

Notes: Algebraic Groups

SS 20, ETH Z

August 16, 2020

Contents

1	Introduction	4
2	Algebraic Groups and Hopf Algebras	6
2.1	An Aside on the General Group	9
3	Actions	11
4	Connected Components	15
5	Jordan Decomposition	21
6	Non-Commutative Algebra	34
7	Unipotent Groups	39
8	Tori	43
9	Trigonalization	52
10	Commutative Groups	55
11	Connected Solvable Groups	59
12	Semisimple Elements of nilpotent Groups	64

13 Algebraic Geometry	68
13.1 Projective Algebraic Sets	68
13.2 Flag Varieties	72
13.3 Local Rings and Function Fields	73
13.4 Regular Functions and Morphisms	75
13.5 Dimensions	76
13.6 Images of Morphisms	78
13.7 Borel's Fixed Point Theorem (special case)	81
13.8 Orbits	82
13.9 Borel's Fixed Point Theorem (General Case)	83
13.10 Generic Openness	85
14 Homogenous Spaces	86
14.1 Quotients	87
15 Borel and Parabolic Groups	91
15.1 Radicals	95
16 Reductivity	97
17 Union of Borel Subgroups	100
18 Splitting Solvable Groups	102
18.1 An Aside	105
18.2 Semisimple Elements of Solvable Groups	106
19 All About Tori	110
19.1 Maximal Tori	110
19.2 Centralizers of Tori	112
19.3 Low Dimensional Groups	113
19.4 Characterizing Nilpotent Groups via maximal Tori	116
19.5 Weyl Groups	118
19.6 Normalizers of Borel Subgroups	120
19.7 Borel Subgroups Containing a Given Torus	125
19.8 Groups of Semisimple Rank One	128
19.9 Isogenies	133

20 Root Data	134
20.1 More on Reductive Groups	134
20.2 Root Data – Definition	137
20.3 Lie Algebras	140
20.4 Root Data – Construction	142

1 Introduction

Let k be an algebraically closed field.

Definition 1. For $I \subseteq k[X] := k[X_1, \dots, X_n]$, we define its **vanishing set** by

$$V(I) := \{p \in k^n \mid \forall f \in I : f(p) = 0\}.$$

A set $S \subset k^n$ is called **algebraic**, if

$$S = V(I)$$

for some $I \subseteq k[X]$.

Example 1. The group $\mathrm{GL}_n(k)$ is not an algebraic subset of $k^{n \times n}$. But, we can identify it with an algebraic subset of $(k^{n \times n})^2$ by

$$\mathrm{GL}_n(k) \cong \{(x, y) \in k^{n \times n} \mid xy = 1_n\} = V(X \cdot Y - 1_n).$$

Definition 2. Let $\iota : \mathrm{GL}_n(k) \hookrightarrow k^{n \times n^2}$ be the injection

$$A \mapsto (A, A^{-1}).$$

A **linear algebraic group** over k is a subgroup $U \subseteq \mathrm{GL}_n(k)$ s.t. $\iota(k)$ is an algebraic subset of k^{2n^2} .

I.e., a linear algebraic group is a matrix-group which can be defined by polynomials over the entries of a matrix and its inverse.

Example 2. The following groups are linear algebraic groups:

1. The multiplicative group $\mathcal{G}_m(k) := k^\times = k \setminus \{0\} = \mathrm{GL}_1(k)$.
2. The general linear group $\mathrm{GL}_n(k)$.
3. The special linear group

$$\mathrm{SL}_n(k) := \{A \in \mathrm{GL}_n(k) \mid \det(A) = 1\}.$$

4. The orthogonal group

$$\mathcal{O}_n(k) := \{A \in \mathrm{GL}_n(k) \mid A^T \cdot A = 1\}.$$

5. The special orthogonal group

$$\mathrm{SO}_n(k) := \mathcal{O}_n(k) \cap \mathrm{SL}_n(k).$$

6. The upper triangle-matrix group

$$\left\{ \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ & \ddots & \vdots \\ & & a_{n,n} \end{pmatrix} \mid a_{i,j} \in k \right\} \cap \mathrm{GL}_n(k).$$

7. The normed upper triangle-matrix group

$$\left\{ \begin{pmatrix} 1 & \cdots & a_{1,n} \\ & \ddots & \vdots \\ & & 1 \end{pmatrix} \mid a_{i,j} \in k \right\} \cap \mathrm{GL}_n(k).$$

8. The group of n -th roots of unity

$$\mu_n(k) := \{x \in k \mid x^n = 1\}.$$

9. The additive group $(k, +)$ is not a subgroup of $\mathrm{GL}_n(k)$, but it can be identified with the linear algebraic group

$$\left\{ \begin{pmatrix} 1 & a \\ & 1 \end{pmatrix} \mid a \in k \right\} \subset \mathrm{GL}_2(k)$$

10. For $k = \mathbb{C}$, the unit sphere and the unitary groups are NOT linear algebraic groups.

2 Algebraic Groups and Hopf Algebras

Definition 3. A **morphism** $f : X \rightarrow Y$ of algebraic sets $X \subset k^m, Y \subset k^n$ is a map which is coordinatewise described by polynomials.

Definition 4. An **algebraic group** is an algebraic set $G \subset k^n$ together with a fixed element $e \in G$ and morphisms $m : G \times G \rightarrow G, i : G \rightarrow G$ s.t. (G, m, i, e) is a group.

A **morphism of algebraic groups** is a morphism of algebraic sets that is also a group homomorphism.

Definition 5. Let $V \subset k^n$ be any subset. Then, we define the vanishing ideal of V by

$$I(V) := \{f \in k[x] \mid f(V) = 0\}.$$

Definition 6. For a commutative ring R we define the **radical** of an ideal $I \subseteq R$ by

$$\sqrt{I} := \{r \in R \mid r^m \in I \text{ for some } m \in \mathbb{N}_0\}.$$

R is called **reduced**, if $\sqrt{0} = 0$.

Lemma 1 (Zariskis Lemma). *Let $L \supseteq k$ be fields. If L is finitely generated as a k -algebra, then the extension $L \supseteq k$ is finite, i.e., L is a finitely-generated k -vector space.*

Theorem 1 (Hilberts Nullstellensatz). *For any ideal $I \subseteq k[x]$, we have*

$$I(V(I)) = \sqrt{I}.$$

Proof. It is easy to see that

$$I \subset \sqrt{I} \subset I(V(I)).$$

Now, let $f \in I(V(I))$ and assume – for the sake of contradiction – that $f \notin \sqrt{I}$. Since \sqrt{I} is the intersection of its upper prime ideals, there is a prime ideal $p \supset I$, s.t. $f \notin p$. Now, define the zero divisor-free ring

$$R := (k[x]/p)[f^{-1}].$$

And let $\phi : k[x] \rightarrow R$ be the corresponding ring homomorphism.

Let $m \subseteq R$ be a maximal ideal in R . Then, R/m is a field, which contains k and is finitely generated as k -algebra. According to Zariski's lemma, R/m is a finite (ergo algebraic) extension of k . Since k is algebraically closed, we have $R/m = k$. Let $\pi_m : R \rightarrow k$ be the corresponding ring homomorphism.

Now, for x_1, \dots, x_n , set

$$t_i := \pi_m(\phi(x_i)).$$

Then, $t = (t_1, \dots, t_n) \in k^n$. We now have

1. $t \in V(I)$: For each $g \in I$, we have $\phi(g) = 0$. On the other hand

$$g(t) = g(\pi_m \circ \phi(x)) = \pi_m \circ \phi(g) = 0.$$

2. $f(t) \neq 0$: $\phi(f)$ is invertible in R , therefore $\phi(f) \neq 0$ and $\phi(f) \notin m$. Ergo

$$f(t) = \pi_m \circ \phi(f) \neq 0.$$

Ergo, there is a point $t \in V(I)$ s.t. $f(t) \neq 0$. This yields a contradiction, since we assumed $f \in I(V(I))$. \square

Definition 7. For an algebraic set $X \subset k^n$, we define its **coordinate ring** by

$$k[X] := k[x_1, \dots, x_n]/I(X).$$

Lemma 2. For a morphism $f : X \rightarrow Y$ of algebraic sets define the following homomorphism of k -algebras.

$$\begin{aligned} f^* : k[Y] &\longrightarrow k[X] \\ p &\longmapsto p \circ f. \end{aligned}$$

We have a contravariant functor $_*$ from the categories of algebraic sets over k to the category of k -algebras:

$$\begin{aligned} X &\longmapsto k[X] \\ \text{Hom}(X, Y) &\longmapsto \text{Hom}_k(k[Y], k[X]) \\ f &\longmapsto f^*. \end{aligned}$$

Lemma 3. We have

$$k[X \times Y] \cong k[X] \otimes k[Y].$$

Proof.

$$k[X] \otimes k[Y] = k[x]/I(X) \otimes_k k[y]/I(Y) = k[x, y]/I(X) \otimes k[y] + k[x] \otimes I(Y).$$

But

$$V(I(X) \otimes k[y] + k[x] \otimes I(Y)) = V(I(X) \otimes k[y]) \cap V(k[x] \otimes I(Y)) = X \times Y.$$

\square

Theorem 2. *Every finitely generated reduced k -algebra A is isomorphic to some $k[X]$ for some algebraic X .*

Proof. Choose some $\pi : k[x_1, \dots, x_n] \twoheadrightarrow A$ and set $X := V(\ker \pi)$. Then $\ker \pi = I(X)$, since π 's kernel is radical since A is reduced. \square

Corollary 1. *The contravariant functor $_* : \mathcal{C}_{\text{algSets}} \rightarrow \mathcal{C}_{k\text{-alg.s}}$ gives an antiequivalence of categories.*

Lemma 4. *An algebraic set X is isomorphic to some algebraic subset of Y iff there is an epimorphism $k[Y] \twoheadrightarrow k[X]$.*

Lemma 5. *Let $G \subset k^n$ be an algebraic group. Then, we have maps*

$$\begin{aligned} m : G \times G &\longrightarrow G \\ i : G &\longrightarrow G \\ e : * &\longrightarrow G. \end{aligned}$$

They induce dual maps in the category of k -algebras:

$$\begin{aligned} \Delta &:= m^* : k[G] \longrightarrow k[G] \otimes_k k[G] \\ \iota &:= i^* : k[G] \longrightarrow k[G] \\ \varepsilon &:= e^* : k[G] \longrightarrow k \end{aligned}$$

Definition 8. A **Hopf-algebra** over k is a (reduced?!) k -algebra together with maps $\Delta, \varepsilon, \iota$ as above s.t. the following holds:

$$\begin{aligned} (\Delta \otimes \text{Id})\Delta &= (\text{Id} \otimes \Delta)\Delta \\ s^* \circ (\iota \otimes \text{Id})\Delta &= s^* \circ (\text{Id} \otimes \iota)\Delta = \varepsilon \\ (\varepsilon \otimes \text{Id})\Delta &= (\text{Id} \otimes \varepsilon)\Delta = \text{Id} \end{aligned}$$

where $s : G \rightarrow G \times G, g \mapsto (g, g)$ is the diagonal map.

A morphism of Hopf-algebras is a homomorphism of k -algebra $F : A \rightarrow B$ s.t.

$$\Delta \circ F = (F \otimes F) \circ \Delta.$$

Theorem 3. *The contravariant functor $_*$ gives an anti-equivalence of the categories of algebraic groups and the categories of finitely generated Hopf-algebras over k .*

Example 3. 1. Let $G = \mathcal{G}_a = (k, +)$. Then, $k[G] = k[x]$, since $I(x) = 0$. Then, we have

$$\begin{aligned}\Delta(x) &= x \otimes 1 + 1 \otimes x \\ \iota(x) &= -x \\ \varepsilon(x) &= 0.\end{aligned}$$

2. Let $G = \mathcal{G}_m = \{(a, a^{-1}) \mid a \neq 0\} \cong k^\times$. Then, $k[G] = k[x, y]/(xy - 1) = k[x, x^{-1}]$. Then, we have

$$\begin{aligned}\Delta(x) &= x \otimes x \\ \iota(x) &= x^{-1} \\ \varepsilon(x) &= 1.\end{aligned}$$

3. Let $G = \mathrm{GL}_n(k)$. Then, $k[G] = k[x, y]/(xy - 1_n) = k[x_{i,j}, \frac{1}{\det}]$. Then, we have

$$\begin{aligned}\Delta(x_{i,j}) &= \sum_k x_{i,k} \otimes x_{k,j} \\ \Delta\left(\frac{1}{\det(x)}\right) &= \frac{1}{\det(x)} \otimes \frac{1}{\det(x)} \\ \iota(x_{i,j}) &= (x^{-1})_{i,j} \\ \varepsilon(x_{i,j}) &= \delta_{i,j}.\end{aligned}$$

2.1 An Aside on the General Group

Let $G = \mathrm{GL}_n(k) = \{(x, y) \mid xy = \mathrm{Id}_n\}$. Since we have

$$x^{-1} = \frac{1}{\det(x)} \cdot \mathrm{adj}(x)$$

where the adjoint $\mathrm{adj}(x)$ can be expressed by polynomials in the entries of x , we have isomorphisms

$$\begin{aligned}k[x, y]/(xy - 1_n) &\longrightarrow k[x, 1/\det(x)] = k[x, t]/(\det(x) \cdot t = 1) \\ (x, y) &\longmapsto (x, \det(y))\end{aligned}$$

and

$$\begin{aligned}k[x, 1/\det(x)] &\longrightarrow k[x, y]/(xy - 1_n) \\ (x, t) &\longmapsto (x, t \cdot \mathrm{adj}(x)).\end{aligned}$$

Lemma 6.

$$k[\mathbf{GL}_n(k)] \cong k[x_{i,j}, \frac{1}{\det(x)}].$$

Lemma 7. *Let V be a finite-dimensional k -vector space. If we choose a basis for V , we get an isomorphism $\mathbf{GL}(V) \cong \mathbf{GL}_n(k) \subset k^{n \times n}$. Hence, $\mathbf{GL}(V)$ is an algebraic group whose structure is up to a unique isomorphism independent of the choice of basis.*

3 Actions

Remark 1. Let $G \curvearrowright M$ be a group action of algebraic sets, then the morphism

$$G \times M \longrightarrow M$$

yields an homomorphism

$$\Delta : k[M] \rightarrow k[G] \otimes k[M].$$

This turns $k[M]$ to a **comodule** of the Hopf-Algebra $k[G]$.

Definition 9. Let V be vector space and G an algebraic group. A morphism $r_V : G \rightarrow \mathrm{GL}(V)$ of groups is called **representation** of G , if there is a linear map

$$\Delta : V \rightarrow V \otimes_k k[G] \subseteq \mathrm{Hom}_{alg}(G, V)$$

s.t. we have for each $v \in V$ and $g \in G$

$$r_V(g) \cdot v = \sum_i v_i \cdot f_i(g)$$

where $\Delta v = \sum_i v_i \otimes f_i$.

That is, V is a comodule for $k[G]$.

A map $\phi : V \rightarrow W$ is called **equivariant** for two representations r_V, r_W of G , if

$$\phi(r_V(g)v) = r_W(g)\phi(v)$$

for all g, v .

Example 4. Let $G = \mathrm{GL}_n(k)$, $V = k^n$ and r_V be the canonical representation. For an orthonormal basis $(b_i)_{i=1, \dots, n}$, we for example can set

$$\Delta v = \sum_{i=1}^n b_i \otimes f_i$$

where

$$f_i(A) := b_i^T A v.$$

Then, we have

$$r_V(A) \cdot v = A \cdot v = \sum_{i=1}^n b_i \cdot b_i^T A v = \Delta(v)(A).$$

Example 5. Let M be a right G -set. Then, G also acts on $k[M]$, therefore we have a map

$$\rho : G \rightarrow \mathrm{GL}(k[M])$$

by, for $v \in k[M]$,

$$(\rho(g)v)(m) := v(m.g).$$

Further, we have an algebra morphism

$$\Delta : k[M] \rightarrow k[M] \otimes k[G] = k[M \times G]$$

with

$$(\Delta v)(m, g) = v(m.g).$$

With $\Delta v = \sum_i v_i \otimes f_i$

$$\rho(g)v(m) = v(mg) = \Delta v(m, g) = \sum_{i=1} f_i(g)v_i(m).$$

Ergo, ρ is a representation of G .

When $M = G$ with action given by the right translation, then $\rho : G \rightarrow \mathrm{GL}(k[G])$ is called the **right regular representation** of G .

Lemma 8. *Let G be an algebraic group and V a finite-dimensional k -vector space. Then $\rho : G \rightarrow \mathrm{GL}(V)$ is morphism of algebraic groups iff it is a representation.*

Definition 10. Let G be an algebraic group and V a representation of G . A subspace $W \subset V$ is called **invariant** or **subrepresentation**, if we have $W.G = W$.

Lemma 9. *The following are equivalent:*

1. W is invariant.
2. $\Delta(W) \subseteq W \otimes k[G]$.

Lemma 10. *Any representation V is a filtered union of its finite-dim. subrepresentations:*

1. Each $v \in V$ is contained in some fin.-dim. subrep.
2. Any two finite-dim. subrep. are contained in some bigger fin.-dim. subrep.

Theorem 4. *Every algebraic group G is isomorphic to a linear algebraic group.*

Proof. Let $\rho : G \rightarrow \mathbf{GL}(k[G])$ be the right regular representation. $k[G]$ is a finitely-generated k -algebra. Then, there is a finite-dim. subrepresentation $V \subseteq k[G]$ s.t. V generates $k[G]$ as k -algebra. Then

$$\phi : G \longrightarrow \mathbf{GL}(V)$$

is morphism of algebraic groups.

Consider the dual map

$$\phi^* : k[\mathbf{GL}(V)] \rightarrow k[G].$$

We need to show that ϕ^* is surjective. It is enough to show that $V \subset \text{Im} \phi^*$. Define

$$\begin{aligned} l : V \subset k[G] &\longrightarrow k \\ f &\longmapsto f(e). \end{aligned}$$

Let $f \in V$ and set $a(g) := l(g \cdot f)$ for $g \in \mathbf{GL}(V)$. Then $a \in k[\mathbf{GL}(V)]$ is regular. Further,

$$\phi^*(a)(g) = a(\rho(g)) = l(\rho(g)f) = f(eg) = f(g).$$

Therefore, $f = \phi^*(a) \in \text{Im}(\phi^*)$. Since V generates $k[G]$, the surjectivity of ϕ^* follows. \square

Theorem 5. *Let H be an algebraic subgroup of an algebraic group G . There is a finite-dim. representation V of G and a line $L \subset V$ s.t. H is the stabilizer in G of L , i.e.*

$$H = \{g \in G \mid L.g = L\}.$$

Proof. Let V be like in the previous proof. Consider

$$I \hookrightarrow k[G] \twoheadrightarrow k[H].$$

We can now set $L' := V \cap I$. We then have for $g \in G$.

$$L'.g \subset L' \iff I.g \subseteq I \iff g \in H.$$

Now, in general L' is not of dimension one. Set $d = \dim(L')$ and consider the one-dimensional subspace $L := \Lambda^d(L') \subseteq \Lambda^d(V)$. G acts on $\Lambda^d(V)$ in the natural way.

It is clear, that H stabilizes L . For the other direction, let $g \notin H$ and let e_1, \dots, e_n be a basis of V s.t. $L' = \langle e_1, \dots, e_d \rangle$. Then,

$$L = \langle e_1 \wedge \dots \wedge e_d \rangle$$

and, since g does not stabilize L' , w.l.o.g. we can assume $e_1.g = e_{d+1}$. Then, we have $g(e_1 \wedge \dots \wedge e_d) = g(e_1) \wedge \dots \wedge g(e_d) =: v$. Now, v cannot be zero and it cannot lie in L because $e_1.g = e_{d+1}$. Therefore, $g \notin H$ does not stabilize L . \square

Theorem 6. *Let H be a normal algebraic subgroup of an algebraic group G . Then, there is a finite-dimensional $\rho : G \rightarrow \mathbf{GL}(V)$ s.t. $H = \ker(\rho)$.*

Proof. Let V, L and $\phi : G \rightarrow \mathbf{GL}(V)$ be like in the preceding theorem. Set

$$V_H := \{v \in V \mid H.v \subset \langle v \rangle\}.$$

Then, V_H is G -invariant, since

$$h.(g.v) = (hg).v = (gh').v = g.(h'v) = g.(\kappa \cdot v) = \kappa \cdot g.v$$

for all $g \in G, h \in H, v \in V_H$ and fitting $h' \in H, \kappa \in k^\times$. W.l.o.g. we have $V = V_H$. V is not trivial, because $L \subset V$.

Let χ range through all homomorphism $H \rightarrow k^\times$, then we have

$$V = \bigotimes_{\chi} V_{\chi}$$

where

$$V_{\chi} = \{v \in V \mid h.v = \chi(h) \cdot v\}.$$

Then each $g \in G$ permutes those eigenspaces by

$$g.V_{\chi} = V_{\chi(g^{-1} \cdot g)}.$$

Now, let $W := \bigoplus_{\chi} \text{End}(V_{\chi}) \subset \text{End}(V)$. For $g \in G$ and $\lambda \in \text{End}(V)$, define

$$\begin{aligned} \tilde{\gamma} : G &\longrightarrow \mathbf{GL}(\text{End}(V)) \\ g &\longmapsto \tilde{\gamma}(g) : [\lambda \mapsto \phi(g) \circ \lambda \circ \phi(g)^{-1}]. \end{aligned}$$

The action $\tilde{\gamma}(g)$ stabilizes W , since each $\phi(g)$ just permutes the V_{χ} and $\phi(g)^{-1}$ permutes them back. Therefore, we have a subrepresentation

$$\gamma : G \rightarrow \mathbf{GL}(W).$$

We now have to show

$$\ker(\gamma) = H.$$

Since elements of H don't permute V_{χ} , we have $\gamma(H) = \text{Id}$.

One the other side, let $g \in G$ with $\gamma(g) = \text{Id}$. Then, we can choose the projection $\pi : V \twoheadrightarrow L$ in W and get

$$\phi(g) \circ \pi = \pi \circ \phi(g).$$

Therefore, g leaves each L invariant. But now, we have $g \in H$. □

4 Connected Components

Lemma 11. *Let $I_1, I_2, I_\lambda \subset k[x]$ be ideals, then*

$$\begin{aligned} V(I_1 \cap I_2) &= V(I_1) \cup V(I_2) \\ V\left(\bigcup_{\lambda} I_{\lambda}\right) &= \bigcap_{\lambda} V(I_{\lambda}). \end{aligned}$$

Definition 11. A topological space X is called **connected**, if any of the following equivalent condition holds:

- There is no pair of non-empty closed subsets $Z_1, Z_2 \subseteq X$, s.t. $X = Z_1 \dot{\cup} Z_2$.
- There is no pair of non-empty open closed subsets $U_1, U_2 \subseteq X$, s.t. $X = U_1 \dot{\cup} U_2$.

Definition 12. A topological space X is called **irreducibel**, if any of the following equivalent condition holds:

- There is no pair of proper closed subsets $Z_1, Z_2 \subsetneq X$, s.t. $X = Z_1 \cup Z_2$.
- For each pair $U_1, U_2 \subseteq X$ of non-empty open subsets we have $U_1 \cap U_2 \neq \emptyset$.
- Each nonempty open subset of X is dense.

Example 6. $V(xy)$ is connected but not irreducible.

Recall: Last time we introduced the **Zariski-Topology** on X .

There, algebraic sets equal closed sets.

We called a set X **irreducible** iff each open subset lies dense in X .

Lemma 12. *For an algebraic set X , the following are equivalent:*

- (1) X is irreducible.
- (2) $k[X] = k[x_1, \dots, x_n]/I(X)$ is an (integral) domain.
- (3) $I(X)$ is a prime ideal.

The proof of (2) \iff (3) is a basic algebraic result.

Lemma 13. *An open base for the Zariski-Topology on an algebraic set X is given by sets:*

$$D(f) := \{p \in X \mid f(p) \neq 0\}$$

for each $f \in k[X]$. We call the $D(f)$ **basic open sets**.

Proof. Suppose $U \subseteq X$ is nonempty and open. Set

$$Z := X \setminus U$$

then Z is closed. Thus

$$Z = \{x \in X \mid f(x) = 0 \ \forall f \in I\} = V(I)$$

for some ideal $I \subseteq k[X]$. Let $p \in U$, then there is an $f \in I$ s.t.

$$f(p) \neq 0.$$

Also, $D(f) \cap Z = \emptyset$, thus $p \in D(f) \subseteq U$. □

Proof: Lemma 1. It is clear that (2) is equivalent to (3).

(1) is equivalent to

$$\forall \text{ nonempty, open } U_1, U_2 \subset X : U_1 \cap U_2 \neq \emptyset$$

$$\stackrel{\text{Lemma 2}}{\iff} \forall \text{ nonempty, basic open } D(f_1), D(f_2) \subset X : D(f_1) \cap D(f_2) \neq \emptyset$$

Since $D(f_1) \cap D(f_2) = D(f_1 f_2)$, this is equivalent to the statement

$$\forall f_1, f_2 \in k[X] : f_1, f_2 \neq 0 \implies f_1 f_2 \neq 0.$$

Which states that $k[X]$ is a domain. □

Lemma 14. *Let X be an algebraic set. We have bijections*

$$\{\text{closed subsets } Z \subseteq X\} \leftrightarrow \{\text{radical ideals } I \subset k[X]\}$$

and

$$\{\text{irreducible, closed subsets } Z \subseteq X\} \leftrightarrow \{\text{prime ideals } I \subset k[X]\}$$

and

$$\{\text{points of } X\} \leftrightarrow \{\text{maximum ideals } I \subset k[X]\}.$$

Lemma 15 (Primary Decompositions, Atiyah, Macdonald Ch. 4). *For an ideal I we call $P \supseteq I$ a **minimal prime** of I , if P is a prime ideal and we have for each prime ideal Q :*

$$P \supseteq Q \supseteq I \implies P = Q.$$

*Any radical ideal I of $k[x_1, \dots, x_n]$ has only finitely many **minimal** primes P_1, \dots, P_r . In particular,*

$$I = \bigcap_{i=1}^r P_i$$

and for each i

$$P_i \not\supseteq \bigcap_{j:j \neq i} P_j.$$

Definition 13. An **(irreducible) component** Z of X is a maximal irreducible closed subset, i.e., an irreducible closed $Z \subseteq X$ s.t. there does not exist an irreducible closed $Y \subset X$ s.t. $Y \supsetneq Z$.

