in M\}.$$

Note, that  $\text{End}_R(M)$  is a ring.

**Lemma 27** (Schur's Lemma). *If  $M$  is simple, then  $\text{End}_R(M)$  is a division ring.*

**Lemma 28.** *Let  $k$  be a field. Then,  $M_n(k)$  has no nontrivial twosided ideals.*

**Theorem 12** (Jacobson Density Theorem (Double Commutant Theorem)). *Suppose  $M$  is a simple left module which is finitely generated as a right  $D$ -module for  $D = \text{End}_R(M)$ .*

*Assume that  $R$  acts faithfully on  $M$ , i.e.  $R \rightarrow \text{End}_R(M)$  is injective.*

*Then, the map  $R \rightarrow \text{End}_D(M)$  is an isomorphism.*

---

<sup>2</sup>If  $ar = rb = 1$ , then  $a = arb = b$ .

**Recap:**

- Basics: definitions, Hopf-algebras, ...
- Jordan decomposition
- Primer on non-commutative algebra
  - Jacobson density theorem
- Unipotent groups
- Tori

We had last week

$$\text{End}_D(M) := \{\phi \in \text{End}(M) \mid \phi \circ d = d \circ \phi \ \forall d \in D\}.$$

Let  $k$  be an algebraically closed field,  $V$  a non-trivial finite-dimensional  $k$ -vector space and let  $G$  be a subgroup of  $\text{GL}(V)$  that acts **irreducibly** on  $V$ , i.e.,  $V$  is  **$G$ -irreducible**, i.e., the only  $G$ -invariant subspaces of  $V$  are 0 and  $V$ .

Set

$$D := \{d \in \text{End}_k(V) \mid dg = gd \ \forall g \in G\}$$

and

$$R := \text{span}_k(G) = \left\{ \sum_{i=1}^n c_i g_i \mid c_i \in k, g_i \in G, n \in \mathbb{N}_0 \right\}.$$

Then,

$$D = \text{End}_R(V)$$

where  $R$  is the  $k$ -subalgebra of  $\text{End}(V)$  that is generated by  $G$ .

**Lemma 29** (Schur's Lemma). *We understand  $k \hookrightarrow \text{End}(V)$  as the inclusion of operations which operate by scalar multiplication*

$$k \xrightarrow{\cong} \{\phi : V \rightarrow V \mid \phi : v \mapsto t \cdot v \text{ for some } t \in k\}.$$

Let  $V$  be  $G$ -irreducible. Then, we have

$$D \cong k.$$

*Proof.* Let  $d \in D$ . Since  $V \neq 0$ , there is an eigenspace  $V_\lambda \neq 0$  for  $d$ . Observe that  $V_\lambda$  has to be  $G$ -invariant:  
if  $g \in G$  and  $v \in V_\lambda$ , then  $gv \in V_\lambda$ , since

$$dgv = gdv = g(\lambda v) = \lambda gv.$$

Since  $V_\lambda$  is a non-trivial  $G$ -invariant subspace and  $V$  is irreducible under  $G$ , we have

$$V_\lambda = V.$$

Ergo  $d = \lambda$  in the sense of  $k \hookrightarrow \text{End}(V)$ . □

**Consequence of the Jacobson Density Theorem:**  $R = \text{End}_k(V)$ , i.e.,  $G$  generates all linear operations on  $V$ , if  $V$  is  $G$ -irreducible.

We will prove this after a lemma.

**Lemma 30.** *Let  $V$  be  $G$ -irreducible.*

*Let  $n \in \mathbb{N}$ . Set*

$$V^n := V \oplus V \oplus \dots \oplus V = V_1 \oplus \dots \oplus V_n$$

*where each  $V_i = V$ .*

*Let  $v = (v_1, \dots, v_n) \in V^n$  and set*

$$Rv := \{(rv_1, \dots, rv_n) \mid r \in R\} = \text{span}\{(gv_1, \dots, gv_n) \mid g \in G\}.$$

*Then,  $Rv \neq V^n$  iff the  $v_j$  are linearly dependent over  $k$ .*

**Consequence:** Take  $n := \dim(V)$ . Let  $\{e_1, \dots, e_n\}$  be a basis of  $V$  and set

$$e := (e_1, \dots, e_n) \in V^n.$$

Since the  $(e_i)_i$  are linearly independent, the lemma states that  $Re = V^n$ .

Now, let  $x \in \text{End}_k(V)$ . Choose  $r \in R$  s.t.

$$re = (xe_1, \dots, xe_n).$$

Then  $re_i = xe_i$  for all  $i$ , thus  $x = r$ . Hence,  $R = \text{End}_k(V)$ .

*Proof.* For  $v = (v_1, \dots, v_n) \in V^n$  choose  $J \in \{1, \dots, n\}$  as large as possible with

$$Rv + V_1 + V_2 + \dots + V_{J-1} =: U \neq V^n.$$

Such an  $J$  does exist, since we know that  $Rv \neq V^n$ .

Then,  $V_J \not\subseteq U$ , otherwise we may increase  $J$ . Also,  $U$  is invariant by the diagonal action of  $G$  on  $V^n$ . Thus,  $V_J \cap U \subseteq V_J$  is a proper  $G$ -invariant subspace of the  $G$ -irreducible  $V_J \cong V$ . Therefore,  $V_J \cap U = 0$ .

On the other hand, by maximality of  $J$ , we have

$$U \oplus V_J = V^n.$$

Ergo, the map (composition)

$$V \cong V_J \hookrightarrow V^n \twoheadrightarrow V^n/U$$

is a  $G$ -equivariant isomorphism, since  $U$  is  $G$ -invariant.

Let  $z : V^n/U \xrightarrow{\cong} V$  be the inverse isomorphism. Let  $l$  be the  $G$ -equivariant map given by

$$\begin{array}{ccc} V^n & \xrightarrow{l} & V \\ \downarrow & \nearrow z & \\ V^n/U & & \end{array}$$

and let  $l_j$  be the  $G$ -equivariant maps by restricting  $l$  on  $V_j$ . Then  $l_j \in D \cong k$ .

Say  $l_j = t_j \in k$ . Then,

$$l(w) = t_1 w_1 + \dots + t_n w_n.$$

Since  $z$  is an isomorphism,  $l$  is nonzero and  $(t_1, \dots, t_n) \neq (0, \dots, 0)$ .

Since  $l|_U = 0$ , we can deduce for all  $u \in U$

$$t_1 u_1 + \dots + t_n u_n = 0.$$

But  $v \in Rv \subseteq U$ , so we may conclude – as required – that the  $(v_i)_i$  are linearly dependent ( $l(v) = 0$ ).  $\square$

## 7 Unipotent Groups

Let  $G$  be a subgroup of  $\mathrm{GL}(V)$  where  $V$  is a finite-dimensional vector space and  $k$  an algebraically closed field.

**Definition 24.** We say that  $G$  is **unipotent** if one of the following equivalent conditions hold for each  $g \in G$ :

- $g$  is unipotent (i.e.  $(g - 1)^n = 0$  for some  $n \in \mathbb{N}$ ).
- all eigenvalues of  $g$  are 1.
- $g$  is conjugate to  $\begin{pmatrix} 1 & \star & \star \\ 0 & \ddots & \star \\ 0 & 0 & 1 \end{pmatrix}$ .

**Theorem 13.** Any unipotent subgroup of  $\mathrm{GL}_n(k)$  is conjugate to a subgroup of

$$U_n := \left\{ \begin{pmatrix} 1 & \star & \star \\ 0 & \ddots & \star \\ 0 & 0 & 1 \end{pmatrix} \right\} = \left\{ U \in M_n(k) \mid U_{i,j} = \begin{cases} 0, & \text{if } i > j \\ 1, & \text{if } i = j \\ \text{arbitrary,} & \text{otherwise.} \end{cases} \right\}.$$

**Definition 25.** For two subgroups  $G, H$  of some common supergroup, define their **commutator** by

$$[G, H] := \langle ghg^{-1}h^{-1} \mid g \in G, h \in H \rangle.$$

A group  $G$  is called **nilpotent**, if one of its commutators is trivial, i.e. if we set

$$G_0 := G \text{ and } G_{i+1} := [G_i, G],$$

then  $G$  is called nilpotent iff there is an  $j \in \mathbb{N}$  with  $G_j = 1$ .

**Corollary 4.** Any unipotent subgroup of  $\mathrm{GL}(V)$  is nilpotent.

**Definition 26.** A group  $G$  is called **solvable**, if  $G^{(n)} = 1$  for some  $n$  where

$$\begin{aligned} G^{(0)} &:= G, \\ G^{(i+1)} &:= [G^{(i)}, G^{(i)}]. \end{aligned}$$

Note that nilpotent groups are solvable, since  $G^{(i)} \subset G_i$ .

**Notation 1.** In the following, we will write  $G' := [G, G]$ .

**Definition 27.** Let  $n := \dim(V)$ . A **complete flag** is a maximal strictly increasing chain of subspaces

$$0 = V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_n = V.$$

Any complete flag is of the form

$$V_j := \text{span}\{e_1, \dots, e_j\}$$

for some basis  $e_1, \dots, e_n$  of  $V$ .

Let  $B$  be the basis of some flag  $0 = V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_n = V$ . For  $x \in \text{End}(V)$ , we have that  $x$  is upper-triangular with respect to  $B$  iff  $x$  leaves each member  $V_i$  of the flag invariant, i.e.  $xV_i \subseteq V_i$ .

**Proposition 2** (Key Proposition). *Let  $G$  be a unipotent subgroup of  $GL(V)$ . Then there is a complete flag  $V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_n$  consisting of  $G$ -invariant subspaces, i.e., each  $V_i$  is  $G$ -invariant.*

*Proof.* Recall, that  $G$  is a unipotent subgroup of  $GL_n(V)$ . We will give an induction on  $n = \dim V$ .

If  $n = 0$ , there is nothing to show.

Let  $n \geq 1$ . We may assume that  $V$  is  $G$ -irreducible. Because, if not, there is a  $G$ -invariant subspace  $0 \neq W \subset V$  s.t.  $W$  and  $V/W$  have dimension  $< n$ . Then there exist complete  $G$ -invariant flags in  $W$  and  $V/W$  and the claim – that there is a complete  $G$ -invariant flag in  $V$  – follows by the induction hypothesis.

By Jacobson Density Theorem, we have

$$R := \text{span}(G) = \text{End}(V) := \text{End}_k(V).$$

Since  $G$  is unipotent, we have for each  $g \in G$

$$\text{trace}(g) = n.$$

Ergo, for  $g, h \in G$

$$\text{trace}(gh) = \text{trace}(h)$$

and

$$\text{trace}((g - 1)h) = \text{trace}(gh) - \text{trace}(h) = 0.$$

Since  $\text{span}(G) = \text{End}(V)$ , it now in particular follows for all  $g \in G, \phi \in \text{End}(V)$

$$\text{trace}((g - 1)\phi) = 0.$$



Since the above holds for all  $\phi \in \mathbf{End}(V)$ , it must hold

$$g - 1 = 0$$

for all  $g \in G$  (take for example the elementary matrices  $\phi = E_{i,j}$ ). Ergo,  $G$  is trivial. Then, any complete flag is trivially  $G$ -invariant.  $\square$

*Remark 3.* This gives the group analogue of Engel's Theorem.

*Proof Goal Theorem.* Let  $B$  be a basis of  $V$  s.t.  $G$  leaves each subspace in the corresponding flag invariant. Then,  $G$  is upper-triangle with respect to this basis.

On the other hand, each  $g \in G$  is unipotent, hence its diagonal elements are all 1. Thus, with respect to  $B$

$$G \subseteq \left\{ \begin{pmatrix} 1 & \star & \star \\ 0 & \ddots & \star \\ 0 & 0 & 1 \end{pmatrix} \right\} = U_n.$$

$\square$

*Remark 4.* Tori are of the form  $(k^\times)^n$ . In the case  $k = \mathbb{C}$ ,  $(\mathbb{C}^\times)^n$  are the complexifications of  $U(1)^n$ . This equals tori in the topological sense. Now,

$$\begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix} \subseteq \mathbf{GL}_2(\mathbb{C})$$

is a non-algebraic unipotent group.

**Exercise.** (To be discussed next time.)

It would have sufficed to prove the Goal theorem in the special case that  $G$  is algebraic.

**Corollary of Proof:** If  $G \subset \mathbf{GL}(V)$  (with  $V \neq 0$ ) is unipotent and  $V$  is  $G$ -irreducible, then  $G = 1$ ,  $\dim V = 1$ .

**Answer to last Exercise:** Recall that the main point was to show that any unipotent subgroup  $G \subseteq \mathrm{GL}(V)$  leaves invariant some complete flag  $\mathcal{F} = (V_0 \subset V_1 \dots)$ . But by some homework (problem 1), the group

$$\mathrm{GL}(V)_{\mathcal{F}} := \{g \in \mathrm{GL}(V) \mid g\mathcal{F} = \mathcal{F}\}$$

is algebraic.

**Proof:** If  $\mathcal{F}$  is the standard flag with  $V_i = \mathrm{span}(e_1, \dots, e_i)$  for the standard basis  $\{e_1, \dots, e_n\}$ , then

$$\mathrm{GL}(V)_{\mathcal{F}} = \{A \in \mathrm{GL}(V) \mid A \text{ is upper-triangle}\}.$$

The condition that  $A$  is upper triangle can be realized by polynomials. □

Thus,

$$\begin{aligned} & G \text{ fixes } \mathcal{F} \\ \iff & G \subseteq \mathrm{GL}(V)_{\mathcal{F}} \\ \overset{\mathrm{GL}(V)_{\mathcal{F}} \text{ is algebraic}}{\iff} & \overline{G} \subseteq \mathrm{GL}(V)_{\mathcal{F}} \\ \iff & \overline{G} \text{ fixes } \mathcal{F}. \end{aligned}$$

Now, the Zariski-Closure  $\overline{G}$  of any group  $G$  is an algebraic group (shown in some homework).

Further, if  $G$  is unipotent, then  $\overline{G}$  is unipotent.

## 8 Tori

**Definition 28.** A **torus** is an algebraic group that is isomorphic to  $\mathcal{G}_m^n$  for some  $n \in \mathbb{N}_0$  where  $\mathcal{G}_m = k^\times = \mathrm{GL}_1(k)$  is the unit group of  $k$ .

We think of  $\mathcal{G}_m^n \subseteq \mathrm{GL}_n(k)$  as the subgroup of diagonal matrices.

**Lemma 31.** *Let  $G$  be a commutative algebraic group. Then the following are equivalent:*

- (i) *each  $g \in G$  is semisimple.*
- (ii) *for each finite-dimensional representation  $V$  of  $G$  and for each  $g \in G$ , the operator  $r_V(g)$  is diagonalizable.*
- (iii) *for all finite-dimensional representations  $V$  of  $G$ , there is a basis of common eigenvectors for  $r_V(G)$ , i.e. a basis s.t.*

$$r_V(G) \subseteq \mathcal{G}_m^n.$$

- (iv)  *$G$  is isomorphic to an algebraic subgroup of a torus.*

*Proof.* We show:

- (i)  $\iff$  (ii): This follows from the Jordan decomposition and definition of semisimple.
- (ii)  $\implies$  (iii) : This is homework. Note that any commutative subset  $S$  of  $\mathrm{GL}(V)$  consisting of semisimple operators may be diagonalized simultaneously.
- (iii)  $\implies$  (iv) : Take any faithful representation  $V$  of  $G$  and diagonalize it simultaneously. Then,  $G \cong r_V(G) \subseteq \mathcal{G}_m^n$ .
- (iv)  $\implies$  (i) : Any diagonal matrix is semisimple.

□

**Definition 29.** A commutative algebraic group  $G$  is called **diagonalizable**, if it satisfies one of the above equivalent conditions.

**Definition 30.** A **character**  $\chi$  of an algebraic group  $G$  is an element  $\chi \in \mathrm{Hom}_{\mathrm{alg.grp.}}(G, k^\times)$ , i.e., a homomorphism  $\chi : G \rightarrow k^\times$  of algebraic groups.

**Notation 2.** For an algebraic group  $G$ , set  $\mathfrak{X}(G) := \mathrm{Hom}_{\mathrm{alg.grp.}}(G, k^\times)$ .

Also denote now by  $\mathcal{O}(X) := k[T]/I(X)$  the coordinate ring of an algebraic set  $X$  (rather than  $k[X]$ ).

**Lemma 32.** *There is a bijection*

$$\mathfrak{X}(G) = \{\text{characters } \chi \text{ of } G\} \longleftrightarrow \{x \in \mathcal{O}(G)^\times \mid \Delta(x) = x \otimes x\}.$$

*Proof.* Note, that any  $x \in \mathcal{O}(G)^\times$  can be thought of as a map  $x : G \rightarrow k^\times \subset k$ .

We have

$$\begin{aligned} \text{Hom}_{\text{alg.grp.}}(G, \mathcal{G}_m) &= \{\phi \in \text{Hom}_{\text{alg.sets}}(G, \mathcal{G}_m) \mid \phi(gh) = \phi(g)\phi(h) \ \forall g, h\} \\ &= \{\phi \in \text{Hom}_{k\text{-alg.}}(\mathcal{O}(\mathcal{G}_m), \mathcal{O}(G)) \mid (\phi \otimes \phi) \circ \Delta = \Delta \circ \phi\}. \end{aligned}$$

**Recall:**  $\mathcal{O}(\mathcal{G}_m) \cong k[t, \frac{1}{t}]$  with  $\Delta(t) = t \otimes t$ .

Thus for any  $k$ -algebra  $A$ ,  $\text{Hom}_{k\text{-alg.}}(\mathcal{O}(\mathcal{G}_m), A) \xrightarrow{A^\times} \cong$  via

$$[t \mapsto a, (t^{-1} \mapsto a^{-1})] \longleftrightarrow a.$$

Thus,

$$\text{Hom}_{\text{alg.grp.}}(G, \mathcal{G}_m) \cong \{a \in \mathcal{O}(G)^\times \mid a \otimes a = \Delta(a)\}.$$

Therefore, it suffices to test the condition  $(\phi \otimes \phi) \circ \Delta = \Delta \circ \phi$  on the generators  $t, t^{-1}$  of  $\mathcal{O}(\mathcal{G}_m)$ . Now, the above isomorphism is given by

$$\phi \mapsto a = \phi(t)$$

which is equivalent or regarding  $\chi : G \rightarrow \mathcal{G}_m$  as a map  $\chi : G \rightarrow k$ . □

**Example 13.** Let  $G = \mathcal{G}_m$ , then  $\mathcal{O}(G) = k[t, \frac{1}{t}]$ .

Which  $x = \sum_{m \in \mathbb{Z}} c_m t^m \in \mathcal{O}(G)$  – with almost all  $c_m = 0$ , but not all of them – have the property

$$\Delta(x) = x \otimes x?$$

We have

$$\begin{aligned} x \otimes x &= \sum_{m, n \in \mathbb{Z}} c_m c_n t^m \otimes t^n, \\ \Delta(x) &= \sum_{m \in \mathbb{Z}} c_m t^m \otimes t^m. \end{aligned}$$

Those sums equal, if

$$\begin{aligned} c_m c_n &= 0 \text{ for all } m \neq n, \\ c_m^2 &= c_m \text{ for all } m. \end{aligned}$$

By those conditions, it follows

$$x = t^m.$$

Therefore

$$\mathfrak{X}(G) = \{\chi_m \mid m \in \mathbb{Z}\} \cong \mathbb{Z}$$

with

$$\chi_m(y) = y^m.$$

**Example 14.** Let  $T \cong \mathcal{G}_m^n$  be a torus. Then,

$$\mathfrak{X}(T) = \{\chi_m \mid m \in \mathbb{Z}^n\} \cong \mathbb{Z}^n$$

where  $\chi_m(y) = y^m = y_1^{m_1} \cdots y_n^{m_n}$ .

**Note:** For each algebraic group  $G$ ,  $\mathfrak{X}(G)$  is naturally an abelian group:

$$(\chi_1 \cdot \chi_2)(g) := \chi_1(g) \cdot \chi_2(g).$$

Given a morphism of algebraic groups  $f : G \rightarrow H$ , we get a morphism of abelian groups

$$\begin{aligned} f^* : \mathfrak{X}(H) &\longrightarrow \mathfrak{X}(G) \\ \chi &\longmapsto \chi \circ f =: f^*(\chi). \end{aligned}$$

This induces a contravariant functor from the category of algebraic groups to the category of abelian groups.

**Lemma 33.** *Let  $G$  be a diagonalizable algebraic group. Then,  $\mathfrak{X}(G)$  is a  $k$ -vector space basis for  $\mathcal{O}(G)$ .*

**Example 15.** Let  $G = \mathcal{G}_m^n$  be a torus. Then, we have the embedding

$$\begin{aligned} \mathfrak{X}(G) &\hookrightarrow \mathcal{O}(G) \\ \chi_{(m_1, \dots, m_n)} &\longmapsto t^{(m_1, \dots, m_n)}. \end{aligned}$$

The lemma is obvious in this case: each element of  $\mathcal{O}(G) = k[t_1, \dots, t_n, t_1^{-1}, \dots, t_n^{-1}]$  can be written uniquely as a linear combination of monomials.

*Proof.* (i)  $\mathfrak{X}(G)$  spans  $\mathcal{O}(G)$ :

Choose an embedding  $G \subset \mathcal{G}_m^n$  of algebraic groups. Then, by restriction, we get

$$\mathcal{O}(\mathcal{G}_m^n) \twoheadrightarrow \mathcal{O}(G).$$

Since the  $\chi_m, m \in \mathbb{Z}^n$ , span  $\mathcal{O}(\mathcal{G}_m^n)$ , their images  $\chi_m|_G \in \mathfrak{X}(G)$  span  $\mathcal{O}(G)$ .

(ii)  $\mathfrak{X}(G)$  is linearly independent:

Suppose otherwise and let  $\phi_1, \dots, \phi_m$  be a linearly dependent subset of  $\mathfrak{X}(G)$  with  $m \geq 1$  chosen minimally, with  $c_1, \dots, c_m \in k^\times$  s.t.

$$\sum_{i=1}^m c_i \phi_i = 0.$$

We distinguish the following cases:

$m = 1$ : In this case, we have  $\phi_1 = 0$ , but  $\phi_1(1) = 1$ , a contradiction.

$m > 1$ : We can assume  $\phi_1 \neq \phi_2$ , so there is an  $h \in G$  s.t.  $\phi_1(h) \neq \phi_2(h)$ . Then,

$$\phi_1(h) \sum_{i=1}^m c_i \phi_i = 0,$$

but also for all  $h, g \in G$

$$\sum_{i=1}^m c_i \phi_i(hg) = \sum_{i=1}^m c_i \phi_i(h) \phi_i(g) = 0.$$

This implies

$$\sum_{i=1}^m c_i \phi_i(h) \phi = 0.$$

Ergo

$$\sum_{i=1}^m c_j (\phi_i(h) - \phi_1(h)) \phi_i = \sum_{i=2}^m c_j (\phi_i(h) - \phi_1(h)) \phi_i = 0.$$

Now,  $\phi_i(h) - \phi_1(h)$  is zero if  $i = 1$  and non-zero, if  $i = 2$ . Therefore, this yields a shorter linear dependency for the elements

$$\phi_2, \dots, \phi_m,$$

which contradicts our requirement. □

**Definition 31.** Let  $M$  be an abelian group. The **group algebra** on  $M$  is the  $k$ -algebra  $k[M]$  (not a coordinate ring!) defined as follows:

$$\begin{aligned} k[M] &:= \text{the } k\text{-vectorspace with basis } M \\ &:= \left\{ \sum_{m \in M} c_m \cdot m \mid c_m \in k, \text{ almost all } c_m = 0 \right\}, \end{aligned}$$

where the multiplication on  $k[M]$  extends that on  $M$ :

$$\left(\sum_{m \in M} c_m m\right) \left(\sum_{n \in M} d_n n\right) = \sum_{m, n \in M} c_m d_n mn.$$

**Corollary 5.** *For a diagonalizable  $G$ , we have*

$$\mathcal{O}(G) \cong k[\mathfrak{X}(G)].$$

**Fact:** For an abelian group  $M$ , there is exactly one Hopf algebra structure on  $k[M]$  given by  $\Delta(m) = m \otimes m$  for all  $m \in M$ .