Then, we have the bijection

$$\{\text{irreducible components of } X\} \leftrightarrow \{\text{minimal primes of } I(X)\}.$$

Lemma 16. *Any algebraic set X has finitely many irreducible components Z_1, \dots, Z_r . We have*

$$X = Z_1 \cup \dots \cup Z_r$$

and for each i

$$Z_i \not\subset \bigcup_{j:j \neq i} Z_j.$$

Example 7. 1. Let $X = V(x \cdot y) \subset k^2$. Then $X = Z_1 \cup Z_2$ where $Z_1 = V(x)$, $Z_2 = V(y)$.

X is connected, but not irreducible ($D(x)$ does not lie dense in X).

2. Let X be a **finite** algebraic set. It is easy to check that every subset of X is closed:

$$\{p\} = V(x_1 - p_1, \dots, x_n - p_n)$$

for each $p \in X$. Further

$$X = \{p_1\} \cup \dots \cup \{p_r\}.$$

Moreover: Any function $f : X \rightarrow k$ is regular (i.e. given by polynomials).

Lemma 17. *We call an element $e \in k[X]$ **idempotent** iff $e^2 = e$.*

Let X be an algebraic set. Then

$$\begin{aligned} X \text{ connected} &\iff \text{the only idempotents } e \in k[X] \text{ are } 0 \text{ and } 1 \\ &\iff k[X] \not\cong A \times B \text{ for any } k\text{-algebras } A, B. \end{aligned}$$

Lemma 18. *Morphisms of algebraic sets are continuous.*

Proof. Let $\phi : X \rightarrow Y$ be a morphism. It suffices to show that for all closed $Z \subset Y$ that $\phi^{-1}(Z) \subset X$ is closed.

But, if

$$Z = V_Y(S) := \{q \in Y \mid f(q) = 0 \ \forall f \in S\}$$

for some ideal $S \subset k[Y]$, then

$$\phi^{-1}(Z) = V_X(\phi^*(S)) = \{q \in X \mid f(\phi(q)) = 0 \ \forall f \in S\}.$$

□

Lemma 19. *Isomorphisms of algebraic sets are homeomorphisms. In particular, any isomorphism of algebraic sets $\phi : X \rightarrow X$ permutes the irreducible components Z_1, \dots, Z_r of X :*

$$\forall i \exists j : \phi(Z_i) = Z_j.$$

Theorem 7. *Let G be an algebraic group.*

- (i) *There is a unique irreducible component G^0 of G with $e \in G^0$.*
- (ii) *Every irreducible component Z of G is a coset gG^0 of G for some $g \in Z$.*
- (iii) *G^0 is a normal algebraic subgroup of G .*
- (iv) *G^0 is of finite index, i.e.*

$$[G : G^0] = \#(G/G^0) < \infty.$$

(v) *The irreducible components are also the connected components.*

Proof. Let $G = Z_1 \cup \dots \cup Z_r$ be the decomposition into components. We may assume that $e \in Z_1$.

Recall that $Z_1 \not\subset \bigcup_{j \geq 2} Z_j$. Then, there is an $x \in Z_1 \setminus \bigcup_{j \geq 2} Z_j$. Thus, for all algebraic set isomorphisms $\phi : G \rightarrow G$, we have by some previous lemma that $\phi(x)$ is likewise contained in some unique component of G . For example, we may take ϕ to be

$$\begin{aligned}\phi_g : G &\rightarrow G \\ y &\mapsto gy\end{aligned}$$

for any $g \in G$. Then, for all $g \in G$, the element $gx = \phi_g(x)$ is contained in only one component of G . Ergo, each $g \in G$ is contained in exactly one component.

(i) Take $g = e$.

(iii) G^0 is an algebraic subset, by construction. Denote by $m : G \times G \rightarrow G$ and $i : G \rightarrow G$ the continuous multiplication and inversion map on G . **Why is G^0 a subgroup?** We need to show

$$\begin{aligned}m(G^0 \times G^0) &\subseteq G^0. \\ i(G^0) &\subseteq G^0.\end{aligned}$$

We know that $i(G^0)$ is some component of G , since i is an isomorphism. But it contains the identity e , since $e^{-1} = e$. Therefore, $i(G^0) = G^0$.

If $g \in G$, then gG^0 is some component of G . Suppose $g \in G^0$. Then $gG^0 \cap G^0 \supseteq \{g\}$, therefore $gG^0 = G^0$. Ergo, G^0 is closed under multiplication.

Why is G^0 a normal? If $g \in G$, then gG^0g^{-1} is a component that contains e , therefore $G^0 = gG^0g^{-1}$.

(Alternative proof that $m(G^0 \times G^0) = G^0$: Consider

- any continuous image of an irreducible set is irreducible.
- the closure of any irreducible set is irreducible.

Ergo $\overline{m(G^0 \times G^0)}$ is a closed irreducible set containing e . Ergo, $\overline{m(G^0 \times G^0)} = G^0$.

(ii) Let $Z \subset G$ be a component. Let $g \in Z$. Then $g \in (gG^0 \cap Z)$, so $gG^0 = Z$.

- (iv) This follows from some previous lemma.
- (v) This is left as a topological exercise. It is true whenever the irreducible components do not intersect.

□

It now follows:

$$\{\text{finite algebraic groups}\} \longleftrightarrow \{\text{finite groups}\}$$

where the above arrow is an equivalence of categories.

Example 8. • Let $G = \{g_1, \dots, g_r\}$ be a finite algebraic group. Then,

$$G^0 = \{e\}.$$

- Without proofs:

$$G \in \{\mathrm{GL}_n(k), \mathrm{SO}_n(k), \mathrm{SL}_n(k)\} \implies G^0 = G.$$

Further,

$$G = \mathrm{O}_n(k) \implies G^0 = \mathrm{SO}_n(k).$$

And if $-1 = 1$ i.e. $\mathrm{char} k = 2$, then $[G : G^0] = 1$. Otherwise $[G : G^0] = 2$.

5 Jordan Decomposition

As usual, $k = \bar{k}$ is an algebraically closed field.

Definition 14. Let V be a finite-dimensional vector space.

An element $x \in \text{End}(V)$ is **semisimple**, if it is diagonalizable, i.e. it has a basis of eigenvectors, or equivalently, if the minimal polynomial of x is square-free.

Then, there is a decomposition $V = \bigoplus_{i=1}^r V_i$ and distinct elements $\lambda_1, \dots, \lambda_n \in k$ s.t.

$$x|_{V_i} = \lambda_i.$$

If $\dim(V_i) = n_i$, then

$$\text{characteristic polynomial of } x = \prod_{i=1}^r (T_i - \lambda_i)^{n_i} \in k[T]$$

and

$$\text{minimal polynomial of } x = \prod_{i=1}^r (T_i - \lambda_i) \in k[T].$$

(Where the minimal polynomial of x is defined as the least degree monic polynomial $m \in k[T]$ s.t. $m(x) = 0$.)

Remark 2. Let $m(T) \in k[T]$ be the minimal polynomial of $x \in k^{n \times n}$.

The theorem of Cayley and Hamilton states that we have for each $p \in k[T]$:

$$p(x) = 0 \implies m|p.$$

Definition 15. $x \in \text{End}(V)$ is **nilpotent** if $x^n = 0$ for some n .

x is **unipotent**, if $x - 1$ is nilpotent.

Lemma 20. x is nilpotent iff the characteristic polynomial of x is $T^{\dim(V)}$. (Use Cayley-Hamilton for one of the directions).

Lemma 21. If x is semisimple and nilpotent, then $x = 0$.

If x is semisimple and unipotent, then $x = 1$.

Lemma 22. If x, y are commuting elements, that are semisimple resp. unipotent resp. nilpotent, then so is xy .

Proof. It is easy to see, that this is true for nilpotent x, y .

Now, let x, y be unipotent and commuting. Then, we have

$$xy - 1 = (x + 1)(y - 1) + (x - y).$$

Since x, y commute, $(x+1)(y-1)$ must be nilpotent. $(x-y)$ must be nilpotent because the sum of commuting nilpotent elements must be nilpotent. Because everything commutes, also $xy - 1$ as the sum of two commuting, nilpotent elements must be nilpotent.

Now, let $A, B \in k^{n \times n}$ be two diagonalizable and commuting matrices. Let $\lambda_1, \dots, \lambda_r$ be different eigenvalues of A and let E_i be the corresponding eigenspaces. We then have

$$A \cdot (BE_i) = BAE_i = \lambda_i \cdot BE_i.$$

Ergo, each E_i is invariant under B . Since $B|_{E_i}$ stays diagonalizable, we can simply choose a basis of eigenvectors $b_1, \dots, b_n \in \bigcup_i E_i$ of B . Since each b_i lies in a E_j , those vectors are also eigenvectors for A . Therefore, b_1, \dots, b_n is basis of eigenvectors for both matrices. \square

Theorem 8 (Goal). *For all algebraic groups G and for all $g \in G$, there exist unique group elements $g_s, g_u \in G$ s.t.*

$$g = g_s g_u = g_u g_s$$

and for all finite-dimensional representations $\rho : G \rightarrow GL(V)$, $\rho(g_s)$ is semisimple and $\rho(g_u)$ is unipotent.

Example 9. If $g = \begin{pmatrix} \lambda & 1 & 0 \\ & \lambda & 1 \\ & & \lambda \end{pmatrix} \in G = GL_3(k)$, then $g_s = \begin{pmatrix} \lambda & 0 & 0 \\ & \lambda & 0 \\ & & \lambda \end{pmatrix}$, $g_u = \begin{pmatrix} 1 & \lambda^{-1} & 0 \\ & 1 & \lambda^{-1} \\ & & 1 \end{pmatrix}$.

Theorem 9 (Goal Theorem). *Let G an algebraic group. For all $g \in G$ there is exactly one pair $g_s, g_u \in G$ s.t.*

$$g = g_s g_u = g_u g_s$$

and for all finite-dimensional representations $r : G \rightarrow GL_n(V)$, the element $r(g_s)$ resp. $r(g_u)$ is semisimple resp. unipotent.

Last time, we saw:

Lemma 23. • *If g, h are commuting and semisimple resp. commuting and unipotent then so is gh .*

• *If g is semisimple and unipotent, then $g = 1$.*

Proposition 1. *Let V be a finite-dimensional vector space and $g \in GL(V)$. There exist unique elements $g_s, g_u \in GL(V)$ s.t.*

$$g = g_s g_u = g_u g_s$$

and g_s is semisimple and g_u is unipotent.

Moreover, $g_s, g_u \in k[g] = \{\sum_{i=1}^m a_i g^i \mid a_i \in k\} \subseteq \text{End}(V)$.

Proof. Existence (Sketch): Say

$$g = \begin{pmatrix} \lambda & 1 & 0 \\ & \lambda & 1 \\ & & \lambda \end{pmatrix}$$

then take

$$g_s = \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{pmatrix}, \quad g_u = \begin{pmatrix} 1 & \lambda^{-1} & 0 \\ & 1 & \lambda^{-1} \\ & & 1 \end{pmatrix}.$$

For $\lambda \in k$, define the **generalized λ -eigenspace** of g by

$$V_\lambda := \{v \in V \mid \exists n \in \mathbb{N}_0 : (g - \lambda)^n v = 0\}.$$

Then

$$V = \bigoplus_{\lambda \in k} V_\lambda.$$

Here V_λ = sum of domains of all Jordan blocks with λ s on the diagonal. (It follows from the Jordan Decomposition for matrices that such a decomposition exist.)

Let's define $g_s \in \text{GL}(V)$ by

$$g_s|_{V_\lambda} = \lambda \cdot \text{Id}.$$

Note that $gV_\lambda \subset V_\lambda$, hence g commutes with g_s , hence g, g_s commutes with $g_u := gg_s^{-1}$. Then, $g = g_s g_u = g_u g_s$.

Write $\det(T - g) = \prod_\lambda (T - \lambda)^{n(\lambda)}$, $n(\lambda) = \dim(V_\lambda)$. Since the polynomials $T - \lambda$ for $\lambda \in k$ are coprime, the chinese remainder theorem implies that there is a $Q \in k[T]$ s.t.

$$Q \equiv \lambda \pmod{(T - \lambda)^{n(\lambda)}}$$

for each $\lambda \in k$.

We claim that

$$Q(g) = g_s.$$

Indeed, since $gV_\lambda \subseteq V_\lambda$, we have

$$Q(g)V_\lambda \subseteq V_\lambda.$$

So, it suffices to show for all $v \in V_\lambda$

$$Q(g)v = g_s v = \lambda v.$$

Note that, by Cayley-Hamilton,

$$V_\lambda = \{v \in V \mid (g - \lambda)^{n(\lambda)} v = 0.\}$$

Write

$$Q = \lambda + R \cdot (T - \lambda)^{n(\lambda)}$$

for some $R \in k[T]$. Since $(g - \lambda)^{n(\lambda)} v = 0$, deduce that $Q(g)v = \lambda v$, as required.

If $P \equiv \lambda^{-1} \pmod{(T - \lambda)^{n(\lambda)}}$, then $P(g) = g_s^{-1}$.

Therefore,

$$g_u = g \cdot P(g)$$

for $T \cdot P(T) \in k[T]$.

Uniqueness: Suppose given some other decomposition

$$g = g'_s g'_u = g'_u g'_s$$

with g'_s semisimple and g'_u unipotent. Then g'_s commutes with g'_s and g'_u , hence with g , hence also with any element in $k[g]$. Ergo, g'_s commutes with g_s and g_u . Similarly, g'_u commutes with g_s and g_u .

Consider

$$h := g'_s g_s^{-1} = g'_s g'_u (g'_u)^{-1} g_s^{-1} = g(g'_u)^{-1} g_s^{-1} = g_u (g'_u)^{-1}.$$

Then $h = g'_s g_s^{-1}$ is a product of semisimple elements and $h = g_u (g'_u)^{-1}$ is a product of unipotent elements. By proceeding lemmas, h is semisimple and unipotent, ergo trivial. It follows $g'_s = g_s$ and $g'_u = g_u$. \square

Corollary 2. *Let $g \in GL(V)$, let $W \subset V$ be any g -invariant subspace, i.e. $gW \subseteq W$.*

Then, W is g_s -invariant and g_u -invariant.

Proof. This is clear, since g_s and g_u are algebraically generated by g over g . \square

Lemma 24. *Let $\phi : V \rightarrow W$ be a linear map between finite-dimensional vector spaces.*

Let $\alpha \in GL(W)$ and $\beta \in GL(W)$ s.t.

$$\begin{array}{ccc} V & \xrightarrow{\alpha} & V \\ \downarrow \phi & & \downarrow \phi \\ W & \xrightarrow{\beta} & W, \end{array}$$

i.e. $\phi \circ \alpha = \beta \circ \phi$.

Then,

$$\begin{aligned} \phi \circ \alpha_s &= \beta_s \circ \phi, \\ \phi \circ \alpha_u &= \beta_u \circ \phi. \end{aligned}$$

Proof. Write $V = \bigoplus_{\lambda \in k} V_\lambda$, $W = \bigoplus_{\lambda \in k} W_\lambda$ where V_λ are the generalized α -eigenspaces and W_λ are the generalized β -eigenspaces.

We claim that

$$\phi(V_\lambda) \subset W_\lambda.$$

Indeed, let $v \in V_\lambda$, then

$$(\beta - \lambda)^n \phi(v) = \phi((\alpha - \lambda)^n v) = 0.$$

Since $(\alpha - \lambda)^n v = 0$, the claim follows.

Since, $\alpha_s|_{V_\lambda} = \lambda \text{Id}$ and $\beta_s|_{W_\lambda} = \lambda \text{Id}$, deduce that

$$\phi \circ \alpha_s = \beta_s \circ \phi.$$

Indeed, both sides are given on V_λ by $\lambda \cdot \phi$. Thus

$$\begin{aligned}\phi \circ \alpha_u &= \phi \circ \alpha \alpha_s^{-1} \\ &= \beta \beta_s^{-1} \circ \phi \\ &= \beta_u \circ \phi.\end{aligned}$$

□

Lemma 25. *Let $\alpha \in GL(V)$, $\beta \in GL(W)$. Then the **tensor** $\alpha \otimes \beta \in GL(V \otimes W)$ is defined by*

$$(\alpha \otimes \beta)(u \otimes v) = \alpha(u) \otimes \beta(v).$$

Then, we have

$$\begin{aligned}(\alpha \otimes \beta)_s &\stackrel{(1)}{=} \alpha_s \otimes \beta_s \\ (\alpha \otimes \beta)_u &\stackrel{(2)}{=} \alpha_u \otimes \beta_u.\end{aligned}$$

Proof. It suffices to prove (1), since

$$\begin{aligned}(\alpha \otimes \beta)_u &= (\alpha \otimes \beta) \circ (\alpha \otimes \beta)_s^{-1} \\ &\stackrel{(1)}{=} (\alpha \otimes \beta) \circ (\alpha_s \otimes \beta_s)^{-1} \\ &= \alpha \alpha_s^{-1} \otimes \beta \beta_s^{-1} \\ &= \alpha_u^{-1} \otimes \beta_u^{-1}\end{aligned}$$

For (1), consider

$$\begin{aligned}V &= \bigoplus_{\lambda \in k} V_\lambda, \\ W &= \bigoplus_{\lambda \in k} W_\lambda.\end{aligned}$$

It follows

$$V \otimes W = \bigoplus_{\lambda, \mu \in k} V_\lambda \otimes W_\mu.$$

Now,

$$\alpha_s \otimes \beta_s|_{V_\lambda \otimes W_\mu} = \lambda \mu \cdot \text{Id}.$$

Ergo, $\alpha_s \otimes \beta_s$ is semisimple. By Proposition, we reduce to checking that $\alpha_u \otimes \beta_u$ is unipotent. Indeed,

$$\alpha_u \otimes \beta_u - 1 = (\alpha_u - 1) \otimes (\beta_u - 1) + 1 \otimes (\beta_u - 1) + (\alpha_u - 1) \otimes 1$$

is nilpotent (You can also check that $\alpha_u \otimes \beta_u = (\alpha_u \otimes 1) \circ (1 \otimes \beta_u)$ is unipotent.) □

Example 10. Let $1 \in GL(V)$. Then $1_s = 1$ and $1_u = 1$.

Summary : Let G be an algebraic group. Let $r_V : G \rightarrow \mathrm{GL}(V)$ be a finite-dimensional representation. Also, fix $g \in G$.

Let $\lambda_V := r_V(g)_s$ (or $r_V(g)_u$).

We get a family of operators $\lambda_V \in \mathrm{End}(V)$ with the following properties:

- (i) if $V = k$ and $r_V(g') = 1$ for all $g' \in G$, then $\lambda_V = 1$.
- (ii) for any two representations in V and W , we have

$$\lambda_{V \otimes W} = \lambda_V \otimes \lambda_W.$$

- (iii) for all G -equivariant $\phi : V \rightarrow W$ we have

$$\phi \circ \lambda_V = \lambda_W \circ \phi.$$

Theorem 10. *Let G be an algebraic group. Let $\lambda_V \in \mathrm{End}(V)$ (i.e. $V = (r_V, V)$ is a finite-dim. representation of G) be a family of operations satisfying (i), (ii), (iii).*

Then, there is exactly one $g \in G$ s.t. $\lambda_V = r_V(g)$ for all V .

Note, that this theorem implies our goal theorem.

Applying the theorem to $\lambda_V = r_V(g)_s$ implies

$$\exists_1 g_s \in G : r_V(g_s) = r_V(g)_s$$

and

$$\exists_1 g_u \in G : r_V(g_u) = r_V(g)_u.$$

Proof of Goal Theorem. There exist unique $g_s, g_u \in G$ s.t.

$$g \stackrel{(*)}{=} g_u g_s = g_s g_u,$$

Then, $r_V(g) = r_V(g_s)r_V(g_u) = r_V(g_u)r_V(g_s)$.

Since $r_V(g_u)$ is unipotent and $r_V(g_s)$ is semisimple, it follows $r_V(g_u) = r_V(g)_u$ and $r_V(g_s) = r_V(g)_s$.

To deduce $(*)$, take any $r_V : G \hookrightarrow \mathrm{GL}(V)$. We know for each V

$$r_V(g) = r_V(g_s)r_V(g_u) = r_V(g_u)r_V(g_s). \quad \square$$

Proof of Theorem 10. We first extend the assignment

$$V \mapsto \lambda_V$$

to all¹ representations of G .

Say $V = \bigcup_j W_j$ where each W_j is a finite-dimensional G -invariant subspace. Try to define $\lambda_V \in \text{End}(V)$ by

$$\lambda_V|_{W_j} := \lambda_{W_j}.$$

For this to be well-defined, we need to show for each i, j

$$\lambda_{W_i}|_{W_i \cap W_j} \stackrel{(*)}{=} \lambda_{W_j}|_{W_i \cap W_j}.$$

Proof of (*): Apply assumption (iii) to the G -equivariant linear maps

$$\begin{aligned} W_i \cap W_j &\xrightarrow{\phi} W_i, \\ W_i \cap W_j &\xrightarrow{\phi'} W_j. \end{aligned}$$

Then,

$$\begin{aligned} \lambda_{W_i}|_{W_i \cap W_j} &= \lambda_{W_i} \circ \phi \\ &\stackrel{(iii)}{=} \phi \circ \lambda_{W_i \cap W_j} \\ &= \phi' \circ \lambda_{W_i \cap W_j} \end{aligned}$$

and

$$\lambda_{W_j}|_{W_i \cap W_j} = \lambda_{W_j} \circ \phi' = \phi' \circ \lambda_{W_i \cap W_j}.$$

Recall here that any finite-dimensional G -invariant $W \subset V$ is a representation. ...

¹Not necessarily finite-dimensional, but may be written as a filtered union of finite-dimensional G -invariant subspaces of W .

Let G be an algebraic group.

Easy Exercise : If V_1, V_2 are representations r_1, r_2 of G , then $V_1 \otimes V_2$ is also a representation with

$$r = r_1 \otimes r_2 : G \rightarrow \mathbf{GL}(V_1 \otimes V_2)$$

given by

$$r(g)(v_1 \otimes v_2) = (r_1(g)v_1) \otimes (r_2(g)v_2).$$

Proof. Given $\Delta_j : V_j \rightarrow V_j \otimes k[G]$, define

$$\Delta : V_1 \otimes V_2 \longrightarrow V_1 \otimes V_2 \otimes k[G]$$

by: if

$$\Delta_1 u = \sum_i u_i \otimes f_i, \quad \Delta_2 v = \sum_j v_j \otimes h_j,$$

then

$$\Delta(u \otimes v) = \sum_i \sum_j u_i \otimes v_j \otimes f_i h_j.$$

Set $A := k[G]$, then

$$r_A := \text{right regular representation with } r_A(g)f(x) = f(xg).$$

The map

$$\begin{aligned} A \otimes A &\xrightarrow{m} A \\ f_1 \otimes f_2 &\longmapsto f_1 f_2 \end{aligned}$$

defines a morphism of representations

$$(A, r_A) \otimes (A, r_A) \rightarrow (A, r_A).$$

Indeed,

$$\begin{aligned} m((r_A \otimes r_A)(g)(f_1 \otimes f_2))(x) &= f_1(xg)f_2(xg), \\ &= f_1 f_2(xg) = r_A(g)(m(f_1 \otimes f_2))(x), \end{aligned}$$

since $f_1(_g) \otimes f_2(_g) = (r_A \otimes r_A)(g)(f_1 \otimes f_2)$.

Ergo $m \circ (r_A \otimes r_A)(g) = r_A(g) \circ m$. □

Recall: We wanted to prove the following theorem

Theorem 11. *Let $\lambda_V \in \text{End}(V)$ be given s.t. for all finite-dim. rep.s V of G s.t.:*

$$(i) \lambda_k = 1$$

$$(ii) \lambda_{V \otimes W} = \lambda_V \otimes \lambda_W$$

(iii) *for all morphisms of rep.s $\phi : V \rightarrow W$ we have*

$$\phi \circ \lambda_V = \lambda_W \circ \phi.$$

Then, there is exactly one $g \in G$ s.t. $\lambda_V = r_V(g)$ for all V .

Proof. Last time, we saw that any such family $V \mapsto \lambda_V$ extends to **all** rep.s V of G .

Let's note also that, if (V_0, r_0) is any representation of G with trivial action, i.e. $r(g) = 1$ for all g , then $\lambda_{V_0} = 1$. Indeed, let $v \in V_0$. We must check that $\lambda_{V_0} v = v$. Since the action is trivial, any subspace of V_0 is G -invariant.

Consider the map

$$\begin{aligned} \phi : k &\longrightarrow V_0 \\ \alpha &\longmapsto \alpha v \end{aligned}$$

where $v = \phi(1)$. Then, ϕ is a morphism of rep.s because the action is trivial.

Thus,

$$\lambda_V v = (\lambda_V \circ \phi)(1) \stackrel{(iii)}{=} (\phi \circ \lambda_k)(1) \stackrel{(i)}{=} \phi(1) = v.$$

Consider $\lambda_A \in \text{End}(A)$. Then,

$$\lambda_{A \otimes A} = \lambda_A \otimes \lambda_A.$$

It is an easy exercise to see that $m : (A, r_A) \otimes (A, r_A) \rightarrow (A, r_A)$ is a morphism of rep.s.

By (iii) it follows, $m \circ (\lambda_A \otimes \lambda_A) = \lambda_A \circ m$, i.e.

$$\lambda_A(f_1 f_2) = \lambda_A(f_1) \lambda_A(f_2)$$

for all $f_1, f_2 \in A$. Thus, λ_A is an algebra morphism (check, using the morphism $k \hookrightarrow A$, that $\lambda_A(1) = 1$).

Thus, $\lambda_A = \phi^*$ for some unique morphism ϕ of algebraic sets $\phi : G \rightarrow G$.

We claim that ϕ commutes with left multiplication i.e.

$$\phi(hx) = h\phi(x)$$

for all $h, x \in G$. Indeed, let's consider the map

$$\begin{aligned} A &\longrightarrow A \\ f &\longmapsto f(h \cdot _). \end{aligned}$$

This induces a morphism

$$(A, r_A) \xrightarrow{\psi} (A, r_A).$$

By (ii), $\psi \circ \lambda_A = \lambda_A \circ \psi$.

Since $\lambda_A = \phi^*$, this implies the claim.

Now, set $g := \phi(e)$. Then for all $h \in G$,

$$\phi(h) = \phi(h e) = h g.$$

Thus, $\lambda_A = \phi^* = r_A(g)$.

(It remains to show that

$$\lambda_V = r_V(g)$$

for each finite-dim. rep. V .)

Let $V = (V, r)$ be any rep. This induces a map

$$\Delta : V \longrightarrow V \otimes A.$$

If $\Delta v = \sum v_i \otimes f_i$, then

$$h v = \sum f_i(h) \otimes v_i.$$

Let

$$\begin{aligned} \varepsilon : V \otimes A &\longrightarrow V \\ v \otimes f &\longmapsto f(1)v. \end{aligned}$$

It follows $\varepsilon \circ \Delta : V \rightarrow V$ is the identity map.

Let (V_0, r_0) be the representation of G with $V_0 := V$ and r_0 the trivial action. Then, $\Delta : V \rightarrow V_0 \otimes A$ is a morphism of representations.

(Indeed, if $\Delta v = \sum v_i \otimes f_i$, then

$$\Delta(r(h)v) = (r_0(h) \otimes r_A(h))\Delta v$$

since

$$\begin{aligned}
\Delta v &= \sum v_i \otimes f_i \\
\iff xv &= \sum f_i(x_i)v_i \quad \forall x \in G \\
\iff xhv &= \sum f_i(xh)v_i \quad \forall x, h \in G.
\end{aligned}$$

Since $r(h)v = hv$, it follows

$$\Delta(hv) = \sum v_i \otimes f_i(_ \cdot h).$$

We want to show

$$\lambda_V = r_V(g).$$

We have

$$\begin{aligned}
\Delta \circ \lambda_V &\stackrel{(iii)}{=} \lambda_{V_0 \otimes A} \circ \Delta \\
&\stackrel{(ii)}{=} (\lambda_{V_0} \otimes \lambda_A) \circ \Delta \\
&= (1 \otimes \lambda_A) \circ \Delta = (1 \otimes r_A(g)) \circ \Delta.
\end{aligned}$$

This implies

$$\Delta \circ \lambda_V = (1 \otimes r_A(g)) \circ \Delta$$

but also

$$\Delta \circ r_V(g) = (1 \otimes r_A(g)) \circ \Delta.$$

Because of the injectivity of Δ it now follows

$$\lambda_V = r_V(g).$$

□

Corollary 3. *Let $\phi : G \rightarrow H$ be any morphism of algebraic groups. Then, for all $g \in G$*

$$\begin{aligned}
\phi(g)_s &= \phi(g_s) \\
\phi(g)_u &= \phi(g_u).
\end{aligned}$$

Proof. Let V be any **faithful** representation of H , i.e. $r_V : H \rightarrow \mathbf{GL}(V)$ is injective, (for a finite-dim. V).

Then, $r_V \circ \phi$ is a rep. of G . To prove (i), it suffices to show

$$r_V(\phi(g)_s) = r_V(\phi(g_s))$$

since H operates faithfully on V .

We know that

$$r_V(\phi(g)_s) = r_V(\phi(g))_s$$

(characterizing property of h_s for $h \in H$). On the other hand,

$$r_V(\phi(g_s)) = (r_V \circ \phi)(g_s) = r_V(\phi(g))_s.$$

Therefore, claim (i) follows. (ii) works analogously. \square

Definition 16. Let $g \in G$ where G is an algebraic group. We call g **semisimple**, if $g = g_s$.

We call g **unipotent**, if $g = g_u$.

Lemma 26. For $g \in G$, the following are equivalent:

- (i) g is semisimple.
- (ii) $r_V(g)$ is semisimple for all finite-dim. rep. V .
- (iii) $r_V(g)$ is semisimple for at least one faithful f.d. rep. V of G .

We get an analogous lemma for unipotent group elements.

Proof. We have

$$\begin{aligned}
(i) &\iff g = g_s \\
&\stackrel{\text{Def. of } g_s \text{ by goal thm.}}{\iff} r_V(g) = r_V(g)_s \forall \text{ f.d. } V \\
&\iff r_V(g) \text{ is semisimple} \\
&\iff (ii) \implies (iii).
\end{aligned}$$

On the other hand,

$$(iii) \implies \exists \text{ faithful f.d. } V \text{ s.t. } r_V(g) = r_V(g)_s = r_V(g_s) \implies g = g_s. \square$$

6 Non-Commutative Algebra

Definition 17. A ring R (for now) is unital, associative but not necessarily commutative.

Example 11. The ring of matrices over some field or ring.

Definition 18. A **left ideal** $I \subset R$ is a subset that is an abelian subgroup of $(R, +)$ s.t. $ra \in I$ for all $r \in R, a \in I$.

A **right ideal** $I \subset R$ is a subset that is an abelian subgroup with

$$IR \subset I.$$

A two-sided ideal I is a subset that is a left and a right ideal of R .

It is easy to check that for any homomorphism of rings $\phi : R \rightarrow S$, $\text{Kern}\phi$ is a two-sided ideal. Also, if $J \subset R$ is any two-sided ideal, then there exists a unique ring structure on R/J s.t. the projection $R \rightarrow R/J$ is a ring homomorphism.

Definition 19. A **left module** M for R is an abelian group equipped with a ring homomorphism

$$R \xrightarrow{\alpha} \text{End}(M)$$

where $\text{End}(M)$ acts on the left of M . We write

$$rm := \alpha(r)m.$$

We have

$$(r_1 r_2)(m) = r_1(r_2(m)).$$

If R acts on M by the left, we write

$$R \curvearrowright M.$$

Example 12. $M_n(k) \curvearrowright k^n$ where k^n is the space of column vectors.

If k^n denotes the space of row vectors, we have $k^n \curvearrowleft M_n(k)$.

Definition 20. A **(left) submodule** $N \subset M$ is an algebraic subgroup s.t.

$$RN \subset N.$$

It follows that N itself is a left module.

Definition 21. A (left) module M of R is **simple** (or irreducible) if it has exactly the two submodules: $0 = \{0\}$ and M .

Definition 22. A ring R is a **division ring** (aka **skew field**) if it satisfies any of the following equivalent requirements:

- (i) $R^\times = R \setminus \{0\}$ where² $R^\times = \{r \in R \mid \exists a, b \in R : ar = rb = 1.\}$
- (ii) R has no nontrivial left or right ideals.

Definition 23. If $R \curvearrowright M$, then we can define

$$\text{End}_R(M) := \{\phi \in \text{End}(M) \mid \phi(rm) = r\phi(m) \forall r \in R, m \in M\}.$$

Note, that $\text{End}_R(M)$ is a ring.

Lemma 27 (Schur's Lemma). *If M is simple, then $\text{End}_R(M)$ is a division ring.*

Lemma 28. *Let k be a field. Then, $M_n(k)$ has no nontrivial twosided ideals.*

Theorem 12 (Jacobson Density Theorem (Double Commutant Theorem)). *Suppose M is a simple left module which is finitely generated as a right D -module for $D = \text{End}_R(M)$.*

Assume that R acts faithfully on M , i.e. $R \rightarrow \text{End}_R(M)$ is injective.

Then, the map $R \rightarrow \text{End}_D(M)$ is an isomorphism.

²If $ar = rb = 1$, then $a = arb = b$.

Recap:

- Basics: definitions, Hopf-algebras, ...
- Jordan decomposition
- Primer on non-commutative algebra
 - Jacobson density theorem
- Unipotent groups
- Tori

We had last week

$$\text{End}_D(M) := \{\phi \in \text{End}(M) \mid \phi \circ d = d \circ \phi \ \forall d \in D\}.$$

Let k be an algebraically closed field, V a non-trivial finite-dimensional k -vector space and let G be a subgroup of $\text{GL}(V)$ that acts **irreducibly** on V , i.e., V is **G -irreducible**, i.e., the only G -invariant subspaces of V are 0 and V .

Set

$$D := \{d \in \text{End}_k(V) \mid dg = gd \ \forall g \in G\}$$

and

$$R := \text{span}_k(G) = \left\{ \sum_{i=1}^n c_i g_i \mid c_i \in k, g_i \in G, n \in \mathbb{N}_0 \right\}.$$

Then,

$$D = \text{End}_R(V)$$

where R is the k -subalgebra of $\text{End}(V)$ that is generated by G .

Lemma 29 (Schur's Lemma). *We understand $k \hookrightarrow \text{End}(V)$ as the inclusion of operations which operate by scalar multiplication*

$$k \xrightarrow{\cong} \{\phi : V \rightarrow V \mid \phi : v \mapsto t \cdot v \text{ for some } t \in k\}.$$

Let V be G -irreducible. Then, we have

$$D \cong k.$$

Proof. Let $d \in D$. Since $V \neq 0$, there is an eigenspace $V_\lambda \neq 0$ for d . Observe that V_λ has to be G -invariant:

if $g \in G$ and $v \in V_\lambda$, then $gv \in V_\lambda$, since

$$dgv = gdv = g(\lambda v) = \lambda gv.$$

Since V_λ is a non-trivial G -invariant subspace and V is irreducible under G , we have

$$V_\lambda = V.$$

Ergo $d = \lambda$ in the sense of $k \hookrightarrow \text{End}(V)$. □

Consequence of the Jacobson Density Theorem: $R = \text{End}_k(V)$, i.e., G generates all linear operations on V , if V is G -irreducible.

We will prove this after a lemma.

Lemma 30. *Let V be G -irreducible.*

Let $n \in \mathbb{N}$. Set

$$V^n := V \oplus V \oplus \dots \oplus V = V_1 \oplus \dots \oplus V_n$$

where each $V_i = V$.

Let $v = (v_1, \dots, v_n) \in V^n$ and set

$$Rv := \{(rv_1, \dots, rv_n) \mid r \in R\} = \text{span}\{(gv_1, \dots, gv_n) \mid g \in G\}.$$

Then, $Rv \neq V^n$ iff the v_j are linearly dependent over k .

Consequence: Take $n := \dim(V)$. Let $\{e_1, \dots, e_n\}$ be a basis of V and set

$$e := (e_1, \dots, e_n) \in V^n.$$

Since the $(e_i)_i$ are linearly independent, the lemma states that $Re = V^n$.

Now, let $x \in \text{End}_k(V)$. Choose $r \in R$ s.t.

$$re = (xe_1, \dots, xe_n).$$

Then $re_i = xe_i$ for all i , thus $x = r$. Hence, $R = \text{End}_k(V)$.

Proof. For $v = (v_1, \dots, v_n) \in V^n$ choose $J \in \{1, \dots, n\}$ as large as possible with

$$Rv + V_1 + V_2 + \dots + V_{J-1} =: U \neq V^n.$$

Such an J does exist, since we know that $Rv \neq V^n$.

Then, $V_J \not\subseteq U$, otherwise we may increase J . Also, U is invariant by the diagonal action of G on V^n . Thus, $V_J \cap U \subseteq V_J$ is a proper G -invariant subspace of the G -irreducible $V_J \cong V$. Therefore, $V_J \cap U = 0$.

On the other hand, by maximality of J , we have

$$U \oplus V_J = V^n.$$

Ergo, the map (composition)

$$V \cong V_J \hookrightarrow V^n \twoheadrightarrow V^n/U$$

is a G -equivariant isomorphism, since U is G -invariant.

Let $z : V^n/U \xrightarrow{\cong} V$ be the inverse isomorphism. Let l be the G -equivariant map given by

$$\begin{array}{ccc} V^n & \xrightarrow{l} & V \\ \downarrow & \nearrow z & \\ V^n/U & & \end{array}$$

and let l_j be the G -equivariant maps by restricting l on V_j . Then $l_j \in D \cong k$.

Say $l_j = t_j \in k$. Then,

$$l(w) = t_1 w_1 + \dots + t_n w_n.$$

Since z is an isomorphism, l is nonzero and $(t_1, \dots, t_n) \neq (0, \dots, 0)$.

Since $l|_U = 0$, we can deduce for all $u \in U$

$$t_1 u_1 + \dots + t_n u_n = 0.$$

But $v \in Rv \subseteq U$, so we may conclude – as required – that the $(v_i)_i$ are linearly dependent ($l(v) = 0$). \square

7 Unipotent Groups

Let G be a subgroup of $\mathrm{GL}(V)$ where V is a finite-dimensional vector space and k an algebraically closed field.

Definition 24. We say that G is **unipotent** if one of the following equivalent conditions hold for each $g \in G$:

- g is unipotent (i.e. $(g - 1)^n = 0$ for some $n \in \mathbb{N}$).
- all eigenvalues of g are 1.
- g is conjugate to $\begin{pmatrix} 1 & \star & \star \\ 0 & \ddots & \star \\ 0 & 0 & 1 \end{pmatrix}$.

Theorem 13. Any unipotent subgroup of $\mathrm{GL}_n(k)$ is conjugate to a subgroup of

$$U_n := \left\{ \begin{pmatrix} 1 & \star & \star \\ 0 & \ddots & \star \\ 0 & 0 & 1 \end{pmatrix} \right\} = \left\{ U \in M_n(k) \mid U_{i,j} = \begin{cases} 0, & \text{if } i > j \\ 1, & \text{if } i = j \\ \text{arbitrary,} & \text{otherwise.} \end{cases} \right\}.$$

Definition 25. For two subgroups G, H of some common supergroup, define their **commutator** by

$$[G, H] := \langle ghg^{-1}h^{-1} \mid g \in G, h \in H \rangle.$$

A group G is called **nilpotent**, if one of its commutators is trivial, i.e. if we set

$$G_0 := G \text{ and } G_{i+1} := [G_i, G],$$

then G is called nilpotent iff there is an $j \in \mathbb{N}$ with $G_j = 1$.

Corollary 4. Any unipotent subgroup of $\mathrm{GL}(V)$ is nilpotent.

Definition 26. A group G is called **solvable**, if $G^{(n)} = 1$ for some n where

$$\begin{aligned} G^{(0)} &:= G, \\ G^{(i+1)} &:= [G^{(i)}, G^{(i)}]. \end{aligned}$$

Note that nilpotent groups are solvable, since $G^{(i)} \subset G_i$.

Notation 1. In the following, we will write $G' := [G, G]$.

Definition 27. Let $n := \dim(V)$. A **complete flag** is a maximal strictly increasing chain of subspaces

$$0 = V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_n = V.$$

Any complete flag is of the form

$$V_j := \text{span}\{e_1, \dots, e_j\}$$

for some basis e_1, \dots, e_n of V .

Let B be the basis of some flag $0 = V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_n = V$. For $x \in \text{End}(V)$, we have that x is upper-triangular with respect to B iff x leaves each member V_i of the flag invariant, i.e. $xV_i \subseteq V_i$.

Proposition 2 (Key Proposition). *Let G be a unipotent subgroup of $GL(V)$. Then there is a complete flag $V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_n$ consisting of G -invariant subspaces, i.e., each V_i is G -invariant.*

Proof. Recall, that G is a unipotent subgroup of $GL_n(V)$. We will give an induction on $n = \dim V$.

If $n = 0$, there is nothing to show.

Let $n \geq 1$. We may assume that V is G -irreducible. Because, if not, there is a G -invariant subspace $0 \neq W \subset V$ s.t. W and V/W have dimension $< n$. Then there exist complete G -invariant flags in W and V/W and the claim – that there is a complete G -invariant flag in V – follows by the induction hypothesis.

By Jacobson Density Theorem, we have

$$R := \text{span}(G) = \text{End}(V) := \text{End}_k(V).$$

Since G is unipotent, we have for each $g \in G$

$$\text{trace}(g) = n.$$

Ergo, for $g, h \in G$

$$\text{trace}(gh) = \text{trace}(h)$$

and

$$\text{trace}((g - 1)h) = \text{trace}(gh) - \text{trace}(h) = 0.$$

Since $\text{span}(G) = \text{End}(V)$, it now in particular follows for all $g \in G, \phi \in \text{End}(V)$

$$\text{trace}((g - 1)\phi) = 0.$$

Since the above holds for all $\phi \in \mathbf{End}(V)$, it must hold

$$g - 1 = 0$$

for all $g \in G$ (take for example the elementary matrices $\phi = E_{i,j}$). Ergo, G is trivial. Then, any complete flag is trivially G -invariant. \square

Remark 3. This gives the group analogue of Engel's Theorem.

Proof Goal Theorem. Let B be a basis of V s.t. G leaves each subspace in the corresponding flag invariant. Then, G is upper-triangle with respect to this basis.

On the other hand, each $g \in G$ is unipotent, hence its diagonal elements are all 1. Thus, with respect to B

$$G \subseteq \left\{ \begin{pmatrix} 1 & \star & \star \\ 0 & \ddots & \star \\ 0 & 0 & 1 \end{pmatrix} \right\} = U_n.$$

\square

Remark 4. Tori are of the form $(k^\times)^n$. In the case $k = \mathbb{C}$, $(\mathbb{C}^\times)^n$ are the complexifications of $U(1)^n$. This equals tori in the topological sense. Now,

$$\begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix} \subseteq \mathbf{GL}_2(\mathbb{C})$$

is a non-algebraic unipotent group.

Exercise. (To be discussed next time.)

It would have sufficed to prove the Goal theorem in the special case that G is algebraic.

Corollary of Proof: If $G \subset \mathbf{GL}(V)$ (with $V \neq 0$) is unipotent and V is G -irreducible, then $G = 1$, $\dim V = 1$.

Answer to last Exercise: Recall that the main point was to show that any unipotent subgroup $G \subseteq \mathrm{GL}(V)$ leaves invariant some complete flag $\mathcal{F} = (V_0 \subset V_1 \dots)$. But by some homework (problem 1), the group

$$\mathrm{GL}(V)_{\mathcal{F}} := \{g \in \mathrm{GL}(V) \mid g\mathcal{F} = \mathcal{F}\}$$

is algebraic.

Proof: If \mathcal{F} is the standard flag with $V_i = \mathrm{span}(e_1, \dots, e_i)$ for the standard basis $\{e_1, \dots, e_n\}$, then

$$\mathrm{GL}(V)_{\mathcal{F}} = \{A \in \mathrm{GL}(V) \mid A \text{ is upper-triangle}\}.$$

The condition that A is upper triangle can be realized by polynomials. □

Thus,

$$\begin{aligned} & G \text{ fixes } \mathcal{F} \\ \iff & G \subseteq \mathrm{GL}(V)_{\mathcal{F}} \\ \overset{\mathrm{GL}(V)_{\mathcal{F}} \text{ is algebraic}}{\iff} & \overline{G} \subseteq \mathrm{GL}(V)_{\mathcal{F}} \\ \iff & \overline{G} \text{ fixes } \mathcal{F}. \end{aligned}$$

Now, the Zariski-Closure \overline{G} of any group G is an algebraic group (shown in some homework).

Further, if G is unipotent, then \overline{G} is unipotent.

8 Tori

Definition 28. A **torus** is an algebraic group that is isomorphic to \mathcal{G}_m^n for some $n \in \mathbb{N}_0$ where $\mathcal{G}_m = k^\times = \mathrm{GL}_1(k)$ is the unit group of k .

We think of $\mathcal{G}_m^n \subseteq \mathrm{GL}_n(k)$ as the subgroup of diagonal matrices.

Lemma 31. *Let G be a commutative algebraic group. Then the following are equivalent:*

- (i) *each $g \in G$ is semisimple.*
- (ii) *for each finite-dimensional representation V of G and for each $g \in G$, the operator $r_V(g)$ is diagonalizable.*
- (iii) *for all finite-dimensional representations V of G , there is a basis of common eigenvectors for $r_V(G)$, i.e. a basis s.t.*

$$r_V(G) \subseteq \mathcal{G}_m^n.$$

- (iv) *G is isomorphic to an algebraic subgroup of a torus.*

Proof. We show:

- (i) \iff (ii): This follows from the Jordan decomposition and definition of semisimple.
- (ii) \implies (iii) : This is homework. Note that any commutative subset S of $\mathrm{GL}(V)$ consisting of semisimple operators may be diagonalized simultaneously.
- (iii) \implies (iv) : Take any faithful representation V of G and diagonalize it simultaneously. Then, $G \cong r_V(G) \subseteq \mathcal{G}_m^n$.
- (iv) \implies (i) : Any diagonal matrix is semisimple.

□

Definition 29. A commutative algebraic group G is called **diagonalizable**, if it satisfies one of the above equivalent conditions.

Definition 30. A **character** χ of an algebraic group G is an element $\chi \in \mathrm{Hom}_{\mathrm{alg.grp.}}(G, k^\times)$, i.e., a homomorphism $\chi : G \rightarrow k^\times$ of algebraic groups.

Notation 2. For an algebraic group G , set $\mathfrak{X}(G) := \mathrm{Hom}_{\mathrm{alg.grp.}}(G, k^\times)$.

Also denote now by $\mathcal{O}(X) := k[T]/I(X)$ the coordinate ring of an algebraic set X (rather than $k[X]$).

Lemma 32. *There is a bijection*

$$\mathfrak{X}(G) = \{\text{characters } \chi \text{ of } G\} \longleftrightarrow \{x \in \mathcal{O}(G)^\times \mid \Delta(x) = x \otimes x\}.$$

Proof. Note, that any $x \in \mathcal{O}(G)^\times$ can be thought of as a map $x : G \rightarrow k^\times \subset k$.

We have

$$\begin{aligned} \text{Hom}_{\text{alg.grp.}}(G, \mathcal{G}_m) &= \{\phi \in \text{Hom}_{\text{alg.sets}}(G, \mathcal{G}_m) \mid \phi(gh) = \phi(g)\phi(h) \ \forall g, h\} \\ &= \{\phi \in \text{Hom}_{k\text{-alg.}}(\mathcal{O}(\mathcal{G}_m), \mathcal{O}(G)) \mid (\phi \otimes \phi) \circ \Delta = \Delta \circ \phi\}. \end{aligned}$$

Recall: $\mathcal{O}(\mathcal{G}_m) \cong k[t, \frac{1}{t}]$ with $\Delta(t) = t \otimes t$.

Thus for any k -algebra A , $\text{Hom}_{k\text{-alg.}}(\mathcal{O}(\mathcal{G}_m), A) \cong^{A^\times}$ via

$$[t \mapsto a, (t^{-1} \mapsto a^{-1})] \longleftrightarrow a.$$

Thus,

$$\text{Hom}_{\text{alg.grp.}}(G, \mathcal{G}_m) \cong \{a \in \mathcal{O}(G)^\times \mid a \otimes a = \Delta(a)\}.$$

Therefore, it suffices to test the condition $(\phi \otimes \phi) \circ \Delta = \Delta \circ \phi$ on the generators t, t^{-1} of $\mathcal{O}(\mathcal{G}_m)$. Now, the above isomorphism is given by

$$\phi \mapsto a = \phi(t)$$

which is equivalent or regarding $\chi : G \rightarrow \mathcal{G}_m$ as a map $\chi : G \rightarrow k$. □

Example 13. Let $G = \mathcal{G}_m$, then $\mathcal{O}(G) = k[t, \frac{1}{t}]$.

Which $x = \sum_{m \in \mathbb{Z}} c_m t^m \in \mathcal{O}(G)$ – with almost all $c_m = 0$, but not all of them – have the property

$$\Delta(x) = x \otimes x?$$

We have

$$\begin{aligned} x \otimes x &= \sum_{m, n \in \mathbb{Z}} c_m c_n t^m \otimes t^n, \\ \Delta(x) &= \sum_{m \in \mathbb{Z}} c_m t^m \otimes t^m. \end{aligned}$$

Those sums equal, if

$$\begin{aligned} c_m c_n &= 0 \text{ for all } m \neq n, \\ c_m^2 &= c_m \text{ for all } m. \end{aligned}$$

By those conditions, it follows

$$x = t^m.$$

Therefore

$$\mathfrak{X}(G) = \{\chi_m \mid m \in \mathbb{Z}\} \cong \mathbb{Z}$$

with

$$\chi_m(y) = y^m.$$

Example 14. Let $T \cong \mathcal{G}_m^n$ be a torus. Then,

$$\mathfrak{X}(T) = \{\chi_m \mid m \in \mathbb{Z}^n\} \cong \mathbb{Z}^n$$

where $\chi_m(y) = y^m = y_1^{m_1} \cdots y_n^{m_n}$.

Note: For each algebraic group G , $\mathfrak{X}(G)$ is naturally an abelian group:

$$(\chi_1 \cdot \chi_2)(g) := \chi_1(g) \cdot \chi_2(g).$$

Given a morphism of algebraic groups $f : G \rightarrow H$, we get a morphism of abelian groups

$$\begin{aligned} f^* : \mathfrak{X}(H) &\longrightarrow \mathfrak{X}(G) \\ \chi &\longmapsto \chi \circ f =: f^*(\chi). \end{aligned}$$

This induces a contravariant functor from the category of algebraic groups to the category of abelian groups.