With this definition, the above isomorphism is one of Hopf algebras.

**Lemma 34.** *If  $G, H$  are diagonalizable algebraic groups, then*

$$\mathrm{Hom}_{\mathrm{alg.grp.s}}(G, H) \xrightarrow{f \mapsto f^*} \mathrm{Hom}_{\mathrm{grp.s}}(\mathfrak{X}(H), \mathfrak{X}(G))$$

*is a bijection.*

*Proof.*

$$\begin{aligned} \mathrm{Hom}(G, H) &\cong \mathrm{Hom}_{\mathrm{Hopf-alg.}}(\mathcal{O}(H), \mathcal{O}(G)) \\ &\cong \{\phi \in \mathrm{Hom}_{k\text{-alg.}}(\mathcal{O}(H), \mathcal{O}(G)) \mid (\phi \otimes \phi) \circ \Delta = \Delta \circ \phi\}. \end{aligned}$$

Since  $\mathrm{Hom}_{k\text{-alg.}}(\mathcal{O}(H), \mathcal{O}(G)) \cong \mathrm{Hom}(k[\mathfrak{X}(H)], k[\mathfrak{X}(G)])$ , this reduces to the following lemma:

**Lemma 35.** *Let  $M_1, M_2$  be two abelian groups. Then*

$$\begin{aligned} \mathrm{Hom}(M_1, M_2) &\xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Hopf-alg.}}(k[M_1], k[M_2]) \\ \phi &\longmapsto \left[ \sum c_m m \mapsto \sum c_m \phi(m) \right]. \end{aligned}$$

*Proof.* We have to show that

$$M = \{x \in K[M]^\times \mid \Delta(x) = x \otimes x\}.$$

Then, by this, it follows for each  $\phi \in \mathrm{Hom}_{\mathrm{Hopf-alg.}}(k[M_1], k[M_2])$ ,

$$\phi(M_1) \subseteq M_2.$$

Ergo,  $\phi|_{M_1} \in \mathbf{Hom}(M_1, M_2)$ . Therefore, the surjectivity of the claimed bijection is shown. The injectivity is clear, since  $M$  generates  $k[M]$  as a  $k$ -algebra.

To show

$$M = \{x \in K[M]^\times \mid \Delta(x) = x \otimes x\},$$

let

$$\begin{aligned} x &= \sum c_m m \in K[M]^\times \\ \Delta(x) &= \sum c_m m \otimes m \\ x \otimes x &= \sum c_m c_n m \otimes n. \end{aligned}$$

If  $\Delta(x) = x \otimes x$ , then it follows

$$x = m$$

for some  $m \in M$ .

□

□



**Recall:** We have seen that for diagonalizable algebraic groups  $G, H$

$$\mathrm{Hom}(G, H) \cong \mathrm{Hom}(\mathfrak{X}(H), \mathfrak{X}(G)).$$

If  $G$  is diagonalizable, then

$$\mathcal{O}(G) \cong k[\mathfrak{X}(G)].$$

**Theorem 14.** *The functor*

$$\begin{aligned} G &\longrightarrow \mathfrak{X}(G) \\ f &\longmapsto f^* \end{aligned}$$

*defines an equivalence of categories:*

$$\{\text{diagonalizable alg. groups}\} \cong \{\text{finite-dim. abelian groups with no char}(k)\text{-torsion}\}.$$

This amounts to the bijection above between Hom-spaces and the following lemma.

**Lemma 36.** *(i) Let  $G$  be a diagonalizable alg. group. Then,  $\mathfrak{X}(G)$  is a finitely generated abelian group with no char}(k)\text{-torsion}.*

*(ii) Let  $\Gamma$  be a finitely generated abelian group with no char}(k)\text{-torsion}. Then, there is a diagonalizable algebraic group  $G$  s.t.  $\mathfrak{X}(G) \cong \Gamma$ .*

*Proof.* We will use the following facts:

- Let  $n \in \mathbb{N}$ . Then,  $t^n - 1$  is square-free in  $k[t]$  iff the ideal  $(t^n - 1)$  is radical in  $k[t]$  iff  $t^n - 1$  has not repetitive root iff either  $\mathrm{char}(k) = 0$  or  $\mathrm{char}(k) = p > 0$  and  $p \nmid n$ .

(Proof: Galois Theory, separable/inseparable extensions.)

- Let  $M := \mathbb{Z}/n\mathbb{Z}$ . Then, the  $k$ -group-algebra generated by  $M$

$$k[M] \cong k[t]/(t^n - 1)$$

is reduced iff either  $\mathrm{char}(k) = 0$  or  $\mathrm{char}(k) = p > 0, p \nmid n$ .

- If  $M_1, M_2$  are abelian groups, then we have the following isomorphism of Hopf algebras

$$\begin{aligned} k[M_1] \otimes_k k[M_2] &\xrightarrow{\cong} k[M_1 \oplus M_2] \\ m_1 \otimes m_2 &\longmapsto m_1 m_2 \end{aligned}$$

where  $M_1 \oplus M_2 \cong M_1 \times M_2$ .

- (i) Embed  $G \hookrightarrow T := \mathcal{G}_m^n$  for some  $n$ . Then, we have a surjection  $\mathbb{Z}^n \cong \mathfrak{X}(T) \twoheadrightarrow \mathfrak{X}(G)$ . Ergo,  $\mathfrak{X}(G)$  is finitely generated.

Suppose  $\text{char}(k) = p > 0$ . Let  $\chi \in \mathfrak{X}(G)$  with  $\chi^p = 1$ . Then, for all  $g \in G$ ,  $\chi^p(g) = \chi(g^p) = 1$ . The unit group  $k^\times$  has no  $p$ -torsion, therefore  $G \hookrightarrow T = (k^\times)^n$  has also no  $p$ -torsion. Therefore, the Frobenius  $g \mapsto g^p$  is an isomorphism on  $G$ . Therefore,  $\chi = 1$  is a trivial character. Ergo  $\mathfrak{X}(G)$  has no  $p$ -torsion.

- (ii) Let  $\Gamma$  be a finitely generated abelian group with no  $\text{char}(k)$ -torsion. Then,

$$\Gamma \cong \mathbb{Z}^r \oplus \mathbb{Z}/n_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/n_l\mathbb{Z}$$

where  $\text{char}(k) \nmid n_1, \dots, n_l$ . We may reduce to the cases:

- (a)  $\Gamma = \mathbb{Z}$ : take  $G = \mathcal{G}_m$ , then  $\Xi(G) \cong \mathbb{Z} \cong \Gamma$ .  
(b)  $\Gamma = \mathbb{Z}/n\mathbb{Z}$  with  $\text{char}(k) =: p \nmid n$ :  
take  $G := \mu_n := \{y \in k^\times \mid y^n = 1\}$ . Then, since  $p \nmid n$ ,  $(t^n - 1)$  is radical. So,

$$\mathcal{O}(\mu_n) \stackrel{\text{Nullstellensatz}}{=} k[t]/(t^n - 1) \stackrel{\text{as Hopf algebras}}{\cong} k[\Gamma]$$

where  $t$  gets mapped to the generator of  $\Gamma$ .

□

**Corollary 6.** *We have the bijection*

$$\{\text{tori}\} \cong \{\text{finitely generated free abelian groups}(\cong \mathbb{Z}^n)\}.$$

*Remark 5.*

$$\{\text{algebraic group schemes}/k\} \stackrel{\text{not necessarily natural}}{\cong} \{\text{f.g. Hopf algebras}\}.$$

by

$$G \mapsto \mathcal{O}(G)$$

and

$$\{\text{diagonalizable algebraic group schemes}/k\} \cong \{\text{f.g. abelian groups}\}.$$

by

$$G \mapsto \mathfrak{X}(G).$$

## 9 Trigonalization

We say a representation  $r : G \rightarrow \mathrm{GL}(V)$  of a group  $G$  on a finite-dimensional  $k$ -vectorspace  $V$  is **trigonalizable** if it admits a basis with respect to which  $r(V)$  is upper-triangular:

$$r(G) \subseteq \left\{ \begin{pmatrix} * & \dots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & * \end{pmatrix} \right\}$$

**Definition 32.** We call a subgroup  $G \subseteq \mathrm{GL}(V)$  **trigonalizable**, if the identity representation is.

**Lemma 37.** *Let  $G$  be an algebraic group. The following are equivalent:*

- (i) *Every finite-dimensional representation  $r : G \rightarrow \mathrm{GL}(V)$  is trigonalizable.*
- (ii) *Every irreducible representation of  $G$  is 1-dimensional.*
- (iii)  *$G$  is isomorphic to an algebraic subgroup of*

$$B_n := \left\{ \begin{pmatrix} * & \dots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & * \end{pmatrix} \right\} \subseteq \mathrm{GL}_n(k).$$

- (iv) *There is a normal unipotent algebraic subgroup  $U$  of  $G$  s.t.  $G/U$  is diagonalizable.*

*Proof.* We prove as follows:

- (i)  $\implies$  (ii): Let  $V$  be an irreducible representation. Then,  $V \neq 0$ . Choose a basis  $e_1, \dots, e_n$  of  $V$  s.t.

$$r(G) \subseteq B_n.$$

Then,  $r(G)e_1 \subseteq ke_1$ , so  $V_0 := ke_1$  is  $G$ -invariant. Ergo  $V = V_0$  is 1-dimensional.

- (ii)  $\implies$  (i): Let  $V$  be a f.d. representation. We show by induction on  $\dim(V)$  that  $r : G \rightarrow \mathrm{GL}(V)$  is trigonalizable:

In the cases  $\dim(V) = 0, 1$ , there is nothing to show.

In the case  $\dim(V) \geq 2$ , assume that  $V$  is not irreducible. Then, there is a  $G$ -invariant  $V_0$  with  $0 \neq V_0 \neq V$ .

By the induction hypothesis,  $V_0$  and  $V/V_0$  are trigonalizable. Ergo,  $V$  is trigonalizable.

(**Recall:** we used this criterion above in the proof that unipotent groups are trigonalizable by showing that every ??? of each  $G$  is trivial.)

(i)  $\implies$  (iii): Choose a faithful representation  $V$  of  $G$ . Then,  $G \cong r(G)$ . Since  $r$  is trigonalizable, there is a basis of  $V$  s.t.

$$r(G) \subseteq B_n \subseteq \mathrm{GL}_n(k).$$

(iii)  $\implies$  (ii): Suppose  $G \subseteq B_n \subseteq \mathrm{GL}_n(k)$ . Set

$$A_n := \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & * \end{pmatrix} \right\} \subseteq \mathrm{GL}_n(k),$$

$$U_n := \left\{ \begin{pmatrix} 1 & \dots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & 1 \end{pmatrix} \right\} \subseteq \mathrm{GL}_n(k) \text{ normal algebraic subgroup of } B_n,$$

$$U := G \cap U_n \text{ normal unipotent algebraic subgroup of } G.$$

Let  $V$  be an irreducible representation of  $G$ , then  $V$  is not zero. Consider the subspace of  $V$  fixed by  $U$

$$V^U := \{v \in V \mid r(u)v = v \forall u \in U\}.$$

Then, we get a representation

$$r|_U : U \longrightarrow \mathrm{GL}(V).$$

Then,  $r(U)$  is a unipotent algebraic group of  $\mathrm{GL}(V)$ . Ergo,

$$r(U) \subseteq \left\{ \begin{pmatrix} 1 & \dots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Ergo,  $V^U \neq 0$ . Since  $U$  is normal in  $G$ , the subspace  $V^U$  of  $V$  is  $G$ -invariant: if  $v \in V^U, g \in G$ , then for all  $u \in U$  we have

$$r(u)r(g)v = r(g)r(g^{-1}ug)v = r(g)v$$

since  $v \in V^U$ . Ergo  $r(g)v \in V^U$ .

Since  $V$  is irreducible,  $V = V^U$ , i.e.  $U$  acts trivially on  $V$ . Ergo,  $r$  descends to a representation of the group  $G/U$ .

But  $G/U \hookrightarrow B_n/U_n \cong A_n$ . Therefore,  $G/U$  and  $r(G)$  are commutative. Moreover, for all  $g \in G$ ,  $r(g) \in \text{GL}(V)$  is semisimple:

if  $g = g_s g_u$ , then  $g_u \in U$ , because  $U_n$  is the group of unipotent elements of  $B_n$ .

Hence,  $r(g) = r(g_s)r(g_u) = r(g_s)$  is semisimple.

It follows that  $r(G)$  is commutative and consists of semisimple elements. By some HW:  $r(G)$  is diagonalizable. It is easy to show now that  $V$  is one-dimensional. (Since  $V$  is irreducible and  $ke_1$  is  $G$ -invariant.)

□

**Definition 33.**  $G$  is **trigonalizable**, if it satisfies one of the above equivalent conditions.

Later, we will see, that if  $G$  is connected, then being trigonalizable implies being solvable.

## 10 Commutative Groups

Let  $G$  be an algebraic group. Denote by  $G_s$  resp.  $G_u$  the subsets of semisimple resp. unipotent elements of  $G$ .

Then,  $G_u$  is always algebraic i.e. closed: if  $G \hookrightarrow \mathrm{GL}_n(k)$ , then  $G_u = \{g \mid (g - 1)^n = 0\}$ .  $G_u$  does not need to be closed under multiplication (for example, take  $G = \mathrm{SL}_2(k)$ ,  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  and  $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ ).

$G_s$  needs not to be algebraic: for example, take  $G = \mathrm{SL}_2(k)$  and if  $G_s$  were algebraic, then

$$\left\{ \lambda \in k^\times \mid \begin{pmatrix} \lambda & 1 \\ 0 & \lambda^{-1} \end{pmatrix} \in G_s \right\} = \{ \lambda \mid \lambda \neq \lambda^{-1} \}$$

but the last set is not algebraic. Also,  $G_s$  does not need to be a group.

We have the surjective map of sets

$$\begin{aligned} G_s \times G_u &\longrightarrow G \\ (g_1, g_2) &\longmapsto g_1 g_2. \end{aligned}$$

**Example 16** (Non-Example). Take generic  $g \in G_s, h \in G_u$  for  $G = \mathrm{SL}_2(k)$ . Then,  $g, h$  do not commute and we have

$$((gh)_s, (gh)_u) \neq (g, h)$$

because Jordan components don't commute.

**Theorem 15.** *Let  $G$  be a commutative algebraic group. Then:*

- (i)  $G_s, G_u$  are closed subgroups and the multiplicative map  $G_s \times G_u \rightarrow G$  is an isomorphism of algebraic groups.
- (ii)  $G$  is trigonalizable. Moreover, for each finite dimensional representation  $r : G \rightarrow \mathrm{GL}(V)$  there is a basis s.t.

$$r(G_s) \subseteq \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & * \end{pmatrix} \right\} \quad r(G_u) \subseteq \left\{ \begin{pmatrix} 1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

- (iii)  $G_s$  is diagonalizable.

*Proof.* (ii) Let  $V$  be any irreducible representation of  $G$ . We have seen that commuting semisimple operators may be simultaneously diagonalizable, then

$$V = \bigoplus_{\chi: G_s \rightarrow \mathcal{G}_m} V_\chi$$

where

$$V_\chi = \{v \in V \mid r(h)v = \chi(h)v \ \forall h \in G_s\}.$$

Since  $G$  is commutative, each subspace  $V_\chi$  is  $G$ -invariant ( $r(h)r(g)v = r(g)r(h)v = r(g)\chi(h)v = \chi(h)r(g)v$ ).

Since  $V$  is irreducible, we must have  $V = V_\chi$  for some  $\chi$ .

Recall that  $G \cong G_s \times G_u$  as abstract groups. We have seen that  $r(G_s) \subseteq \mathcal{G}_m^n$ . We proved a while ago that any unipotent group, such as  $G_u$ , is trigonalizable. Ergo,  $V$  is trigonalizable. Since  $V$  is irreducible, we have  $\dim V = 1$ .



If we apply the same argument without assuming that  $V$  is irreducible, then we see that  $V$  is the coproduct of  $V_\chi$ 's as above and that each  $V_\chi$  admits a basis s.t.

$$r(G_s)|_{V_\chi} \subseteq \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & * \end{pmatrix} \right\} \quad r(G_u)|_{V_\chi} \subseteq \left\{ \begin{pmatrix} 1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

This yields the same conclusion for  $V$ .

- (i) We have to show that  $G_s$  and  $G_u$  are closed and  $j : G_s \times G_u \rightarrow G$  is an isomorphism of groups. Take any faithful representation

$$G \xrightarrow{\cong, r} r(G) \subseteq \mathbf{GL}(V)$$

and apply (ii). Then we have

$$\begin{aligned} r(G) &\subseteq \left\{ \begin{pmatrix} * & * & * \\ 0 & \ddots & * \\ 0 & 0 & * \end{pmatrix} \right\} =: B \\ B_u &= \left\{ \begin{pmatrix} 1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & 1 \end{pmatrix} \right\}, \\ r(G_s) &\subseteq \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & * \end{pmatrix} \right\} =: A. \end{aligned}$$

In fact,  $r(G_s) = r(G) \cap A$ , because if  $g \in G$  with  $r(g) \in A$ , then  $r(g)$  is semisimple, so  $g \in G_s$ .

Therefore,  $G_s$  is closed in  $G$ . Ergo,  $G_s$  and  $G_u$  are closed subgroups.

Then, the map  $j$  is a morphism of algebraic groups.

We need to show that  $j^{-1}$  is a morphism of algebraic groups. For this, it suffices to verify that the projection  $G \rightarrow G_s$  is a morphism. But this map is given under  $r$  by the morphism:

$$t := \begin{pmatrix} a_1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & a_n \end{pmatrix} \mapsto \begin{pmatrix} a_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_n \end{pmatrix} =: t_s.$$

This suffices because if  $g = g_s g_u$ , then  $g_u = g_s^{-1} g$ , so if the map  $g \mapsto g_s$  is a morphism, so is  $g \mapsto g g_s^{-1} = g_u$ , hence so is  $g \mapsto (g_s, g_u)$ .

- (iii) We have seen that  $G_s$  is a closed subgroup. Hence  $G_s$  is a commutative algebraic group where elements are semisimple. Ergo,  $G_s$  is diagonalizable.

□

## 11 Connected Solvable Groups

**Theorem 16** (Lie-Kolchin). *Let  $G$  be a connected solvable algebraic group. Then  $G$  is trigonalizable.*

(By comparison, recall that we have seen so far that, if  $G$  is commutative or unipotent, then  $G$  is trigonalizable.) We can reformulate this theorem as: Any connected solvable subgroup of  $\mathrm{GL}(V)$  stabilizes some complete flag  $\mathcal{F} = (V_0 \subsetneq \dots \subsetneq V_n)$ .

**Generalization (Borel's Fixed Point Theorem):** Any connected algebraic group  $G$  acting on a projective variety  $X$  has a fixed point in  $X$ .

We get a relation between complete flags and projective varieties.

*Proof.* Induct on the number  $n$  s.t.  $G^{(n)} = 1$ .

For  $n = 0$ , there is nothing to show.

If  $n = 1$ ,  $(G, G) = 1$ , then  $G$  is commutative, ergo trigonalizable.

Let  $n \geq 2$ . Then, we have  $G' := (G, G) \neq 1$ . We will show the following lemma: □

**Lemma 38.** *Let  $G \subseteq \mathrm{GL}(V)$  be a subgroup.*

*If  $G$  is connected, then the group  $G'$  with the induced topology is connected ( $\iff$  the Zariski Closure of  $G'$  is connected).*

*Proof.* We have the following facts:

- An increasing union of connected spaces is connected.
- A continuous image of a connected space is connected.

We have

$$\begin{aligned} G' &= \langle (g, h) := ghg^{-1}h^{-1} \mid g, h \in G \rangle \\ &= \bigcup_{j \geq 0} \bigcup_{g_1, h_1, \dots, g_j, h_j \in G} \{(g_1, h_1) \cdots (g_j, h_j)\}. \end{aligned}$$

Since

$$\bigcup_{g_1, h_1, \dots, g_j, h_j \in G} \{(g_1, h_1) \cdots (g_j, h_j)\} = \mathrm{img} \phi_j$$

for some continuous map  $\phi_j : G^{2j} \rightarrow G$ , the claim follows. Ergo,  $G'$  is connected. □

*Remark 6.* It is equivalent to show that (\*) any subgroup  $G$  of  $\mathrm{GL}(V)$  – s.t.  $G$  is connected and solvable – is trigonalizable in  $\mathrm{GL}(V)$ .

Indeed, the theorem implies (\*): the Zariski closure of  $G$  is a connected algebraic group that is solvable (which extends by continuity). If  $Zcl(G)$  is trigonalizable, then also  $G$  is trigonalizable.

On the other hand: (\*) implies the theorem, since if  $G$  is given as in the theorem, apply (\*) to  $r(G) \subseteq \mathrm{GL}(V)$ .

*Proof of Theorem.* If  $G^{(n)} = 1$ , then  $(G')^{(n-1)} = G^{(n)} = 1$ . By induction, we may assume that  $G'$  satisfies the following:

For all finite dimensional representations  $r : G \rightarrow \mathrm{GL}(V)$ ,  $r(G')$  is trigonalizable.

Our aim is to show that any irreducible representation  $V$  of  $G$  has dimension 1.

The induction hypothesis implies that  $r(G')$  is trigonalizable. In particular, there exists an eigenspace  $V_\chi \subseteq V$  for  $G'$  for some character  $\chi : G' \rightarrow k^\times$ . Since  $G'$  is normal in  $G$  we know that  $G$  acts from the left on

$$\{\text{eigenspaces } V_\chi \text{ in } V \text{ for } G'\}.$$

Ergo,  $\bigoplus_{\chi: G' \rightarrow k^\times} V_\chi$  is  $G$ -invariant. Since  $V$  is  $G$ -irreducible, we have

$$V = \bigoplus_{\chi: G' \rightarrow k^\times} V_\chi = \bigoplus_{\chi \in \mathfrak{X}'} V_\chi$$

for some finite subset  $\mathfrak{X}' = \{\chi \mid V_\chi \neq 0\}$  of  $\mathrm{Hom}(G', \mathcal{G}_m)$ , since  $V$  is finite dimensional.

**Claim:** Let  $h \in G'$ . Then, the map

$$\begin{aligned} G &\longrightarrow \mathrm{GL}(V) \\ g &\longmapsto r(ghg^{-1}) \end{aligned}$$

has a finite image.

*Proof.* Denote by  $\chi \mapsto \chi^g$  the action of  $g \in G$  in  $\mathrm{Hom}(G', \mathcal{G}_m)$  given by  $\chi^g(h) := \chi(ghg^{-1})$ . This is an action, since  $G'$  is normal.

Note, that  $\mathfrak{X}' \subseteq \mathrm{Hom}(G', \mathcal{G}_m)$  is a finite subset.

Also note, that the action  $\chi \mapsto \chi^g$  descends to an action  $G \curvearrowright \mathfrak{X}'$ .

Now, let  $\mathfrak{X}' = \{\chi_1, \dots, \chi_r\}$ . The matrix  $r(h)$  is totally determined by the values  $\chi_1(h), \dots, \chi_r(h)$ . Then, the element  $r(ghg^{-1})$  is totally determined by the values  $\chi_1^g(h), \dots, \chi_r^g(h)$ . It follows

$$\#\{r(ghg^{-1}) \mid g \in G\} \leq r!.$$

□

The following lemma is easy to show:

**Lemma 39.** *Let  $G$  be an algebraic set. Then,  $G$  is connected iff for each finite algebraic set  $X$ , and for each morphism  $f : G \rightarrow X$  of algebraic sets, we have that  $f$  is constant.*

Claim with the Lemma implies that the map  $g \mapsto t(ghg^{-1})$  is constant. This implies that  $r(ghg^{-1}) = r(h)$  for all  $g \in G, h \in G'$ . Ergo,  $G$  stabilizes each eigenspace  $V_\chi$  for  $G'$ . Ergo,  $V = V_{\chi_0}$ , since  $V$  is irreducible. □

**Lemma 40.** *Let  $G$  be any group with a finite dimensional representation  $r : G \rightarrow GL(V)$ . Then, the subspaces  $V_\chi$  for  $\chi \in \text{Hom}(G, k^\times)$  are linearly independent, i.e., the map*

$$\oplus V_\chi \longrightarrow V$$

*is injective.*

*Proof.* The spaces  $V_\chi$  are  $G$ -invariant. Suppose, there exist distinct  $\chi_1, \dots, \chi_n$  of non-zero  $v_j \in V_{\chi_j}$  s.t.  $\sum_j v_j = 0$ .

We may assume that  $n$ , the number of  $v_j$ , is minimal. W.l.o.g.,  $n \geq 2$ .

Choose  $g \in G$  s.t.  $\chi_1(g) \neq \chi_2(g)$ . Use that  $0 = g \sum_j v_j = \sum_j g v_j$  and take the linear combination as in the proof of linear independence of characters to contradict the minimality of  $n$ .

( $g - \chi_1(g)$  is not zero, but reduces  $\sum_j v_j$  by one summand.) □

*Finishing Proof of Theorem.* Since  $G' = \langle ghg^{-1}h^{-1} \mid g, h \in G \rangle$ , so  $\det(r(G')) = 1$ .

On the other hand, for each  $g \in G'$ , we have

$$r(g) = \begin{pmatrix} \chi_0(g) & & \\ & \ddots & \\ & & \chi_0(g) \end{pmatrix}$$

since  $V = V_{\chi_0}$ . This implies

$$1 = \det(r(g)) = \chi_0(g)^d.$$

Ergo,  $\chi_0$  defines a morphism

$$\chi_0 : G' \longrightarrow \mu_d \subseteq \mathcal{G}_m.$$

But  $G'$  is connected and  $\mu_d$  is finite. Since  $\chi_0$  is a morphism,  $\chi_0$  must be constant, ergo the trivial character.

As a consequence, we get  $r(G') = 1$  on  $V = V_{\chi_0}$ .

**Lemma 41.** *Let  $G$  be an algebraic group,  $r : G \rightarrow \mathrm{GL}(V)$  a representation.  $v \in V$  shall be a simultaneous non-zero eigenvector for  $r(G)$ .*

*Then, for each  $g \in G$ , there is a value  $\chi(g) \in k^\times$  s.t.*

$$r(g)v =: \chi(g)v.$$

*Then, the mapping  $\chi : G \rightarrow \mathcal{G}_m$  is a morphism of algebraic groups.*

Therefore,  $r$  descends to a representation of the commutative group

$$\bar{r} : G/G' \longrightarrow \mathrm{GL}(V).$$

Ergo,  $r(G/G') = r(G)$  is commutative and therefore trigonalizable (because of irreducibility).

□

**Example 17** (Non-Example). • Take  $G = D_4 \hookrightarrow \mathrm{GL}_2(\mathbb{C})$  which is solvable and has an irreducible and faithful representation over  $\mathbb{C}^2$ .

- Consider the solvable group

$$G = \left\langle \begin{pmatrix} \pm 1 & \\ & \pm 1 \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \right\rangle$$

which is a finite subgroup of  $\mathrm{GL}_2(\mathbb{C})$ , s.t.  $\mathbb{C}^2$  defines an irreducible representation of  $G$ .

**Lemma 42** (Form of Schur's Lemma). *Let  $S$  be any commutative subset of  $\mathrm{GL}(V)$  for a finite-dimensional  $0 \neq V$  over an algebraically closed field  $k$ . Let  $V$  be  $S$ -irreducible. Then,  $\dim V = 1$ .*

*Proof.* There is nothing to show if  $S$  is empty.

Let  $s \in S$  and denote by  $V_\lambda \subseteq V$  the  $\lambda$ -eigenspace for  $s$ . Then, since  $S$  is commutative,  $V_\lambda$  is  $S$ -invariant. Therefore,  $V = V_\lambda$  for one  $\lambda \in k^\times$ .

Thus, every  $s \in S$  acts by scaling, therefore every subspace of  $V$  is  $S$ -invariant. Since  $V$  is invariant, we get  $\dim V = 1$ .  $\square$

**Corollary 7.** *Let  $G$  be a connected algebraic group. Then,  $G$  is solvable iff  $G$  is trigonalizable.*

**Proposition 3.** *If  $G$  is trigonalizable, then  $G_u$  is a normal algebraic subgroup.*

*Proof.* We have

$$G \hookrightarrow B := \left\{ \begin{pmatrix} * & \dots & * \\ & \ddots & \vdots \\ & & * \end{pmatrix} \right\} \subseteq \mathrm{GL}_n(k).$$

$B$  has the normal subgroup  $U := \left\{ \begin{pmatrix} 1 & \dots & * \\ & \ddots & \vdots \\ & & 1 \end{pmatrix} \right\}$  and we have  $G_u = G \cap U$ . Now,

$U$  is the kernel of the multiplicative morphism

$$\begin{pmatrix} a_1 & \dots & * \\ & \ddots & \vdots \\ & & a_n \end{pmatrix} \mapsto \begin{pmatrix} a_1 & & \\ & \dots & \\ & & a_n \end{pmatrix}.$$

$\square$

**Corollary 8.** *If  $G$  is connected and solvable, then  $G_u$  is a normal algebraic subgroup.*

## 12 Semisimple Elements of nilpotent Groups

**Theorem 17.** *Let  $G$  be a connected nilpotent algebraic group. Then, we have*

$$G_s \subseteq Z(G)$$

where  $Z(G)$  denotes the **center** of  $G$ , i.e.

$$Z(G) = \{g \in G \mid \forall h \in G : gh = hg\}.$$

**Theorem 18** (Lie-algebraic Analogue). *Let  $V$  be a finite-dimensional vectorspace. Let  $\mathfrak{g}$  be the Lie-Subalgebra of  $\text{End}(V)$ , i.e.  $\mathfrak{g}$  is a subspace s.t. we have for each  $x, y \in \mathfrak{g}$*

$$[x, y] := xy - yx \in \mathfrak{g}.$$

*Assume that  $\mathfrak{g}$  is nilpotent, i.e. there is an  $n \in \mathbb{N}_0$  s.t.*

$$[x_1, [x_2, [\dots, [x_{n-1}, x_n]]]] = 0$$

*for all  $x_1, \dots, x_n \in \mathfrak{g}$ .*

*Then, any semisimple (semisimple in  $\text{End}(V)$  that is)  $x \in \mathfrak{g}$  is **central** in  $\mathfrak{g}$ , i.e.  $[x, y] = 0$  for each  $y \in \mathfrak{g}$ .*

*Remark 7.* The Lie-algebraic Analogue implies the general theorem if – for example –  $k = \mathbb{C}$ .

*Proof.* Let  $g \in G_s$ . We want to show  $Z_G(g) = G$ .

**Fact from the theory of Lie-Algebras:** For the Lie-Algebra  $\text{Lie}Z_G(g)$  we have

$$\text{Lie}Z_G(g) = \ker(\text{Ad}(g))$$

where  $\text{Ad}$  is the map

$$\begin{aligned} \text{Ad} : G &\longrightarrow \text{GL}(\mathfrak{g}) \\ x &\longmapsto gxg^{-1}. \end{aligned}$$

Since  $G$  is connected, it suffices to verify

$$\ker(\text{Ad}(g)) = \mathfrak{g}$$

i.e.  $\text{Ad}(g) = 1$ .



Since  $g$  is semisimple, we have for suitable basis

$$g = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$$

with  $a_j \in \mathbb{C}^\times$ . This is  $\exp(x)$  for a suitable diagonal matrix  $x = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} \in \mathrm{GL}_n(\mathbb{C})$ .

**Fact:** We may assume that  $x \in \mathfrak{g} := \mathrm{Lie}(G)$ .

Since  $G$  is nilpotent, it can be shown that  $\mathfrak{g}$  is nilpotent.

By the theorem,  $x$  is central in  $\mathfrak{g}$ . By the properties of  $\exp$  we have

$$\mathrm{Ad}(g) = \exp(\mathrm{ad}(g)) = 1$$

ergo  $\mathrm{ad}(x) = 0$  where  $\mathrm{ad} : \mathfrak{g} \rightarrow \mathfrak{g}$  is defined by

$$\mathrm{ad}(x) \cdot y := [x, y].$$

□

*Proof.* If  $\mathfrak{g}$  is nilpotent, then  $\mathrm{ad}(x) \in \mathrm{End}(\mathfrak{g})$  is nilpotent.

Since  $x$  is semisimple,  $\mathrm{ad}(x)$  is semisimple, because  $\mathrm{ad}(x)$  is the restriction to  $\mathfrak{g}$  of the map

$$\begin{aligned} \mathrm{End}(V) &\longrightarrow \mathrm{End}(V) \\ y &\longmapsto [x, y] \end{aligned}$$

and, if  $e_1, \dots, e_n$  are a basis of eigenvectors for  $x$ , then  $E_{i,j}$  is a basis of eigenvectors for  $\ell$ .

So,  $\mathrm{ad}(x)$  is nilpotent and semisimple, therefore  $\mathrm{ad}(x) = 0$ . □

*Proof Theorem.* Let  $G$  be a connected nilpotent algebraic group,  $G \hookrightarrow \mathrm{GL}(V)$ .

Let  $g \in G_s$ , we want to show that  $g \in Z(G)$ .

Assume otherwise, then we have a  $h \in G$  s.t.  $(g, h) = ghg^{-1}h^{-1} \neq 1$ .

Since  $G$  is connected and nilpotent (ergo solvable), we know by Lie-Kolchin that  $G$  stabilizes some complete flag  $V_0 \subset \dots \subset V_n$ .

We have  $g|_{V_i}, h|_{V_i} \in \mathrm{GL}(V_i)$ . They commute, if  $i = 0$ , but not if  $i = n$ .

So, there is an  $i$  s.t.  $g|_{V_i}, h|_{V_i}$  commute but  $g|_{V_{i+1}}, h|_{V_{i+1}}$  don't commute. W.l.o.g.  $V = V_{i+1}, g = g|_{V_{i+1}}, h = h|_{V_{i+1}}$ . Set  $a := g|_{V_i}, b := h|_{V_i} \in \text{GL}(V_i)$ .  $a$  will be semisimple, since  $g$  is.

Since  $g$  is semisimple, there is an eigenvector  $v \in V_{i+1}$  for  $g$  s.t.

$$V_{i+1} = V_i \oplus \langle v \rangle.$$

We have an isomorphism of vector spaces

$$\text{End}(V_{i+1}) \cong \text{End}(V_i) \oplus \text{Hom}(\langle v \rangle, V_i) \oplus \text{Hom}(V_i, \langle v \rangle) \oplus \text{End}(\langle v \rangle)$$

with

$$\text{End}(\langle v \rangle) \cong k \text{ and } \text{Hom}(\langle v \rangle, V_i) \cong V_i.$$

So, we can write  $g|_{V_{i+1}}, h|_{V_{i+1}}$  as

$$g = \begin{pmatrix} a & \\ & * \in k \end{pmatrix} \text{ and } h = \begin{pmatrix} b & c \in V_i \\ & * \end{pmatrix}.$$

We may replace  $g, h$  with scalar multiples to reduce to the case that  $* = 1$ . Then, we can write  $g|_{V_{i+1}}, h|_{V_{i+1}}$  as

$$g = \begin{pmatrix} a & \\ & 1 \end{pmatrix} \text{ and } h = \begin{pmatrix} b & c \\ & 1 \end{pmatrix}.$$

Then,

$$h \neq ghg^{-1} = \begin{pmatrix} b & ac \\ & 1 \end{pmatrix}.$$

Ergo,  $c \neq ac$ , i.e.  $c \notin \ker(a - 1)$ . Define

$$h_1 := h^{-1}ghg^{-1} = \begin{pmatrix} 1 & b^{-1}(a-1)c \\ & 1 \end{pmatrix}.$$

We claim that  $h_1$  does not commute with  $g$ . This claim implies the theorem, since we can iterate the claim to obtain elements  $h_i$  by  $h_{i+1} := h_i^{-1}gh_i g^{-1}$ . Then,  $h_i$  does not commute with  $g$ . But  $G$  is nilpotent, therefore  $h_i = 1$  for some large enough  $i$ .

We can prove the claim as follows: By some calculation as for  $h$  and  $g$ , we see, that  $h_1$  and  $g$  don't commute iff  $b^{-1}(a-1)c \notin \ker(a-1)$ . This is equivalent to

$$\begin{aligned} &\iff (a-1)b^{-1}(a-1)c \neq 0 \\ &\iff b^{-1}(a-1)^2c \neq 0 \\ &\iff (a-1)^2c \neq 0 \\ &\iff c \notin \ker((a-1)^2). \end{aligned}$$

But  $a$  being semisimple implies  $a - 1$  being semisimple, therefore

$$\ker((a - 1)^2) = \ker(a - 1).$$

So  $h_1, g$  don't commute iff  $c \in \ker(a - 1)$  iff  $h, g$  don't commute. □

## 13 Algebraic Geometry

### 13.1 Projective Algebraic Sets

Let  $V$  be a finite-dimensional vector space. Then  $\mathcal{G}_m = k^\times$  acts on  $V$  by scalar multiplication.  $\{0\}$  is a  $\mathcal{G}_m$ -invariant subspace of  $V$ . We are interested on the orbits of  $\mathcal{G}_m$  on  $V \setminus \{0\}$ .

Define the **projective space** over  $V$  by

$$\mathbb{P}V := \mathcal{G}_m \backslash (V - 0) = (V - 0) / \sim \cong \{\text{lines in } V\}$$

where for  $a, b \in V - 0$  we set

$$a \sim b : \iff \exists \lambda \in k^\times : \lambda a = b.$$

If  $V = k^{n+1}$ , we denote the  $n$ -dimensional projective space by  $\mathbb{P}^n := \mathbb{P}V$ .

Given  $a = (a_0, a_1, \dots, a_n) \in k^{n+1} - 0$ , we denote the  $\sim$ -class of  $a$  by

$$[a] = [a_0, \dots, a_n] \in \mathbb{P}^n.$$

Define  $S$  to be the graded algebra of polynomials in  $k$

$$S := k[x_0, \dots, x_n] = \bigoplus_{d \geq 0} S_d$$

where each  $S_d$  is the space of homogenous polynomials of degree  $d$ , i.e.

$$S_d = \bigoplus_{i_1, \dots, i_d \in \{0, \dots, n\}} k \cdot x_{i_1} \cdots x_{i_d}.$$

We identify  $k$  with the space of constant polynomials  $S_0 \subseteq S$ .

We have

$$S_d = \{f \in S \mid f(\lambda X) = \lambda^d f(X) \ \forall \lambda \in k^\times\}.$$

Given  $f \in S_d$ , the set

$$\{a \in k^{n+1} \mid f(a) = 0\}$$

is  $\mathcal{G}_m$ -invariant. In other words, given  $a \in \mathbb{P}^n$  and  $f \in S^d$ , it is well-defined to state  $f(a) = 0$  and  $f(a) \neq 0$ .

**Definition 34.** A **projective algebraic subset**  $X \subseteq \mathbb{P}^n$  is a set of the form

$$X = V(\Sigma) := V_{\mathbb{P}^n}(\Sigma)$$

where  $\Sigma$  is a collection of homogenous elements of  $S$ , where

$$V_{\mathbb{P}^n}(\Sigma) := \{a \in \mathbb{P}^n \mid f(a) = 0 \ \forall f \in \Sigma\}.$$

**Facts:**

- Hilbert's basis theorem states

$$V(\Sigma) = V(f_1, \dots, f_m)$$

for some finite collection  $f_1, \dots, f_m \in \Sigma$ .

- It is useful to extend the meaning of " $f(a) = 0$ " for  $a \in \mathbb{P}^n$  to *general* elements  $f \in S$  by requiring that  $f(a') = 0$  for each  $a' \in [a]$ .

If we write  $f = \sum_{d \geq 0} f_d$ ,  $f_d \in S_d$ , then we have

$$f(a) = 0 \iff f_d(a) = 0 \ \forall d \geq 0.$$

Therefore, we can extend the definition of  $V(\Sigma)$  to any  $\Sigma \subseteq S$ .

- We have  $V(\Sigma) = V((\Sigma))$  where  $(\Sigma)$  is the ideal generated by some finite subset of  $\Sigma$ .
- We call an ideal  $I \subseteq S$  **homogenous** if it is the direct sum of its  $d$ -homogeneous components, i.e.

$$I = \sum_{d \geq 0} I_d$$

where  $I_d = \{f \in I \mid f \text{ is homogenous of degree } d\}$ .

$I$  is homogeneous iff it is generated by homogeneous elements.

- We have the following *Nullstellensatz*:

For any  $X \subseteq \mathbb{P}^n$ , set  $I(X)$  to be the ideal generated by all homogeneous polynomials of  $S$  vanishing on  $X$ .

Let  $I \subseteq S$  be a *homogeneous* ideal which is *not equal* to  $(x_0, \dots, x_n)$ . Then, we have

$$I(V_{\mathbb{P}^n}(I)) = \sqrt{I}.$$

**Example 18** (Anti-example). The second property is necessary:

Set  $I = (x_0, \dots, x_n)$ . Then  $V_{k^{n+1}}(I) = 0$ . Therefore,  $V_{\mathbb{P}^n}(I) = \emptyset$ . However,

$$I(V_{\mathbb{P}^n}(I)) = S.$$

- The above point induces a bijection between algebraic subsets of  $\mathbb{P}^n$  and radical ideals  $I \subset S$  which are not  $(x_0, \dots, x_n)$ .

For  $i = 0, \dots, n$ , set  $D(x_i) := \{a \in \mathbb{P}^n \mid a_i \neq 0\}$ .  $D(x_i)$  is an open set homeomorphic to  $k^n$  by mapping

$$\phi_i : a \mapsto \left( \frac{a_0}{a_i}, \dots, \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i}, \dots, \frac{a_n}{a_i} \right).$$

The  $D(x_i)$  cover  $\mathbb{P}^n = \bigcup_i D(x_i)$ .

Given a projective algebraic subset  $X \subset \mathbb{P}^n$ , define  $X^{(i)} \subset k^n$  by

$$X^{(i)} := \phi_i(X \cap D(x_i)).$$

If  $X = V_{\mathbb{P}^n}(I)$ , then

$$X^{(i)} = V_{k^n}(I^{(i)})$$

where

$$I^{(i)} := \{f^{(i)} \mid f \in I\}$$

where  $f^{(i)}(t_1, \dots, t_n) := f(t_1, \dots, t_{i-1}, 1, t_i, \dots, t_n)$ . Thus,  $X^{(i)}$  is an algebraic subset of  $k^n$ .

**Definition 35.** The **Zariski topology** on  $\mathbb{P}^n$  is defined by setting the set of closed sets to be the set of projective algebraic sets.

**Facts:**

- $D(x_i)$  is open in  $\mathbb{P}^n$ , since  $D(x_i) = \mathbb{P}^n - V(x_i)$ .
- The bijections  $D(x_i) \cong k^n$  are homeomorphisms.

**Definition 36.** A **quasi-projective** algebraic set  $Y$  is an open subset of a projective algebraic set  $X \subseteq \mathbb{P}^n$ .

**Example 19.** Any algebraic set in  $k^n$  is quasi-projective.

**Definition 37.** A **quasi-projective variety** is defined as an irreducible quasi-projective algebraic set.

**Lemma 43** (Products). *Define the **Segre-embedding** by*

$$\begin{aligned} S^{n,m} : \mathbb{P}^n \times \mathbb{P}^m &\hookrightarrow \mathbb{P}^{nm+n+m} \\ (a, b) &\mapsto [(a_i b_j)_{i,j=0,\dots,n}]. \end{aligned}$$

*We have:*

1.  $S^{n,m}$  is injective.
2.  $S^{n,m}$  has a closed image.
3.  $k^n \times k^m \cong D(z_{00}) \cap S^{n,m}(\mathbb{P}^n \times \mathbb{P}^m) = S^{n,m}(D(x_0) \times D(y_0))$ .

**Definition 38.** For quasi-projective algebraic sets  $X \subset \mathbb{P}^n, Y \subset \mathbb{P}^m$ , we define their product by

$$X \times Y := S^{n,m}(X, Y) \subseteq \mathbb{P}^{nm+n+m}.$$

Then,  $X \times Y$  is a quasi-projective algebraic subset of  $\mathbb{P}^{nm+n+m}$ .

## 13.2 Flag Varieties

**Definition 39.** We define the **Grassmannian manifold** by

$$G(n, d) := \{W \subset k^n \mid W \text{ is a } d\text{-dimensional subvectorspace}\}.$$

Then, we have the **Plücker-embedding** by

$$\begin{aligned} P_d : G(n, d) &\longrightarrow \mathbb{P} \left( \bigwedge^d k^n \right) = \mathbb{P}^{\binom{n}{d}-1} \\ W &\longmapsto [w_1 \wedge \dots \wedge w_d] \end{aligned}$$

where  $w_1, \dots, w_d$  is a basis of  $W$ .

**Lemma 44.**  $P_d$  is injective and has a closed image.

Therefore, we can see  $G(n, d)$  as a projective algebraic set.

**Definition 40.** Let  $V$  be a finite-dimensional vector space of dimension  $n$ . Set

$$\mathrm{Gr}_d(V) := \{d\text{-dim. subspaces of } V\} \cong G(n, d).$$

Define further the **flag manifold** to be

$$\mathrm{Flag}(V) := \{\text{complete flags } \mathcal{F} = (0 = V_0 \subseteq V_1 \subseteq \dots \subseteq V_n = V)\}.$$

Then, we have a map

$$\begin{aligned} P_v : \mathrm{Flag}(V) &\longrightarrow \mathrm{Gr}_0(V) \times \dots \times \mathrm{Gr}_n(V) \\ \mathcal{F} &\longmapsto (V_0, \dots, V_n). \end{aligned}$$

**Lemma 45.**  $P_v$  has a closed image and is injective.

Thus, we can see  $\mathrm{Flag}(V)$  as a projective algebraic set.

**Lemma 46.**  $\mathrm{Gr}_d(V)$  and  $\mathrm{Flag}(V)$  are both irreducible, hence projective alg. varieties.

$\mathrm{Flag}(V)$  is called the **variety of complete flags**.



### 13.3 Local Rings and Function Fields

**Definition 41.** An **affine variety** is an irreducible algebraic subset of  $k^n$ .

**Definition 42.** If  $X$  is an affine variety, then the coordinate ring  $\mathcal{O}(X)$  is a domain. Define the **function field** of  $X$  by

$$k(X) := \text{Frac}(\mathcal{O}(X)) := \left\{ \frac{a}{b} \mid a, b \in \mathcal{O}(X), b \neq 0 \right\}.$$

**Definition 43.** Let  $p \in X$ . We define the **local ring** of  $\mathcal{O}(X)$  at  $p$  by

$$\mathcal{O}_{X,p} := \left\{ \frac{a}{b} \mid a \in \mathcal{O}(X), 0 \neq b \in \mathcal{O}(X), b(p) \neq 0 \right\} \subset k(X).$$

We have an **evaluation** map

$$\begin{aligned} \text{eval}_p : \mathcal{O}_{X,p} &\longrightarrow k \\ \frac{a}{b} &\longmapsto \frac{a(p)}{b(p)}. \end{aligned}$$

**Lemma 47.** Let  $X$  be an affine variety. Then

$$\mathcal{O}(X) = \bigcap_{p \in X} \mathcal{O}_{X,p}.$$

**Definition 44.** Let  $X \subset \mathbb{P}^n$  be a projective variety. Denote by  $I_{\mathbb{P}}(X)$  its homogenous vanishing ideal.