Lemma 33. *Let G be a diagonalizable algebraic group. Then, $\mathfrak{X}(G)$ is a k -vector space basis for $\mathcal{O}(G)$.*

Example 15. Let $G = \mathcal{G}_m^n$ be a torus. Then, we have the embedding

$$\begin{aligned} \mathfrak{X}(G) &\hookrightarrow \mathcal{O}(G) \\ \chi_{(m_1, \dots, m_n)} &\longmapsto t^{(m_1, \dots, m_n)}. \end{aligned}$$

The lemma is obvious in this case: each element of $\mathcal{O}(G) = k[t_1, \dots, t_n, t_1^{-1}, \dots, t_n^{-1}]$ can be written uniquely as a linear combination of monomials.

Proof. (i) $\mathfrak{X}(G)$ spans $\mathcal{O}(G)$:

Choose an embedding $G \subset \mathcal{G}_m^n$ of algebraic groups. Then, by restriction, we get

$$\mathcal{O}(\mathcal{G}_m^n) \twoheadrightarrow \mathcal{O}(G).$$

Since the $\chi_m, m \in \mathbb{Z}^n$, span $\mathcal{O}(\mathcal{G}_m^n)$, their images $\chi_m|_G \in \mathfrak{X}(G)$ span $\mathcal{O}(G)$.

(ii) $\mathfrak{X}(G)$ is linearly independent:

Suppose otherwise and let ϕ_1, \dots, ϕ_m be a linearly dependent subset of $\mathfrak{X}(G)$ with $m \geq 1$ chosen minimally, with $c_1, \dots, c_m \in k^\times$ s.t.

$$\sum_{i=1}^m c_i \phi_i = 0.$$

We distinguish the following cases:

$m = 1$: In this case, we have $\phi_1 = 0$, but $\phi_1(1) = 1$, a contradiction.

$m > 1$: We can assume $\phi_1 \neq \phi_2$, so there is an $h \in G$ s.t. $\phi_1(h) \neq \phi_2(h)$. Then,

$$\phi_1(h) \sum_{i=1}^m c_i \phi_i = 0,$$

but also for all $h, g \in G$

$$\sum_{i=1}^m c_i \phi_i(hg) = \sum_{i=1}^m c_i \phi_i(h) \phi_i(g) = 0.$$

This implies

$$\sum_{i=1}^m c_i \phi_i(h) \phi = 0.$$

Ergo

$$\sum_{i=1}^m c_j (\phi_i(h) - \phi_1(h)) \phi_i = \sum_{i=2}^m c_j (\phi_i(h) - \phi_1(h)) \phi_i = 0.$$

Now, $\phi_i(h) - \phi_1(h)$ is zero if $i = 1$ and non-zero, if $i = 2$. Therefore, this yields a shorter linear dependency for the elements

$$\phi_2, \dots, \phi_m,$$

which contradicts our requirement.

□

Definition 31. Let M be an abelian group. The **group algebra** on M is the k -algebra $k[M]$ (not a coordinate ring!) defined as follows:

$$\begin{aligned} k[M] &:= \text{the } k\text{-vectorspace with basis } M \\ &:= \left\{ \sum_{m \in M} c_m \cdot m \mid c_m \in k, \text{ almost all } c_m = 0 \right\}, \end{aligned}$$

where the multiplication on $k[M]$ extends that on M :

$$\left(\sum_{m \in M} c_m m\right) \left(\sum_{n \in M} d_n n\right) = \sum_{m, n \in M} c_m d_n mn.$$

Corollary 5. *For a diagonalizable G , we have*

$$\mathcal{O}(G) \cong k[\mathfrak{X}(G)].$$

Fact: For an abelian group M , there is exactly one Hopf algebra structure on $k[M]$ given by $\Delta(m) = m \otimes m$ for all $m \in M$.

With this definition, the above isomorphism is one of Hopf algebras.

Lemma 34. *If G, H are diagonalizable algebraic groups, then*

$$\mathrm{Hom}_{\mathrm{alg. grp.s}}(G, H) \xrightarrow{f \mapsto f^*} \mathrm{Hom}_{\mathrm{grp.s}}(\mathfrak{X}(H), \mathfrak{X}(G))$$

is a bijection.

Proof.

$$\begin{aligned} \mathrm{Hom}(G, H) &\cong \mathrm{Hom}_{\mathrm{Hopf-alg.}}(\mathcal{O}(H), \mathcal{O}(G)) \\ &\cong \{\phi \in \mathrm{Hom}_{k\text{-alg.}}(\mathcal{O}(H), \mathcal{O}(G)) \mid (\phi \otimes \phi) \circ \Delta = \Delta \circ \phi\}. \end{aligned}$$

Since $\mathrm{Hom}_{k\text{-alg.}}(\mathcal{O}(H), \mathcal{O}(G)) \cong \mathrm{Hom}(k[\mathfrak{X}(H)], k[\mathfrak{X}(G)])$, this reduces to the following lemma:

Lemma 35. *Let M_1, M_2 be two abelian groups. Then*

$$\begin{aligned} \mathrm{Hom}(M_1, M_2) &\xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Hopf-alg.}}(k[M_1], k[M_2]) \\ \phi &\longmapsto \left[\sum c_m m \mapsto \sum c_m \phi(m) \right]. \end{aligned}$$

Proof. We have to show that

$$M = \{x \in K[M]^\times \mid \Delta(x) = x \otimes x\}.$$

Then, by this, it follows for each $\phi \in \mathrm{Hom}_{\mathrm{Hopf-alg.}}(k[M_1], k[M_2])$,

$$\phi(M_1) \subseteq M_2.$$

Ergo, $\phi|_{M_1} \in \mathbf{Hom}(M_1, M_2)$. Therefore, the surjectivity of the claimed bijection is shown. The injectivity is clear, since M generates $k[M]$ as a k -algebra.

To show

$$M = \{x \in K[M]^\times \mid \Delta(x) = x \otimes x\},$$

let

$$\begin{aligned} x &= \sum c_m m \in K[M]^\times \\ \Delta(x) &= \sum c_m m \otimes m \\ x \otimes x &= \sum c_m c_n m \otimes n. \end{aligned}$$

If $\Delta(x) = x \otimes x$, then it follows

$$x = m$$

for some $m \in M$.

□

□

Recall: We have seen that for diagonalizable algebraic groups G, H

$$\mathrm{Hom}(G, H) \cong \mathrm{Hom}(\mathfrak{X}(H), \mathfrak{X}(G)).$$

If G is diagonalizable, then

$$\mathcal{O}(G) \cong k[\mathfrak{X}(G)].$$

Theorem 14. *The functor*

$$\begin{aligned} G &\longrightarrow \mathfrak{X}(G) \\ f &\longmapsto f^* \end{aligned}$$

defines an equivalence of categories:

$$\{\text{diagonalizable alg. groups}\} \cong \{\text{finitely gen. abelian groups with no } \mathrm{char}(k)\text{-torsion}\}.$$

This amounts to the bijection above between Hom-spaces and the following lemma.

Lemma 36. (i) *Let G be a diagonalizable alg. group. Then, $\mathfrak{X}(G)$ is a finitely generated abelian group with no $\mathrm{char}(k)$ -torsion.*

(ii) *Let Γ be a finitely generated abelian group with no $\mathrm{char}(k)$ -torsion. Then, there is a diagonalizable algebraic group G s.t. $\mathfrak{X}(G) \cong \Gamma$.*

Proof. We will use the following facts:

- Let $n \in \mathbb{N}$. Then, $t^n - 1$ is square-free in $k[t]$ iff the ideal $(t^n - 1)$ is radical in $k[t]$ iff $t^n - 1$ has not repetitive root iff either $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) = p > 0$ and $p \nmid n$.

(Proof: Galois Theory, separable/inseparable extensions.)

- Let $M := \mathbb{Z}/n\mathbb{Z}$. Then, the k -group-algebra generated by M

$$k[M] \cong k[t]/(t^n - 1)$$

is reduced iff either $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) = p > 0, p \nmid n$.

- If M_1, M_2 are abelian groups, then we have the following isomorphism of Hopf algebras

$$\begin{aligned} k[M_1] \otimes_k k[M_2] &\xrightarrow{\cong} k[M_1 \oplus M_2] \\ m_1 \otimes m_2 &\longmapsto m_1 m_2 \end{aligned}$$

where $M_1 \oplus M_2 \cong M_1 \times M_2$.

- (i) Embed $G \hookrightarrow T := \mathcal{G}_m^n$ for some n . Then, we have a surjection $\mathbb{Z}^n \cong \mathfrak{X}(T) \twoheadrightarrow \mathfrak{X}(G)$. Ergo, $\mathfrak{X}(G)$ is finitely generated.

Suppose $\text{char}(k) = p > 0$. Let $\chi \in \mathfrak{X}(G)$ with $\chi^p = 1$. Then, for all $g \in G$, $\chi^p(g) = \chi(g^p) = 1$. The unit group k^\times has no p -torsion, therefore $G \hookrightarrow T = (k^\times)^n$ has also no p -torsion. Therefore, the Frobenius $g \mapsto g^p$ is an isomorphism on G . Therefore, $\chi = 1$ is a trivial character. Ergo $\mathfrak{X}(G)$ has no p -torsion.

- (ii) Let Γ be a finitely generated abelian group with no $\text{char}(k)$ -torsion. Then,

$$\Gamma \cong \mathbb{Z}^r \oplus \mathbb{Z}/n_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/n_l\mathbb{Z}$$

where $\text{char}(k) \nmid n_1, \dots, n_l$. We may reduce to the cases:

- (a) $\Gamma = \mathbb{Z}$: take $G = \mathcal{G}_m$, then $\Xi(G) \cong \mathbb{Z} \cong \Gamma$.
(b) $\Gamma = \mathbb{Z}/n\mathbb{Z}$ with $\text{char}(k) =: p \nmid n$:
take $G := \mu_n := \{y \in k^\times \mid y^n = 1\}$. Then, since $p \nmid n$, $(t^n - 1)$ is radical. So,

$$\mathcal{O}(\mu_n) \stackrel{\text{Nullstellensatz}}{=} k[t]/(t^n - 1) \stackrel{\text{as Hopf algebras}}{\cong} k[\Gamma]$$

where t gets mapped to the generator of Γ .

□

Corollary 6. *We have the bijection*

$$\{\text{tori}\} \cong \{\text{finitely generated free abelian groups}(\cong \mathbb{Z}^n)\}.$$

Remark 5.

$$\{\text{algebraic group schemes}/k\} \stackrel{\text{not necessarily natural}}{\cong} \{\text{f.g. Hopf algebras}\}.$$

by

$$G \mapsto \mathcal{O}(G)$$

and

$$\{\text{diagonalizable algebraic group schemes}/k\} \cong \{\text{f.g. abelian groups}\}.$$

by

$$G \mapsto \mathfrak{X}(G).$$

9 Trigonalization

We say a representation $r : G \rightarrow \mathrm{GL}(V)$ of a group G on a finite-dimensional k -vectorspace V is **trigonalizable** if it admits a basis with respect to which $r(V)$ is upper-triangular:

$$r(G) \subseteq \left\{ \begin{pmatrix} * & \dots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & * \end{pmatrix} \right\}$$

Definition 32. We call a subgroup $G \subseteq \mathrm{GL}(V)$ **trigonalizable**, if the identity representation is.

Lemma 37. *Let G be an algebraic group. The following are equivalent:*

- (i) *Every finite-dimensional representation $r : G \rightarrow \mathrm{GL}(V)$ is trigonalizable.*
- (ii) *Every irreducible representation of G is 1-dimensional.*
- (iii) *G is isomorphic to an algebraic subgroup of*

$$B_n := \left\{ \begin{pmatrix} * & \dots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & * \end{pmatrix} \right\} \subseteq \mathrm{GL}_n(k).$$

- (iv) *There is a normal unipotent algebraic subgroup U of G s.t. G/U is diagonalizable.*

Proof. We prove as follows:

- (i) \implies (ii): Let V be an irreducible representation. Then, $V \neq 0$. Choose a basis e_1, \dots, e_n of V s.t.

$$r(G) \subseteq B_n.$$

Then, $r(G)e_1 \subseteq ke_1$, so $V_0 := ke_1$ is G -invariant. Ergo $V = V_0$ is 1-dimensional.

- (ii) \implies (i): Let V be a f.d. representation. We show by induction on $\dim(V)$ that $r : G \rightarrow \mathrm{GL}(V)$ is trigonalizable:

In the cases $\dim(V) = 0, 1$, there is nothing to show.

In the case $\dim(V) \geq 2$, assume that V is not irreducible. Then, there is a G -invariant V_0 with $0 \neq V_0 \neq V$.

By the induction hypothesis, V_0 and V/V_0 are trigonalizable. Ergo, V is trigonalizable.

(**Recall:** we used this criterion above in the proof that unipotent groups are trigonalizable by showing that every ??? of each G is trivial.)

(i) \implies (iii): Choose a faithful representation V of G . Then, $G \cong r(G)$. Since r is trigonalizable, there is a basis of V s.t.

$$r(G) \subseteq B_n \subseteq \mathrm{GL}_n(k).$$

(iii) \implies (ii): Suppose $G \subseteq B_n \subseteq \mathrm{GL}_n(k)$. Set

$$A_n := \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & * \end{pmatrix} \right\} \subseteq \mathrm{GL}_n(k),$$

$$U_n := \left\{ \begin{pmatrix} 1 & \dots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & 1 \end{pmatrix} \right\} \subseteq \mathrm{GL}_n(k) \text{ normal algebraic subgroup of } B_n,$$

$$U := G \cap U_n \text{ normal unipotent algebraic subgroup of } G.$$

Let V be an irreducible representation of G , then V is not zero. Consider the subspace of V fixed by U

$$V^U := \{v \in V \mid r(u)v = v \forall u \in U\}.$$

Then, we get a representation

$$r|_U : U \longrightarrow \mathrm{GL}(V).$$

Then, $r(U)$ is a unipotent algebraic group of $\mathrm{GL}(V)$. Ergo,

$$r(U) \subseteq \left\{ \begin{pmatrix} 1 & \dots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Ergo, $V^U \neq 0$. Since U is normal in G , the subspace V^U of V is G -invariant: if $v \in V^U, g \in G$, then for all $u \in U$ we have

$$r(u)r(g)v = r(g)r(g^{-1}ug)v = r(g)v$$

since $v \in V^U$. Ergo $r(g)v \in V^U$.

Since V is irreducible, $V = V^U$, i.e. U acts trivially on V . Ergo, r descends to a representation of the group G/U .

But $G/U \hookrightarrow B_n/U_n \cong A_n$. Therefore, G/U and $r(G)$ are commutative. Moreover, for all $g \in G$, $r(g) \in \text{GL}(V)$ is semisimple:

if $g = g_s g_u$, then $g_u \in U$, because U_n is the group of unipotent elements of B_n .

Hence, $r(g) = r(g_s)r(g_u) = r(g_s)$ is semisimple.

It follows that $r(G)$ is commutative and consists of semisimple elements. By some HW: $r(G)$ is diagonalizable. It is easy to show now that V is one-dimensional. (Since V is irreducible and ke_1 is G -invariant.)

□

Definition 33. G is **trigonalizable**, if it satisfies one of the above equivalent conditions.

Later, we will see, that if G is connected, then being trigonalizable implies being solvable.

10 Commutative Groups

Let G be an algebraic group. Denote by G_s resp. G_u the subsets of semisimple resp. unipotent elements of G .

Then, G_u is always algebraic i.e. closed: if $G \hookrightarrow \mathrm{GL}_n(k)$, then $G_u = \{g \mid (g - 1)^n = 0\}$. G_u does not need to be closed under multiplication (for example, take $G = \mathrm{SL}_2(k)$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$).

G_s needs not to be algebraic: for example, take $G = \mathrm{SL}_2(k)$ and if G_s were algebraic, then

$$\left\{ \lambda \in k^\times \mid \begin{pmatrix} \lambda & 1 \\ 0 & \lambda^{-1} \end{pmatrix} \in G_s \right\} = \{ \lambda \mid \lambda \neq \lambda^{-1} \}$$

but the last set is not algebraic. Also, G_s does not need to be a group.

We have the surjective map of sets

$$\begin{aligned} G_s \times G_u &\longrightarrow G \\ (g_1, g_2) &\longmapsto g_1 g_2. \end{aligned}$$

Example 16 (Non-Example). Take generic $g \in G_s, h \in G_u$ for $G = \mathrm{SL}_2(k)$. Then, g, h do not commute and we have

$$((gh)_s, (gh)_u) \neq (g, h)$$

because Jordan components don't commute.

Theorem 15. *Let G be a commutative algebraic group. Then:*

- (i) G_s, G_u are closed subgroups and the multiplicative map $G_s \times G_u \rightarrow G$ is an isomorphism of algebraic groups.
- (ii) G is trigonalizable. Moreover, for each finite dimensional representation $r : G \rightarrow \mathrm{GL}(V)$ there is a basis s.t.

$$r(G_s) \subseteq \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & * \end{pmatrix} \right\} \quad r(G_u) \subseteq \left\{ \begin{pmatrix} 1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

- (iii) G_s is diagonalizable.

Proof. (ii) Let V be any irreducible representation of G . We have seen that commuting semisimple operators may be simultaneously diagonalizable, then

$$V = \bigoplus_{\chi: G_s \rightarrow \mathcal{G}_m} V_\chi$$

where

$$V_\chi = \{v \in V \mid r(h)v = \chi(h)v \ \forall h \in G_s\}.$$

Since G is commutative, each subspace V_χ is G -invariant ($r(h)r(g)v = r(g)r(h)v = r(g)\chi(h)v = \chi(h)r(g)v$).

Since V is irreducible, we must have $V = V_\chi$ for some χ .

Recall that $G \cong G_s \times G_u$ as abstract groups. We have seen that $r(G_s) \subseteq \mathcal{G}_m^n$. We proved a while ago that any unipotent group, such as G_u , is trigonalizable. Ergo, V is trigonalizable. Since V is irreducible, we have $\dim V = 1$.

If we apply the same argument without assuming that V is irreducible, then we see that V is the coproduct of V_χ 's as above and that each V_χ admits a basis s.t.

$$r(G_s)|_{V_\chi} \subseteq \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & * \end{pmatrix} \right\} \quad r(G_u)|_{V_\chi} \subseteq \left\{ \begin{pmatrix} 1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

This yields the same conclusion for V .

- (i) We have to show that G_s and G_u are closed and $j : G_s \times G_u \rightarrow G$ is an isomorphism of groups. Take any faithful representation

$$G \xrightarrow{\cong, r} r(G) \subseteq \mathbf{GL}(V)$$

and apply (ii). Then we have

$$\begin{aligned} r(G) &\subseteq \left\{ \begin{pmatrix} * & * & * \\ 0 & \ddots & * \\ 0 & 0 & * \end{pmatrix} \right\} =: B \\ B_u &= \left\{ \begin{pmatrix} 1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & 1 \end{pmatrix} \right\}, \\ r(G_s) &\subseteq \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & * \end{pmatrix} \right\} =: A. \end{aligned}$$

In fact, $r(G_s) = r(G) \cap A$, because if $g \in G$ with $r(g) \in A$, then $r(g)$ is semisimple, so $g \in G_s$.

Therefore, G_s is closed in G . Ergo, G_s and G_u are closed subgroups.

Then, the map j is a morphism of algebraic groups.

We need to show that j^{-1} is a morphism of algebraic groups. For this, it suffices to verify that the projection $G \rightarrow G_s$ is a morphism. But this map is given under r by the morphism:

$$t := \begin{pmatrix} a_1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & a_n \end{pmatrix} \mapsto \begin{pmatrix} a_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_n \end{pmatrix} =: t_s.$$

This suffices because if $g = g_s g_u$, then $g_u = g_s^{-1} g$, so if the map $g \mapsto g_s$ is a morphism, so is $g \mapsto g g_s^{-1} = g_u$, hence so is $g \mapsto (g_s, g_u)$.

- (iii) We have seen that G_s is a closed subgroup. Hence G_s is a commutative algebraic group where elements are semisimple. Ergo, G_s is diagonalizable.

□

11 Connected Solvable Groups

Theorem 16 (Lie-Kolchin). *Let G be a connected solvable algebraic group. Then G is trigonalizable.*

(By comparison, recall that we have seen so far that, if G is commutative or unipotent, then G is trigonalizable.) We can reformulate this theorem as: Any connected solvable subgroup of $\mathrm{GL}(V)$ stabilizes some complete flag $\mathcal{F} = (V_0 \subsetneq \dots \subsetneq V_n)$.

Generalization (Borel's Fixed Point Theorem): Any connected algebraic group G acting on a projective variety X has a fixed point in X .

We get a relation between complete flags and projective varieties.

Proof. Induct on the number n s.t. $G^{(n)} = 1$.

For $n = 0$, there is nothing to show.

If $n = 1$, $(G, G) = 1$, then G is commutative, ergo trigonalizable.

Let $n \geq 2$. Then, we have $G' := (G, G) \neq 1$. We will show the following lemma: □

Lemma 38. *Let $G \subseteq \mathrm{GL}(V)$ be a subgroup.*

If G is connected, then the group G' with the induced topology is connected (\iff the Zariski Closure of G' is connected).

Proof. We have the following facts:

- An increasing union of connected spaces is connected.
- A continuous image of a connected space is connected.

We have

$$\begin{aligned} G' &= \langle (g, h) := ghg^{-1}h^{-1} \mid g, h \in G \rangle \\ &= \bigcup_{j \geq 0} \bigcup_{g_1, h_1, \dots, g_j, h_j \in G} \{(g_1, h_1) \cdots (g_j, h_j)\}. \end{aligned}$$

Since

$$\bigcup_{g_1, h_1, \dots, g_j, h_j \in G} \{(g_1, h_1) \cdots (g_j, h_j)\} = \mathrm{img} \phi_j$$

for some continuous map $\phi_j : G^{2j} \rightarrow G$, the claim follows. Ergo, G' is connected. □

Remark 6. It is equivalent to show that (*) any subgroup G of $\mathrm{GL}(V)$ – s.t. G is connected and solvable – is trigonalizable in $\mathrm{GL}(V)$.

Indeed, the theorem implies (*): the Zariski closure of G is a connected algebraic group that is solvable (which extends by continuity). If $Zcl(G)$ is trigonalizable, then also G is trigonalizable.

On the other hand: (*) implies the theorem, since if G is given as in the theorem, apply (*) to $r(G) \subseteq \mathrm{GL}(V)$.

Proof of Theorem. If $G^{(n)} = 1$, then $(G')^{(n-1)} = G^{(n)} = 1$. By induction, we may assume that G' satisfies the following:

For all finite dimensional representations $r : G \rightarrow \mathrm{GL}(V)$, $r(G')$ is trigonalizable.

Our aim is to show that any irreducible representation V of G has dimension 1.

The induction hypothesis implies that $r(G')$ is trigonalizable. In particular, there exists an eigenspace $V_\chi \subseteq V$ for G' for some character $\chi : G' \rightarrow k^\times$. Since G' is normal in G we know that G acts from the left on

$$\{\text{eigenspaces } V_\chi \text{ in } V \text{ for } G'\}.$$

Ergo, $\bigoplus_{\chi: G' \rightarrow k^\times} V_\chi$ is G -invariant. Since V is G -irreducible, we have

$$V = \bigoplus_{\chi: G' \rightarrow k^\times} V_\chi = \bigoplus_{\chi \in \mathfrak{X}'} V_\chi$$

for some finite subset $\mathfrak{X}' = \{\chi \mid V_\chi \neq 0\}$ of $\mathrm{Hom}(G', \mathcal{G}_m)$, since V is finite dimensional.

Claim: Let $h \in G'$. Then, the map

$$\begin{aligned} G &\longrightarrow \mathrm{GL}(V) \\ g &\longmapsto r(ghg^{-1}) \end{aligned}$$

has a finite image.

Proof. Denote by $\chi \mapsto \chi^g$ the action of $g \in G$ in $\mathrm{Hom}(G', \mathcal{G}_m)$ given by $\chi^g(h) := \chi(ghg^{-1})$. This is an action, since G' is normal.

Note, that $\mathfrak{X}' \subseteq \mathrm{Hom}(G', \mathcal{G}_m)$ is a finite subset.

Also note, that the action $\chi \mapsto \chi^g$ descends to an action $G \curvearrowright \mathfrak{X}'$.

Now, let $\mathfrak{X}' = \{\chi_1, \dots, \chi_r\}$. The matrix $r(h)$ is totally determined by the values $\chi_1(h), \dots, \chi_r(h)$. Then, the element $r(ghg^{-1})$ is totally determined by the values $\chi_1^g(h), \dots, \chi_r^g(h)$. It follows

$$\#\{r(ghg^{-1}) \mid g \in G\} \leq r!.$$

□

The following lemma is easy to show:

Lemma 39. *Let G be an algebraic set. Then, G is connected iff for each finite algebraic set X , and for each morphism $f : G \rightarrow X$ of algebraic sets, we have that f is constant.*

Claim with the Lemma implies that the map $g \mapsto t(ghg^{-1})$ is constant. This implies that $r(ghg^{-1}) = r(h)$ for all $g \in G, h \in G'$. Ergo, G stabilizes each eigenspace V_χ for G' . Ergo, $V = V_{\chi_0}$, since V is irreducible. □

Lemma 40. *Let G be any group with a finite dimensional representation $r : G \rightarrow GL(V)$. Then, the subspaces V_χ for $\chi \in \text{Hom}(G, k^\times)$ are linearly independent, i.e., the map*

$$\oplus V_\chi \longrightarrow V$$

is injective.

Proof. The spaces V_χ are G -invariant. Suppose, there exist distinct χ_1, \dots, χ_n of non-zero $v_j \in V_{\chi_j}$ s.t. $\sum_j v_j = 0$.