Define its **function field** by

$$k(X) := R/M,$$

where

$$\begin{aligned} R &:= \left\{ \frac{f}{g} \mid f, g \in k[x_0, \dots, x_n] \text{ homogen.}, \deg f = \deg g, g \notin I_{\mathbb{P}}(X) \right\}, \\ M &:= \left\{ \frac{f}{g} \in R \mid f \in I_{\mathbb{P}}(X) \right\}. \end{aligned}$$

**Lemma 48.**  $M$  is a maximal ideal in  $R$  and  $R/M$  is a field.

**Lemma 49.** If  $X$  is a projective variety, then  $X^{(i)} \subset k^n$  is an affine variety.

If  $X^{(i)} \neq \emptyset$ , then

$$k(X) \cong k(X^{(i)}).$$

**Definition 45.** Let  $X$  be a projective variety. For  $p \in X$ , we define its **local ring** at  $p$  by

$$\mathcal{O}_{X,p} := \left\{ \frac{f}{g} \in k(X) \mid g(p) \neq 0 \right\} \subset k(X).$$

**Lemma 50.** For a projective variety  $X$ , we have:

1. For  $p \in X^{(i)}$ :  $\mathcal{O}_{X,p} \cong \mathcal{O}_{X^{(i)},p}$ .
2. For  $p \in X^{(i)} \cap X^{(j)}$ :  $\mathcal{O}_{X^{(j)},p} \cong \mathcal{O}_{X^{(i)},p}$ .

**Definition 46.** If  $X \subset \mathbb{P}^n$  is quasi-projective variety, there is a minimal projective variety  $\overline{X} \subset \mathbb{P}^n$  which contains  $X$  as an open subset.

Then, we can set

$$\begin{aligned} k(X) &:= k(\overline{X}) \\ \mathcal{O}_{X,p} &:= \mathcal{O}_{\overline{X},p}. \end{aligned}$$

### 13.4 Regular Functions and Morphisms

**Definition 47.** Let  $X$  be quasi-projective variety. Let  $U \subseteq X$  be open. Then, we define the **ring of regular functions** on  $U$  by

$$\mathcal{O}(U) := \bigcap_{P \in U} \mathcal{O}_{X,P} \subseteq k(X).$$

**Definition 48.** Let  $X, Y$  be two quasi-projective varieties. A map  $f : X \rightarrow Y$  is called a **morphism**, if  $f$  is continuous and we have

$$f^* \mathcal{O}(U) := \{h \circ f \mid h \in \mathcal{O}(U)\} \subseteq \mathcal{O}(f^{-1}(U)).$$

*Remark 8.* Let  $X, Y$  be affine varieties and  $f : X \rightarrow Y$  be a map. Then we have

$$f^* \mathcal{O}(U) \subseteq \mathcal{O}(f^{-1}(U))$$

iff  $f$  is given by polynomials.

**Lemma 51.** *Let  $X$  be a quasi-projective variety and let  $p \in X$ .*

*Then there is an open neighborhood  $U$  of  $p$  in  $X$  s.t.  $U$  is isomorphic (as quasi-projective varieties) to an affine variety  $U' \subset k^n$ .*

*Proof.* Let  $Y$  be a projective variety s.t.  $X$  lies open in  $Y$ . By replacing  $X \hookrightarrow Y$  with  $X^{(i)} \hookrightarrow Y^{(i)}$ , we may assume that  $X$  is an open subset of an affine variety  $Y$  in  $k^n$ .

Since the sets  $D(f)$  give an open basis of  $k^n$ , there is a  $f \in \mathcal{O}(Y)$  s.t.

$$p \in D_Y(f) := \{y \in Y \mid f(y) \neq 0\} \subset X.$$

Now,  $D_Y(f)$  is affine, because the map

$$\begin{aligned} D_Y(f) &\longrightarrow \{(q, r) \in k^{n+1} \mid q \in Y, f(q)r = 1\} \\ q &\longmapsto (q, \frac{1}{f(q)}) \end{aligned}$$

is an isomorphism of quasi-projective varieties. □

## 13.5 Dimensions

**Definition 49.** Let  $X$  be a quasi-projective variety. We define its **dimension** as the transcendency degree of its function field, i.e.

$$\dim(X) := \text{tr.-deg}_k(k(X)).$$

*Remark 9.* If  $X$  is affine, then

$$\begin{aligned} \dim(X) &= \text{tr.-deg}_k(k(X)) \\ &= \dim_{\text{Krull}}(\mathcal{O}(X)) \\ &= \sup \{n \in \mathbb{N}_0 \mid P_0 \subsetneq P_1 \subsetneq \dots \subsetneq P_n, P_i \text{ prime in } \mathcal{O}(X)\} \\ &= \sup \{n \in \mathbb{N}_0 \mid Z_n \subsetneq \dots \subsetneq Z_0, Z_i \text{ closed, irreducible in } X\} \end{aligned}$$

*Remark 10.* If  $U \subset X$  is open, then  $k(U) = k(X)$  and  $\dim(U) = \dim(X)$ .

**Lemma 52.** Let  $\phi : X \rightarrow Y$  be a surjective morphism of quasi-projective varieties. Then,

$$\dim X \geq \dim Y.$$

*Proof.*  $\phi$  induces an inclusion

$$\begin{aligned} \phi^* : k(Y) &\hookrightarrow k(X) \\ [(U_i, \alpha_i)_i] &\longmapsto [(\phi^{-1}(U_i), \alpha_i \circ \phi)]. \end{aligned}$$

This map is indeed injective, since  $\phi$  is surjective. Therefore, the claim follows.  $\square$

**Lemma 53.** Let  $X$  be a quasi-projective variety and  $Y$  a proper, closed subvariety. Then,

$$\dim(Y) < \dim(X).$$

*Proof.* By going from  $X$  to its closure  $\tilde{X}$  and from there to  $\tilde{X}^{(i)}$ , we can assume that  $X$  is affine.

Then,  $I_X(Y)$  is a non-trivial prime ideal in  $\mathcal{O}(X)$ . Therefore, we have

$$\dim_{\text{Krull}}(\mathcal{O}(Y)) = \dim_{\text{Krull}}(\mathcal{O}(X)/I_X(Y)) \leq \dim_{\text{Krull}}(\mathcal{O}(X)) - 1,$$

since  $A$  is a finitely generated  $k$ -algebra and a domain, and since  $I_X(Y)$  is a prime ideal.  $\square$

**Lemma 54.** *Let  $X$  be an affine variety and  $f \in \mathcal{O}(X)$  be non-zero.*

*Then, the set*

$$V_X(f) := \{p \in X \mid f(p) = 0\}$$

*is a proper, closed subset of  $X$  and we can decompose it into irreducible components*

$$V_X(f) = Z_1 \cup \dots \cup Z_l.$$

*For each of those  $Z_i$ , we have*

$$\dim(Z_i) = \dim(X) - 1.$$

*Proof.* The  $Z_i$  correspond to minimal prime ideals  $P_i$  in  $\mathcal{O}(X)$  which contain  $(f)$ . Since, they are minimal, we have

$$\text{height}(P_i) = 1. \quad \square$$

**Lemma 55.** *Let  $X$  be an quasi-projective algebraic set. Then – as in the affine case – we may write*

$$X = Z_1 \cup \dots \cup Z_l$$

*where each  $Z_i \subseteq X$  is an **irreducible component**, i.e. a maximal closed irreducible subset.*

*We then define*

$$\dim(X) := \max_i \dim(Z_j).$$

**Lemma 56.** *Let  $\phi : X \rightarrow Y$  be a morphism of quasi-projective varieties. Further, let  $\phi$  be **dominant**, i.e.,  $\text{Im}\phi$  is dense.*

*Then, for all  $p \in \text{Im}(\phi)$ , we have the following for the fiber of  $\phi$  along  $p$ :*

$$\dim(\phi^{-1}(p)) \geq \dim(X) - \dim(Y).$$

### 13.6 Images of Morphisms

**Lemma 57.** *Let  $Y$  be a quasi-projective set. Then for each  $p \in Y$ , there is an open, affine neighborhood  $Y_0 \subset Y$  which contains  $p$ .*

*Proof.* Denote by  $\bar{Y}$  the algebraic closure of  $Y$  in  $\mathbb{P}^n$ . Assume that  $p_i \neq 0$ . Then, the affine sets  $Y^{(i)} = Y \cap D(x_i)$  and  $\bar{Y}^{(i)} = \bar{Y} \cap D(x_i)$  lie dense in  $Y$  and  $\bar{Y}$ .

Now,  $Y^{(i)}$  is open in  $\bar{Y}^{(i)}$ . Since the  $D_{\bar{Y}^{(i)}}(f)$ ,  $f \in \mathcal{O}(\bar{Y}^{(i)})$ , give a basis of the topology of  $\bar{Y}^{(i)}$ , there is an  $f \in \mathcal{O}(\bar{Y}^{(i)})$  s.t.

$$p \in D_{\bar{Y}^{(i)}}(f) \subseteq Y^{(i)} \subset Y.$$

The neighborhood  $D_{\bar{Y}^{(i)}}(f)$  is, in particular, affine. □

**Lemma 58.** *Let  $Y$  be a quasi-projective algebraic set. Then the diagonal*

$$\Delta Y := \{(y, y) \mid y \in Y\}$$

*is closed in  $Y \times Y$ .*

*Proof.* If we cover  $Y$  by affine open subsets, then, we can reduce the claim to the case, where  $Y$  is affine, i.e. closed in  $k^n$ .

Then,  $\Delta Y = (Y \times Y) \cap \Delta k^n \subset k^n \times k^n$ . Since  $Y \times Y$  is algebraic, it suffices to show that  $\Delta k^n$  is algebraic. And, indeed,

$$\Delta k^n = \{(x, y) \mid x - y = 0\}.$$

□

**Theorem 19** (Thm2). *Let  $X$  be a projective variety and  $Y$  be a quasi-projective variety. Then, the projection*

$$\pi_Y : X \times Y \longrightarrow Y$$

*is **closed**, i.e.  $\pi_Y(Z)$  is closed for each  $Z \subseteq X \times Y$  closed.*

*Proof.* Since  $X \hookrightarrow \mathbb{P}^n$  is a closed map (since  $X$  is closed in  $\mathbb{P}^n$ ), it suffices to show the claim for  $X = \mathbb{P}^n$ .

Actually, at this point, we are done, since  $\mathbb{P}^n$  with the Zariski-topology is topologically quasi-compact. □

**Theorem 20** (Thm1). *Let  $X$  be a projective variety and  $Y$  be a quasi-projective variety. Then, for each morphism  $\phi : X \rightarrow Y$ , the image  $\phi(X)$  is closed in  $Y$ .*

*Proof.* First, we show that

$$\Gamma := \{(x, y) \in X \times Y \mid \phi(x) = y\}$$

is closed in  $X \times Y$ . In fact, we have

$$\Gamma = (\phi \times 1)^{-1}(\Delta Y)$$

where  $\Delta Y \subseteq Y \times Y$  is closed.

Now, we can consider the chain

$$X \xrightarrow{\text{Id} \times \phi} X \times Y \xrightarrow{\pi_Y} Y.$$

We have  $\phi(X) = \pi_Y(\Gamma)$ . Since  $\pi_Y$  and  $\Gamma$  are closed, the claim follows.  $\square$

**Example 20.** 1. The condition that  $X$  is a *projective* variety is necessary. Consider

$$\pi_x : \{(x, y) \mid xy = 1\} \longrightarrow k.$$

The image  $k^\times = \pi_x(\{(x, y) \mid xy = 1\})$  is not closed in  $k$ .

2. Let  $Y = k \subset_o \mathbb{P}^1$ . Then any morphism  $\phi : X \rightarrow k$  is constant.

This is, because  $\phi(X)$  must be closed in  $\mathbb{P}^1$ , ergo a finite set. Now, this finite set cannot contain multiple elements. Otherwise,  $X$  would not be irreducible.

**Corollary 9.** *Let  $X$  be a projective variety and  $Y$  be an affine variety. Then, any morphism  $X \rightarrow Y$  is constant.*

*Proof.* We have the chain

$$X \longrightarrow Y \hookrightarrow k^m \xrightarrow{\pi_i} k.$$

For each  $\pi_i$  this chain must be constant.  $\square$

**Theorem 21** (Thm3). *Let  $\phi : X \rightarrow Y$  be a morphism of quasi-projective varieties. Assume that  $\phi$  is **dominant**, i.e.  $\phi(X)$  is dense in  $Y$ .*

*Then,  $\phi(X)$  contains a nonempty open (hence dense) subset of  $Y$ .*

*Proof.* Postponed...  $\square$

**Corollary 10.** *Let  $\phi : G \rightarrow H$  be a morphism of algebraic groups. Then,  $\phi(G)$  is closed.*

*Proof.* Since  $G$  can be reduced to finite many irreducible components and since  $\phi(G) = \bigcup_i \phi(g_i)\phi(G^o)$ , it suffices to show the claim in the case where  $G = G^o$  is irreducible.

Set  $Y = \overline{\phi(G)}$ .  $Y$  is irreducible and closed. Further,  $Y$  is a subgroup of  $H$ .

We are finished, if we can show  $\phi(G) = \overline{\phi(G)}$ .

By the previous theorem,  $\phi(G)$  contains a nonempty open subset  $U$  of  $Y$ , hence  $\phi(G)$  is dense in  $Y$ . Now, assume there are any  $h \in Y - \phi(G)$ . The map  $y \mapsto hy$  is an isomorphism, hence  $h\phi(G)$  lies dense in  $Y$ . Ergo

$$\phi(G) \cap (h\phi(G)) \neq \emptyset.$$

Take  $u_1, u_2 \in \phi(G)$  s.t.

$$u_1 = hu_2.$$

Then, it follow  $h = u_1u_2^{-1} \in \phi(G)$ . A contradiction. □



### 13.7 Borel's Fixed Point Theorem (special case)

**Theorem 22.** *Let  $G$  be a connected solvable algebraic subgroup of  $GL(V)$ , where  $V$  is a finite-dimensional non-trivial vector space.*

*Then,  $G$  acts algebraically on  $\mathbb{P}(V)$ .*

*Let  $X \subseteq \mathbb{P}(V)$  be a non-empty, closed  $G$ -stable subset. Then,  $G$  has a fixed point in  $X$ .*

*Proof.* We prove this by an induction on  $n = \dim(V)$ :

- $n = 1$ :

In this case,  $\mathbb{P}(V)$  contains only one element.

- $n = 2$ :

We have  $\mathbb{P}V \cong \mathbb{P}^1$ . If  $X = \mathbb{P}(V)$ , then there is a complete invariant flag  $0 \subset \langle v \rangle \subset V$  which is  $G$ -stable.

Then,  $[v]$  is fixed by  $G$ .

If  $X$  is finite, let  $x \in X$ . Then,  $G.x$  is a connected subset of  $X$ , hence  $G.x = \{x\}$ .

- $n \geq 3$ :

Take again a complete  $G$ -stable flag  $0 \subset \langle v \rangle \subset \dots \subset V$ . If  $[v] \in X$ , we are done.

Otherwise, consider the morphism

$$\phi : X \longrightarrow \mathbb{P}(V/\langle v \rangle).$$

Since  $\langle v \rangle$  is  $G$ -invariant,  $G$  acts on  $\mathbb{P}(V/\langle v \rangle)$  and  $\phi$  is  $G$ -equivariant.

The image  $\phi(X)$  is closed by a theorem in the preceding subsection. By the induction hypothesis, there is a fixed point  $[w + \langle v \rangle] \in \phi(X) \subseteq \mathbb{P}(V/\langle v \rangle)$ .

In particular,  $[w + \langle v \rangle]$  has a preimage  $[w]$  in  $X$ . Consider the subset

$$W := \langle w, v \rangle \subseteq V.$$

$W$  is  $G$ -stable and we have  $\mathbb{P}W \cap X \neq \emptyset$ . Since  $\mathbb{P}W \cap X$  is closed in  $\mathbb{P}W \cong \mathbb{P}^2$ , it follows from a previous case that there is a  $G$ -fixed point in  $\mathbb{P}W \cap X$ .

□

## 13.8 Orbits

**Definition 50.** Let  $G$  be an algebraic group and  $Y$  a quasi-projective variety.

An **action**  $G \curvearrowright Y$  is an action described by a morphism<sup>3</sup>

$$\phi : G \times Y \longrightarrow Y.$$

**Lemma 59.** *Let  $G$  be an algebraic group which acts on a quasi-projective algebraic set  $Y$ . For an orbit  $O \subset Y$ , we have that  $O$  is open in  $\overline{O}$ .*

*Proof.* Let  $G_i$  be an irreducible component of  $G$ . For a point  $p \in O$ , the map

$$\begin{aligned} G_i &\longrightarrow \overline{G_i \cdot p} \\ g &\longmapsto g \cdot p \end{aligned}$$

is dominant. Ergo,  $G_i \cdot p$  contains a nonempty open subset of  $\overline{G_i \cdot p}$ . Ergo, the set  $O = G \cdot p$  contains a nonempty open subset  $U$  of  $\overline{O} = \overline{G \cdot p}$ .

Now, for  $q \in O$ , there is some isomorphism  $g \in G$  s.t.  $q \in g \cdot U$ . Ergo,  $O$  is open.  $\square$

**Definition 51.** If  $O$  is a  $G$ -orbit in a quasi-projective variety  $Y$ , we can consider it to be a quasi-projective set. Therefore, the notion of the dimension of an orbit  $O$  is well-defined.

**Lemma 60** (Minimal Orbit Lemma). *Let  $G$  be an algebraic group. Let  $Y$  be a quasi-projective variety s.t.  $Y$  is projective or affine.*

*Let  $O$  be a  $G$ -orbit in  $Y$  s.t. the dimension of  $O$  is minimal among all  $G$ -orbits in  $Y$ .*

*Then,  $O$  is closed.*

*Proof.* Since the action of an element of  $G$  does not change the dimension of a quasi-projective set, we can reduce the claim to the case that  $G$  is connected.

Then,  $O$  is irreducible. Further  $\overline{O}$  is closed and, because of the previous lemma,  $\overline{O} - O$  is closed. It is easy to see, that  $G$  operates on  $\overline{O} - O$ .

Let  $Z$  be an irreducible component of  $\overline{O} - O$ . Since  $Z$  is a proper closed subset of  $\overline{O}$ , we have

$$\dim(Z) < \dim(\overline{O}) = \dim(O).$$

Since  $O$  is dimensionally minimal, we must have  $Z = \emptyset$ . Ergo,  $O = \overline{O}$ .  $\square$

**Corollary 11.** *Let  $G$  be an algebraic group. Let  $Y$  be a quasi-projective variety s.t.  $Y$  is projective or affine.*

*Then  $G$  has a closed orbit in  $Y$ .*

---

<sup>3</sup>If  $G$  is connected,  $\phi$  shall be a morphism of quasi-projective varieties. Otherwise, we just require that  $G^\circ \times Y \rightarrow Y$  is a morphism of quasi-projective varieties.

### 13.9 Borel's Fixed Point Theorem (General Case)

**Theorem 23.** *Let  $G$  be a connected solvable algebraic group which acts on a projective variety  $X$ .*

*Then, there exists a  $G$ -fixed point in  $X$ .*

*Proof.* Since orbits of minimal dimensions are closed, we can replace  $X$  by a  $G$ -orbit. That is, we can assume that  $G$  acts transitively on  $X$ .

For  $p \in X$ , the  $G$ -stabilizer set

$$\text{Stab}_G(p) = \{g \in G \mid g.p = p\}$$

is a closed subgroup in  $G$ , since it is the preimage of  $p$  under the continuous map  $g \mapsto g.p$ .

We showed earlier, that there exist a finite-dimensional representation  $\rho : G \rightarrow \text{GL}(V)$  with a one-dimensional subspace  $L \subset V$  s.t.

$$G_p = \{g \in G \mid gL = L\}.$$

Let  $q = [L] \in \mathbb{P}(V)$ . Then  $G$  operates on  $\mathbb{P}V$  and

$$G_q := \text{Stab}_G(q) = \{g \in G \mid g.q = q\} = G_p.$$

Now, define

$$\begin{aligned} Y &:= G.q \subset \mathbb{P}V \\ Z &:= G.(p, q) \subset X \times \mathbb{P}V. \end{aligned}$$

$Y$  and  $Z$  are quasi-projective varieties, since  $G$  is connected. We then have a  $G$ -equivariant diagram of quasi-projective varieties:

$$X \longleftarrow Z \xrightarrow{\pi} Y$$

via

$$X \longleftarrow X \times \mathbb{P}(V) \xrightarrow{\pi} \mathbb{P}(V).$$

Since  $X$  is projective,  $\pi$  is closed. Since  $G_p = G_q$ , the maps are bijective. Since all maps are bijective and  $G$ -equivariant, we need only to show that  $Y$  has a fixed point.

Since  $\pi$  is closed, the existence of a  $G$ -fixed point  $Y$  follows by the closedness of  $Z$ , because of Borel's special fixed point theorem.

The closedness of  $Z$  in  $X \times \mathbb{P}(V)$  follows, if we can show that  $Z$  is an orbit of minimal dimension in  $X \times \mathbb{P}(V)$ . Indeed, we have.

- If  $O \subset X \times \mathbb{P}(V)$  is a  $G$ -orbit, the projection  $O \rightarrow X$  is  $G$ -equivariant and surjective, because  $X$  is a  $G$ -orbit. Since  $X, Y$  are quasi-projective varieties, it then follows

$$\dim(X) \leq \dim(O).$$

- The map  $Z \rightarrow X$  is bijective, hence

$$\dim(*) \geq \dim(Z) - \dim(X).$$

Ergo

$$\dim(Z) \leq \dim(X).$$

□

### 13.10 Generic Openness

**Proposition 4.** *Let  $\phi : X \rightarrow Y$  be a dominant morphism of quasi-projective varieties.*

*Then, there is an open nonempty set  $U \subset X$ , s.t.,  $\phi|_U$  is open, that is, it maps open sets to open sets.*

**Corollary 12.** *Let  $G$  be a connected algebraic group.*

*Then,  $[G, G] = \langle aba^{-1}b^{-1} \mid a, b \in G \rangle$  is a closed subgroup of  $G$ .*

*Proof.* For  $a, b \in G$ , set

$$[a, b] = aba^{-1}b^{-1}.$$

For  $n \geq 0$ , define

$$\begin{aligned} \phi_n : G^{2n} &\longrightarrow G \\ (a_1, b_1, \dots, a_n, b_n) &\longmapsto [a_1, b_1] \cdots [a_n, b_n]. \end{aligned}$$

Let  $Z_n := \overline{\text{Im} \phi_n}$ . Then, we have an ascending chain

$$Z_1 \subseteq Z_2 \subseteq \dots$$

Each  $Z_n$  is closed and irreducible, because  $G^{2n}$  is connected.

Then, at some point the chains of  $Z_i$ 's must become stationary, because  $\dim(G) < \infty$ , because  $\mathcal{O}(G)$  is a finitely generated  $k$ -algebra.