We may assume that n , the number of v_j , is minimal. W.l.o.g., $n \geq 2$.

Choose $g \in G$ s.t. $\chi_1(g) \neq \chi_2(g)$. Use that $0 = g \sum_j v_j = \sum_j g v_j$ and take the linear combination as in the proof of linear independence of characters to contradict the minimality of n .

($g - \chi_1(g)$ is not zero, but reduces $\sum_j v_j$ by one summand.) □

Finishing Proof of Theorem. Since $G' = \langle ghg^{-1}h^{-1} \mid g, h \in G \rangle$, so $\det(r(G')) = 1$.

On the other hand, for each $g \in G'$, we have

$$r(g) = \begin{pmatrix} \chi_0(g) & & \\ & \ddots & \\ & & \chi_0(g) \end{pmatrix}$$

since $V = V_{\chi_0}$. This implies

$$1 = \det(r(g)) = \chi_0(g)^d.$$

Ergo, χ_0 defines a morphism

$$\chi_0 : G' \longrightarrow \mu_d \subseteq \mathcal{G}_m.$$

But G' is connected and μ_d is finite. Since χ_0 is a morphism, χ_0 must be constant, ergo the trivial character.

As a consequence, we get $r(G') = 1$ on $V = V_{\chi_0}$.

Lemma 41. *Let G be an algebraic group, $r : G \rightarrow \mathrm{GL}(V)$ a representation. $v \in V$ shall be a simultaneous non-zero eigenvector for $r(G)$.*

Then, for each $g \in G$, there is a value $\chi(g) \in k^\times$ s.t.

$$r(g)v =: \chi(g)v.$$

Then, the mapping $\chi : G \rightarrow \mathcal{G}_m$ is a morphism of algebraic groups.

Therefore, r descends to a representation of the commutative group

$$\bar{r} : G/G' \longrightarrow \mathrm{GL}(V).$$

Ergo, $r(G/G') = r(G)$ is commutative and therefore trigonalizable (because of irreducibility).

□

Example 17 (Non-Example). • Take $G = D_4 \hookrightarrow \mathrm{GL}_2(\mathbb{C})$ which is solvable and has an irreducible and faithful representation over \mathbb{C}^2 .

- Consider the solvable group

$$G = \left\langle \begin{pmatrix} \pm 1 & \\ & \pm 1 \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \right\rangle$$

which is a finite subgroup of $\mathrm{GL}_2(\mathbb{C})$, s.t. \mathbb{C}^2 defines an irreducible representation of G .

Lemma 42 (Form of Schur's Lemma). *Let S be any commutative subset of $\mathrm{GL}(V)$ for a finite-dimensional $0 \neq V$ over an algebraically closed field k . Let V be S -irreducible. Then, $\dim V = 1$.*

Proof. There is nothing to show if S is empty.

Let $s \in S$ and denote by $V_\lambda \subseteq V$ the λ -eigenspace for s . Then, since S is commutative, V_λ is S -invariant. Therefore, $V = V_\lambda$ for one $\lambda \in k^\times$.

Thus, every $s \in S$ acts by scaling, therefore every subspace of V is S -invariant. Since V is invariant, we get $\dim V = 1$. \square

Corollary 7. *Let G be a connected algebraic group. Then, G is solvable iff G is trigonalizable.*

Proposition 3. *If G is trigonalizable, then G_u is a normal algebraic subgroup.*

Proof. We have

$$G \hookrightarrow B := \left\{ \begin{pmatrix} * & \dots & * \\ & \ddots & \vdots \\ & & * \end{pmatrix} \right\} \subseteq \mathrm{GL}_n(k).$$

B has the normal subgroup $U := \left\{ \begin{pmatrix} 1 & \dots & * \\ & \ddots & \vdots \\ & & 1 \end{pmatrix} \right\}$ and we have $G_u = G \cap U$. Now,

U is the kernel of the multiplicative morphism

$$\begin{pmatrix} a_1 & \dots & * \\ & \ddots & \vdots \\ & & a_n \end{pmatrix} \mapsto \begin{pmatrix} a_1 & & \\ & \dots & \\ & & a_n \end{pmatrix}.$$

\square

Corollary 8. *If G is connected and solvable, then G_u is a normal algebraic subgroup.*

12 Semisimple Elements of nilpotent Groups

Theorem 17. *Let G be a connected nilpotent algebraic group. Then, we have*

$$G_s \subseteq Z(G)$$

where $Z(G)$ denotes the **center** of G , i.e.

$$Z(G) = \{g \in G \mid \forall h \in G : gh = hg\}.$$

Theorem 18 (Lie-algebraic Analogue). *Let V be a finite-dimensional vectorspace. Let \mathfrak{g} be the Lie-Subalgebra of $\text{End}(V)$, i.e. \mathfrak{g} is a subspace s.t. we have for each $x, y \in \mathfrak{g}$*

$$[x, y] := xy - yx \in \mathfrak{g}.$$

Assume that \mathfrak{g} is nilpotent, i.e. there is an $n \in \mathbb{N}_0$ s.t.

$$[x_1, [x_2, [\dots, [x_{n-1}, x_n]]]] = 0$$

for all $x_1, \dots, x_n \in \mathfrak{g}$.

*Then, any semisimple (semisimple in $\text{End}(V)$ that is) $x \in \mathfrak{g}$ is **central** in \mathfrak{g} , i.e. $[x, y] = 0$ for each $y \in \mathfrak{g}$.*

Remark 7. The Lie-algebraic Analogue implies the general theorem if – for example – $k = \mathbb{C}$.

Proof. Let $g \in G_s$. We want to show $Z_G(g) = G$.

Fact from the theory of Lie-Algebras: For the Lie-Algebra $\text{Lie}Z_G(g)$ we have

$$\text{Lie}Z_G(g) = \ker(\text{Ad}(g))$$

where Ad is the map

$$\begin{aligned} \text{Ad} : G &\longrightarrow \text{GL}(\mathfrak{g}) \\ x &\longmapsto gxg^{-1}. \end{aligned}$$

Since G is connected, it suffices to verify

$$\ker(\text{Ad}(g)) = \mathfrak{g}$$

i.e. $\text{Ad}(g) = 1$.

Since g is semisimple, we have for suitable basis

$$g = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$$

with $a_j \in \mathbb{C}^\times$. This is $\exp(x)$ for a suitable diagonal matrix $x = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} \in \mathrm{GL}_n(\mathbb{C})$.

Fact: We may assume that $x \in \mathfrak{g} := \mathrm{Lie}(G)$.

Since G is nilpotent, it can be shown that \mathfrak{g} is nilpotent.

By the theorem, x is central in \mathfrak{g} . By the properties of \exp we have

$$\mathrm{Ad}(g) = \exp(\mathrm{ad}(g)) = 1$$

ergo $\mathrm{ad}(x) = 0$ where $\mathrm{ad} : \mathfrak{g} \rightarrow \mathfrak{g}$ is defined by

$$\mathrm{ad}(x) \cdot y := [x, y].$$

□

Proof. If \mathfrak{g} is nilpotent, then $\mathrm{ad}(x) \in \mathrm{End}(\mathfrak{g})$ is nilpotent.

Since x is semisimple, $\mathrm{ad}(x)$ is semisimple, because $\mathrm{ad}(x)$ is the restriction to \mathfrak{g} of the map

$$\begin{aligned} \mathrm{End}(V) &\longrightarrow \mathrm{End}(V) \\ y &\longmapsto [x, y] \end{aligned}$$

and, if e_1, \dots, e_n are a basis of eigenvectors for x , then $E_{i,j}$ is a basis of eigenvectors for ℓ .

So, $\mathrm{ad}(x)$ is nilpotent and semisimple, therefore $\mathrm{ad}(x) = 0$. □

Proof Theorem. Let G be a connected nilpotent algebraic group, $G \hookrightarrow \mathrm{GL}(V)$.

Let $g \in G_s$, we want to show that $g \in Z(G)$.

Assume otherwise, then we have a $h \in G$ s.t. $(g, h) = ghg^{-1}h^{-1} \neq 1$.

Since G is connected and nilpotent (ergo solvable), we know by Lie-Kolchin that G stabilizes some complete flag $V_0 \subset \dots \subset V_n$.

We have $g|_{V_i}, h|_{V_i} \in \mathrm{GL}(V_i)$. They commute, if $i = 0$, but not if $i = n$.

So, there is an i s.t. $g|_{V_i}, h|_{V_i}$ commute but $g|_{V_{i+1}}, h|_{V_{i+1}}$ don't commute. W.l.o.g. $V = V_{i+1}, g = g|_{V_{i+1}}, h = h|_{V_{i+1}}$. Set $a := g|_{V_i}, b := h|_{V_i} \in \text{GL}(V_i)$. a will be semisimple, since g is.

Since g is semisimple, there is an eigenvector $v \in V_{i+1}$ for g s.t.

$$V_{i+1} = V_i \oplus \langle v \rangle.$$

We have an isomorphism of vector spaces

$$\text{End}(V_{i+1}) \cong \text{End}(V_i) \oplus \text{Hom}(\langle v \rangle, V_i) \oplus \text{Hom}(V_i, \langle v \rangle) \oplus \text{End}(\langle v \rangle)$$

with

$$\text{End}(\langle v \rangle) \cong k \text{ and } \text{Hom}(\langle v \rangle, V_i) \cong V_i.$$

So, we can write $g|_{V_{i+1}}, h|_{V_{i+1}}$ as

$$g = \begin{pmatrix} a & \\ & * \in k \end{pmatrix} \text{ and } h = \begin{pmatrix} b & c \in V_i \\ & * \end{pmatrix}.$$

We may replace g, h with scalar multiples to reduce to the case that $* = 1$. Then, we can write $g|_{V_{i+1}}, h|_{V_{i+1}}$ as

$$g = \begin{pmatrix} a & \\ & 1 \end{pmatrix} \text{ and } h = \begin{pmatrix} b & c \\ & 1 \end{pmatrix}.$$

Then,

$$h \neq ghg^{-1} = \begin{pmatrix} b & ac \\ & 1 \end{pmatrix}.$$

Ergo, $c \neq ac$, i.e. $c \notin \ker(a - 1)$. Define

$$h_1 := h^{-1}ghg^{-1} = \begin{pmatrix} 1 & b^{-1}(a-1)c \\ & 1 \end{pmatrix}.$$

We claim that h_1 does not commute with g . This claim implies the theorem, since we can iterate the claim to obtain elements h_i by $h_{i+1} := h_i^{-1}gh_i g^{-1}$. Then, h_i does not commute with g . But G is nilpotent, therefore $h_i = 1$ for some large enough i .

We can prove the claim as follows: By some calculation as for h and g , we see, that h_1 and g don't commute iff $b^{-1}(a-1)c \notin \ker(a-1)$. This is equivalent to

$$\begin{aligned} &\iff (a-1)b^{-1}(a-1)c \neq 0 \\ &\iff b^{-1}(a-1)^2c \neq 0 \\ &\iff (a-1)^2c \neq 0 \\ &\iff c \notin \ker((a-1)^2). \end{aligned}$$

But a being semisimple implies $a - 1$ being semisimple, therefore

$$\ker((a - 1)^2) = \ker(a - 1).$$

So h_1, g don't commute iff $c \in \ker(a - 1)$ iff h, g don't commute. □

13 Algebraic Geometry

13.1 Projective Algebraic Sets

Let V be a finite-dimensional vector space. Then $\mathcal{G}_m = k^\times$ acts on V by scalar multiplication. $\{0\}$ is a \mathcal{G}_m -invariant subspace of V . We are interested on the orbits of \mathcal{G}_m on $V \setminus \{0\}$.

Define the **projective space** over V by

$$\mathbb{P}V := \mathcal{G}_m \backslash (V - 0) = (V - 0) / \sim \cong \{\text{lines in } V\}$$

where for $a, b \in V - 0$ we set

$$a \sim b : \iff \exists \lambda \in k^\times : \lambda a = b.$$

If $V = k^{n+1}$, we denote the n -dimensional projective space by $\mathbb{P}^n := \mathbb{P}V$.

Given $a = (a_0, a_1, \dots, a_n) \in k^{n+1} - 0$, we denote the \sim -class of a by

$$[a] = [a_0, \dots, a_n] \in \mathbb{P}^n.$$

Define S to be the graded algebra of polynomials in k

$$S := k[x_0, \dots, x_n] = \bigoplus_{d \geq 0} S_d$$

where each S_d is the space of homogenous polynomials of degree d , i.e.

$$S_d = \bigoplus_{i_1, \dots, i_d \in \{0, \dots, n\}} k \cdot x_{i_1} \cdots x_{i_d}.$$

We identify k with the space of constant polynomials $S_0 \subseteq S$.

We have

$$S_d = \{f \in S \mid f(\lambda X) = \lambda^d f(X) \ \forall \lambda \in k^\times\}.$$

Given $f \in S_d$, the set

$$\{a \in k^{n+1} \mid f(a) = 0\}$$

is \mathcal{G}_m -invariant. In other words, given $a \in \mathbb{P}^n$ and $f \in S_d$, it is well-defined to state $f(a) = 0$ and $f(a) \neq 0$.

Definition 34. A **projective algebraic subset** $X \subseteq \mathbb{P}^n$ is a set of the form

$$X = V(\Sigma) := V_{\mathbb{P}^n}(\Sigma)$$

where Σ is a collection of homogenous elements of S , where

$$V_{\mathbb{P}^n}(\Sigma) := \{a \in \mathbb{P}^n \mid f(a) = 0 \ \forall f \in \Sigma\}.$$

Facts:

- Hilbert's basis theorem states

$$V(\Sigma) = V(f_1, \dots, f_m)$$

for some finite collection $f_1, \dots, f_m \in \Sigma$.

- It is useful to extend the meaning of " $f(a) = 0$ " for $a \in \mathbb{P}^n$ to *general* elements $f \in S$ by requiring that $f(a') = 0$ for each $a' \in [a]$.

If we write $f = \sum_{d \geq 0} f_d$, $f_d \in S_d$, then we have

$$f(a) = 0 \iff f_d(a) = 0 \ \forall d \geq 0.$$

Therefore, we can extend the definition of $V(\Sigma)$ to any $\Sigma \subseteq S$.

- We have $V(\Sigma) = V((\Sigma))$ where (Σ) is the ideal generated by some finite subset of Σ .
- We call an ideal $I \subseteq S$ **homogenous** if it is the direct sum of its d -homogeneous components, i.e.

$$I = \sum_{d \geq 0} I_d$$

where $I_d = \{f \in I \mid f \text{ is homogenous of degree } d\}$.

I is homogeneous iff it is generated by homogeneous elements.

- We have the following *Nullstellensatz*:

For any $X \subseteq \mathbb{P}^n$, set $I(X)$ to be the ideal generated by all homogeneous polynomials of S vanishing on X .

Let $I \subseteq S$ be a *homogeneous* ideal which is *not equal* to (x_0, \dots, x_n) . Then, we have

$$I(V_{\mathbb{P}^n}(I)) = \sqrt{I}.$$

Example 18 (Anti-example). The second property is necessary:

Set $I = (x_0, \dots, x_n)$. Then $V_{k^{n+1}}(I) = 0$. Therefore, $V_{\mathbb{P}^n}(I) = \emptyset$. However,

$$I(V_{\mathbb{P}^n}(I)) = S.$$

- The above point induces a bijection between algebraic subsets of \mathbb{P}^n and radical ideals $I \subset S$ which are not (x_0, \dots, x_n) .

For $i = 0, \dots, n$, set $D(x_i) := \{a \in \mathbb{P}^n \mid a_i \neq 0\}$. $D(x_i)$ is an open set homeomorphic to k^n by mapping

$$\phi_i : a \mapsto \left(\frac{a_0}{a_i}, \dots, \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i}, \dots, \frac{a_n}{a_i} \right).$$

The $D(x_i)$ cover $\mathbb{P}^n = \bigcup_i D(x_i)$.

Given a projective algebraic subset $X \subset \mathbb{P}^n$, define $X^{(i)} \subset k^n$ by

$$X^{(i)} := \phi_i(X \cap D(x_i)).$$

If $X = V_{\mathbb{P}^n}(I)$, then

$$X^{(i)} = V_{k^n}(I^{(i)})$$

where

$$I^{(i)} := \{f^{(i)} \mid f \in I\}$$

where $f^{(i)}(t_1, \dots, t_n) := f(t_1, \dots, t_{i-1}, 1, t_i, \dots, t_n)$. Thus, $X^{(i)}$ is an algebraic subset of k^n .

Definition 35. The **Zariski topology** on \mathbb{P}^n is defined by setting the set of closed sets to be the set of projective algebraic sets.

Facts:

- $D(x_i)$ is open in \mathbb{P}^n , since $D(x_i) = \mathbb{P}^n - V(x_i)$.
- The bijections $D(x_i) \cong k^n$ are homeomorphisms.

Definition 36. A **quasi-projective** algebraic set Y is an open subset of a projective algebraic set $X \subseteq \mathbb{P}^n$.

Example 19. Any algebraic set in k^n is quasi-projective.

Definition 37. A **quasi-projective variety** is defined as an irreducible quasi-projective algebraic set.

Lemma 43 (Products). *Define the **Segre-embedding** by*

$$\begin{aligned} S^{n,m} : \mathbb{P}^n \times \mathbb{P}^m &\hookrightarrow \mathbb{P}^{nm+n+m} \\ (a, b) &\mapsto [(a_i b_j)_{i,j=0,\dots,n}]. \end{aligned}$$

We have:

1. $S^{n,m}$ is injective.
2. $S^{n,m}$ has a closed image.
3. $k^n \times k^m \cong D(z_{00}) \cap S^{n,m}(\mathbb{P}^n \times \mathbb{P}^m) = S^{n,m}(D(x_0) \times D(y_0))$.

Definition 38. For quasi-projective algebraic sets $X \subset \mathbb{P}^n, Y \subset \mathbb{P}^m$, we define their product by

$$X \times Y := S^{n,m}(X, Y) \subseteq \mathbb{P}^{nm+n+m}.$$

Then, $X \times Y$ is a quasi-projective algebraic subset of \mathbb{P}^{nm+n+m} .

13.2 Flag Varieties

Definition 39. We define the **Grassmannian manifold** by

$$G(n, d) := \{W \subset k^n \mid W \text{ is a } d\text{-dimensional subvectorspace}\}.$$

Then, we have the **Plücker-embedding** by

$$\begin{aligned} P_d : G(n, d) &\longrightarrow \mathbb{P} \left(\bigwedge^d k^n \right) = \mathbb{P}^{\binom{n}{d}-1} \\ W &\longmapsto [w_1 \wedge \dots \wedge w_d] \end{aligned}$$

where w_1, \dots, w_d is a basis of W .

Lemma 44. P_d is injective and has a closed image.

Therefore, we can see $G(n, d)$ as a projective algebraic set.

Definition 40. Let V be a finite-dimensional vector space of dimension n . Set

$$\mathrm{Gr}_d(V) := \{d\text{-dim. subspaces of } V\} \cong G(n, d).$$

Define further the **flag manifold** to be

$$\mathrm{Flag}(V) := \{\text{complete flags } \mathcal{F} = (0 = V_0 \subseteq V_1 \subseteq \dots \subseteq V_n = V)\}.$$

Then, we have a map

$$\begin{aligned} P_v : \mathrm{Flag}(V) &\longrightarrow \mathrm{Gr}_0(V) \times \dots \times \mathrm{Gr}_n(V) \\ \mathcal{F} &\longmapsto (V_0, \dots, V_n). \end{aligned}$$

Lemma 45. P_v has a closed image and is injective.

Thus, we can see $\mathrm{Flag}(V)$ as a projective algebraic set.

Lemma 46. $\mathrm{Gr}_d(V)$ and $\mathrm{Flag}(V)$ are both irreducible, hence projective alg. varieties.

$\mathrm{Flag}(V)$ is called the **variety of complete flags**.

13.3 Local Rings and Function Fields

Definition 41. An **affine variety** is an irreducible algebraic subset of k^n .

Definition 42. If X is an affine variety, then the coordinate ring $\mathcal{O}(X)$ is a domain. Define the **function field** of X by

$$k(X) := \text{Frac}(\mathcal{O}(X)) := \left\{ \frac{a}{b} \mid a, b \in \mathcal{O}(X), b \neq 0 \right\}.$$

Definition 43. Let $p \in X$. We define the **local ring** of $\mathcal{O}(X)$ at p by

$$\mathcal{O}_{X,p} := \left\{ \frac{a}{b} \mid a \in \mathcal{O}(X), 0 \neq b \in \mathcal{O}(X), b(p) \neq 0 \right\} \subset k(X).$$

We have an **evaluation** map

$$\begin{aligned} \text{eval}_p : \mathcal{O}_{X,p} &\longrightarrow k \\ \frac{a}{b} &\longmapsto \frac{a(p)}{b(p)}. \end{aligned}$$

Lemma 47. Let X be an affine variety. Then

$$\mathcal{O}(X) = \bigcap_{p \in X} \mathcal{O}_{X,p}.$$

Definition 44. Let $X \subset \mathbb{P}^n$ be a projective variety. Denote by $I_{\mathbb{P}}(X)$ its homogenous vanishing ideal.

Define its **function field** by

$$k(X) := R/M,$$

where

$$\begin{aligned} R &:= \left\{ \frac{f}{g} \mid f, g \in k[x_0, \dots, x_n] \text{ homogen.}, \deg f = \deg g, g \notin I_{\mathbb{P}}(X) \right\}, \\ M &:= \left\{ \frac{f}{g} \in R \mid f \in I_{\mathbb{P}}(X) \right\}. \end{aligned}$$

Lemma 48. M is a maximal ideal in R and R/M is a field.

Lemma 49. If X is a projective variety, then $X^{(i)} \subset k^n$ is an affine variety.

If $X^{(i)} \neq \emptyset$, then

$$k(X) \cong k(X^{(i)}).$$

Definition 45. Let X be a projective variety. For $p \in X$, we define its **local ring** at p by

$$\mathcal{O}_{X,p} := \left\{ \frac{f}{g} \in k(X) \mid g(p) \neq 0 \right\} \subset k(X).$$

Lemma 50. For a projective variety X , we have:

1. For $p \in X^{(i)}$: $\mathcal{O}_{X,p} \cong \mathcal{O}_{X^{(i)},p}$.
2. For $p \in X^{(i)} \cap X^{(j)}$: $\mathcal{O}_{X^{(j)},p} \cong \mathcal{O}_{X^{(i)},p}$.

Definition 46. If $X \subset \mathbb{P}^n$ is quasi-projective variety, there is a minimal projective variety $\overline{X} \subset \mathbb{P}^n$ which contains X as an open subset.

Then, we can set

$$\begin{aligned} k(X) &:= k(\overline{X}) \\ \mathcal{O}_{X,p} &:= \mathcal{O}_{\overline{X},p}. \end{aligned}$$

13.4 Regular Functions and Morphisms

Definition 47. Let X be quasi-projective variety. Let $U \subseteq X$ be open. Then, we define the **ring of regular functions** on U by

$$\mathcal{O}(U) := \bigcap_{P \in U} \mathcal{O}_{X,P} \subseteq k(X).$$

Definition 48. Let X, Y be two quasi-projective varieties. A map $f : X \rightarrow Y$ is called a **morphism**, if f is continuous and we have

$$f^* \mathcal{O}(U) := \{h \circ f \mid h \in \mathcal{O}(U)\} \subseteq \mathcal{O}(f^{-1}(U)).$$

Remark 8. Let X, Y be affine varieties and $f : X \rightarrow Y$ be a map. Then we have

$$f^* \mathcal{O}(U) \subseteq \mathcal{O}(f^{-1}(U))$$

iff f is given by polynomials.

Lemma 51. *Let X be a quasi-projective variety and let $p \in X$.*

Then there is an open neighborhood U of p in X s.t. U is isomorphic (as quasi-projective varieties) to an affine variety $U' \subset k^n$.

Proof. Let Y be a projective variety s.t. X lies open in Y . By replacing $X \hookrightarrow Y$ with $X^{(i)} \hookrightarrow Y^{(i)}$, we may assume that X is an open subset of an affine variety Y in k^n .

Since the sets $D(f)$ give an open basis of k^n , there is a $f \in \mathcal{O}(Y)$ s.t.

$$p \in D_Y(f) := \{y \in Y \mid f(y) \neq 0\} \subset X.$$

Now, $D_Y(f)$ is affine, because the map

$$\begin{aligned} D_Y(f) &\longrightarrow \{(q, r) \in k^{n+1} \mid q \in Y, f(q)r = 1\} \\ q &\longmapsto (q, \frac{1}{f(q)}) \end{aligned}$$

is an isomorphism of quasi-projective varieties. □

13.5 Dimensions

Definition 49. Let X be a quasi-projective variety. We define its **dimension** as the transcendency degree of its function field, i.e.

$$\dim(X) := \text{tr.-deg}_k(k(X)).$$

Remark 9. If X is affine, then

$$\begin{aligned} \dim(X) &= \text{tr.-deg}_k(k(X)) \\ &= \dim_{\text{Krull}}(\mathcal{O}(X)) \\ &= \sup \{n \in \mathbb{N}_0 \mid P_0 \subsetneq P_1 \subsetneq \dots \subsetneq P_n, P_i \text{ prime in } \mathcal{O}(X)\} \\ &= \sup \{n \in \mathbb{N}_0 \mid Z_n \subsetneq \dots \subsetneq Z_0, Z_i \text{ closed, irreducible in } X\} \end{aligned}$$

Remark 10. If $U \subset X$ is open, then $k(U) = k(X)$ and $\dim(U) = \dim(X)$.