Let  $N$  be s.t.

$$Z_N = Z_{N+1} = \dots$$

Since  $[G, G] = \bigcup_n \text{Im}(\phi_n)$ , we have then

$$Z_N = \bigcup_n Z_n = \overline{[G, G]}.$$

Since  $\phi_n : G^{2n} \rightarrow Z_n$  is dominant and since  $G^{2n}$  and  $Z_n$  are quasi-projective varieties,  $\text{Im} \phi_n = [G, G]$  contains a nonempty open subset  $U$ .

Now, let  $h \in \overline{[G, G]}$ . Then,  $hU \cap U \neq \emptyset$ , since both are nonempty open and  $\overline{[G, G]}$  is irreducible. Therefore, we have  $u_1, u_2 \in U \subseteq [G, G]$  with

$$hu_1 = u_2.$$

Ergo,  $h$  lies in  $[G, G]$ . □

## 14 Homogenous Spaces

**Definition 52.** Let  $G$  be a connected algebraic group. A homogenous space for  $G$  is a quasi-projective variety  $X$  equipped with a **transitive** action  $G \curvearrowright X$ .

Let  $G$  be now disconnected. Then, we only demand that  $X$  is a finite union of irreducible components. Still  $G$  needs to act transitively on  $X$ .

A morphism of  $G$ -homogenous spaces is a **morphism** of quasi-projective varieties/sets which is  $G$ -equivariant.

**Corollary 13.** *If  $\phi : X \rightarrow Y$  is a morphism of  $G$ -homogenous spaces, then  $\phi$  is an open map.*

*Proof.* It suffices, if we show this statement for an irreducible  $X$ . Note, that  $\phi$  must be surjective, ergo dominant.

By a previous proposition,  $X$  must contain an open nonempty subset  $U$  s.t.  $\phi|_U$  is an open map. Since  $G$  acts transitively on  $X$ , we can cover  $X$  with such open sets  $gU$ .  $\square$

**Proposition 5.** *Let  $G$  be an algebraic group and  $H$  a closed subgroup.*

*Then, there is a homogenous space  $X$  for  $G$  and a point  $p \in X$  s.t.*

$$H = \text{Stab}_G(p)$$

*and the map*

$$\begin{aligned} G/H &\longrightarrow X \\ gH &\longmapsto g \cdot p \end{aligned}$$

*is a bijection.*

*Proof.* There is a faithful representation  $\rho : G \rightarrow \text{GL}(V)$  with  $V$  finite-dimensional s.t. there is a one-dimensional subspace  $L \subset V$  with

$$H = \{g \in G \mid gL = L\}.$$

Set  $p := [L] \in \mathbb{P}(V)$ . Then, we can set

$$X := G \cdot p.$$

Then,  $X$  is an orbit of  $G$ , ergo a quasi-projective set/variety.  $\square$

## 14.1 Quotients

**Definition 53.** A (left) **quotient** of an algebraic group  $G$  by a closed group  $H$  is a pair  $(X, \rho)$  s.t.

- (1)  $X$  is a quasi-projective variety.
- (2)  $\rho : G \rightarrow X$  is a morphism with

$$\rho(hg) = \rho(g)$$

for all  $h \in H, g \in G$ .

Further, we demand that a quotient is **initial** in the category of all objects satisfying the above conditions. I.e. for each pair  $(X', \rho')$  there must be a unique morphism  $\phi$  s.t. the following diagram commutes:

$$\begin{array}{ccc} G & & \\ \downarrow \rho & \searrow \rho' & \\ X & \xrightarrow{\phi} & X' \end{array}$$

*Remark 11.* Set theoretically, we just have  $X = G/H$ .

**Lemma 61.** Let  $(X, \rho)$  satisfy conditions (1) and (2) from the above definition. Suppose further

- (i)  $\{\text{fibers of } \rho\} = \{\text{left } H\text{-cosets of } G\},$
- (ii)  $X$  is a  $G$ -homogenous space and  $\rho$  is  $G$ -equivariant,
- (iii) for each open  $U \subset X$  the pullback map

$$\begin{aligned} \rho^* : \mathcal{O}(U) &\longrightarrow \mathcal{O}(\rho^{-1}(U)) \\ f &\longmapsto f \circ \rho \end{aligned}$$

defines an isomorphism

$$\mathcal{O}(U) \cong \{f \in \mathcal{O}(\rho^{-1}(U)) \mid f(Hg) = f(g)\} =: \mathcal{O}(\rho^{-1}(U))^H.$$

Then,  $(X, \rho)$  is a quotient of  $G$  by  $H$ .

*Proof.* We have to show that  $(X, \rho)$  is initial. Let  $(X', \rho')$  be another object satisfying (1), (2). Because of (i), we have a unique settheoretic map  $\phi : X \rightarrow X'$  s.t. the diagram

$$\begin{array}{ccc} G & & \\ \downarrow \rho & \searrow \rho' & \\ X & \xrightarrow{\phi} & X' \end{array}$$

commutes. We need to check that  $\phi$  is a morphism:

- $\phi$  is continuous, since  $\rho'$  is continuous and  $\rho$  is open (since  $X$  is a  $G$ -homogenous space). Therefore,  $\phi = \rho' \circ \rho^{-1}$  is continuous.
- Let  $U' \subset X'$  be open. We need to show

$$\phi^* \mathcal{O}(U') \subseteq \mathcal{O}(\phi^{-1}U').$$

Let  $f \in \mathcal{O}(U')$  and set  $U := \phi^{-1}U'$ . Since  $\rho'$  is a morphism, we have

$$\rho'^*(f) \in \mathcal{O}(\rho'^{-1}U').$$

Because of (iii), we have

$$\mathcal{O}(U) \cong \mathcal{O}(\rho^{-1}U)^H.$$

Therefore, it suffices to show

$$\rho'^*(f) \in \mathcal{O}(\rho^{-1}U)^H.$$

And, indeed

$$f \circ \rho'(hg) = f \circ \rho'(g)$$

for  $g \in G, h \in H$ .

□

**Lemma 62.** *Suppose  $\text{char } k = 0$ . Any injective morphism of quasi-projective varieties with dense image is **birational**, i.e., induces, via pullback an isomorphism*

$$k(X) \cong k(Y).$$

**Theorem 24.** *Let  $G$  be an algebraic group with a closed subgroup  $H$ .*

- *A quotient  $(X, \rho)$  exists and  $X$  is a homogenous space for  $G$  s.t.  $H = \text{Stab}_G(p)$  for some  $p \in X$ .*



- If  $\text{char}(k) = 0$ , then each  $G$ -homogenous space  $X$  together with a point  $p \in X$  s.t.  $H = \text{Stab}_G(p)$  gives a quotient of  $G$  by  $H$ , where  $\rho(g) = g.p$ .

*Proof.* We only prove the theorem for the case  $\text{char} k = 0$ . We construct  $X$  as in a previous proposition, i.e.  $X = G.p$  for a point  $p \in \mathbb{P}(V)$  s.t.  $H = \text{Stab}_G(p)$ .

It is then clear, that conditions (i) and (ii) of the previous lemma are met. We only need to show

$$\rho^* \mathcal{O}(U) = \mathcal{O}(\rho^{-1}U)^H.$$

Naturally,  $\rho^* \mathcal{O}(U)$  is contained in  $\mathcal{O}(\rho^{-1}U)^H$ .

Let  $f \in \mathcal{O}(\rho^{-1}U)^H$ . W.l.o.g., we can assume that  $U$  is affine. Consider the diagram

$$\begin{array}{ccc} \rho^{-1}U & \xrightarrow{f} & k \\ \downarrow \rho & \nearrow g & \\ U & & \end{array}$$

$g := f \circ \rho^{-1}$  is well-defined, because  $f$  is  $H$ -invariant. We need to show, that  $g$  is regular, i.e.  $g \in \mathcal{O}(\rho^{-1}U)$ .

We can blow up the diagram as follows:

$$\begin{array}{ccccc} U \times k & \xleftarrow{\text{open}} & V \supset \text{Im}(\rho \times f) & & \\ & \searrow \pi_2 & \nearrow \pi_2 & & \\ \rho \times f \uparrow & & k & & \downarrow p \\ & \nearrow f & \nwarrow g & & \\ \rho^{-1}U & \xrightarrow{\rho} & U & & \end{array}$$

Then  $V$  is a quasi-projective variety and  $p$  is dominant and injective, hence birational. Therefore, we have

$$k(U) \cong k(V).$$

Since  $X$  is a homogenous space for  $G$ , it is smooth. On a smooth quasi-projective variety, every rational function that fails to be regular must have a pole.

In particular, we do have  $\pi_2 \in \mathcal{O}(V)$  and therefore

$$g = p^*(\pi_2) \in \mathcal{O}(U).$$

□

**Example 21** (Non-Example). The proof of the theorem does not hold, if  $\text{char}(k) = p > 0$ .

Consider,

$$\begin{aligned} G &:= \mathcal{G}_a \\ H &:= 1 \\ V &= k^2 \end{aligned}$$

and

$$\begin{aligned} G &\longrightarrow \text{GL}(V) \\ x &\longmapsto \begin{pmatrix} 1 & x^{p^n} \\ & 1 \end{pmatrix} \end{aligned}$$

for some  $n \in \mathbb{N}_0$ .

For  $q = [1, 0] \in \mathbb{P}(V)$ , we have

$$X := G.q = \{[1, x^{p^n}] \mid x \in k\} \cong k.$$

Define  $\rho$  by

$$\begin{aligned} \rho : G &\longrightarrow X \\ g &\longmapsto g.q. \end{aligned}$$

Then,  $(\rho, X)$  fulfills the conditions of the above theorem, but it is NOT a quotient for  $n \geq 1$ .

Indeed, for  $n_1 \geq n_2$ , we have non-isomorphic maps

$$\begin{aligned} X_{n_2} &\longrightarrow X_{n_1} \\ x &\longmapsto x^{p^{n_1 - n_2}}. \end{aligned}$$

## 15 Borel and Parabolic Groups

Let  $G$  be a connected algebraic group.

**Definition 54.** A subgroup  $B \subset G$  is called **Borel**, if  $B$  is maximal among all connected solvable closed subgroups.

Since  $\dim(G) < \infty$ , Borel subgroups exist.

**Definition 55.** A subgroup  $P \subset G$  is called **parabolic**, if the quasi-projective variety  $G/P$  is **projective**, i.e. closed in  $\mathbb{P}^n$ .

**Lemma 63.** *Let  $G$  connected,  $P$  parabolic,  $B$  Borel. Then,  $P$  contains some conjugate of  $B$ .*

*Proof.*  $B$  acts on the projective variety  $G/P$ . According to Borel's fixed point theorem, there is a fixed point  $gP \in G/P$  s.t.

$$bgP = gP$$

for each  $b \in B$ . Ergo

$$g^{-1}bg \in P$$

for each  $b \in B$ . □

**Theorem 25.** *Let  $G$  be connected.*

*Any two Borel subgroups are conjugate.*

*Proof.* Take a faithful representation  $G \hookrightarrow \mathrm{GL}(V)$  with a finite-dimensional  $V$ . Let  $\mathcal{F} = \mathrm{Flag}(V)$  be the flag variety of  $V$ .

Choose  $F \in \mathcal{F}$  s.t. the orbit  $G.F$  has a minimal dimension. Then,  $G.F$  is closed, hence projective. If we set

$$H := \mathrm{Stab}_G(F),$$

then  $H$  is parabolic. Therefore, each Borel group  $B$  has a conjugate in  $H$ . Since  $B$  is connected, its conjugate is contained in an irreducible component  $H^\circ$  of the neutral element.

Since  $H$  is solvable<sup>4</sup>,  $H^\circ$  is a connected, solvable, closed subgroup. Ergo  $H^\circ$  is the conjugate of  $B$ . □

**Proposition 6.** *Let  $G$  be connected. Then, each Borel group is parabolic.*

---

<sup>4</sup>Why is  $H$  solvable?

*Proof.* Let  $B$  be a Borel subgroup of  $G$ .

Take a representation  $G \rightarrow \mathrm{GL}(V)$  with a finite-dimensional  $V$  s.t. there is a one-dimensional  $L \subseteq V$  s.t.

$$B = \{g \in G \mid gL = L\}.$$

$B$  acts on  $V/L$ . Since  $B$  is connected and solvable there must be a complete  $B$ -invariant flag  $\overline{F}$  in  $V/L$ . We can lift  $\overline{F}$  to a complete flag  $F = (L = V_1 \subset \dots \subset V_n)$  of  $V$ . Then, it is easy to see

$$B = \mathrm{Stab}_G(F).$$

Choose  $F' \in \mathrm{Flag}(V)$  s.t. the orbit  $G.F'$  has a minimal dimension. Then,  $G.F'$  is closed, hence projective. If we set

$$H := \mathrm{Stab}_G(F'),$$

we have (by conjugating)

$$B = H^o.$$

Consider the map

$$G/B = G/H^o \twoheadrightarrow G/H.$$

This map has finite fibers, because  $[H : B] < \infty$ . Ergo

$$\dim(G/B) \leq \dim(G/H).$$

Ergo,  $G/B$  is of minimal dimension, hence closed. Hence,  $B$  is parabolic.  $\square$

**Corollary 14.** *Let  $P$  be an algebraic subgroup of a connected algebraic group  $G$ . Then,  $P$  is parabolic iff it contains a Borel group.*

*Proof.* The direction to the right is known.

Let  $P$  contain a Borel group  $B$ . Consider the maps

$$G/B \twoheadrightarrow G/P \hookrightarrow \mathbb{P}^n.$$

Since  $B$  is parabolic,  $G/B$  is closed. Therefore, the morphism  $G/B \rightarrow \mathbb{P}^n$  has a closed image. But its image is exactly  $G/P$ . Ergo,  $P$  is parabolic.  $\square$

**Corollary 15.** *Let  $B$  be an algebraic subgroup of a connected algebraic group  $G$ . Then,  $B$  is Borel iff it is a minimal parabolic subgroup.*

**Example 22.** If  $G = \mathrm{GL}_n(k)$ , then

$$B = \left\{ \begin{pmatrix} * & \dots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & * \end{pmatrix} \right\}$$

is a Borel group.

Let  $n = n_1 + \dots + n_r$  and set

$$P_{(n_1, \dots, n_r)} := \left\{ \begin{pmatrix} \mathrm{GL}_{n_1}(k) & * & * \\ 0 & \ddots & * \\ 0 & 0 & \mathrm{GL}_{n_r}(k) \end{pmatrix} \right\}.$$

Each  $P_{(n_1, \dots, n_r)}$  is closed, since it is the stabilizer of an incomplete flag.

In fact, each parabolic group is conjugate to one of those  $P_{(n_1, \dots, n_r)}$ .

If  $P \neq G$  is parabolic,  $P$  is called a **proper** parabolic subgroup.

**Example 23.** •  $G = \mathrm{SL}_n(k)$ : In this case parabolic groups are like in the above case, but inside of  $\mathrm{SL}_n(k)$ .

- $G = \mathrm{SO}_n(k)$ : Then, we can embed  $G$  in  $\mathrm{GL}(V)$ . Let  $\langle \cdot | \cdot \rangle$  be (any?) symmetric bilinear form.

A subspace  $W \subset V$  is called **isotopic** iff  $\langle \cdot | \cdot \rangle|_{W \times W} \equiv 0$ .

Then, we have the equivalence

$$\{\text{Borel Group } B \subset G\} \Leftrightarrow \{\text{maximal isotropic flags } \mathcal{F} \text{ in } V\}.$$

- $G = \mathrm{SP}_{2n}$ : The symplectic group is defined by

$$\mathrm{SP}_{2n} := \left\{ A \in \mathrm{GL}_{2n}(k) \mid A^T \cdot \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \cdot A = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \right\}.$$

Embed again  $G$  in  $\mathrm{GL}(V)$ .

Let  $\langle \cdot | \cdot \rangle$  be a **symplectic** form on  $V$ , i.e.,  $\langle \cdot | \cdot \rangle$  is bilinear, alternating ( $\langle v | v \rangle = 0$ ) and nonsingular, i.e.  $\langle v | \_ \rangle \equiv 0 \iff v = 0$ .

Then, again, we have the equivalence

$$\{\text{Borel Group } B \subset G\} \Leftrightarrow \{\text{maximal isotropic flags } \mathcal{F} \text{ in } V\}.$$

Further, we can take a basis  $e_1, \dots, e_n, f_1, \dots, f_n$  of  $V$  with

$$\begin{aligned}\langle e_i \mid e_j \rangle &= \langle f_i \mid f_j \rangle = 0 \\ \langle e_i \mid f_j \rangle &= \delta_{i,j}.\end{aligned}$$

Then, one can for example set

$$V_j = \text{span}\{e_1, \dots, e_j\}$$

to get a flag  $V_0 \subset V_1 \subset \dots$

Vice versa, one can convert each maximal isotropic flag to such a symplectic basis.

## 15.1 Radicals

Let  $G$  be a connected algebraic group.

**Definition 56.** The **radical**  $R(G)$  of  $G$  is defined as the intersection of all Borel subgroups of  $G$  i.e.

$$R(G) := \bigcap_{B \subset G \text{ Borel}} B.$$

The **unipotent radical** is defined by

$$R_u(G) := R(G)_u = \{\text{unipotent elements of } R(G)\}.$$

**Lemma 64.** *Let  $G$  be a connected algebraic group.*

*$R(G)$  is the largest connected solvable normal algebraic subgroup of  $G$ .*

*Proof.* It is clear that  $R(G)$  is connected, solvable, normal and algebraic.

We need to show that each connected solvable normal algebraic subgroup  $H$  of  $G$  is contained in  $R(G)$ .

Clearly,  $H$  is contained in one Borel group  $B$ . Since  $H$  is normal, we have for each  $g \in G$

$$H = gHg^{-1} \subset gBg^{-1}.$$

Since  $gBg^{-1}$  is a Borel group and all Borel groups are conjugated, it follows  $H$  is contained in each Borel group, ergo it is contained in  $R(G)$ .  $\square$

**Definition 57.** We call  $G$  **semisimple** iff  $R(G) = 1$ .

We call  $G$  **reductive** iff  $R_u(G) = 1$  (iff  $R(G)$  is a torus).

**Example 24.** • Let  $n \geq 1$  and  $G = \text{GL}_n(k)$ .  $G$  is reductive, but not semisimple:

$G$  has two Borel groups:

$$B = \left\{ \begin{pmatrix} * & * \\ & * \end{pmatrix} \right\} \qquad B' = \left\{ \begin{pmatrix} * & \\ * & * \end{pmatrix} \right\}.$$

Ergo, we have for the radical

$$R(G) \subset B \cap B' = \left\{ \begin{pmatrix} * & \\ & * \end{pmatrix} \right\} =: T.$$

But, now we have

$$\{t \in T \mid gtg^{-1} \in T \ \forall g \in G\} = k^\times.$$

Ergo,

$$R(G) = k^\times.$$

Let  $G = \mathrm{SL}_n(k)$ .  $G$  is semisimple and reductive:

As above, one can compute

$$Z = G \cap \left\{ \begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{pmatrix} \right\}.$$

However,  $Z$  is not connected. In particular

$$R(G) = Z^o = 1.$$

$G = \mathcal{G}_m^n$  is a torus: It is easy to see that  $R(G) = G$  in this case.

$G$  is solvable (and connected): Trivially, we have then  $R(G) = G$ .

$G$  is unipotent: In this case, we know that  $G$  is solvable. Further, we even have  $R_u(G) = G$ .

If  $G$  is  $\mathrm{SO}_n$  or  $\mathrm{SP}_{2n}$ , then  $R(G) = R_u(G) = 1$ .



## 16 Reductivity

Let  $G$  be a connected algebraic group which acts on an affine variety  $X$ .

**Definition 58.** A **quotient** of  $X$  by  $G$  is a pair  $(Y, \rho)$  s.t.

1.  $Y$  is an affine variety
2. and  $\rho : X \rightarrow Y$  is a morphism which is constant on  $G$ -orbits.

Further, we demand that a quotient is initial in the category of all objects which fulfill the above conditions. I.e.

$$\begin{array}{ccc} X & & \\ \downarrow \rho & \searrow \rho' & \\ Y & \xrightarrow{\exists_1 \phi} & Y' \end{array}$$

*Remark 12.* • Such quotients need not to exist.

- Even when such quotients exist, they don't need to describe orbits. I.e.,  $G \backslash X$  must not be related to  $Y$ .

**Example 25.** Consider the action of  $G = \mathcal{G}_m$  on  $X = k^1$ . This action has two orbits: the open orbit  $k \setminus \{0\}$  and the closed orbit  $\{0\}$ .

Then the quotient of  $X$  by  $G$  is given by  $(Y, \rho) = (\{0\}, x \mapsto 0)$ .

Note, if  $f : X \rightarrow k$  is regular and constant on  $G$ -orbits, then  $f$  is constant on  $X$ , because  $k \setminus \{0\}$  lies dense in  $k$ .

**Definition 59.** Let  $G$  be a connected algebraic group.

We call  $G$  **geometrically reductive** if we have for each finite-dimensional representation  $V$  of  $G$ :

$$\forall v \in V^G \exists f : V \rightarrow k : f \text{ is a homogenous } G\text{-invariant polynomial s.t. } f(v) \neq 0$$

where

$$V^G = \{v \in V \mid g.v = v \ \forall g \in G\}.$$

*Remark 13.*  $G$  is geometrically reductive iff for each affine  $X$  on which  $G$  operates and for each pair of closed  $G$ -invariant disjoint subsets  $W_1, W_2 \subset X$  there is an  $f \in \mathcal{O}(X)^G$  s.t.

$$\begin{aligned} f|_{W_1} &\equiv 1, \\ f|_{W_2} &\equiv 0. \end{aligned}$$

Is this easy to see? I only see the backwards direction (take  $X = V, W_1 = 0, W_2 = v$ ).

**Theorem 26.** *Let  $G$  be a connected algebraic group.*

*Then,  $G$  is reductive iff  $G$  is geometrically reductive.*

**Theorem 27.** *Let  $G$  be a connected algebraic group which is geometrically reductive and acts on an affine set  $X$ .*

*Then, there is a quotient  $(Y, \rho)$  of  $X$  by  $G$ .*

*Moreover,  $\rho$  induces a bijection*

$$\{\text{closed } G\text{-orbits in } X\} \Longleftrightarrow Y.$$

**Definition 60.** Let  $G$  be a connected algebraic group.

We call  $G$  **linearly reductive** if we have for each finite-dimensional representation  $V$  of  $G$ :

$$\forall v \in V^G \setminus \{0\} \exists f : V \rightarrow k : f \text{ is a linear } G\text{-invariant polynomial s.t. } f(v) \neq 0.$$

*Remark 14.* Naturally, linear reductivity implies geometrical reductivity. The converse does hold iff  $\text{char } k = 0$ .

*Remark 15.*  $\text{GL}_n(k)$  is linear reductive.

*Remark 16.*  $G$  is linear reductive iff every finite-dimensional representation  $V$  of  $G$  is completely **reducible**, i.e.

$$V = \bigoplus_i V_i$$

where each  $V_i$  is irreducible.

## 17 Union of Borel Subgroups

**Theorem 28.** *Let  $G$  be a connected algebraic group. Then,*

$$G = \bigcup_{B \text{ Borel}} B.$$

Because of Jordan Decomposition, it is clear that the theorem holds for  $\mathrm{GL}_n(k)$ . We will prove it only for the case  $k = \mathbb{C}$ .

**Lemma 65.** *Let  $k$  be any (not necessarily algebraically closed) field. Let  $B$  be some Borel subgroup.*

*Then,  $X := \bigcup_{g \in G} gBg^{-1}$  is closed in  $G$ .*

*Proof.* Our intuition is as follows:

$gBg^{-1}$  only depends on  $gB \in G/B$ . Since  $B$  is Borel, ergo parabolic,  $G/B$  is projective, ergo somewhat 'compact'. Then,  $X = \bigcup_{g \in G} gBg^{-1}$  is a union of 'compactly-many' closed sets.