Lemma 52. Let $\phi : X \rightarrow Y$ be a surjective morphism of quasi-projective varieties. Then,

$$\dim X \geq \dim Y.$$

Proof. ϕ induces an inclusion

$$\begin{aligned} \phi^* : k(Y) &\hookrightarrow k(X) \\ [(U_i, \alpha_i)_i] &\longmapsto [(\phi^{-1}(U_i), \alpha_i \circ \phi)]. \end{aligned}$$

This map is indeed injective, since ϕ is surjective. Therefore, the claim follows. \square

Lemma 53. Let X be a quasi-projective variety and Y a proper, closed subvariety. Then,

$$\dim(Y) < \dim(X).$$

Proof. By going from X to its closure \tilde{X} and from there to $\tilde{X}^{(i)}$, we can assume that X is affine.

Then, $I_X(Y)$ is a non-trivial prime ideal in $\mathcal{O}(X)$. Therefore, we have

$$\dim_{\text{Krull}}(\mathcal{O}(Y)) = \dim_{\text{Krull}}(\mathcal{O}(X)/I_X(Y)) \leq \dim_{\text{Krull}}(\mathcal{O}(X)) - 1,$$

since A is a finitely generated k -algebra and a domain, and since $I_X(Y)$ is a prime ideal. \square

Lemma 54. *Let X be an affine variety and $f \in \mathcal{O}(X)$ be non-zero.*

Then, the set

$$V_X(f) := \{p \in X \mid f(p) = 0\}$$

is a proper, closed subset of X and we can decompose it into irreducible components

$$V_X(f) = Z_1 \cup \dots \cup Z_l.$$

For each of those Z_i , we have

$$\dim(Z_i) = \dim(X) - 1.$$

Proof. The Z_i correspond to minimal prime ideals P_i in $\mathcal{O}(X)$ which contain (f) . Since, they are minimal, we have

$$\text{height}(P_i) = 1. \quad \square$$

Lemma 55. *Let X be an quasi-projective algebraic set. Then – as in the affine case – we may write*

$$X = Z_1 \cup \dots \cup Z_l$$

*where each $Z_i \subseteq X$ is an **irreducible component**, i.e. a maximal closed irreducible subset.*

We then define

$$\dim(X) := \max_i \dim(Z_j).$$

Lemma 56. *Let $\phi : X \rightarrow Y$ be a morphism of quasi-projective varieties. Further, let ϕ be **dominant**, i.e., $\text{Im}\phi$ is dense.*

Then, for all $p \in \text{Im}(\phi)$, we have the following for the fiber of ϕ along p :

$$\dim(\phi^{-1}(p)) \geq \dim(X) - \dim(Y).$$

13.6 Images of Morphisms

Lemma 57. *Let Y be a quasi-projective set. Then for each $p \in Y$, there is an open, affine neighborhood $Y_0 \subset Y$ which contains p .*

Proof. Denote by \bar{Y} the algebraic closure of Y in \mathbb{P}^n . Assume that $p_i \neq 0$. Then, the affine sets $Y^{(i)} = Y \cap D(x_i)$ and $\bar{Y}^{(i)} = \bar{Y} \cap D(x_i)$ lie dense in Y and \bar{Y} .

Now, $Y^{(i)}$ is open in $\bar{Y}^{(i)}$. Since the $D_{\bar{Y}^{(i)}}(f)$, $f \in \mathcal{O}(\bar{Y}^{(i)})$, give a basis of the topology of $\bar{Y}^{(i)}$, there is an $f \in \mathcal{O}(\bar{Y}^{(i)})$ s.t.

$$p \in D_{\bar{Y}^{(i)}}(f) \subseteq Y^{(i)} \subset Y.$$

The neighborhood $D_{\bar{Y}^{(i)}}(f)$ is, in particular, affine. □

Lemma 58. *Let Y be a quasi-projective algebraic set. Then the diagonal*

$$\Delta Y := \{(y, y) \mid y \in Y\}$$

is closed in $Y \times Y$.

Proof. If we cover Y by affine open subsets, then, we can reduce the claim to the case, where Y is affine, i.e. closed in k^n .

Then, $\Delta Y = (Y \times Y) \cap \Delta k^n \subset k^n \times k^n$. Since $Y \times Y$ is algebraic, it suffices to show that Δk^n is algebraic. And, indeed,

$$\Delta k^n = \{(x, y) \mid x - y = 0\}.$$

□

Theorem 19 (Thm2). *Let X be a projective variety and Y be a quasi-projective variety. Then, the projection*

$$\pi_Y : X \times Y \longrightarrow Y$$

*is **closed**, i.e. $\pi_Y(Z)$ is closed for each $Z \subseteq X \times Y$ closed.*

Proof. Since $X \hookrightarrow \mathbb{P}^n$ is a closed map (since X is closed in \mathbb{P}^n), it suffices to show the claim for $X = \mathbb{P}^n$.

Actually, at this point, we are done, since \mathbb{P}^n with the Zariski-topology is topologically quasi-compact. □

Theorem 20 (Thm1). *Let X be a projective variety and Y be a quasi-projective variety. Then, for each morphism $\phi : X \rightarrow Y$, the image $\phi(X)$ is closed in Y .*

Proof. First, we show that

$$\Gamma := \{(x, y) \in X \times Y \mid \phi(x) = y\}$$

is closed in $X \times Y$. In fact, we have

$$\Gamma = (\phi \times 1)^{-1}(\Delta Y)$$

where $\Delta Y \subseteq Y \times Y$ is closed.

Now, we can consider the chain

$$X \xrightarrow{\text{Id} \times \phi} X \times Y \xrightarrow{\pi_Y} Y.$$

We have $\phi(X) = \pi_Y(\Gamma)$. Since π_Y and Γ are closed, the claim follows. \square

Example 20. 1. The condition that X is a *projective* variety is necessary. Consider

$$\pi_x : \{(x, y) \mid xy = 1\} \longrightarrow k.$$

The image $k^\times = \pi_x(\{(x, y) \mid xy = 1\})$ is not closed in k .

2. Let $Y = k \subset_o \mathbb{P}^1$. Then any morphism $\phi : X \rightarrow k$ is constant.

This is, because $\phi(X)$ must be closed in \mathbb{P}^1 , ergo a finite set. Now, this finite set cannot contain multiple elements. Otherwise, X would not be irreducible.

Corollary 9. *Let X be a projective variety and Y be an affine variety. Then, any morphism $X \rightarrow Y$ is constant.*

Proof. We have the chain

$$X \longrightarrow Y \hookrightarrow k^m \xrightarrow{\pi_i} k.$$

For each π_i this chain must be constant. \square

Theorem 21 (Thm3). *Let $\phi : X \rightarrow Y$ be a morphism of quasi-projective varieties. Assume that ϕ is **dominant**, i.e. $\phi(X)$ is dense in Y .*

Then, $\phi(X)$ contains a nonempty open (hence dense) subset of Y .

Proof. Postponed... \square

Corollary 10. *Let $\phi : G \rightarrow H$ be a morphism of algebraic groups. Then, $\phi(G)$ is closed.*

Proof. Since G can be reduced to finite many irreducible components and since $\phi(G) = \bigcup_i \phi(g_i)\phi(G^o)$, it suffices to show the claim in the case where $G = G^o$ is irreducible.

Set $Y = \overline{\phi(G)}$. Y is irreducible and closed. Further, Y is a subgroup of H .

We are finished, if we can show $\phi(G) = \overline{\phi(G)}$.

By the previous theorem, $\phi(G)$ contains a nonempty open subset U of Y , hence $\phi(G)$ is dense in Y . Now, assume there are any $h \in Y - \phi(G)$. The map $y \mapsto hy$ is an isomorphism, hence $h\phi(G)$ lies dense in Y . Ergo

$$\phi(G) \cap (h\phi(G)) \neq \emptyset.$$

Take $u_1, u_2 \in \phi(G)$ s.t.

$$u_1 = hu_2.$$

Then, it follow $h = u_1u_2^{-1} \in \phi(G)$. A contradiction. □

13.7 Borel's Fixed Point Theorem (special case)

Theorem 22. *Let G be a connected solvable algebraic subgroup of $GL(V)$, where V is a finite-dimensional non-trivial vector space.*

Then, G acts algebraically on $\mathbb{P}(V)$.

Let $X \subseteq \mathbb{P}(V)$ be a non-empty, closed G -stable subset. Then, G has a fixed point in X .

Proof. We prove this by an induction on $n = \dim(V)$:

- $n = 1$:

In this case, $\mathbb{P}(V)$ contains only one element.

- $n = 2$:

We have $\mathbb{P}V \cong \mathbb{P}^1$. If $X = \mathbb{P}(V)$, then there is a complete invariant flag $0 \subset \langle v \rangle \subset V$ which is G -stable.

Then, $[v]$ is fixed by G .

If X is finite, let $x \in X$. Then, $G.x$ is a connected subset of X , hence $G.x = \{x\}$.

- $n \geq 3$:

Take again a complete G -stable flag $0 \subset \langle v \rangle \subset \dots \subset V$. If $[v] \in X$, we are done.

Otherwise, consider the morphism

$$\phi : X \longrightarrow \mathbb{P}(V/\langle v \rangle).$$

Since $\langle v \rangle$ is G -invariant, G acts on $\mathbb{P}(V/\langle v \rangle)$ and ϕ is G -equivariant.

The image $\phi(X)$ is closed by a theorem in the preceeding subsection. By the induction hypothesis, there is a fixed point $[w + \langle v \rangle] \in \phi(X) \subseteq \mathbb{P}(V/\langle v \rangle)$.

In particular, $[w + \langle v \rangle]$ has a preimage $[w]$ in X . Consider the subset

$$W := \langle w, v \rangle \subseteq V.$$

W is G -stable and we have $\mathbb{P}W \cap X \neq \emptyset$. Since $\mathbb{P}W \cap X$ is closed in $\mathbb{P}W \cong \mathbb{P}^2$, it follows from a previous case that there is a G -fixed point in $\mathbb{P}W \cap X$.

□

13.8 Orbits

Definition 50. Let G be an algebraic group and Y a quasi-projective variety.

An **action** $G \curvearrowright Y$ is an action described by a morphism³

$$\phi : G \times Y \longrightarrow Y.$$

Lemma 59. *Let G be an algebraic group which acts on a quasi-projective algebraic set Y . For an orbit $O \subset Y$, we have that O is open in \overline{O} .*

Proof. Let G_i be an irreducible component of G . For a point $p \in O$, the map

$$\begin{aligned} G_i &\longrightarrow \overline{G_i \cdot p} \\ g &\longmapsto g \cdot p \end{aligned}$$

is dominant. Ergo, $G_i \cdot p$ contains a nonempty open subset of $\overline{G_i \cdot p}$. Ergo, the set $O = G \cdot p$ contains a nonempty open subset U of $\overline{O} = \overline{G \cdot p}$.

Now, for $q \in O$, there is some isomorphism $g \in G$ s.t. $q \in g \cdot U$. Ergo, O is open. \square

Definition 51. If O is a G -orbit in a quasi-projective variety Y , we can consider it to be a quasi-projective set. Therefore, the notion of the dimension of an orbit O is well-defined.

Lemma 60 (Minimal Orbit Lemma). *Let G be an algebraic group. Let Y be a quasi-projective variety s.t. Y is projective or affine.*

Let O be a G -orbit in Y s.t. the dimension of O is minimal among all G -orbits in Y .

Then, O is closed.

Proof. Since the action of an element of G does not change the dimension of a quasi-projective set, we can reduce the claim to the case that G is connected.

Then, O is irreducible. Further \overline{O} is closed and, because of the previous lemma, $\overline{O} - O$ is closed. It is easy to see, that G operates on $\overline{O} - O$.

Let Z be an irreducible component of $\overline{O} - O$. Since Z is a proper closed subset of \overline{O} , we have

$$\dim(Z) < \dim(\overline{O}) = \dim(O).$$

Since O is dimensionally minimal, we must have $Z = \emptyset$. Ergo, $O = \overline{O}$. \square

Corollary 11. *Let G be an algebraic group. Let Y be a quasi-projective variety s.t. Y is projective or affine.*

Then G has a closed orbit in Y .

³If G is connected, ϕ shall be a morphism of quasi-projective varieties. Otherwise, we just require that $G^\circ \times Y \rightarrow Y$ is a morphism of quasi-projective varieties.

13.9 Borel's Fixed Point Theorem (General Case)

Theorem 23. *Let G be a connected solvable algebraic group which acts on a projective variety X .*

Then, there exists a G -fixed point in X .

Proof. Since orbits of minimal dimensions are closed, we can replace X by a G -orbit. That is, we can assume that G acts transitively on X .

For $p \in X$, the G -stabilizer set

$$\text{Stab}_G(p) = \{g \in G \mid g.p = p\}$$

is a closed subgroup in G , since it is the preimage of p under the continuous map $g \mapsto g.p$.

We showed earlier, that there exist a finite-dimensional representation $\rho : G \rightarrow \text{GL}(V)$ with a one-dimensional subspace $L \subset V$ s.t.

$$G_p = \{g \in G \mid gL = L\}.$$

Let $q = [L] \in \mathbb{P}(V)$. Then G operates on $\mathbb{P}V$ and

$$G_q := \text{Stab}_G(q) = \{g \in G \mid g.q = q\} = G_p.$$

Now, define

$$\begin{aligned} Y &:= G.q \subset \mathbb{P}V \\ Z &:= G.(p, q) \subset X \times \mathbb{P}V. \end{aligned}$$

Y and Z are quasi-projective varieties, since G is connected. We then have a G -equivariant diagram of quasi-projective varieties:

$$X \longleftarrow Z \xrightarrow{\pi} Y$$

via

$$X \longleftarrow X \times \mathbb{P}(V) \xrightarrow{\pi} \mathbb{P}(V).$$

Since X is projective, π is closed. Since $G_p = G_q$, the maps are bijective. Since all maps are bijective and G -equivariant, we need only to show that Y has a fixed point.

Since π is closed, the existence of a G -fixed point Y follows by the closedness of Z , because of Borel's special fixed point theorem.

The closedness of Z in $X \times \mathbb{P}(V)$ follows, if we can show that Z is an orbit of minimal dimension in $X \times \mathbb{P}(V)$. Indeed, we have.

- If $O \subset X \times \mathbb{P}(V)$ is a G -orbit, the projection $O \rightarrow X$ is G -equivariant and surjective, because X is a G -orbit. Since X, Y are quasi-projective varieties, it then follows

$$\dim(X) \leq \dim(O).$$

- The map $Z \rightarrow X$ is bijective, hence

$$\dim(*) \geq \dim(Z) - \dim(X).$$

Ergo

$$\dim(Z) \leq \dim(X).$$

□

13.10 Generic Openness

Proposition 4. *Let $\phi : X \rightarrow Y$ be a dominant morphism of quasi-projective varieties.*

Then, there is an open nonempty set $U \subset X$, s.t., $\phi|_U$ is open, that is, it maps open sets to open sets.

Corollary 12. *Let G be a connected algebraic group.*

Then, $[G, G] = \langle aba^{-1}b^{-1} \mid a, b \in G \rangle$ is a closed subgroup of G .

Proof. For $a, b \in G$, set

$$[a, b] = aba^{-1}b^{-1}.$$

For $n \geq 0$, define

$$\begin{aligned} \phi_n : G^{2n} &\longrightarrow G \\ (a_1, b_1, \dots, a_n, b_n) &\longmapsto [a_1, b_1] \cdots [a_n, b_n]. \end{aligned}$$

Let $Z_n := \overline{\text{Im} \phi_n}$. Then, we have an ascending chain

$$Z_1 \subseteq Z_2 \subseteq \dots$$

Each Z_n is closed and irreducible, because G^{2n} is connected.

Then, at some point the chains of Z_i 's must become stationary, because $\dim(G) < \infty$, because $\mathcal{O}(G)$ is a finitely generated k -algebra.

Let N be s.t.

$$Z_N = Z_{N+1} = \dots$$

Since $[G, G] = \bigcup_n \text{Im}(\phi_n)$, we have then

$$Z_N = \bigcup_n Z_n = \overline{[G, G]}.$$

Since $\phi_n : G^{2n} \rightarrow Z_n$ is dominant and since G^{2n} and Z_n are quasi-projective varieties, $\text{Im} \phi_n = [G, G]$ contains a nonempty open subset U .

Now, let $h \in \overline{[G, G]}$. Then, $hU \cap U \neq \emptyset$, since both are nonempty open and $\overline{[G, G]}$ is irreducible. Therefore, we have $u_1, u_2 \in U \subseteq [G, G]$ with

$$hu_1 = u_2.$$

Ergo, h lies in $[G, G]$. □

14 Homogenous Spaces

Definition 52. Let G be a connected algebraic group. A homogenous space for G is a quasi-projective variety X equipped with a **transitive** action $G \curvearrowright X$.

Let G be now disconnected. Then, we only demand that X is a finite union of irreducible components. Still G needs to act transitively on X .

A morphism of G -homogenous spaces is a **morphism** of quasi-projective varieties/sets which is G -equivariant.

Corollary 13. *If $\phi : X \rightarrow Y$ is a morphism of G -homogenous spaces, then ϕ is an open map.*

Proof. It suffices, if we show this statement for an irreducible X . Note, that ϕ must be surjective, ergo dominant.

By a previous proposition, X must contain an open nonempty subset U s.t. $\phi|_U$ is an open map. Since G acts transitively on X , we can cover X with such open sets gU . \square

Proposition 5. *Let G be an algebraic group and H a closed subgroup.*

Then, there is a homogenous space X for G and a point $p \in X$ s.t.

$$H = \text{Stab}_G(p)$$

and the map

$$\begin{aligned} G/H &\longrightarrow X \\ gH &\longmapsto g \cdot p \end{aligned}$$

is a bijection.

Proof. There is a faithful representation $\rho : G \rightarrow \text{GL}(V)$ with V finite-dimensional s.t. there is a one-dimensional subspace $L \subset V$ with

$$H = \{g \in G \mid gL = L\}.$$

Set $p := [L] \in \mathbb{P}(V)$. Then, we can set

$$X := G \cdot p.$$

Then, X is an orbit of G , ergo a quasi-projective set/variety. \square

14.1 Quotients

Definition 53. A (left) **quotient** of an algebraic group G by a closed group H is a pair (X, ρ) s.t.

- (1) X is a quasi-projective variety.
- (2) $\rho : G \rightarrow X$ is a morphism with

$$\rho(hg) = \rho(g)$$

for all $h \in H, g \in G$.

Further, we demand that a quotient is **initial** in the category of all objects satisfying the above conditions. I.e. for each pair (X', ρ') there must be a unique morphism ϕ s.t. the following diagram commutes:

$$\begin{array}{ccc} G & & \\ \downarrow \rho & \searrow \rho' & \\ X & \xrightarrow{\phi} & X' \end{array}$$

Remark 11. Set theoretically, we just have $X = G/H$.

Lemma 61. Let (X, ρ) satisfy conditions (1) and (2) from the above definition. Suppose further

- (i) $\{\text{fibers of } \rho\} = \{\text{left } H\text{-cosets of } G\}$,
- (ii) X is a G -homogenous space and ρ is G -equivariant,
- (iii) for each open $U \subset X$ the pullback map

$$\begin{aligned} \rho^* : \mathcal{O}(U) &\longrightarrow \mathcal{O}(\rho^{-1}(U)) \\ f &\longmapsto f \circ \rho \end{aligned}$$

defines an isomorphism

$$\mathcal{O}(U) \cong \{f \in \mathcal{O}(\rho^{-1}(U)) \mid f(Hg) = f(g)\} =: \mathcal{O}(\rho^{-1}(U))^H.$$

Then, (X, ρ) is a quotient of G by H .

Proof. We have to show that (X, ρ) is initial. Let (X', ρ') be another object satisfying (1), (2). Because of (i), we have a unique settheoretic map $\phi : X \rightarrow X'$ s.t. the diagram

$$\begin{array}{ccc} G & & \\ \downarrow \rho & \searrow \rho' & \\ X & \xrightarrow{\phi} & X' \end{array}$$

commutes. We need to check that ϕ is a morphism:

- ϕ is continuous, since ρ' is continuous and ρ is open (since X is a G -homogenous space). Therefore, $\phi = \rho' \circ \rho^{-1}$ is continuous.
- Let $U' \subset X'$ be open. We need to show

$$\phi^* \mathcal{O}(U') \subseteq \mathcal{O}(\phi^{-1}U').$$

Let $f \in \mathcal{O}(U')$ and set $U := \phi^{-1}U'$. Since ρ' is a morphism, we have

$$\rho'^*(f) \in \mathcal{O}(\rho'^{-1}U').$$

Because of (iii), we have

$$\mathcal{O}(U) \cong \mathcal{O}(\rho^{-1}U)^H.$$

Therefore, it suffices to show

$$\rho'^*(f) \in \mathcal{O}(\rho^{-1}U)^H.$$

And, indeed

$$f \circ \rho'(hg) = f \circ \rho'(g)$$

for $g \in G, h \in H$.

□

Lemma 62. *Suppose $\text{char } k = 0$. Any injective morphism of quasi-projective varieties with dense image is **birational**, i.e., induces, via pullback an isomorphism*

$$k(X) \cong k(Y).$$

Theorem 24. *Let G be an algebraic group with a closed subgroup H .*

- *A quotient (X, ρ) exists and X is a homogenous space for G s.t. $H = \text{Stab}_G(p)$ for some $p \in X$.*

- If $\text{char}(k) = 0$, then each G -homogenous space X together with a point $p \in X$ s.t. $H = \text{Stab}_G(p)$ gives a quotient of G by H , where $\rho(g) = g.p$.

Proof. We only prove the theorem for the case $\text{char} k = 0$. We construct X as in a previous proposition, i.e. $X = G.p$ for a point $p \in \mathbb{P}(V)$ s.t. $H = \text{Stab}_G(p)$.

It is then clear, that conditions (i) and (ii) of the previous lemma are met. We only need to show

$$\rho^* \mathcal{O}(U) = \mathcal{O}(\rho^{-1}U)^H.$$

Naturally, $\rho^* \mathcal{O}(U)$ is contained in $\mathcal{O}(\rho^{-1}U)^H$.

Let $f \in \mathcal{O}(\rho^{-1}U)^H$. W.l.o.g., we can assume that U is affine. Consider the diagram

$$\begin{array}{ccc} \rho^{-1}U & \xrightarrow{f} & k \\ \downarrow \rho & \nearrow g & \\ U & & \end{array}$$

$g := f \circ \rho^{-1}$ is well-defined, because f is H -invariant. We need to show, that g is regular, i.e. $g \in \mathcal{O}(\rho^{-1}U)$.

We can blow up the diagram as follows:

$$\begin{array}{ccccc} U \times k & \xleftarrow{\text{open}} & V \supset \text{Im}(\rho \times f) & & \\ & \searrow \pi_2 & \swarrow \pi_2 & & \\ \rho \times f \uparrow & & k & & \downarrow p \\ & \nearrow f & \nwarrow g & & \\ \rho^{-1}U & \xrightarrow{\rho} & U & & \end{array}$$

Then V is a quasi-projective variety and p is dominant and injective, hence birational. Therefore, we have

$$k(U) \cong k(V).$$

Since X is a homogenous space for G , it is smooth. On a smooth quasi-projective variety, every rational function that fails to be regular must have a pole.

In particular, we do have $\pi_2 \in \mathcal{O}(V)$ and therefore

$$g = p^*(\pi_2) \in \mathcal{O}(U).$$

□

Example 21 (Non-Example). The proof of the theorem does not hold, if $\text{char}(k) = p > 0$.

Consider,

$$\begin{aligned} G &:= \mathcal{G}_a \\ H &:= 1 \\ V &= k^2 \end{aligned}$$

and

$$\begin{aligned} G &\longrightarrow \text{GL}(V) \\ x &\longmapsto \begin{pmatrix} 1 & x^{p^n} \\ & 1 \end{pmatrix} \end{aligned}$$

for some $n \in \mathbb{N}_0$.

For $q = [1, 0] \in \mathbb{P}(V)$, we have

$$X := G.q = \{[1, x^{p^n}] \mid x \in k\} \cong k.$$

Define ρ by

$$\begin{aligned} \rho : G &\longrightarrow X \\ g &\longmapsto g.q. \end{aligned}$$

Then, (ρ, X) fulfills the conditions of the above theorem, but it is NOT a quotient for $n \geq 1$.

Indeed, for $n_1 \geq n_2$, we have non-isomorphic maps

$$\begin{aligned} X_{n_2} &\longrightarrow X_{n_1} \\ x &\longmapsto x^{p^{n_1 - n_2}}. \end{aligned}$$

15 Borel and Parabolic Groups

Let G be a connected algebraic group.

Definition 54. A subgroup $B \subset G$ is called **Borel**, if B is maximal among all connected solvable closed subgroups.

Since $\dim(G) < \infty$, Borel subgroups exist.

Definition 55. A subgroup $P \subset G$ is called **parabolic**, if the quasi-projective variety G/P is **projective**, i.e. closed in \mathbb{P}^n .

Lemma 63. *Let G connected, P parabolic, B Borel. Then, P contains some conjugate of B .*

Proof. B acts on the projective variety G/P . According to Borel's fixed point theorem, there is a fixed point $gP \in G/P$ s.t.

$$bgP = gP$$

for each $b \in B$. Ergo

$$g^{-1}bg \in P$$

for each $b \in B$. □

Theorem 25. *Let G be connected.*

Any two Borel subgroups are conjugate.

Proof. Take a faithful representation $G \hookrightarrow \mathrm{GL}(V)$ with a finite-dimensional V . Let $\mathcal{F} = \mathrm{Flag}(V)$ be the flag variety of V .

Choose $F \in \mathcal{F}$ s.t. the orbit $G.F$ has a minimal dimension. Then, $G.F$ is closed, hence projective. If we set

$$H := \mathrm{Stab}_G(F),$$

then H is parabolic. Therefore, each Borel group B has a conjugate in H . Since B is connected, its conjugate is contained in an irreducible component H° of the neutral element.