Now, the actual proof works as follows: We want to use that  $G/B \times G \rightarrow G$  is a closed map. Consider the chain

$$G \times B \xrightarrow{\phi(g,b)=(g,gbg^{-1})} G \times G \longrightarrow G/B \times G \longrightarrow G.$$

$X$  is the image of the composition  $(g, b) \mapsto gbg^{-1}$ . It therefore suffices to show that the image of

$$\pi \times \mathrm{Id} : G \times G \longrightarrow G/B \times G$$

is closed.

Set

$$Y := (\pi \times \mathrm{Id})(\phi(G \times B)).$$

If we can show, that  $(\pi \times \mathrm{Id})^{-1}(Y)$  is closed, then  $Y$  is closed, because  $\pi \times \mathrm{Id}$  is, as a morphism of homogenous spaces, open. However, we have

$$(\pi \times \mathrm{Id})^{-1}(Y) = \mathrm{Im} \phi.$$

Now,  $\mathrm{Im} \phi$  is closed, since morphisms of algebraic groups have closed images. □

**Lemma 66.** *Let  $k = \mathbb{C}$ .*

*Then,  $X = \bigcup_{g \in G} gBg^{-1}$  is dense in  $G$ .*

*Proof idea.* We want to show  $\overleftarrow{X} = G$ .

Since  $G$  is connected, it would suffice to show that  $X$  contains an Euclidean neighborhood of  $1 \in G$ .

Let  $\mathfrak{g} := \text{Lie}(G)$  be the Lie-algebra of  $G$ . A Borel-subalgebra  $\mathfrak{b} \subset \mathfrak{g}$  is a maximal solvable subalgebra.

Then, one can show, that for each Borel-subalgebra  $\mathfrak{b} \subset \mathfrak{g}$  there is a Borel-subgroup  $B \subset G$  s.t.  $\mathfrak{b} = \text{Lie}(B)$ .

Is easy to see, that each  $x \in \mathfrak{g}$  is contained in some Borel-subalgebra, since  $\mathbb{C} \cdot x$  is a solvable subalgebra.

With the two above facts, it follows that  $X$  contains a small euclidean neighborhood of 1.  $\square$

## 18 Splitting Solvable Groups

Let  $B$  be a connected solvable algebraic group. (Then,  $B$  is trigonalizable.)

Then,  $U := B_u$  is a unipotent normal algebraic subgroup (since  $U = R_u(B)$ , since  $B = R(B)$ ).

**Lemma 67.** *The group  $B/U$  is a torus.*

*Proof.* We have an injective morphism

$$B \hookrightarrow \left\{ \begin{pmatrix} * & \dots & * \\ & \ddots & \vdots \\ & & * \end{pmatrix} \right\}$$

with

$$U = B \cap \left\{ \begin{pmatrix} 1 & \dots & * \\ & \ddots & \vdots \\ & & 1 \end{pmatrix} \right\}.$$

Therefore, we get an injection

$$B/U \hookrightarrow \left\{ \begin{pmatrix} * & \dots & * \\ & \ddots & \vdots \\ & & * \end{pmatrix} \right\} / \left\{ \begin{pmatrix} 1 & \dots & * \\ & \ddots & \vdots \\ & & 1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \right\}.$$

Ergo,  $B/U$  is diagonalizable. Since  $B$  is connected,  $B/U$  is connected, too. It follows that  $B/U$  is a torus.  $\square$

**Theorem 29.** *Let  $B$  be a connected solvable algebraic group.*

*Then, there is a torus  $T \subset B$  s.t. the composition*

$$T \hookrightarrow B \twoheadrightarrow B/U$$

*is an isomorphism.*

Before, we can prove the theorem we need some lemmata:

**Lemma 68.** *Suppose  $\text{char} k = 0$ . Let  $T$  be a torus.*

*Then, there is an  $s \in T$  s.t.*

$$\overline{\langle s \rangle} = T.$$

*$s$  is called the **generator** of  $T$ .*

*Remark 17.* The lemma does not hold, if  $\text{char } k > 0$ .

*Proof.* Recall that we have the following correspondence:

$$\{\text{tori}\} \xleftrightarrow{T \mapsto \mathfrak{X}(T)} \{\text{f.g. free } \mathbb{Z}\text{-modules}\}.$$

and in particular for each torus  $T$ :

$$\begin{aligned} \{\text{alg. subgroups } H \text{ of } T\} &\longleftrightarrow \{\text{submodules } \Gamma \text{ of } \mathfrak{X}(T)\} \\ H &\longmapsto \{\chi \in \mathfrak{X}(T) \mid \chi \in \mathfrak{X}(T) : \chi|_H \equiv 1\} \\ \{t \in T \mid \chi(t) = 1 \ \forall \chi \in \Gamma\} &\longleftarrow \Gamma. \end{aligned}$$

So, we have  $\overline{\langle s \rangle}$  iff

$$\chi(s) \neq 1$$

for each  $1 \neq \chi \in \mathfrak{X}(T)$ .

W.l.o.g.  $T = (k^\times)^n$ . Then

$$\mathfrak{X}(T) = \{\chi_m \mid m \in \mathbb{Z}^n\}$$

with

$$\chi_m(t_1, \dots, t_n) = t_1^{m_1} \dots t_n^{m_n}.$$

We can then pick

$$s = (2, 3, 5, 7, \dots).$$

□

**Lemma 69.** *If  $\text{char } k = 0$ , then any bijective morphism of algebraic groups is an isomorphism of algebraic groups.*

*Remark 18.* This does not need to hold for non-zero characteristic. If  $\text{char } k = p$ , then

$$\begin{aligned} k &\longrightarrow k \\ x &\longmapsto x^p \end{aligned}$$

is bijective without being isomorphic.

*Proof of Theorem.* We only show the theorem in case  $\text{char } k = 0$ .

Let  $B$  be a connected solvable algebraic group.

We need to show, that there is a torus  $T \subset B$  s.t. the composition

$$T \hookrightarrow B \twoheadrightarrow B/U$$

is an isomorphism where

$$U = B_u.$$

We know, that  $B/U$  is a torus. Take  $s' \in B/U$  s.t.

$$\overline{\langle s' \rangle} = B/U.$$

Take a preimage  $g \in B$  s.t.  $\pi(g) = s'$ .

We can decompose  $g = su$  into a semisimple and a unipotent element. We then have

$$\phi(g) = \phi(s) \cdot \phi(u) = \phi(s),$$

since  $\phi(u)$  must be unipotent, ergo trivial.

Set

$$T = \overline{\langle s \rangle}.$$

Since  $s$  is semisimple,  $T$  must be diagonalizable. Ergo

$$T \cap U = 1.$$

Ergo, the chain

$$T \hookrightarrow B \twoheadrightarrow B/U$$

must be bijective, hence an isomorphism, since  $\text{char } k = 0$ . □

The theorem gives the structure of a semidirect product of algebraic groups:

$$B = U \rtimes T$$

(where  $T \curvearrowright U$  by conjugation.)

**Definition 61.** Let  $G_1, G_2$  be algebraic groups. Let  $G_2$  act algebraically on  $G_1$  via  $b : G_2 \rightarrow \text{Aut}(G_1)$  s.t. the map

$$\begin{aligned} G_2 \times G_1 &\longrightarrow G_1 \\ (g_2, g_1) &\longmapsto b(g_2)(g_1) \end{aligned}$$

is a morphism.

Their semidirect group  $G_1 \rtimes_b G_2$  is an algebraic group which is:

- set-theoretically  $G_1 \times G_2$ ,
- group-theoretically the semidirect product  $G_1 \rtimes_b G_2$ . I.e. multiplication works by

$$(g_1, g_2) \cdot (h_1, h_2) = (g_1 \cdot b(g_2)(h_1), g_2 h_2).$$

*Remark 19.* Even if we are given an algebraic group  $G$  with closed subgroups  $G_1, G_2$  s.t.

$$G = G_1 \rtimes G_2$$

as abstract groups, it does not need to be the case that

$$G_1 \rtimes G_2 \longrightarrow G$$

is an isomorphism. (However, it is the case, if  $\text{char } k = 0$ .)

## 18.1 An Aside

Let  $G$  be an algebraic group and  $H$  a normal algebraic subgroup.

Then,  $G/H$  is a quasi-projective variety equipped with a  $G$ -action. Ergo, we have an algebraic group structure on  $G/H$ .

**Theorem 30.**  *$G/H$  is an affine algebraic group.*

*Proof.* We need to show that  $G/H$  is affine.

We showed in a lemma long before, that there is a finite-dimensional representation  $V, \rho$  s.t.

$$H = \ker \rho.$$

Therefore, we can simply set

$$G/H := \text{Im}(\rho) \subset \text{GL}(V)$$

which is closed as  $\rho$  is a morphism of algebraic groups. □



## 18.2 Semisimple Elements of Solvable Groups

**Theorem 31.** *Let  $B = U \rtimes T$  as before be a solvable connected algebraic group. Let  $s \in B$  be semisimple. Then  $s$  is conjugated to one element in  $T$ .*

**Corollary 16.** *Let  $G$  be a connected algebraic group. Then, every semisimple element of  $G$  is contained in some torus.*

*Proof.* Let  $s \in G$  be semisimple and choose a Borel group  $B \subset G$  which contains  $s$ .  $B$  is of the form  $U \rtimes T$ , ergo  $s \in b^{-1}Tb$  for some  $b \in B$ .  $\square$

**Lemma 70.** *Suppose  $\text{char } k = 0$ .*

(i) *Let  $g \in GL_n(k)$  be unipotent and set  $G(g) := \overline{\langle g \rangle}$ . Then, we have the following isomorphism of algebraic groups*

$$G(g) = \{g^t \mid t \in k\} \cong k$$

where

$$\begin{aligned} g^t &:= \exp(t \cdot \log(g)) \\ -\log(1 - X) &= \sum_{n=1}^{\infty} \frac{X^n}{n} \\ \exp(Y) &= \sum_{k=0}^{\infty} \frac{Y^k}{k!}. \end{aligned}$$

(ii) *Any unipotent algebraic group is connected. (This does not hold if  $\text{char } k > 0$ .)*

(iii) *Any unipotent commutative algebraic group is isomorphic to some vector space.*

*Proof.* (i) We will not prove this, but the idea is that  $\mathbb{Z}$  is dense in  $k$ .

(ii) Let  $g, h \in G$  be unipotent. Then the subgroups  $G(g), G(h)$  are connected and share a common point  $(e)$ , ergo  $g, h$  are contained in the same component.

(iii) Since all elements commute  $\log$  gives an isomorphism into an additive group, on which  $k$  acts.  $\square$

*Proof of Theorem.* We only prove the theorem in case  $\text{char } k = 0$ .

Let  $s \in B = U \rtimes T$  be semisimple. Since  $\text{char } k = 0$ ,  $U$  is connected.

We induct on  $\dim(U)$ :

- $\dim(U) = 0$ : In this case  $U = 1$  and  $s \in G = T$ .
- $\dim(U) = 1$ : This is the crucial case.

Write

$$s = ut$$

with  $u \in U$  and  $t \in T$ .

If  $u$  and  $t$  commute, then  $ut$  is a Jordan decomposition and we have  $u = 1$ , ergo  $s \in T$ .

Assume therefore, that  $u, t$  don't commute. We claim:

**Claim:** For each  $h \in sU = Us$ , we have for the  $B$ -conjugacy class  $C(h) = \{ghg^{-1} \mid g \in B\}$

$$C(h) = sU.$$

The claim implies the theorem, because we then have

$$t = su \in sU = C(s)$$

ergo  $t = gsg^{-1}$ .

**Proof of Claim:**

- First note, that  $B$  acts by conjugation on  $Us = sU$ . This is because  $G/U$  is commutative and  $U$  is normal. In fact, we have for  $g \in B, u \in U$

$$gsug^{-1} = s \cdot (s^{-1}gsug^{-1}) = s \cdot (s^{-1}gsg^{-1}) \cdot u'.$$

Now,  $(s^{-1}gsg^{-1})$  must lie in  $U$  because  $G/U$  is commutative.

- Since  $\dim(U) = 1$ , we have

$$U = \{v^k \mid k \in K\} \cong k.$$

- $h \in sU$  does not commute with  $u$ , since – otherwise –  $s, t$  would commute with  $u$ .

Ergo,  $h \neq u^{-1}hu$ , which means  $C(h) \supseteq \{h, u^{-1}hu\}$  contains at least two different elements.

- Note, that  $C(h)$  is a  $B$ -orbit and therefore connected and **locally closed** (that is a closed subset of an open subset of  $G$ ). Since  $G/U$  is commutative, we have

$$C(h) \subset sU = hU \cong k.$$

Now, the only connected, locally closed subset of  $k$  are singletons and complements of finite sets.

Since  $C(h)$  is not a singleton, we have

$$C(h) = sU - \Sigma$$

for a finite set  $\Sigma$ .

We claim that  $\Sigma$  is empty. Note, that  $B$  acts by conjugation on  $sU$  and  $C(h)$ , ergo also on  $\Sigma$ . If we pick  $h' \in \Sigma \subset sU$ , then  $C(h')$  must be finite, connected and contain two different elements. This is a contradiction.

- $\dim(U) \geq 2$ :

We want to reduce this case to the case  $\dim(U) = 1$ . We need therefore, to show a lemma:

**Lemma 71.** *Let  $B = U \rtimes T$  as above and suppose again  $\text{char } k = 0$ .*

*Then, there is an algebraic subgroup  $V \subset U$  s.t.  $V$  is normal in  $B$  and*

$$\dim(U/V) = 1.$$

*Proof.*  $U$  is nilpotent, since it is unipotent. Consider the chain

$$U = U_0 \supset U_1 \supset \dots \supset U_n \supset 1$$

where

$$U_{i+1} := [U_i, U].$$

Since  $U$  is normal in  $B$ , each  $U_i$  is also normal in  $B$ . In particular,  $B$  acts on each  $U_i$  by conjugation.

Now,  $U/U_1$  is unipotent and commutative, hence isomorphic to a vector space.

Further,  $T$  acts on  $U/U_1$  by conjugation. Note, that is diagonalizable, ergo reductive. Therefore,  $U/U_1$  must be completely reducible and we can decompose it

$$U/U_1 = \bigoplus_j V_j.$$

Since  $T$  is diagonalizable and each  $V_j$  is  $T$ -invariant, each  $V_j$  must be one-dimensional. Set

$$\bar{V} := \bigoplus_{j \geq 2} V_j.$$

And now set

$$V := \pi^{-1}(\bar{V}) = \{u \in U \mid uU_1 \in \bar{V}\}.$$

Then, we have

$$U/V = (U/U_1)/(V/U_1) = (U/U_1)/\bar{V} \cong V_1 \cong k.$$

$V$  is normal in  $U$ , since  $U_1$  is normal in  $U$  and  $T$  acts on  $V$  and  $\bar{V}$  by conjugation.  $\square$

Let  $s \in B$  be semisimple and  $\dim(U) \geq 2$ . Choose  $V \subset U$  s.t.  $\dim(U/V) = 1$  and  $V$  is normal in  $B$ . Set

$$\begin{aligned} B' &:= B/V \\ U' &:= U/V. \end{aligned}$$

Then,  $B'$  is a connected algebraic group with

$$\begin{aligned} (B')_u &= U' \\ B'/U &\cong B/U = T \\ B' &= U' \rtimes T. \end{aligned}$$

Since  $\dim(U') = 1$ , we know that  $\pi_V(s) \in B$  is contained in a conjugacy class of  $T$ . Let  $s' \in B$  be the conjugate of  $s \in B$  s.t.  $\pi_V(s') \in T$ . Then,

$$s' \in TV.$$

But  $TV$  is a connected solvable algebraic group and we have

$$TV \cong V \rtimes T \subset U \times T.$$

Since  $(TV)_u = V$  and  $\dim(V) = \dim(U) - 1$ , the induction hypothesis does also hold in  $TV$ . Ergo,  $s'$  is conjugated to some element in  $T$ , as we wanted.  $\square$

## 19 All About Tori

### 19.1 Maximal Tori

**Definition 62.** A **maximal torus**  $T \subset G$  is a torus that is not contained in any larger torus.

Since  $G$  has a finite dimension, it always has at least one maximal torus.

**Example 26.** If  $G = \mathrm{GL}_n(k)$ , then  $k^\times$  is its maximal torus.

**Lemma 72.** *Let  $G$  be a connected algebraic group. Let  $s \in G$  be semisimple. Then, there is a maximal torus in  $G$  s.t.*

$$C(s) \cap T \neq \emptyset$$

where  $C(s) = \{gsg^{-1} \mid g \in G\}$ .

*Proof.* Choose a Borel group  $B \subset G$  s.t.  $s \in B$ . Then, we can decompose

$$B \cong U \rtimes T.$$

$T$  is a torus. By the previous theorem, we know that a conjugate of  $s$  is contained in  $T$ . The claim follows, if we enlarge  $T$  to a maximal torus in  $G$ .  $\square$

**Theorem 32.** *Let  $G$  be a connected algebraic group with a maximal torus  $T$ . Let  $s \in G$  be semisimple. Then,*

$$C(s) \cap T \neq \emptyset.$$

*Proof.* We only show the theorem in case  $\mathrm{char} k = 0$ .

Since  $T$  is connected and solvable, it is contained in some Borel group  $B$ .

Choose further a Borel group  $B' = U \rtimes S$  s.t.  $s \in B'$  is conjugate to some element  $s' \in S$ .

Now,  $B'$  is conjugate to  $B$ . If we choose a generator  $t \in T$  s.t.

$$\overline{\langle t \rangle} = T,$$

then  $t$  is conjugated to some element  $t' \in S$ . The torus  $T' = \overline{\langle t' \rangle}$  generated by  $t'$  is again maximal, therefore

$$T' = S.$$

Since  $s' \in T'$ , the claim follows.  $\square$

**Corollary 17.** *Let  $G$  be a connected algebraic group.*

(i) Any two maximal tori are conjugate.

(ii) For each torus  $S$  and each maximal torus  $T$  exists a  $g \in G$  s.t.

$$gSg^{-1} \subset T.$$

*Proof.* (i) follows, if we can show (ii).

Let  $S$  be torus with a generator  $s$ . Then there is a  $g \in G$  s.t.

$$gs g^{-1} \in T.$$

Ergo

$$gSg^{-1} = \overline{\langle gs g^{-1} \rangle} \subset T.$$

□

**Corollary 18.** *Let  $s$  be a central semisimple element of  $G$  (i.e.  $s$  commutates with each other element). Then  $s$  is contained in every maximal torus. In other words*

$$Z(G)_s \subset \bigcap_{T \text{ max. torus}} T.$$

*Proof.* This is clear, since a conjugate of  $s$  must be contained in each maximal torus and  $s$  commutes with each element. □

**Corollary 19.** *Let  $T$  be a torus in a connected group  $G$ . Then,  $T$  is maximal iff its dimension is maximal among all dimensions of tori in  $G$ .*

## 19.2 Centralizers of Tori

**Lemma 73.** *Let  $G$  be a connected algebraic group. Let  $S \subset T$  be a torus. Let  $g \in G$  be a semisimple element which commutes with each element of  $S$ .*

*Then,  $S \cup \{g\}$  is contained in some torus of  $G$ .*

*Proof.* Set  $H := Z_G(g)^\circ$ . Then,  $H$  is a connected algebraic group that contains  $S$ . Then,

$$g \in Z(H)_s \subset \bigcap_{T \text{ maximal tori in } H} T.$$

In particular, there must be some maximal torus of  $H$  which contains  $S$ . □

**Theorem 33.** *Let  $G$  be a connected algebraic group. Let  $S \subset T$  be a torus.*

*Then,  $Z_G(S)$  is connected.*

*Proof.* We assume  $\text{char } k = 0$ .

Let  $g \in Z_G(s)$ . Decompose  $g = g_s g_u$ . Then, we need too show the claim in case:

(i)  $g$  is semisimple:

By the previous lemma, there is a torus  $T \subset Z_G(S)^\circ$  which contains  $S$  and  $g$ .

(ii)  $g$  is unipotent:

Since  $k$  has characteristic zero, the group

$$\overline{\langle g \rangle} = g^k \cong \begin{cases} k, & g \neq 1 \\ 1, & g = 1 \end{cases}$$

is connected. □

### 19.3 Low Dimensional Groups

**Lemma 74.** *Let  $G$  be a connected algebraic group with a Borel subgroup  $B$ . If  $B$  is nilpotent, then  $G$  is solvable i.e.  $B = G$ .*

*Proof.* We induct on  $\dim(B)$ :

- $\dim(B) = 0$ :  
In this case, we have  $B = 1$ . Then,  $G = G/B$  must be projective and connected. Therefore, we must have

$$\mathcal{O}(G) = \mathcal{O}(\mathbb{P}^n)/I(G) = k$$

since  $\mathcal{O}(\mathbb{P}^n) = k$ . On the other side,  $G$  affine. Therefore, we have

$$G = 1.$$

Or:  $G = \bigcup_{g \in G} gBg^{-1}$ , since  $G$  is connected. Since  $B = 1$ , it follows  $G = 1$ .

- $\dim(B) \geq 1$ :  
Since  $B$  is nilpotent, we have a descending chain

$$B = B_0 \supsetneq \dots \supsetneq B_n \supsetneq 1$$

where

$$B_{i+1} = [B, B_i].$$

Note, that each  $B_i$  is connected, since  $B$  is connected. Let  $Z(B) = \{b \in B \mid \forall g \in B : gb = bg\}$  be the center of  $B$  and let  $Z := Z(B)^\circ$  be the component of the neutral element. Then, we have

$$B_n \subset Z.$$

Ergo,  $Z$  is not the trivial subgroup.

We want to show

$$Z \subset Z(G).$$

Let  $z \in Z$  and consider the morphism

$$\begin{aligned} \phi : G/B &\longrightarrow G \\ gB &\longmapsto gzg^{-1}. \end{aligned}$$



$\phi$  is well-defined, because  $z \in Z(B)$ . Since  $\phi$  is a morphism from a projective variety to an affine variety,  $\phi$  must be constant. Thus,

$$Z \subset Z(G).$$

In particular,  $Z$  is normal in  $G$ . We now get an inclusion of quotient groups

$$B/Z \hookrightarrow G/Z.$$

It is clear that

$$\dim(B/Z) < \dim(B).$$

Further,  $B/Z$  is parabolic, since

$$(G/Z)/(B/Z) = G/B$$

is projective. Ergo,  $B/Z$  is Borel. By the induction hypothesis, we get

$$G/Z = B/Z.$$

Ergo,  $B = G$ .

□

**Theorem 34.** *Let  $G$  be connected with  $\dim(G) \leq 2$ . Then,  $G$  is solvable.*

**Example 27** (Non-Example). The condition  $\dim(G) \leq 2$  is necessary. Consider e.g.  $G = \mathrm{SL}_2(k)$  which has a dimension of 3.

*Proof.* Let  $B \subset G$  be a Borel subgroup. We want to show

$$B = G.$$

Assume otherwise. Then,  $B$  is of dimension 1. The key here is, that Borel groups of dimension 1 are nilpotent.

Decompose  $B = U \rtimes T$ , then we have:

- $U \neq 1$ : Then,  $\dim(T) = \dim(B/U) = 0$ , ergo  $T = 1$ . Hence

$$B = U.$$

Since unipotent groups are nilpotent,  $B$  is nilpotent.

- $U = 1$ : In this case, we have

$$B = T.$$

Now  $B$  as a torus is commutative, ergo nilpotent.

Now, the above lemma states

$$B = G$$

since  $B$  is nilpotent. Ergo,  $G$  is solvable. □

**Corollary 20.** *Let  $G$  be connected with  $\dim(G) = 1$ . Then,  $G$  is commutative.*

*Proof.* Because of the theorem,  $G$  is solvable. Therefore,  $[G, G]$  is a closed proper subgroup of  $G$ . Hence,  $\dim([G, G]) = 0$ . Since  $[G, G]$  is connected, it follows  $[G, G] = 1$ . □

*Remark 20.* If  $G$  is commutative, it decomposes nicely into semisimple and unipotent elements

$$G = G_s \times G_u.$$

So, if  $\dim(G) = 1$  and if  $G$  is connected, then  $G = G_s \cong \mathcal{G}_m$  is a torus, or  $G = G_u \cong \mathcal{G}_a$  is unipotent.

Further, we can consider

$$G = \left\{ \begin{pmatrix} * & * \\ & 1 \end{pmatrix} \right\}.$$

$G$  is connected and of dimension 2. It decomposes

$$G = G_s \times G_u$$

into two groups of dimension 1.

## 19.4 Characterizing Nilpotent Groups via maximal Tori

**Lemma 75.** *Let  $T$  be a diagonalizable algebraic group. Then,*

$$T = \overline{\bigcup_{n \geq 1} T[n]}$$

where

$$T[n] := \{t \in T \mid t^n = 1\}.$$

*Proof.* If the claim is true for any  $T_1, T_2$ , then it is also true for  $T_1 \times T_2$ .

Therefore, we can reduce the claim to the cases where  $T$  is finite or  $T = \mathcal{G}_m$ .

A finite  $T$  is contained in some  $T[n]$ .

Let  $T = \mathcal{G}_m$ . Then, we have

$$\mathcal{O}(T) = k[x, \frac{1}{x}].$$

We need to show for each  $f \in \mathcal{O}(T)$ :

$$f(\alpha) = 0 \ \forall \alpha \in k, \in \mathbb{N}_0 \text{ s.t. } \alpha^n = 1 \implies f = 0.$$

So, let  $f \in \mathcal{O}(T)$ . By multiplying with a large enough  $x^r$ , we can assume  $f \in k[x]$ .

If  $f \neq 0$ , then  $f$  has finitely many roots. However the set

$$\{\alpha \in k \mid \exists n \in \mathbb{N}_0 : \alpha^n = 1\}$$

has infinitely many elements. Indeed we have

$$\# \{\alpha \in k \mid \alpha^n = 1\} = n,$$

if  $\text{char} k = 0$  or if  $\gcd(\text{char} k, n) = 1$ . □

*Remark 21.* If  $\text{char} k = 0$  and if  $U$  is unipotent, then

$$U[n] = 1$$

for all  $n$ .

**Theorem 35.** *Let  $G$  be a connected algebraic group. Then, the following are equivalent:*

(i)  $G$  is nilpotent.

(ii) Each maximal torus  $T$  of  $G$  satisfies  $T \subset Z(G)$ .

(iii)  $G$  has a unique maximal torus.

*Proof.* We show:

(i)  $\implies$  (ii): We have shown, if  $G$  is nilpotent and connected, then each semisimple element is central.

(ii)  $\implies$  (iii): Any two maximal tori are conjugated.

(iii) + (ii)  $\implies$  (i): Since each semisimple element is contained in some torus, we have that each semisimple element is central.

Let  $B = U \rtimes T' \subset G$  be a Borel subgroup. Since  $T' \subset T$  is central in  $G$ , we have  $B = U \times T'$ . So,  $B$  is nilpotent. We showed in this case that

$$G = B.$$

(iii)  $\implies$  (ii): Let  $T$  be the maximal torus of  $G$ . We have to show that  $T$  is central.

Since  $T$  is unique, it must be normal in  $G$ . Then,  $G$  acts via conjugation on  $T$  and each  $T[n]$ . Therefore, we get a morphism for each  $t \in T[n]$

$$\begin{aligned} G &\longrightarrow T[n] \\ g &\longmapsto gtg^{-1}. \end{aligned}$$

Since  $G$  is connected, this morphism must be constant, ergo trivial. Ergo, each  $T[n]$  is central in  $G$ . Since

$$T = \overline{\bigcup_{n \in \mathbb{N}} T[n]},$$

$T$  must be central in  $G$ .

□

## 19.5 Weyl Groups

**Definition 63.** Let  $G$  be a connected algebraic group with a torus  $T$ . Define the **normalizer** of  $T$  in  $G$  by

$$N_G(T) := \{g \in G \mid gT = Tg\}.$$

Then, the centralizer  $Z_G(T)$  is a normal subgroup in  $N_G(T)$ . Define the **Weyl group** of  $T$  as the quotient

$$W(G, T) := N_G(T)/Z_G(T).$$

If  $T$  is maximal, then the Weyl group  $W(G, T)$  is up to conjugation independent of  $T$ .

**Proposition 7.** (i)  $\#W < \infty$ .

(ii) For each tori  $S \subset G$ , we have

$$N_G(S)^o = Z_G(S)^o.$$

*Proof.* It is clear, that (ii) implies (i).

Let  $S \subset G$  be a torus. We want to show

$$N_G(S)^o \subset Z_G(S).$$

Note, that  $N_G(S)^o$  acts by conjugation on  $S$ . Now, it is clear that

$$S = \bigcup_n \overline{S[n]}$$

where  $S[n]$  denotes the subgroup of  $n$ -th roots of unity.  $N_G(S)^o$  acts on each  $S[n]$ . As before, this gives for each  $s \in S[n]$  a group morphism  $\phi_s : N_G(S)^o \rightarrow S[n]$ . Since  $S[n]$  is finite and  $N_G(S)^o$  is connected,  $\phi_s$  must be trivial. Ergo

$$N_G(S)^o \subset Z_G(S).$$

□

*Remark 22.* In general,  $W(G, T)$  acts on  $T$  by conjugation and induces an inclusion

$$W(G, T) \hookrightarrow \text{Aut}(T).$$

**Example 28.** Let  $G = \mathrm{GL}_n(k)$  with the maximal torus

$$T = \left\{ \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \right\}.$$

Denote by  $S(n)$  the group of all permutation matrices of  $G$ .

Then, we have

$$\begin{aligned} Z_G(T) &= T \\ N_G(T) &= T \cdot S(n) \\ W(G, T) &\cong S(n). \end{aligned}$$

## 19.6 Normalizers of Borel Subgroups

**Lemma 76.** *Let  $G$  be a connected algebraic group,  $B \subset G$  a Borel subgroup and  $S \subset B$  any torus.*

*Then,  $Z_B(S)$  is a Borel subgroup of  $Z_G(S)$ .*

*Proof.* We showed before, that  $Z_G(S)$  is connected, if  $G$  is connected. Set

$$U := B_u.$$

- We claim

$$Z_G(S)B = \{g \in G \mid \forall s \in S : g^{-1}sg \in sU\} =: A.$$

It is easy to see, that

$$Z_G(S) \subset A.$$

For  $b \in B$ , we have

$$b^{-1}sb \in sU,$$

since  $B/U$  is commutative.

Now, let  $g \in A$ . Then,

$$g^{-1}Sg \subset sU \subset B.$$

One can extend  $S$  to a maximal torus  $T$  of  $B$ . Then,

$$B = U \rtimes T \supset sU = U \rtimes S.$$

Since  $S$  is closed in  $T$ ,  $sU$  is closed in  $B$ . Further,  $g^{-1}Sg$  and  $S$  are maximal tori in  $sU$ . Then, there is a  $b \in B$  s.t.

$$b(g^{-1}Sg)b^{-1} = S.$$

Set

$$z := gb^{-1}.$$

We need to show, that  $z$  lies in  $Z_G(S)$ .

Since  $B/U$  is commutative, we have for each  $s \in S$

$$z^{-1}sz = b(g^{-1}sg)b^{-1} \in g^{-1}sgU = sU,$$

since  $g \in A$ . Now, we have for each  $s \in S$

$$z^{-1}sz \in sU \cap S = \{s\}.$$

Ergo,  $z \in Z_G(S)$ .

- We showed that

$$Z_G(S)B = \{g \in G \mid \forall s \in S : g^{-1}sg \in sU\} = \{g \in G \mid \forall s \in S : [g, s] \in U\}.$$

Then,  $Z_G(S)B$  is closed. Since

$$\pi : G \twoheadrightarrow G/B$$

is an open and surjective map, it is easy to see that

$$Z_G(S)/Z_B(S) \cong \pi(Z_G(S))$$

is closed. Since  $Z_B(S)$  is closed,  $Z_B(S)$  is a parabolic subgroup of  $Z_G(S)$ . Since  $Z_B(S)$  is contained in  $B$ , it is solvable, hence a Borel subgroup.

□

**Example 29.**

$$\begin{aligned} G &= \mathrm{GL}_5(k) \\ B &= \left\{ \begin{pmatrix} * & * & * & * & * \\ & * & * & * & * \\ & & * & * & * \\ & & & * & * \\ & & & & * \end{pmatrix} \right\} \\ S &= \left\{ \begin{pmatrix} t_1 & & & & \\ & t_1 & & & \\ & & t_1 & & \\ & & & t_2 & \\ & & & & t_2 \end{pmatrix} \mid t_1, t_2 \in k^\times \right\} \\ Z_G(S) &= \left\{ \begin{pmatrix} * & * & * & & \\ * & * & * & & \\ * & * & * & & \\ & & & * & * \\ & & & * & * \end{pmatrix} \right\} = \mathrm{GL}_3(k) \times \mathrm{GL}_2(k) \\ Z_G(S) &= \left\{ \begin{pmatrix} * & * & * & & \\ & * & * & & \\ & & * & & \\ & & & * & * \\ & & & & * \end{pmatrix} \right\}. \end{aligned}$$



**Theorem 36.** *Let  $G$  be a connected algebraic group with a Borel subgroup  $B \subset G$ . Then,*

$$N_G(B) = B.$$

*Proof.* We induct on  $\dim(G)$ :

- $\dim(G) = 0$ : In this case, we have  $G = 1$ .
- $\dim(G) = 1$ : We have seen, that in this case  $G$  is commutative.
- $\dim(G) \geq 2$ : Let  $T$  be a maximal torus in  $B$ . Let  $x \in N_G(B)$ .

Then,  $xTx^{-1}$  is again a maximal torus in  $B$ . Since all maximal tori in  $B$  are related via  $B$ -conjugation, there is  $b \in B$  s.t.

$$xTx^{-1} = bTb^{-1}.$$

We therefore replace  $x$  by  $b^{-1}x$  to achieve

$$xTx^{-1} = T.$$

Now, consider the map

$$\begin{aligned} \rho : T &\longrightarrow T \\ t &\longmapsto txt^{-1}x^{-1}. \end{aligned}$$

We distinguish two cases:

- $\rho$  is not surjective:  
Since all tori are irreducible, we have then

$$\begin{aligned} \dim(\text{Im}(\rho)) &< T \\ \dim(\text{Kern}(\rho)^o) &> 0. \end{aligned}$$

If we set  $S := \text{Kern}(\rho)^o$ , then  $S$  is a non-trivial torus in  $T$ .

Since  $S \subset \text{Kern}(\rho)$ ,  $x$  centralizes  $S$  and normalizes  $B$ . Hence,  $x$  normalizes  $Z_B(S)$ .

Because of the previous lemma,  $Z_B(S)$  is Borel subgroup of  $Z_G(S)$ . If  $Z_G(S) \neq G$ , then the induction hypothesis implies

$$x \in N_{Z_G(S)}(Z_B(S)) = Z_B(S) \subset B.$$

Otherwise, if  $Z_G(S) = G$ , then  $B/S$  is a Borel subgroup of  $G/S$ . So the induction hypothesis implies

$$xS \in N_{G/S}(B/S) = B/S,$$

ergo  $x \in B$ .

–  $\rho$  is surjective:

Then,

$$T = \text{Im } \rho \subset [N_G(B), N_G(B)].$$

Set  $H := N_G(B)$ . We have to show

$$H = B.$$

Choose a finite-dimensional representation

$$G \hookrightarrow \text{GL}(V)$$

and a line  $L \subset V$  s.t.

$$H = \{g \in G \mid gL = L\}.$$

Then, we have a morphism of algebraic groups

$$\gamma : H \longrightarrow \text{GL}(L) = \mathcal{G}_m(k).$$

Since the right side is a torus, we have

$$\gamma|_{H_u} \equiv 1$$

$$\gamma|_{[H, H]} \equiv 1.$$

Ergo,  $\gamma(T) = 1$  and, since  $B = B_u \rtimes T$ ,  $\gamma(B) = 1$ .

Choose a non-zero element  $v \in L$  and consider

$$\begin{aligned} \phi : G/B &\longrightarrow V \\ gB &\longmapsto gv. \end{aligned}$$

Since  $G/B$  is a projective variety, while  $V$  is an affine variety,  $\phi$  must be constant. Therefore, we have for each  $g \in G$

$$gv \in L.$$

Ergo,  $G = H$  and  $B$  is normal in  $G$ . But, now

$$G = \bigcup_{g \in G} gBg^{-1} = B.$$

Ergo  $H = B$ . □

**Corollary 21.** *We have a bijection:*

$$\begin{aligned} G/B &\longrightarrow \{\text{Borel Subgroups of } G\} \\ gB &\longmapsto gBg^{-1}. \end{aligned}$$

## 19.7 Borel Subgroups Containing a Given Torus

Let  $G$  be a connected algebraic group with a maximal torus  $T$ . Set

$$\mathcal{B}^T := \{B \subset G \text{ Borel} \mid T \subset B\}.$$

Then,  $N_G(T)$  acts on  $\mathcal{B}^T$  by conjugation.

**Example 30.** Let  $G = \mathrm{GL}_2(k)$  with  $T = \left\{ \begin{pmatrix} * & \\ & * \end{pmatrix} \right\}$ . Then,

$$\mathcal{B}^T = \left\{ \begin{pmatrix} * & * \\ & * \end{pmatrix}, \begin{pmatrix} * & \\ * & * \end{pmatrix} \right\}.$$

**Lemma 77.** *The action of  $Z_G(T)$  on  $\mathcal{B}^T$  by conjugation is trivial.*

*Equivalently (since  $B = N_G(B)$ ),  $Z_G(T) \subset B$  for each  $B \in \mathcal{B}^T$ .*

*Proof.* We know, that  $Z_G(T)$  is connected, since  $T$  is a torus. Further, since  $T \subset Z_G(T)$  is central and a maximal torus, it must be the unique maximal torus in  $Z_G(T)$ . We showed before, that this is equivalent to  $Z_G(T)$  being nilpotent. Thus,  $Z_G(T)$  is contained in some Borel group  $B_0 \in \mathcal{B}^T$ .

Let  $B \in \mathcal{B}^T$  and choose  $g \in G$  s.t.

$$B = gB_0g^{-1}.$$

Since maximal tori in  $B$  are  $B$ -conjugated, we can choose  $g \in G$  s.t.  $g \in N_G(T)$ . (Otherwise, we can replace  $g$  by  $bg$  s.t.  $bgTg^{-1}b^{-1} = T$ .)

One can show that

$$g \in N_G(T) \implies g \in N_G(Z_G(T)).$$

Thus

$$g^{-1}Z_G(T)g = Z_G(T) \subset B_0$$

which implies

$$Z_G(T) \subset gB_0g^{-1} = B.$$

□

**Corollary 22.** *The action  $N_G(T) \curvearrowright \mathcal{B}^T$  induces an action by the Weyl group  $W(G, T) = N_G(T)/Z_G(T)$  on  $\mathcal{B}^T$ .*

**Corollary 23.** *In the proof, we could see that  $N_G(T)$  and  $W(G, T)$  act transitively on  $\mathcal{B}^T$ .*

**Corollary 24.**

$$\#\mathcal{B}^T \leq \#W < \infty.$$

**Theorem 37.**  $W$  acts *simply-transitively* on  $\mathcal{B}^T$ , i.e., for each  $B_1, B_2 \in \mathcal{B}^T$  there is exactly one  $g \in G$  s.t.

$$gB_1g^{-1} = B_2.$$

In particular,

$$\#\mathcal{B}^T = \#W.$$

*Proof.* Let  $B \in \mathcal{B}^T$ . We need to show

$$N_G(T) \cap N_G(B) \subset Z_G(T).$$

Note, that

$$N_G(T) \cap N_G(B) = N_G(T) \cap B = N_B(T).$$

Set  $U := B_u$ , then  $B = U \rtimes T$ .

Choose  $b \in N_B(T)$  with  $b = ut$ ,  $u \in U, t \in T$ . Then,

$$T = bTb^{-1} = uTu^{-1}.$$

Since  $t \in Z_G(T)$ , it suffices to show that  $u \in Z_G(T)$ .

Let  $t \in T$  and set  $t' = utu^{-1} \in T$ . Since, we have an isomorphism

$$T \hookrightarrow B \twoheadrightarrow B/U$$

and  $B/U$  is commutative,  $t$  and  $t'$  must be equal in  $T$ . Ergo,  $u \in Z_G(T)$ .  $\square$

**Corollary 25.** Since  $N_B(T) \subset Z_G(T)$  we have for each Borel group  $B$  and maximal torus  $T$  of  $G$

$$W(G, T) = 1.$$

In particular,

$$\mathcal{B}^T = \{B\}.$$

**Proposition 8.** Let  $G$  be a connected non-solvable algebraic group (this implies  $\dim G \geq 3$ ). Let  $B$  be a Borel subgroup with a maximal torus  $T$ . Then,

$$\#W(G, T) \geq 2.$$

Moreover,

$$\#W = 2 \iff \dim(G/B) = 1.$$

*Sketch of Proof.* We have a bijection

$$\{\text{Borel subgroups in } G\} \longleftrightarrow G/B.$$

This can be restricted to a bijection

$$\mathcal{B}^T \longleftrightarrow \{gB \in G/B \mid TgB = gB\}.$$

**Idea:** Show that  $T$  acts non-trivially on  $G/B$ , since  $G$  is non-solvable, and deduce that it has at least two different fixed points:

$$\begin{array}{ccccc}
 \mathcal{G}_m & \hookrightarrow & T & \xrightarrow{\sim} & G/B \\
 \downarrow & & & \nearrow & \\
 \mathbb{P}^1 & & & \ni 0, \infty & 
 \end{array}$$

One can show, that  $0$  and  $\infty$  are mapped to different fixed points of  $G/B$ . □

## 19.8 Groups of Semisimple Rank One

**Definition 64.** Let  $G$  be an algebraic group. The **rank** of  $G$  is defined by

$$\text{rank}(G) := \dim(T)$$

for any maximal torus  $T$ .

*Remark 23.* If  $G$  is connected and of rank zero, then  $G$  is unipotent.

**Definition 65.** The **semisimple rank** of  $G$  is defined by

$$\text{ss-rank}(G) := \text{rank}(G/R(G)).$$

(Note, that  $G/R(G)$  is semisimple.)

**Example 31.** • If  $T$  is a torus, then  $\text{ss-rank}(T) = 0$ .

- Let  $Z = Z_{\text{GL}_n(k)}(\text{GL}_n(k))$  be the centralizer of  $\text{GL}_n(k)$ . Then,  $Z \cong k$ . For a matrix group  $G \subset \text{GL}_n(k)$ , set

$$PG := G/(G \cap Z).$$

Then,  $PG$  acts on  $\mathbb{P}^n$ .

We now have

| $G$            | $\text{ss-rank}(G)$ | $\text{rank}(G)$ |
|----------------|---------------------|------------------|
| $\text{SL}_2$  | 1                   | 1                |
| $\text{PGL}_2$ | 1                   | 1                |
| $\text{GL}_2$  | 1                   | 2                |
| $\text{GL}_n$  | $n - 1$             | $n$              |

- Consider

$$G = \left\{ \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \right\} \subset \text{GL}_3(k).$$

Then,

$$\begin{aligned} \text{rank}(G) &= 3, \\ \text{ss-rank}(G) &= 1. \end{aligned}$$

*Remark 24.* Let  $G$  be a connected algebraic group. Then,  $G$  is of semisimple rank zero iff  $G$  is Borel (because  $R(G)$  is the connected component of 1 in the intersection of all Borel subgroups in  $G$ ).

**Lemma 78** (Fact). *We call  $X$  a **curve**, if  $X$  is a one-dimensional variety.*

*Let  $X$  be a smooth projective curve  $X$  that admits a nontrivial action by a non-trivial connected algebraic group  $H$ . Then, we have*

$$X \cong \mathbb{P}^1.$$

*Idea of Proof.* Reduce to the case  $H = \mathcal{G}_m$  or  $H = \mathcal{G}_a$ . They give a commutative diagram:

$$\begin{array}{ccc} \mathcal{G}_m & \longrightarrow & X \\ \downarrow & \nearrow \exists \phi & \\ \mathbb{P}^1 & & \end{array}$$

Now,  $\phi$  is a non-constant orbit map, therefore

$$k(X) \hookrightarrow k(\mathbb{P}^1) \cong k(T).$$

By Lüroth's Theorem<sup>5</sup>, there is a transcendent  $T' \in k(T)$  s.t.  $k(X) = k(T')$ . Ergo

$$X = \mathbb{P}^1. \quad \square$$

**Lemma 79** (Fact). *We have*

$$\text{Aut}(\mathbb{P}^1) \cong \text{PGL}_2(k).$$

**Proposition 9.** *Let  $G$  be of semisimple rank one. Then, there is a surjective morphism*

$$\rho : G \rightarrow \text{PGL}_2(k)$$

*s.t.  $\text{Ker} \rho = R(G)$ .*

*In particular, if  $G$  is semisimple, then  $\text{Ker} \rho$  is finite, since  $R(G)$  is trivial in this case.*

---

<sup>5</sup>Lüroth's Theorem states that each intermediate field  $L \supset P \supset K$  of a purely transcendental extension  $L \supset K$  of degree 1 is either  $K$  or purely transcendental over  $K$ .



*Proof.* By dividing out  $R(G)$ , we can reduce the proof to the case, in which  $G$  is semisimple and of rank 1. In particular,  $G$  cannot be solvable, ergo  $\dim(G) \geq 3$ .

We have seen for unsolvable groups, that  $\#W(G, T) \geq 2$ . But, since  $T \cong \mathcal{G}_m$

$$W(G, T) = N_G(T)/Z_G(T) \hookrightarrow \text{Aut}(T) = \{\text{Id}, t \mapsto t^{-1}\} \cong \mathbb{Z}/2\mathbb{Z}.$$

Ergo,  $\#W(G, T) = 2$ . We have seen, that we have in this case

$$\dim(G/B) = 1.$$

Ergo,  $G/B$  is a projective one-dimensional variety. Ergo,  $G/B \cong \mathbb{P}^1$ . Now, define

$$\begin{aligned} \rho : G &\longrightarrow \text{Aut}(G/B) \cong \text{Aut}(\mathbb{P}^1) \cong \text{PGL}_2(k) \\ g &\longmapsto [xB \mapsto gx B]. \end{aligned}$$

Clearly,

$$\text{Kern}\rho = \{g \in G \mid gx B = x B \forall x \in G\} = \bigcap_{x \in G} x B x^{-1} = \bigcap \{B \subset G \text{ Borel}\}.$$

Ergo  $(\text{Kern}\rho)^0 = R(G) = 1$ , since  $G$  is semisimple.