We claim that H is solvable. Indeed, H stabilizes a complete flag. Since the embedding $G \hookrightarrow \mathrm{GL}(V)$ is faithful, H must be isomorphic to a trigonalizable subgroup of $\mathrm{GL}(V)$. Since those are solvable, H must be, too.

Since H is solvable, H° is a connected, solvable, closed subgroup. Ergo H° is the conjugate of B . □

Proposition 6. *Let G be connected. Then, each Borel group is parabolic.*

Proof. Let B be a Borel subgroup of G .

Take a representation $G \rightarrow \mathrm{GL}(V)$ with a finite-dimensional V s.t. there is a one-dimensional $L \subseteq V$ s.t.

$$B = \{g \in G \mid gL = L\}.$$

B acts on V/L . Since B is connected and solvable there must be a complete B -invariant flag \overline{F} in V/L . We can lift \overline{F} to a complete flag $F = (L = V_1 \subset \dots \subset V_n)$ of V . Then, it is easy to see

$$B = \mathrm{Stab}_G(F).$$

Choose $F' \in \mathrm{Flag}(V)$ s.t. the orbit $G.F'$ has a minimal dimension. Then, $G.F'$ is closed, hence projective. If we set

$$H := \mathrm{Stab}_G(F'),$$

we have (by conjugating)

$$B = H^o.$$

Consider the map

$$G/B = G/H^o \twoheadrightarrow G/H.$$

This map has finite fibers, because $[H : B] < \infty$. Ergo

$$\dim(G/B) \leq \dim(G/H).$$

Ergo, G/B is of minimal dimension, hence closed. Hence, B is parabolic. \square

Corollary 14. *Let P be an algebraic subgroup of a connected algebraic group G . Then, P is parabolic iff it contains a Borel group.*

Proof. The direction to the right is known.

Let P contain a Borel group B . Consider the maps

$$G/B \twoheadrightarrow G/P \hookrightarrow \mathbb{P}^n.$$

Since B is parabolic, G/B is closed. Therefore, the morphism $G/B \rightarrow \mathbb{P}^n$ has a closed image. But its image is exactly G/P . Ergo, P is parabolic. \square

Corollary 15. *Let B be an algebraic subgroup of a connected algebraic group G . Then, B is Borel iff it is a minimal parabolic subgroup.*

Example 22. If $G = \mathrm{GL}_n(k)$, then

$$B = \left\{ \begin{pmatrix} * & \dots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & * \end{pmatrix} \right\}$$

is a Borel group.

Let $n = n_1 + \dots + n_r$ and set

$$P_{(n_1, \dots, n_r)} := \left\{ \begin{pmatrix} \mathrm{GL}_{n_1}(k) & * & * \\ 0 & \ddots & * \\ 0 & 0 & \mathrm{GL}_{n_r}(k) \end{pmatrix} \right\}.$$

Each $P_{(n_1, \dots, n_r)}$ is closed, since it is the stabilizer of an incomplete flag.

In fact, each parabolic group is conjugate to one of those $P_{(n_1, \dots, n_r)}$.

If $P \neq G$ is parabolic, P is called a **proper** parabolic subgroup.

Example 23. • $G = \mathrm{SL}_n(k)$: In this case parabolic groups are like in the above case, but inside of $\mathrm{SL}_n(k)$.

- $G = \mathrm{SO}_n(k)$: Then, we can embed G in $\mathrm{GL}(V)$. Let $\langle \cdot | \cdot \rangle$ be (any?) symmetric bilinear form.

A subspace $W \subset V$ is called **isotopic** iff $\langle \cdot | \cdot \rangle|_{W \times W} \equiv 0$.

Then, we have the equivalence

$$\{\text{Borel Group } B \subset G\} \Leftrightarrow \{\text{maximal isotropic flags } \mathcal{F} \text{ in } V\}.$$

- $G = \mathrm{SP}_{2n}$: The symplectic group is defined by

$$\mathrm{SP}_{2n} := \left\{ A \in \mathrm{GL}_{2n}(k) \mid A^T \cdot \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \cdot A = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \right\}.$$

Embed again G in $\mathrm{GL}(V)$.

Let $\langle \cdot | \cdot \rangle$ be a **symplectic** form on V , i.e., $\langle \cdot | \cdot \rangle$ is bilinear, alternating ($\langle v | v \rangle = 0$) and nonsingular, i.e. $\langle v | _ \rangle \equiv 0 \iff v = 0$.

Then, again, we have the equivalence

$$\{\text{Borel Group } B \subset G\} \Leftrightarrow \{\text{maximal isotropic flags } \mathcal{F} \text{ in } V\}.$$

Further, we can take a basis $e_1, \dots, e_n, f_1, \dots, f_n$ of V with

$$\begin{aligned}\langle e_i \mid e_j \rangle &= \langle f_i \mid f_j \rangle = 0 \\ \langle e_i \mid f_j \rangle &= \delta_{i,j}.\end{aligned}$$

Then, one can for example set

$$V_j = \text{span}\{e_1, \dots, e_j\}$$

to get a flag $V_0 \subset V_1 \subset \dots$

Vice versa, one can convert each maximal isotropic flag to such a symplectic basis.

15.1 Radicals

Let G be a connected algebraic group.

Definition 56. The **radical** $R(G)$ of G is defined as the intersection of all Borel subgroups of G i.e.

$$R(G) := \bigcap_{B \subset G \text{ Borel}} B.$$

The **unipotent radical** is defined by

$$R_u(G) := R(G)_u = \{\text{unipotent elements of } R(G)\}.$$

Lemma 64. *Let G be a connected algebraic group.*

$R(G)$ is the largest connected solvable normal algebraic subgroup of G .

Proof. It is clear that $R(G)$ is connected, solvable, normal and algebraic.

We need to show that each connected solvable normal algebraic subgroup H of G is contained in $R(G)$.

Clearly, H is contained in one Borel group B . Since H is normal, we have for each $g \in G$

$$H = gHg^{-1} \subset gBg^{-1}.$$

Since gBg^{-1} is a Borel group and all Borel groups are conjugated, it follows H is contained in each Borel group, ergo it is contained in $R(G)$. \square

Definition 57. We call G **semisimple** iff $R(G) = 1$.

We call G **reductive** iff $R_u(G) = 1$ (iff $R(G)$ is a torus).

Example 24. • Let $n \geq 1$ and $G = \text{GL}_n(k)$. G is reductive, but not semisimple:

G has two Borel groups:

$$B = \left\{ \begin{pmatrix} * & * \\ & * \end{pmatrix} \right\} \qquad B' = \left\{ \begin{pmatrix} * & \\ * & * \end{pmatrix} \right\}.$$

Ergo, we have for the radical

$$R(G) \subset B \cap B' = \left\{ \begin{pmatrix} * & \\ & * \end{pmatrix} \right\} =: T.$$

But, now we have

$$\{t \in T \mid gtg^{-1} \in T \ \forall g \in G\} = k^\times.$$

Ergo,

$$R(G) = k^\times.$$

- Let $G = \mathrm{SL}_n(k)$. G is semisimple and reductive:

As above, one can compute

$$Z = G \cap \left\{ \begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{pmatrix} \right\}.$$

However, Z is not connected. In particular

$$R(G) = Z^o = 1.$$

- $G = \mathcal{G}_m^n$ is a torus: It is easy to see that $R(G) = G$ in this case.
- G is solvable (and connected): Trivially, we have then $R(G) = G$.
- G is unipotent: In this case, we know that G is solvable. Further, we even have $R_u(G) = G$.
- If G is SO_n or SP_{2n} , then $R(G) = R_u(G) = 1$.

16 Reductivity

Let G be a connected algebraic group which acts on an affine variety X .

Definition 58. A **quotient** of X by G is a pair (Y, ρ) s.t.

1. Y is an affine variety
2. and $\rho : X \rightarrow Y$ is a morphism which is constant on G -orbits.

Further, we demand that a quotient is initial in the category of all objects which fulfill the above conditions. I.e.

$$\begin{array}{ccc} X & & \\ \downarrow \rho & \searrow \rho' & \\ Y & \xrightarrow{\exists_1 \phi} & Y' \end{array}$$

Remark 12. • Such quotients need not to exist.

- Even when such quotients exist, they don't need to describe orbits. I.e., $G \backslash X$ must not be related to Y .

Example 25. Consider the action of $G = \mathcal{G}_m$ on $X = k^1$. This action has two orbits: the open orbit $k \setminus \{0\}$ and the closed orbit $\{0\}$.

Then the quotient of X by G is given by $(Y, \rho) = (\{0\}, x \mapsto 0)$.

Note, if $f : X \rightarrow k$ is regular and constant on G -orbits, then f is constant on X , because $k \setminus \{0\}$ lies dense in k .

Definition 59. Let G be a connected algebraic group.

We call G **geometrically reductive** if we have for each finite-dimensional representation V of G :

$$\forall v \in V^G \exists f : V \rightarrow k : f \text{ is a homogenous } G\text{-invariant polynomial s.t. } f(v) \neq 0$$

where

$$V^G = \{v \in V \mid g.v = v \ \forall g \in G\}.$$

Remark 13. G is geometrically reductive iff for each affine X on which G operates and for each pair of closed G -invariant disjoint subsets $W_1, W_2 \subset X$ there is an $f \in \mathcal{O}(X)^G$ s.t.

$$\begin{aligned} f|_{W_1} &\equiv 1, \\ f|_{W_2} &\equiv 0. \end{aligned}$$

Proof. \Rightarrow : Let G be geomtrically reductive and connected. W.l.o.g., we can assume $X = W_1 \cup W_2$. According to the next theorem, we then have a quotient (Y, ρ) of X by G .

Y only contains points associated with the closed orbits of X . Each point Y comes either from W_1 or W_2 .

Therefore, we can define a function

$$f' : Y \rightarrow k$$

s.t.

$$f'(\rho(W_1)) = 1 \text{ and } f'(\rho(W_2)) = 0.$$

Now, (for some reason?) f' lies in $\mathcal{O}(Y)$. For $f = f' \circ \rho$, we now have

$$f \in \rho^* \mathcal{O}(Y) = \mathcal{O}(X)^G.$$

\Leftarrow : Let V be a f.d. vector space, on which G acts. Let $v \in V^G$. We need to show, that there is a homogenous, G -invariant $f : V \rightarrow k$ s.t.

$$f(v) \neq 0.$$

Assume $v \neq 0$ and take $X = V, W_1 = 0$ and $W_2 = v^\perp$. Then, there is an $f \in \mathcal{O}(V)^G$ s.t.

$$f(v) = 1 \text{ and } f(v^\perp) = 0$$

Now, because $f(v^\perp) = 0$, f must be linear (or of high degree and homogenous in char $k \neq 0$), ergo homogenous.

□

Theorem 26. *Let G be a connected algebraic group.*

Then, G is reductive iff G is geometrically reductive.

Theorem 27. *Let G be a connected algebraic group which is geometrically reductive and acts on an affine set X .*

Then, there is a quotient (Y, ρ) of X by G .

Moreover, ρ induces a bijection

$$\{\text{closed } G\text{-orbits in } X\} \longleftrightarrow Y.$$

Definition 60. Let G be a connected algebraic group.

We call G **linearly reductive** if we have for each finite-dimensional representation V of G :

$$\forall v \in V^G \setminus \{0\} \exists f : V \rightarrow k : f \text{ is a linear } G\text{-invariant polynomial s.t. } f(v) \neq 0.$$

Remark 14. Naturally, linear reductivity implies geometrical reductivity. The converse does hold iff $\text{char } k = 0$.

Remark 15. $\text{GL}_n(k)$ is linear reductive.

Remark 16. G is linear reductive iff every finite-dimensional representation V of G is completely **reducible**, i.e.

$$V = \bigoplus_i V_i$$

where each V_i is irreducible.

17 Union of Borel Subgroups

Theorem 28. *Let G be a connected algebraic group. Then,*

$$G = \bigcup_{B \text{ Borel}} B.$$

Because of Jordan Decomposition, it is clear that the theorem holds for $\mathrm{GL}_n(k)$. We will prove it only for the case $k = \mathbb{C}$.

Lemma 65. *Let k be any (not necessarily algebraically closed) field. Let B be some Borel subgroup.*

Then, $X := \bigcup_{g \in G} gBg^{-1}$ is closed in G .

Proof. Our intuition is as follows:

gBg^{-1} only depends on $gB \in G/B$. Since B is Borel, ergo parabolic, G/B is projective, ergo somewhat 'compact'. Then, $X = \bigcup_{g \in G} gBg^{-1}$ is a union of 'compactly-many' closed sets.

Now, the actual proof works as follows: We want to use that $G/B \times G \rightarrow G$ is a closed map. Consider the chain

$$G \times B \xrightarrow{\phi(g,b)=(g,gbg^{-1})} G \times G \longrightarrow G/B \times G \longrightarrow G.$$

X is the image of the composition $(g, b) \mapsto gbg^{-1}$.

Set

$$Y := (\pi \times \mathrm{Id})(\phi(G \times B)).$$

If we can show, that $(\pi \times \mathrm{Id})^{-1}(Y)$ is closed, then Y is closed, because $\pi \times \mathrm{Id}$ is, as a morphism of homogenous spaces, open. However, we have

$$(\pi \times \mathrm{Id})^{-1}(Y) = \mathrm{Im} \phi.$$

Now, $\mathrm{Im} \phi$ is closed, since morphisms of algebraic groups have closed images. □

Lemma 66. *Let $k = \mathbb{C}$.*

Then, $X = \bigcup_{g \in G} gBg^{-1}$ is dense in G .

Proof idea. We want to show $\overline{X} = G$.

Since G is connected, it would suffice to show that X contains an Euclidean neighborhood of $1 \in G$.

Let $\mathfrak{g} := \mathrm{Lie}(G)$ be the Lie-algebra of G . A Borel-subalgebra $\mathfrak{b} \subset \mathfrak{g}$ is a maximal solvable subalgebra.

Then, one can show, that for each Borel-subalgebra $\mathfrak{b} \subset \mathfrak{g}$ there is a Borel-subgroup $B \subset G$ s.t. $\mathfrak{b} = \text{Lie}(B)$.

Is easy to see, that each $x \in \mathfrak{g}$ is contained in some Borel-subalgebra, since $\mathbb{C} \cdot x$ is a solvable subalgebra.

With the two above facts, it follows that X contains a small euclidean neighborhood of 1. \square

18 Splitting Solvable Groups

Let B be a connected solvable algebraic group. (Then, B is trigonalizable.)

Then, $U := B_u$ is a unipotent normal algebraic subgroup (since $U = R_u(B)$, since $B = R(B)$).

Lemma 67. *The group B/U is a torus.*

Proof. We have an injective morphism

$$B \hookrightarrow \left\{ \begin{pmatrix} * & \dots & * \\ & \ddots & \vdots \\ & & * \end{pmatrix} \right\}$$

with

$$U = B \cap \left\{ \begin{pmatrix} 1 & \dots & * \\ & \ddots & \vdots \\ & & 1 \end{pmatrix} \right\}.$$

Therefore, we get an injection

$$B/U \hookrightarrow \left\{ \begin{pmatrix} * & \dots & * \\ & \ddots & \vdots \\ & & * \end{pmatrix} \right\} / \left\{ \begin{pmatrix} 1 & \dots & * \\ & \ddots & \vdots \\ & & 1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \right\}.$$

Ergo, B/U is diagonalizable. Since B is connected, B/U is connected, too. It follows that B/U is a torus. \square

Theorem 29. *Let B be a connected solvable algebraic group.*

Then, there is a torus $T \subset B$ s.t. the composition

$$T \hookrightarrow B \twoheadrightarrow B/U$$

is an isomorphism.

Before, we can prove the theorem we need some lemmata:

Lemma 68. *Suppose $\text{char} k = 0$. Let T be a torus.*

Then, there is an $s \in T$ s.t.

$$\overline{\langle s \rangle} = T.$$

*s is called the **generator** of T .*

Remark 17. The lemma does not hold, if $\text{char } k > 0$.

Proof. Recall that we have the following correspondence:

$$\{\text{tori}\} \xleftrightarrow{T \mapsto \mathfrak{X}(T)} \{\text{f.g. free } \mathbb{Z}\text{-modules}\}.$$

and in particular for each torus T :

$$\begin{aligned} \{\text{alg. subgroups } H \text{ of } T\} &\longleftrightarrow \{\text{submodules } \Gamma \text{ of } \mathfrak{X}(T)\} \\ H &\longmapsto \{\chi \in \mathfrak{X}(T) \mid \chi \in \mathfrak{X}(T) : \chi|_H \equiv 1\} \\ \{t \in T \mid \chi(t) = 1 \ \forall \chi \in \Gamma\} &\longleftarrow \Gamma. \end{aligned}$$

So, we have $\overline{\langle s \rangle} = T$ iff

$$\chi(s) \neq 1$$

for each $1 \neq \chi \in \mathfrak{X}(T)$.

W.l.o.g. $T = (k^\times)^n$. Then

$$\mathfrak{X}(T) = \{\chi_m \mid m \in \mathbb{Z}^n\}$$

with

$$\chi_m(t_1, \dots, t_n) = t_1^{m_1} \dots t_n^{m_n}.$$

We can then pick

$$s = (2, 3, 5, 7, \dots).$$

□

Lemma 69. *If $\text{char } k = 0$, then any bijective morphism of algebraic groups is an isomorphism of algebraic groups.*

Remark 18. This does not need to hold for non-zero characteristic. If $\text{char } k = p$, then

$$\begin{aligned} k &\longrightarrow k \\ x &\longmapsto x^p \end{aligned}$$

is bijective without being isomorphic.

Proof of Theorem. We only show the theorem in case $\text{char } k = 0$.

Let B be a connected solvable algebraic group.

We need to show, that there is a torus $T \subset B$ s.t. the composition

$$T \hookrightarrow B \twoheadrightarrow B/U$$

is an isomorphism where

$$U = B_u.$$

We know, that B/U is a torus. Take $s' \in B/U$ s.t.

$$\overline{\langle s' \rangle} = B/U.$$

Take a preimage $g \in B$ s.t. $\pi(g) = s'$.

We can decompose $g = su$ into a semisimple and a unipotent element. We then have

$$\phi(g) = \phi(s) \cdot \phi(u) = \phi(s),$$

since $\phi(u)$ must be unipotent, ergo trivial.

Set

$$T = \overline{\langle s \rangle}.$$

Since s is semisimple, T must be diagonalizable. Ergo

$$T \cap U = 1.$$

Ergo, the chain

$$T \hookrightarrow B \twoheadrightarrow B/U$$

must be bijective, hence an isomorphism, since $\text{char } k = 0$. □

The theorem gives the structure of a semidirect product of algebraic groups:

$$B = U \rtimes T$$

(where $T \curvearrowright U$ by conjugation.)

Definition 61. Let G_1, G_2 be algebraic groups. Let G_2 act algebraically on G_1 via $b : G_2 \rightarrow \text{Aut}(G_1)$ s.t. the map

$$\begin{aligned} G_2 \times G_1 &\longrightarrow G_1 \\ (g_2, g_1) &\longmapsto b(g_2)(g_1) \end{aligned}$$

is a morphism.

Their semidirect group $G_1 \rtimes_b G_2$ is an algebraic group which is:

- set-theoretically $G_1 \times G_2$,
- group-theoretically the semidirect product $G_1 \rtimes_b G_2$. I.e. multiplication works by

$$(g_1, g_2) \cdot (h_1, h_2) = (g_1 \cdot b(g_2)(h_1), g_2 h_2).$$

Remark 19. Even if we are given an algebraic group G with closed subgroups G_1, G_2 s.t.

$$G = G_1 \rtimes G_2$$

as abstract groups, it does not need to be the case that

$$G_1 \rtimes G_2 \longrightarrow G$$

is an isomorphism. (However, it is the case, if $\text{char } k = 0$.)

18.1 An Aside

Let G be an algebraic group and H a normal algebraic subgroup.

Then, G/H is a quasi-projective variety equipped with a G -action. Ergo, we have an algebraic group structure on G/H .

Theorem 30. *G/H is an affine algebraic group.*

Proof. We need to show that G/H is affine.

We showed in a lemma long before, that there is a finite-dimensional representation V, ρ s.t.

$$H = \ker \rho.$$

Therefore, we can simply set

$$G/H := \text{Im}(\rho) \subset \text{GL}(V)$$

which is closed as ρ is a morphism of algebraic groups. □

18.2 Semisimple Elements of Solvable Groups

Theorem 31. *Let $B = U \rtimes T$ as before be a solvable connected algebraic group. Let $s \in B$ be semisimple. Then s is conjugated to one element in T .*

Corollary 16. *Let G be a connected algebraic group. Then, every semisimple element of G is contained in some torus.*

Proof. Let $s \in G$ be semisimple and choose a Borel group $B \subset G$ which contains s . B is of the form $U \rtimes T$, ergo $s \in b^{-1}Tb$ for some $b \in B$. \square

Lemma 70. *Suppose $\text{char } k = 0$.*

(i) *Let $g \in GL_n(k)$ be unipotent and set $G(g) := \overline{\langle g \rangle}$. Then, we have the following isomorphism of algebraic groups*

$$G(g) = \{g^t \mid t \in k\} \cong k$$

where

$$\begin{aligned} g^t &:= \exp(t \cdot \log(g)) \\ -\log(1 - X) &= \sum_{n=1}^{\infty} \frac{X^n}{n} \\ \exp(Y) &= \sum_{k=0}^{\infty} \frac{Y^k}{k!}. \end{aligned}$$

(ii) *Any unipotent algebraic group is connected. (This does not hold if $\text{char } k > 0$.)*

(iii) *Any unipotent commutative algebraic group is isomorphic to some vector space.*

Proof. (i) We will not prove this, but the idea is that \mathbb{Z} is dense in k .

(ii) Let $g, h \in G$ be unipotent. Then the subgroups $G(g), G(h)$ are connected and share a common point (e) , ergo g, h are contained in the same component.

(iii) Since all elements commute \log gives an isomorphism into an additive group, on which k acts. \square

Proof of Theorem. We only prove the theorem in case $\text{char } k = 0$.

Let $s \in B = U \rtimes T$ be semisimple. Since $\text{char } k = 0$, U is connected.

We induct on $\dim(U)$:

- $\dim(U) = 0$: In this case $U = 1$ and $s \in G = T$.
- $\dim(U) = 1$: This is the crucial case.

Write

$$s = ut$$

with $u \in U$ and $t \in T$.

If u and t commute, then ut is a Jordan decomposition and we have $u = 1$, ergo $s \in T$.

Assume therefore, that u, t don't commute. We claim:

Claim: For each $h \in sU = Us$, we have for the B -conjugacy class $C(h) = \{ghg^{-1} \mid g \in B\}$

$$C(h) = sU.$$

The claim implies the theorem, because we then have

$$t = su \in sU = C(s)$$

ergo $t = gsg^{-1}$.

Proof of Claim:

- First note, that B acts by conjugation on $Us = sU$. This is because G/U is commutative and U is normal. In fact, we have for $g \in B, u \in U$

$$gsug^{-1} = s \cdot (s^{-1}gsug^{-1}) = s \cdot (s^{-1}gsg^{-1}) \cdot u'.$$

Now, $(s^{-1}gsg^{-1})$ must lie in U because G/U is commutative.

- Since $\dim(U) = 1$, we have

$$U = \{v^k \mid k \in K\} \cong k.$$

- $h \in sU$ does not commute with u , since – otherwise – s, t would commute with u .

Ergo, $h \neq u^{-1}hu$, which means $C(h) \supseteq \{h, u^{-1}hu\}$ contains at least two different elements.

- Note, that $C(h)$ is a B -orbit and therefore connected and **locally closed** (that is a closed subset of an open subset of G). Since G/U is commutative, we have

$$C(h) \subset sU = hU \cong k.$$

Now, the only connected, locally closed subsets of k are singletons and complements of finite sets.

Since $C(h)$ is not a singleton, we have

$$C(h) = sU - \Sigma$$

for a finite set Σ .

We claim that Σ is empty. Note, that B acts by conjugation on sU and $C(h)$, ergo also on Σ . If we pick $h' \in \Sigma \subset sU$, then $C(h')$ must be finite, connected and contain two different elements. This is a contradiction.

- $\dim(U) \geq 2$:

We want to reduce this case to the case $\dim(U) = 1$. We need therefore, to show a lemma:

Lemma 71. *Let $B = U \rtimes T$ as above and suppose again $\text{char } k = 0$.*

Then, there is an algebraic subgroup $V \subset U$ s.t. V is normal in B and

$$\dim(U/V) = 1.$$

Proof. U is nilpotent, since it is unipotent. Consider the chain

$$U = U_0 \supset U_1 \supset \dots \supset U_n \supset 1$$

where

$$U_{i+1} := [U_i, U].$$

Since U is normal in B , each U_i is also normal in B . In particular, B acts on each U_i by conjugation.

Now, U/U_1 is unipotent and commutative, hence isomorphic to a vector space.