It remains to show, that  $\rho$  is surjective. Indeed, we have

$$\dim(\rho(G)) \geq \dim(G) - \dim(\text{Kern}\rho) \geq 3.$$

Since  $\text{PGL}_2(k)$  is 3-dimensional and connected, we have

$$\rho(G) = \text{PGL}_2(k). \quad \square$$

**Proposition 10.** *Let  $G$  be a reductive algebraic group of semisimple rank one.*

*Let  $\rho : G \twoheadrightarrow \text{Aut}(G/B) \cong \text{PGL}_2(k)$  be as in the previous proposition. Then,  $\ker \rho$  is diagonalizable.*

*Proof.* Let  $T$  be a maximal torus in  $G$ . It suffices to show that

$$\text{Kern}\rho \subset T.$$

By the above proof

$$2 = \#W(G, T) = \#\mathcal{B}^T.$$

Therefore,

$$\mathcal{B}^T = \{B^+, B^-\}.$$

Since  $\text{Kern}\rho \subset \bigcap_{B \subset G \text{ Borel}} B = B^+ \cap B^-$  it suffices to show

$$B^+ \cap B^- = T.$$

Which is equivalent to

$$B_u^+ \cap B_u^- = 1.$$

Since  $G$  has a semisimple rank of one, we have  $G \neq B^\pm$ . Ergo,  $B^\pm$  is not nilpotent (otherwise  $G = B^\pm$ ). Thus,  $B_u^\pm$  is connected and non-trivial. Thus,

$$\dim(B_u^\pm) \geq 1.$$

Also,

$$(\text{Kern}\rho)^o \cap B_u^\pm = R(G) \cap B_u^\pm \subset R_u(G) = 1$$

since  $G$  is reductive. So,  $\text{Kern}(\rho|_{B_u^\pm})$  is finite. And therefore

$$\dim(\rho(B_u^\pm)) \geq 1.$$

But  $\rho(B_u^\pm)$  is a unipotent subgroup of  $PGL_2(k)$ . Therefore

$$\dim(\rho(B_u^\pm)) = 1.$$

Since  $\text{Kern}(\rho|_{B_u^\pm})$  is finite, we have

$$\dim B_u^\pm = 1.$$

Ergo,  $B_u^\pm \cong \mathcal{G}_a$ .

In characteristic zero, we proved that  $T$  acts on  $B_u^\pm$  by conjugation. We have that the composition

$$T \longrightarrow GL(\mathcal{G}_a) \cong \mathcal{G}_m$$

is nontrivial, since  $B^\pm$  is not nilpotent.

We still want to show

$$B_u^+ \cap B_u^- = 1.$$

Assume, for the sake of contradiction, that we have a non-trivial  $x \in B_u^+ \cap B_u^-$ . Then,  $T.x = \{txt^{-1} \mid t \in T\}$  lies dense in  $B_u^+ \cap B_u^-$ , in fact

$$T.x \cong \mathcal{G}_a \setminus \{0\}.$$

Since  $B_u^\pm$  is one-dimensional, it follows  $B_u^+ = B_u^-$ . Therefore,

$$B^+ = B^-.$$

This is a contradiction to  $\#W(G, T) = 2$ . □

**Corollary 26** (Eventual Corollary). *Any connected reductive group  $G$  of semisimple rank one is isomorphic to one of the following:*

$$\begin{aligned} &SL_2(k) \times T \\ &PGL_2(k) \times T \\ &GL_2(k) \times T \end{aligned}$$

*for some torus  $T$ .*

## 19.9 Isogenies

**Definition 66.** A morphism  $\phi : G_1 \rightarrow G_2$  of connected algebraic groups is an **isogeny**, if  $\phi$  is surjective and  $\text{Kern}\phi$  is finite.

An isogeny is called **multiplicative**, if moreover  $\text{Kern}\phi$  is diagonalizable.

Thus, the last result said:

If  $G$  is a semisimple connected group of rank one, then there is a multiplicative isogeny  $G \twoheadrightarrow \text{PGL}_2(k)$ . (Which implies  $G \cong \text{PGL}_2(k)$  or  $G \cong \text{SL}_2(k)$ .)

**Definition 67.**  $G$  is **simply connected**, if every multiplicative isogeny  $\tilde{G} \twoheadrightarrow G$  is an isomorphism.

*Remark 25 (Facts).*      • If  $G$  is semisimple, then there is a simply connected semisimple  $G^{sc}$  and a multiplicative isogeny  $G^{sc} \twoheadrightarrow G$  s.t.  $G^{sc} \twoheadrightarrow G$  is initial in the category of all multiplicative isogenies  $\tilde{G} \twoheadrightarrow G$ .

- $\text{SL}_2(k) \twoheadrightarrow \text{PGL}_2(k)$  is simply connected.

## 20 Root Data

### 20.1 More on Reductive Groups

**Proposition 11.** *Let  $G$  be a connected, reductive algebraic group. Then,*

$$R(G) = Z(G)^\circ.$$

*Proof.*  $Z(G)^\circ$  is connected, normal and commutative, ergo solvable. Ergo  $Z(G)^\circ$  is contained in  $R(G)$ .

Since  $R(G)$  is a normal subgroup of the connected group  $G$ , we have

$$G = N_G(R(G)) = N_G(R(G))^\circ.$$

Since  $R(G)$  is a torus in a connected group  $G$ , we have

$$N_G(R(G))^\circ = Z_G(R(G))^\circ.$$

$G = Z_G(R(G))^\circ$  implies  $R(G) \subset Z(G)$ . □

**Proposition 12.** *Let  $G$  be a connected, reductive algebraic group. Then,*

$$R(G) \cap [G, G]$$

*is finite.*

*Proof.* Take a faithful representation  $G \hookrightarrow \mathrm{GL}(V)$ .  $R(G)$  is a torus in  $G$ , therefore we can decompose  $V$  into eigenspaces of  $R(G)$ :

$$V = \bigoplus_{\chi} V_{\chi}$$

where  $\chi \in \mathfrak{X}(R(G))$  and

$$V_{\chi} = \{v \in V \mid h.v = \chi(h)v \ \forall h \in R(G)\}.$$

Since  $R(G)$  is normal in  $G$ ,  $G$  acts on each  $V_{\chi}$ . Consider the representations

$$\rho_{\chi} : G \longrightarrow \mathrm{GL}(V_{\chi}).$$

It is easy to see that

$$\rho_{\chi}([G, G]) \subseteq \mathrm{SL}(V_{\chi})$$

and

$$\rho_\chi(R(G)) \subseteq Z_{\mathrm{GL}(V_\chi)}(\mathrm{GL}(V_\chi)) = k^\times.$$

Ergo,

$$\rho_\chi(R(G) \cap [G, G]) \subseteq \mu_{\dim(V_\chi)}.$$

And, therefore,

$$\#([G, G] \cap R(G)) \leq \prod_{\chi: V_\chi \neq 0} \dim(V_\chi). \quad \square$$

**Example 32.** Let  $G = \mathrm{GL}_n(k)$ . Then

$$R(G) = k^\times \cdot 1_n.$$

$$[G, G] = \mathrm{SL}_n.$$

Ergo,

$$R(G) \cap [G, G] \cong \mu_n.$$

**Proposition 13.** *Let  $G$  be a connected, reductive algebraic group. Then,  $[G, G]$  is semisimple, i.e.  $R([G, G]) = 1$ .*

*Proof.* If  $B'$  is Borel group in  $[G, G]$ , then  $gBg^{-1}$  stays a Borel group in  $[G, G]$  for each  $g \in G$ . Therefore,  $R([G, G])$  is normal in  $G$ .

Now, take a Borel subgroup  $B$  of  $G$  s.t.

$$R([G, G]) \subset B.$$

Then, we have

$$R([G, G]) \subset \bigcap_{g \in G} gBg^{-1} = R(G).$$

So,

$$R([G, G]) \subset R(G) \cap [G, G].$$

Since  $R([G, G])$  is finite and connected, it is trivial. Ergo,  $[G, G]$  is semisimple.  $\square$

**Proposition 14.** *Let  $G$  be a connected, reductive algebraic group.*

*Let  $S \subset G$  be a torus. Then,  $Z_G(S)$  is reductive.*

*If  $T$  is a maximal torus, then  $Z_G(T) = T$ .*

*Proof.* Since  $S$  is a torus,  $Z_G(S)$  is connected. Note that:

- (i) every Borel subgroup of  $Z_G(S)$  is contained in some Borel subgroup of  $G$ .

(ii) for each Borel  $B \subset G$  which contains  $S$ , the intersection

$$Z_B(S) = Z_G(S) \cap B$$

is a Borel subgroup of  $Z_G(S)$ .

(iii) From the above, it follows

$$R(Z_G(S)) = \bigcap_{S \subset B \subset G \text{ Borel}} Z_B(S) \subset \bigcap_{S \subset B \subset G \text{ Borel}} B.$$

(iv) Since  $R_u(G)$  is connected, we have

$$R_u(G) = \left( \bigcap_{B \subset G \text{ Borel}} B \right)_u^o.$$

(v) One can show for any maximal torus  $T$

$$R_u(G) = \left( \bigcap_{T \subset B \subset G \text{ Borel}} B \right)_u^o.$$

Now, it follows

$$R_u(Z_G(S)) \subset \left( \bigcap_{S \subset B \subset G \text{ Borel}} B \right)_u^o \subset R_u(G) = 1.$$

For the second part:  $T$  is central in  $Z_G(T)$  and a maximal torus in  $Z_G(T)$ . Therefore,  $Z_G(T)$  must be nilpotent. Now,  $W(Z_G(T), T) = 1$ , ergo  $Z_G(T)$  only has one Borel subgroup  $R(Z_G(T))$ . But  $R(Z_G(T))$  must be a torus, hence

$$R(Z_G(T)) = T. \quad \square$$

## 20.2 Root Data – Definition

**Definition 68.** A **root datum** is a tuple

$$\Psi = (X, X^\vee, R, R^\vee)$$

where  $X, X^\vee$  are finitely generated free  $\mathbb{Z}$ -modules equipped with a pairing

$$\begin{aligned} X \times X^\vee &\longrightarrow \mathbb{Z} \\ (x, \xi) &\longmapsto \langle x \mid \xi \rangle \end{aligned}$$

which is **perfect**, i.e.  $\langle \cdot \mid \cdot \rangle$  induces isomorphism  $X \cong \operatorname{Hom}_{\mathbb{Z}}(X^\vee, \mathbb{Z})$  and  $X^\vee \cong \operatorname{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$ .

$R$  and  $R^\vee$  are to be finite subsets  $R \subset X, R^\vee \subset X^\vee$  with a bijective map

$$\begin{aligned} R &\longrightarrow R^\vee \\ \alpha &\longmapsto \alpha^\vee. \end{aligned}$$

The  $(X, X^\vee, R, R^\vee)$  shall meet the following axioms:

(i) For each  $\alpha \in R$ , we have

$$\langle \alpha \mid \alpha^\vee \rangle = 2.$$

(ii) For each  $\alpha \in R$ , define the maps

$$\begin{aligned} S_\alpha : X &\longrightarrow X \\ x &\longmapsto x - \langle x \mid \alpha^\vee \rangle \alpha \\ S_{\alpha^\vee} : X^\vee &\longrightarrow X^\vee \\ \xi &\longmapsto \xi - \langle \alpha \mid \xi \rangle \alpha^\vee. \end{aligned}$$

These maps satisfy for each  $\alpha \in R$

$$S_\alpha(R) \subseteq R \text{ and } S_{\alpha^\vee}(R^\vee) \subset R^\vee.$$

(iii) We call  $\Psi$  **reduced**, if additionally the following is fulfilled:

If  $\alpha, c\alpha \in R$ , for some  $c \in \mathbb{Q}$ , then  $c = \pm 1$ .

Elements  $\alpha \in R$  are called **roots**, while corresponding elements  $\alpha^\vee \in R^\vee$  are called **coroots**.

*Remark 26.* Let  $\Psi = (X, X^\vee, R, R^\vee)$  be a root datum:



1. For  $\alpha \in R$ , we have

$$S_\alpha(\alpha) = -\alpha.$$

2. In particular, we have for  $\alpha \in R$

$$S_\alpha^2 = \text{Id}_R.$$

3. If  $\Psi = (X, X^\vee, R, R^\vee)$  is a root datum, then so is  $\Psi^\vee = (X^\vee, X, R^\vee, R)$ .

4. If we have for  $\alpha, \beta \in R$

$$\langle \_ \mid \alpha^\vee \rangle \equiv \langle \_ \mid \beta^\vee \rangle,$$

then  $\alpha = \beta$ .

**Lemma 80.** *In the definition of a root datum, it would suffice to demand that*

$$\begin{aligned} R &\longrightarrow R^\vee \\ \alpha &\longmapsto \alpha^\vee \end{aligned}$$

*is only surjective.*

*Proof.* Let  $\alpha, \beta \in R$  s.t.

$$\alpha^\vee = \beta^\vee.$$

If  $\alpha, \beta$  are linearly dependent, they must be equal, because of

$$\langle \alpha \mid \alpha^\vee \rangle = 2 = \langle \beta \mid \alpha^\vee \rangle.$$

Assume, therefore, that they are linearly independent. Let  $V$  be the  $\mathbb{Z}$ -module spanned by the basis  $(\alpha, \beta)$ . Regarding this basis, the action of  $S_\alpha$  and  $S_\beta$  can be represented by the following matrices:

$$\begin{aligned} S_\alpha &\hat{=} \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix} \\ S_\beta &\hat{=} \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix}. \end{aligned}$$

It is now easy to show that

$$\begin{aligned} (S_\alpha \circ S_\beta)^2 &\hat{=} \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}, \\ (S_\alpha \circ S_\beta)^n &\hat{=} \begin{pmatrix} 2n+1 & 2n \\ -2n & 1-2n \end{pmatrix}. \end{aligned}$$

But  $R$  must be closed under the action of  $S_\alpha$  and  $S_\beta$ . Since  $R$  must be finite,  $\alpha, \beta$  cannot be linearly independent.  $\square$

**Definition 69.** The **Weyl group**  $W(\Psi)$  of a root datum  $\Psi$  is the subgroup of  $\text{Aut}_\ell(X) \cong \text{GL}_n(\mathbb{Z})$  which is generated by

$$\{S_\alpha \mid \alpha \in R\}.$$

Our goal is to construct for each connected reductive group  $G$  with a maximal torus  $T$  a root datum

$$\Phi(G, T) = \Phi(G)$$

s.t. the Weyl groups  $W(G, T)$  and  $W(\Phi(G))$  are isomorphic (in a canonical way).

**Theorem 38** (Facts). *Suppose, we were given a notion of morphism of root data at this point.*

1.  $\Phi(G) \cong \Phi(G')$  iff  $G \cong G'$ .
2. Every root datum is isomorphic to some  $\Phi(G)$  for a reductive connected group  $G$ .

*Remark 27.* • Obviously, the notion of root data is independent of  $k$ .

- Root data also classify compact connected Lie groups.
- Root data refine the less precise notion of root systems (which classify semisimple Lie algebras and simply connected semisimple algebraic groups).
- Every root system is a finite direct sum of the simple root systems

$$(A_n)_{n \geq 1}, (B_n)_{n \geq 2}, (C_n), (D_n), E_6, E_7, E_8, F_4, G_2.$$

## 20.3 Lie Algebras

**Lemma 81** (Fact). *Let  $G$  be an algebraic group. Take a faithful representation*

$$G \hookrightarrow \mathrm{GL}(V).$$

*Set*

$$I := I(G) \subset \mathcal{O}(\mathrm{GL}(V)).$$

*Consider the nilpotent element  $\varepsilon$  in  $\mathcal{O}(\mathrm{GL}(V))[\varepsilon]/(\varepsilon^2)$  and define the **Lie algebra** of  $G$  by*

$$\begin{aligned} \mathfrak{g} := \mathrm{Lie}(G) &:= \{x \in \mathrm{End}(V) \mid \forall f \in I : f(1 + \varepsilon x) = 0 \pmod{(\varepsilon^2)}\} \\ &= \{x \in \mathrm{End}(V) \mid \forall f \in I : f(1 + \varepsilon x) \in (\varepsilon^2)\} \\ &= \left\{ x \in \mathrm{End}(V) \mid \forall f \in I : \frac{d}{dt} \left( \frac{f(1 + tx)}{t} \right) \Big|_{t=0} = 0. \right\}. \end{aligned}$$

*Then,  $\mathfrak{g}$  is a  $k$ -vector space of dimension*

$$\dim_k(\mathfrak{g}) = \dim(G).$$

*Idea of Proof.* The proof boils down to show that  $G$  is smooth at 1.

However, each variety is generically smooth. Therefore,  $G$  is smooth at some points. Since  $G$  acts on itself via isomorphisms, 1 must look locally identically to one of  $G$ 's smooth points. Ergo,  $G$  is smooth everywhere.  $\square$

**Example 33.** •  $\mathrm{Lie}(\mathrm{GL}(V)) = \mathrm{End}(V)$  because  $I(G) = 0$ .

•

$$\begin{aligned} \mathrm{Lie}(\mathcal{O}_n(k)) &= \{x \in M_n(k) \mid (1 + \varepsilon x)(1 + \varepsilon x^T) = 1 + (\varepsilon^2)\} \\ &= \{x \in M_n(k) \mid x^T = -x\}. \end{aligned}$$

**Definition 70.** To each algebraic group we can attach the **adjoint representation**

$$\begin{aligned} \mathrm{Ad} : G &\longrightarrow \mathrm{GL}(\mathfrak{g}) \\ g &\longmapsto [x \mapsto gxg^{-1}]. \end{aligned}$$

If  $T$  is a maximal torus in  $G$ , we can restrict the adjoint representation

$$\mathrm{Ad} : T \longrightarrow \mathrm{GL}(\mathfrak{g}).$$

Given any representation  $\rho : T \rightarrow \mathrm{GL}(V)$ , we may decompose  $V$

$$V = \bigoplus_{\chi \in \mathfrak{X}(T)} V^\chi$$

where

$$V^\chi = \{v \in V \mid \forall t \in T : t.v = \chi(t)v\}.$$

Therefore,

$$\mathfrak{g} = \bigoplus_{\chi \in \mathfrak{X}(T)} \mathfrak{g}^\chi = \mathfrak{g}^o \oplus \bigoplus_{0 \neq \alpha \in \mathfrak{X}(T)} \mathfrak{g}^\alpha$$

where

$$\mathfrak{g}^\alpha = \{x \in \mathfrak{g} \mid \forall t \in T : txt^{-1} = \alpha(t)x\}$$

and

$$\mathfrak{g}^o = \{x \in \mathfrak{g} \mid \forall t \in T : txt^{-1} = x\}.$$

**Example 34.** Let  $G \subset \mathrm{GL}_n(k)$  with the torus  $T = \mathcal{G}_m^n$ . Then,

$$\mathfrak{g} = M_n(k) = \mathfrak{g}^o \oplus \left( \bigoplus_{0 \neq \alpha} \mathfrak{g}^\alpha \right)$$

where

$$\mathfrak{g}^o = \left\{ \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \right\}.$$

For  $i = 1, \dots, n$  we define  $\chi_i \in \mathfrak{X}(T)$  as follows:

$$\chi_i \left( \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \right) = t_i.$$

Then, we have for each  $i \neq j$

$$\mathfrak{g}^{\chi_i/\chi_j} = kE_{i,j}$$

and for each other  $\chi \in \mathfrak{X}(T)$

$$\mathfrak{g}^\chi = 0.$$

Therefore

$$\mathfrak{g} = \mathfrak{t} \oplus \left( \bigoplus_{i \neq j} kE_{i,j} \right)$$

where

$$\mathfrak{t} = \mathrm{Lie}(T) = \mathfrak{g}^o.$$

## 20.4 Root Data – Construction

Let  $G$  be a connected reductive group with a torus  $T$ . We want to construct a corresponding root datum

$$\Psi = (X, X^\vee, R, R^\vee).$$

- Set  $X$  to be the **character lattice**

$$X = \mathfrak{X}(T) = \text{Hom}(T, \mathcal{G}_m).$$

- Set  $X^\vee$  to be the **cocharacter lattice**

$$X^\vee = \text{Hom}(\mathcal{G}_m, T).$$

- If we are given  $x \in X$  and  $\xi \in X^\vee$ , we define

$$\langle x \mid \xi \rangle := m \in \mathbb{Z}$$

s.t.

$$\begin{aligned} x \circ \xi : \mathcal{G}_m &\longrightarrow \mathcal{G}_m \\ t &\longmapsto t^m. \end{aligned}$$

(Recall,  $\mathfrak{X}(\mathcal{G}_m) = \mathbb{Z}$ .)

•

$$R := \{0 \neq \alpha \in \mathfrak{X}(T) \mid \mathfrak{g}^\alpha \neq 0\}.$$

- Let  $g \in \mathcal{G}_m$  be s.t.  $\mathcal{G}_m = \overline{\langle g \rangle}$ . For  $x \in R$ , we need to choose  $t \in T$  s.t.

$$x(t) = g^2$$

and, if we set

$$x^\vee = [g^n \mapsto t^n]$$

the other axioms of a root datum are fulfilled.

**Example 35.** •  $G = \text{SL}_2(k)$ :

In  $G$  we have the maximal torus

$$T = \left\{ \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \right\}.$$

Consider the character

$$\lambda : T \longrightarrow \mathcal{G}_m$$

$$\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \longmapsto t.$$

Set

$$R = \{\alpha, -\alpha\}$$

with

$$\alpha = 2\lambda : \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \longmapsto t^2$$

$$-\alpha : \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \longmapsto t^{-2}.$$

Then, we have

$$\mathfrak{g}^\alpha = \left\{ \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \right\}$$

$$\mathfrak{g}^{-\alpha} = \left\{ \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix} \right\},$$

since

$$\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} 0 & t^2 \\ 0 & 0 \end{pmatrix}.$$

Define

$$\alpha^\vee := \left[ t \mapsto \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \right]$$

$$(-\alpha)^\vee := -\alpha^\vee \left[ t \mapsto \begin{pmatrix} t^{-1} & \\ & t \end{pmatrix} \right].$$

Then, we have

$$\langle \alpha \mid \alpha^\vee \rangle = 2.$$

- $G = \mathrm{PGL}_2(k)$ :

We have the maximal torus

$$T = \left\{ \begin{pmatrix} t & \\ & 1 \end{pmatrix} \right\}.$$

Set

$$R = \{\alpha, -\alpha\}$$

with

$$\alpha \begin{pmatrix} t & \\ & 1 \end{pmatrix} := t.$$

Define  $\alpha^\vee$  by

$$\alpha^\vee(t) := \begin{pmatrix} t^2 & \\ & 1 \end{pmatrix} = \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}.$$

We have

$$\begin{pmatrix} t & \\ & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} t & \\ & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}.$$

Now, let  $G$  be reductive of semisimple rank one. Then:

- $G/R(G) \cong PGL_2(k)$  with  $R(G) = 1$  or  $R(G) = \mu_n$ .
- $[G, G] \cong SL_2(k)$  is semisimple of semisimple rank 1.
- $\{\text{roots for } G/R(G)\} = \{\text{roots for } G\}$ .
- If  $\alpha$  is a root of  $G$ , then it is also a root of  $[G, G]$ . Hence, define

$$\alpha^\vee : \mathcal{G}_m \rightarrow [G, G] \hookrightarrow G.$$

Let  $G$  be reductive. Take a root  $\alpha \in \mathfrak{X}(T)$  and define the torus of codimension 1 in  $T$

$$S_\alpha := \text{Kern}(\alpha)^o.$$

Then,

$$G_\alpha := Z_G(S_\alpha)$$

is connected and reductive. Then,

$$S_\alpha \subset Z_G(G_\alpha)$$

and

$$S_\alpha \subset R(G_\alpha).$$

Ergo,  $T/S_\alpha$  is a maximal torus of rank 1 in  $G_\alpha/R(G_\alpha)$ , while  $G_\alpha/R(G_\alpha)$  is semisimple.

Ergo,  $G_\alpha$  is reductive of semisimple rank 1.

**Example 36.** Let  $G = \mathrm{GL}_3$  with the maximal torus  $T = \left\{ \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix} \right\}$ .

Let  $\alpha = \chi_1/\chi_2$ . Then,

$$S_\alpha = \left\{ \begin{pmatrix} x & & \\ & x & \\ & & y \end{pmatrix} \right\}.$$



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