Further, T acts on U/U_1 by conjugation. Note, that T is diagonalizable, ergo reductive. Therefore, U/U_1 must be completely reducible and we can decompose it

$$U/U_1 = \bigoplus_j V_j.$$

Since T is diagonalizable and each V_j is T -invariant, each V_j must be one-dimensional. Set

$$\bar{V} := \bigoplus_{j \geq 2} V_j.$$

And now set

$$V := \pi^{-1}(\bar{V}) = \{u \in U \mid uU_1 \in \bar{V}\}.$$

Then, we have

$$U/V = (U/U_1)/(V/U_1) = (U/U_1)/\bar{V} \cong V_1 \cong k.$$

V is normal in U , since U_1 is normal in U and T acts on V and \bar{V} by conjugation. \square

Let $s \in B$ be semisimple and $\dim(U) \geq 2$. Choose $V \subset U$ s.t. $\dim(U/V) = 1$ and V is normal in B . Set

$$\begin{aligned} B' &:= B/V \\ U' &:= U/V. \end{aligned}$$

Then, B' is a connected algebraic group with

$$\begin{aligned} (B')_u &= U' \\ B'/U &\cong B/U = T \\ B' &= U' \rtimes T. \end{aligned}$$

Since $\dim(U') = 1$, we know that $\pi_V(s) \in B'$ is contained in a conjugacy class of T . Let $s' \in B$ be the conjugate of $s \in B$ s.t. $\pi_V(s') \in T$. Then,

$$s' \in TV.$$

But TV is a connected solvable algebraic group and we have

$$TV \cong V \rtimes T \subset U \times T.$$

Since $(TV)_u = V$ and $\dim(V) = \dim(U) - 1$, the induction hypothesis does also hold in TV . Ergo, s' is conjugated to some element in T , as we wanted. \square

19 All About Tori

19.1 Maximal Tori

Definition 62. A **maximal torus** $T \subset G$ is a torus that is not contained in any larger torus.

Since G has a finite dimension, it always has at least one maximal torus.

Example 26. If $G = \mathrm{GL}_n(k)$, then $(k^\times)^n$ is its maximal torus.

Lemma 72. *Let G be a connected algebraic group. Let $s \in G$ be semisimple. Then, there is a maximal torus in G s.t.*

$$C(s) \cap T \neq \emptyset$$

where $C(s) = \{gsg^{-1} \mid g \in G\}$.

Proof. Choose a Borel group $B \subset G$ s.t. $s \in B$. Then, we can decompose

$$B \cong U \rtimes T.$$

T is a torus. By the previous theorem, we know that a conjugate of s is contained in T . The claim follows, if we enlarge T to a maximal torus in G . \square

Theorem 32. *Let G be a connected algebraic group with a maximal torus T . Let $s \in G$ be semisimple. Then,*

$$C(s) \cap T \neq \emptyset.$$

Proof. We only show the theorem in case $\mathrm{char} k = 0$.

Since T is connected and solvable, it is contained in some Borel group B .

Choose further a Borel group $B' = U \rtimes S$ s.t. $s \in B'$ is conjugate to some element $s' \in S$.

Now, B' is conjugate to B . If we choose a generator $t \in T$ s.t.

$$\overline{\langle t \rangle} = T,$$

then t is conjugated to some element $t' \in S$. The torus $T' = \overline{\langle t' \rangle}$ generated by t' is again maximal, therefore

$$T' = S.$$

Since $s' \in T'$, the claim follows. \square

Corollary 17. *Let G be a connected algebraic group.*

(i) Any two maximal tori are conjugate.

(ii) For each torus S and each maximal torus T , there exists a $g \in G$ s.t.

$$gSg^{-1} \subset T.$$

Proof. (i) follows, if we can show (ii).

Let S be torus with a generator s . Then there is a $g \in G$ s.t.

$$gs g^{-1} \in T.$$

Ergo

$$gSg^{-1} = \overline{\langle gs g^{-1} \rangle} \subset T,$$

where we assume that $\text{char } k = 0$. □

Corollary 18. *Let s be a central semisimple element of G (i.e. s commutes with each other element). Then s is contained in every maximal torus. In other words*

$$Z(G)_s \subset \bigcap_{T \text{ max. torus}} T.$$

Proof. This is clear, since a conjugate of s must be contained in each maximal torus and s commutes with each element. □

Corollary 19. *Let T be a torus in a connected group G . Then, T is maximal iff its dimension is maximal among all dimensions of tori in G .*

19.2 Centralizers of Tori

Lemma 73. *Let G be a connected algebraic group. Let $S \subset G$ be a torus. Let $g \in G$ be a semisimple element which commutes with each element of S .*

Then, $S \cup \{g\}$ is contained in some torus of G .

Proof. Set $H := Z_G(g)^\circ$. Then, H is a connected algebraic group that contains S . Then,

$$g \in Z(H)_s \subset \bigcap_{T \text{ maximal tori in } H} T.$$

In particular, there must be some maximal torus of H which contains S . □

Theorem 33. *Let G be a connected algebraic group. Let $S \subset G$ be a torus.*

Then, $Z_G(S)$ is connected.

Proof. We assume $\text{char } k = 0$.

Let $g \in Z_G(s)$. Decompose $g = g_s g_u$. Then, we need too show the claim in case:

(i) g is semisimple:

By the previous lemma, there is a torus $T \subset Z_G(S)^\circ$ which contains S and g .

(ii) g is unipotent:

Since k has characteristic zero, the group

$$\overline{\langle g \rangle} = g^k \cong \begin{cases} k, & g \neq 1 \\ 1, & g = 1 \end{cases}$$

is connected. □

19.3 Low Dimensional Groups

Lemma 74. *Let G be a connected algebraic group with a Borel subgroup B . If B is nilpotent, then G is solvable i.e. $B = G$.*

Proof. We induct on $\dim(B)$:

- $\dim(B) = 0$:
In this case, we have $B = 1$. Then, $G = G/B$ must be projective and connected. Therefore, we must have

$$\mathcal{O}(G) = \mathcal{O}(\mathbb{P}^n)/I(G) = k$$

since $\mathcal{O}(\mathbb{P}^n) = k$. On the other side, G is affine. Therefore, we have

$$G = 1.$$

Or: $G = \bigcup_{g \in G} gBg^{-1}$, since G is connected. Since $B = 1$, it follows $G = 1$.

- $\dim(B) \geq 1$:
Since B is nilpotent, we have a descending chain

$$B = B_0 \supsetneq \dots \supsetneq B_n \supsetneq 1$$

where

$$B_{i+1} = [B, B_i].$$

Note, that each B_i is connected, since B is connected. Let $Z(B) = \{b \in B \mid \forall g \in B : gb = bg\}$ be the center of B and let $Z := Z(B)^\circ$ be the component of the neutral element. Then, we have

$$B_n \subset Z.$$

Ergo, Z is not the trivial subgroup.

We want to show

$$Z \subset Z(G).$$

Let $z \in Z$ and consider the morphism

$$\begin{aligned} \phi : G/B &\longrightarrow G \\ gB &\longmapsto gzg^{-1}. \end{aligned}$$

ϕ is well-defined, because $z \in Z(B)$. Since ϕ is a morphism from a projective variety to an affine variety, ϕ must be constant. Thus,

$$Z \subset Z(G).$$

In particular, Z is normal in G . We now get an inclusion of quotient groups

$$B/Z \hookrightarrow G/Z.$$

It is clear that

$$\dim(B/Z) < \dim(B).$$

Further, B/Z is parabolic, since

$$(G/Z)/(B/Z) = G/B$$

is projective. Ergo, B/Z is Borel. By the induction hypothesis, we get

$$G/Z = B/Z.$$

Ergo, $B = G$.

□

Theorem 34. *Let G be connected with $\dim(G) \leq 2$. Then, G is solvable.*

Example 27 (Non-Example). The condition $\dim(G) \leq 2$ is necessary. Consider e.g. $G = \mathrm{SL}_2(k)$ which has a dimension of 3.

Proof. Let $B \subset G$ be a Borel subgroup. We want to show

$$B = G.$$

Assume otherwise. Then, B is of dimension 1. The key here is, that Borel groups of dimension 1 are nilpotent.

Decompose $B = U \rtimes T$, then we have:

- $U \neq 1$: Then, $\dim(T) = \dim(B/U) = 0$, ergo $T = 1$. Hence

$$B = U.$$

Since unipotent groups are nilpotent, B is nilpotent.

- $U = 1$: In this case, we have

$$B = T.$$

Now B as a torus is commutative, ergo nilpotent.

Now, the above lemma states

$$B = G$$

since B is nilpotent. Ergo, G is solvable. \square

Corollary 20. *Let G be connected with $\dim(G) = 1$. Then, G is commutative.*

Proof. Because of the theorem, G is solvable. Therefore, $[G, G]$ is a closed proper subgroup of G . Hence, $\dim([G, G]) = 0$. Since $[G, G]$ is connected, it follows $[G, G] = 1$. \square

Remark 20. If G is commutative, it decomposes nicely into semisimple and unipotent elements

$$G = G_s \times G_u.$$

So, if $\dim(G) = 1$ and if G is connected, then $G = G_s \cong \mathcal{G}_m$ is a torus, or $G = G_u \cong \mathcal{G}_a$ is unipotent.

Further, we can consider

$$G = \left\{ \begin{pmatrix} * & * \\ & 1 \end{pmatrix} \right\}.$$

G is connected and of dimension 2. It decomposes

$$G = G_s \times G_u$$

into two groups of dimension 1.

19.4 Characterizing Nilpotent Groups via maximal Tori

Lemma 75. *Let T be a diagonalizable algebraic group. Then,*

$$T = \overline{\bigcup_{n \geq 1} T[n]}$$

where

$$T[n] := \{t \in T \mid t^n = 1\}.$$

Proof. If the claim is true for any T_1, T_2 , then it is also true for $T_1 \times T_2$.

Therefore, we can reduce the claim to the cases where T is finite or $T = \mathcal{G}_m$.

A finite T is contained in some $T[n]$.

Let $T = \mathcal{G}_m$. Then, we have

$$\mathcal{O}(T) = k[x, \frac{1}{x}].$$

We need to show for each $f \in \mathcal{O}(T)$:

$$f(\alpha) = 0 \ \forall \alpha \in k, \in \mathbb{N}_0 \text{ s.t. } \alpha^n = 1 \implies f = 0.$$

So, let $f \in \mathcal{O}(T)$. By multiplying with a large enough x^r , we can assume $f \in k[x]$.

If $f \neq 0$, then f has finitely many roots. However the set

$$\{\alpha \in k \mid \exists n \in \mathbb{N}_0 : \alpha^n = 1\}$$

has infinitely many elements. Indeed we have

$$\# \{\alpha \in k \mid \alpha^n = 1\} = n,$$

if $\text{char} k = 0$ or if $\gcd(\text{char} k, n) = 1$. □

Remark 21. If $\text{char} k = 0$ and if U is unipotent, then

$$U[n] = 1$$

for all n .

Theorem 35. *Let G be a connected algebraic group. Then, the following are equivalent:*

(i) G is nilpotent.

(ii) Each maximal torus T of G satisfies $T \subset Z(G)$.

(iii) G has a unique maximal torus.

Proof. We show:

(i) \implies (ii): We have shown, if G is nilpotent and connected, then each semisimple element is central.

(ii) \implies (iii): Any two maximal tori are conjugated.

(iii) + (ii) \implies (i): Since each semisimple element is contained in some torus, we have that each semisimple element is central.

Let $B = U \rtimes T' \subset G$ be a Borel subgroup. Since $T' \subset T$ is central in G , we have $B = U \times T'$. So, B is nilpotent. We showed in this case that

$$G = B.$$

(iii) \implies (ii): Let T be the maximal torus of G . We have to show that T is central.

Since T is unique, it must be normal in G . Then, G acts via conjugation on T and each $T[n]$. Therefore, we get a morphism for each $t \in T[n]$

$$\begin{aligned} G &\longrightarrow T[n] \\ g &\longmapsto gtg^{-1}. \end{aligned}$$

Since G is connected, this morphism must be constant, ergo trivial. Ergo, each $T[n]$ is central in G . Since

$$T = \overline{\bigcup_{n \in \mathbb{N}} T[n]},$$

T must be central in G .

□

19.5 Weyl Groups

Definition 63. Let G be a connected algebraic group with a torus T . Define the **normalizer** of T in G by

$$N_G(T) := \{g \in G \mid gT = Tg\}.$$

Then, the centralizer $Z_G(T)$ is a normal subgroup in $N_G(T)$. Define the **Weyl group** of T as the quotient

$$W(G, T) := N_G(T)/Z_G(T).$$

If T is maximal, then the Weyl group $W(G, T)$ is up to conjugation independent of T .

Proposition 7. (i) $\#W < \infty$.

(ii) For each tori $S \subset G$, we have

$$N_G(S)^o = Z_G(S)^o.$$

Proof. It is clear, that (ii) implies (i).

Let $S \subset G$ be a torus. We want to show

$$N_G(S)^o \subset Z_G(S).$$

Note, that $N_G(S)^o$ acts by conjugation on S . Now, it is clear that

$$S = \overline{\bigcup_n S[n]}$$

where $S[n]$ denotes the subgroup of n -th roots of unity. $N_G(S)^o$ acts on each $S[n]$. As before, this gives for each $s \in S[n]$ a group morphism $\phi_s : N_G(S)^o \rightarrow S[n]$. Since $S[n]$ is finite and $N_G(S)^o$ is connected, ϕ_s must be trivial. Ergo

$$N_G(S)^o \subset Z_G(S).$$

□

Remark 22. In general, $W(G, T)$ acts on T by conjugation and induces an inclusion

$$W(G, T) \hookrightarrow \text{Aut}(T).$$

Example 28. Let $G = \mathrm{GL}_n(k)$ with the maximal torus

$$T = \left\{ \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \right\}.$$

Denote by $S(n)$ the group of all permutation matrices of G .

Then, we have

$$Z_G(T) = T$$

$$N_G(T) = T \cdot S(n)$$

$$W(G, T) \cong S(n).$$

19.6 Normalizers of Borel Subgroups

Lemma 76. *Let G be a connected algebraic group, $B \subset G$ a Borel subgroup and $S \subset B$ any torus.*

Then, $Z_B(S)$ is a Borel subgroup of $Z_G(S)$.

Proof. We showed before, that $Z_G(S)$ is connected, if G is connected. Set

$$U := B_u.$$

- We claim

$$Z_G(S)B = \{g \in G \mid \forall s \in S : g^{-1}sg \in sU\} =: A.$$

It is easy to see, that

$$Z_G(S) \subset A.$$

For $b \in B$, we have

$$b^{-1}sb \in sU,$$

since B/U is commutative.

Now, let $g \in A$. Then,

$$g^{-1}Sg \subset sU \subset B.$$

One can extend S to a maximal torus T of B . Then,

$$B = U \rtimes T \supset sU = U \rtimes S.$$

Since S is closed in T , sU is closed in B . Further, $g^{-1}Sg$ and S are maximal tori in sU . Then, there is a $b \in B$ s.t.

$$b(g^{-1}Sg)b^{-1} = S.$$

Set

$$z := gb^{-1}.$$

We need to show, that z lies in $Z_G(S)$.

Since B/U is commutative, we have for each $s \in S$

$$z^{-1}sz = b(g^{-1}sg)b^{-1} \in g^{-1}sgU = sU,$$

since $g \in A$. Now, we have for each $s \in S$

$$z^{-1}sz \in sU \cap S = \{s\}.$$

Ergo, $z \in Z_G(S)$.

- We showed that

$$Z_G(S)B = \{g \in G \mid \forall s \in S : g^{-1}sg \in sU\} = \{g \in G \mid \forall s \in S : [g, s] \in U\}.$$

Then, $Z_G(S)B$ is closed. Since

$$\pi : G \twoheadrightarrow G/B$$

is an open and surjective map, it is easy to see that

$$Z_G(S)/Z_B(S) \cong \pi(Z_G(S)B)$$

is closed. Since $Z_B(S)$ is closed, $Z_B(S)$ is a parabolic subgroup of $Z_G(S)$. Since $Z_B(S)$ is contained in B , it is solvable, hence a Borel subgroup.

□

Example 29.

$$\begin{aligned} G &= \mathrm{GL}_5(k) \\ B &= \left\{ \begin{pmatrix} * & * & * & * & * \\ & * & * & * & * \\ & & * & * & * \\ & & & * & * \\ & & & & * \end{pmatrix} \right\} \\ S &= \left\{ \begin{pmatrix} t_1 & & & & \\ & t_1 & & & \\ & & t_1 & & \\ & & & t_2 & \\ & & & & t_2 \end{pmatrix} \mid t_1, t_2 \in k^\times \right\} \\ Z_G(S) &= \left\{ \begin{pmatrix} * & * & * & & \\ * & * & * & & \\ * & * & * & & \\ & & & * & * \\ & & & * & * \end{pmatrix} \right\} = \mathrm{GL}_3(k) \times \mathrm{GL}_2(k) \\ Z_G(S) &= \left\{ \begin{pmatrix} * & * & * & & \\ & * & * & & \\ & & * & & \\ & & & * & * \\ & & & & * \end{pmatrix} \right\}. \end{aligned}$$

Theorem 36. *Let G be a connected algebraic group with a Borel subgroup $B \subset G$. Then,*

$$N_G(B) = B.$$

Proof. We induct on $\dim(G)$:

- $\dim(G) = 0$: In this case, we have $G = 1$.
- $\dim(G) = 1$: We have seen, that in this case G is commutative.
- $\dim(G) \geq 2$: Let T be a maximal torus in B . Let $x \in N_G(B)$.

Then, xTx^{-1} is again a maximal torus in B . Since all maximal tori in B are related via B -conjugation, there is $b \in B$ s.t.

$$xTx^{-1} = bTb^{-1}.$$

We therefore replace x by $b^{-1}x$ to achieve

$$xTx^{-1} = T.$$

Now, consider the map

$$\begin{aligned} \rho : T &\longrightarrow T \\ t &\longmapsto txt^{-1}x^{-1}. \end{aligned}$$

We distinguish two cases:

- ρ is not surjective:
Since all tori are irreducible, we have then

$$\begin{aligned} \dim(\text{Im}(\rho)) &< T \\ \dim(\text{Kern}(\rho)^o) &> 0. \end{aligned}$$

If we set $S := \text{Kern}(\rho)^o$, then S is a non-trivial torus in T .

Since $S \subset \text{Kern}(\rho)$, x centralizes S and normalizes B . Hence, x normalizes $Z_B(S)$.

Because of the previous lemma, $Z_B(S)$ is Borel subgroup of $Z_G(S)$. If $Z_G(S) \neq G$, then the induction hypothesis implies

$$x \in N_{Z_G(S)}(Z_B(S)) = Z_B(S) \subset B.$$

Otherwise, if $Z_G(S) = G$, then B/S is a Borel subgroup of G/S . So the induction hypothesis implies

$$xS \in N_{G/S}(B/S) = B/S,$$

ergo $x \in B$.

– ρ is surjective:

Then,

$$T = \text{Im } \rho \subset [N_G(B), N_G(B)].$$

Set $H := N_G(B)$. We have to show

$$H = B.$$

Choose a finite-dimensional representation

$$G \longrightarrow \text{GL}(V)$$

and a line $L \subset V$ s.t.

$$H = \{g \in G \mid gL = L\}.$$

Then, we have a morphism of algebraic groups

$$\gamma : H \longrightarrow \text{GL}(L) = \mathcal{G}_m(k).$$

Since the right side is a torus, we have

$$\gamma|_{H_u} \equiv 1$$

$$\gamma|_{[H, H]} \equiv 1.$$

Ergo, $\gamma(T) = 1$ and, since $B = B_u \rtimes T$, $\gamma(B) = 1$.

Choose a non-zero element $v \in L$ and consider

$$\begin{aligned} \phi : G/B &\longrightarrow V \\ gB &\longmapsto gv. \end{aligned}$$

Since G/B is a projective variety, while V is an affine variety, ϕ must be constant. Therefore, we have for each $g \in G$

$$gv \in L.$$

Ergo, $G = H$ and B is normal in G . But, now

$$G = \bigcup_{g \in G} gBg^{-1} = B.$$

Ergo $H = B$. □

Corollary 21. *We have a bijection:*

$$\begin{aligned} G/B &\longrightarrow \{\text{Borel Subgroups of } G\} \\ gB &\longmapsto gBg^{-1}. \end{aligned}$$

19.7 Borel Subgroups Containing a Given Torus

Let G be a connected algebraic group with a maximal torus T . Set

$$\mathcal{B}^T := \{B \subset G \text{ Borel} \mid T \subset B\}.$$

Then, $N_G(T)$ acts on \mathcal{B}^T by conjugation.

Example 30. Let $G = \mathrm{GL}_2(k)$ with $T = \left\{ \begin{pmatrix} * & \\ & * \end{pmatrix} \right\}$. Then,

$$\mathcal{B}^T = \left\{ \begin{pmatrix} * & * \\ & * \end{pmatrix}, \begin{pmatrix} * & \\ * & * \end{pmatrix} \right\}.$$

Lemma 77. *The action of $Z_G(T)$ on \mathcal{B}^T by conjugation is trivial.*

Equivalently (since $B = N_G(B)$), $Z_G(T) \subset B$ for each $B \in \mathcal{B}^T$.

Proof. We know, that $Z_G(T)$ is connected, since T is a torus. Further, since $T \subset Z_G(T)$ is central and a maximal torus, it must be the unique maximal torus in $Z_G(T)$. We showed before, that this is equivalent to $Z_G(T)$ being nilpotent. Thus, $Z_G(T)$ is contained in some Borel group $B_0 \in \mathcal{B}^T$.

Let $B \in \mathcal{B}^T$ and choose $g \in G$ s.t.

$$B = gB_0g^{-1}.$$

Since maximal tori in B are B -conjugated, we can choose $g \in G$ s.t. $g \in N_G(T)$. (Otherwise, we can replace g by bg s.t. $bgTg^{-1}b^{-1} = T$.)

One can show that

$$g \in N_G(T) \implies g \in N_G(Z_G(T)).$$

Thus

$$g^{-1}Z_G(T)g = Z_G(T) \subset B_0$$

which implies

$$Z_G(T) \subset gB_0g^{-1} = B.$$

□

Corollary 22. *The action $N_G(T) \curvearrowright \mathcal{B}^T$ induces an action by the Weyl group $W(G, T) = N_G(T)/Z_G(T)$ on \mathcal{B}^T .*

Corollary 23. *In the proof, we could see that $N_G(T)$ and $W(G, T)$ act transitively on \mathcal{B}^T .*

Corollary 24.

$$\#\mathcal{B}^T \leq \#W < \infty.$$

Theorem 37. W acts *simply-transitively* on \mathcal{B}^T , i.e., for each $B_1, B_2 \in \mathcal{B}^T$ there is exactly one $g \in G$ s.t.

$$gB_1g^{-1} = B_2.$$

In particular,

$$\#\mathcal{B}^T = \#W.$$

Proof. Let $B \in \mathcal{B}^T$. We need to show

$$N_G(T) \cap N_G(B) \subset Z_G(T).$$

Note, that

$$N_G(T) \cap N_G(B) = N_G(T) \cap B = N_B(T).$$

Set $U := B_u$, then $B = U \rtimes T$.

Choose $b \in N_B(T)$ with $b = ut$, $u \in U, t \in T$. Then,

$$T = bTb^{-1} = uTu^{-1}.$$

Since $t \in Z_G(T)$, it suffices to show that $u \in Z_G(T)$.

Let $t \in T$ and set $t' = utu^{-1} \in T$. Since, we have an isomorphism

$$T \hookrightarrow B \twoheadrightarrow B/U$$

and B/U is commutative, t and t' must be equal in T . Ergo, $u \in Z_G(T)$. \square

Corollary 25. Since $N_B(T) \subset Z_G(T)$ we have for each Borel group B and maximal torus T of B

$$W(B, T) = 1.$$

In particular,

$$\mathcal{B}^T = \{B\}.$$

Proposition 8. Let G be a connected non-solvable algebraic group (this implies $\dim G \geq 3$). Let B be a Borel subgroup with a maximal torus T . Then,

$$\#W(G, T) \geq 2.$$

Moreover,

$$\#W = 2 \iff \dim(G/B) = 1.$$

Sketch of Proof. We have a bijection

$$\{\text{Borel subgroups in } G\} \longleftrightarrow G/B.$$

This can be restricted to a bijection

$$\mathcal{B}^T \longleftrightarrow \{gB \in G/B \mid TgB = gB\}.$$

Idea: Show that T acts non-trivially on G/B , since G is non-solvable, and deduce that it has at least two different fixed points:

$$\begin{array}{ccccc}
 \mathcal{G}_m & \hookrightarrow & T & \xrightarrow{\sim} & G/B \\
 \downarrow & & & \nearrow & \\
 \mathbb{P}^1 & & & \ni 0, \infty &
 \end{array}$$

One can show, that 0 and ∞ are mapped to different fixed points of G/B . □

19.8 Groups of Semisimple Rank One

Definition 64. Let G be an algebraic group. The **rank** of G is defined by

$$\text{rank}(G) := \dim(T)$$

for any maximal torus T .

Remark 23. If G is connected and of rank zero, then G is unipotent.

Definition 65. The **semisimple rank** of G is defined by

$$\text{ss-rank}(G) := \text{rank}(G/R(G)).$$

(Note, that $G/R(G)$ is semisimple.)

Example 31. • If T is a torus, then $\text{ss-rank}(T) = 0$.

- Let $Z = Z_{\text{GL}_n(k)}(\text{GL}_n(k))$ be the centralizer of $\text{GL}_n(k)$. Then, $Z \cong k$. For a matrix group $G \subset \text{GL}_n(k)$, set

$$PG := G/(G \cap Z).$$

Then, PG acts on \mathbb{P}^n .

We now have

G	$\text{ss-rank}(G)$	$\text{rank}(G)$
SL_2	1	1
PGL_2	1	1
GL_2	1	2
GL_n	$n - 1$	n

- Consider

$$G = \left\{ \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \right\} \subset \text{GL}_3(k).$$

Then,

$$\begin{aligned} \text{rank}(G) &= 3, \\ \text{ss-rank}(G) &= 1. \end{aligned}$$

Remark 24. Let G be a connected algebraic group. Then, G is of semisimple rank zero iff G is Borel (because $R(G)$ is the connected component of 1 in the intersection of all Borel subgroups in G).

Lemma 78 (Fact). *We call X a **curve**, if X is a one-dimensional variety.*

Let X be a smooth projective curve X that admits a nontrivial action by a non-trivial connected algebraic group H . Then, we have

$$X \cong \mathbb{P}^1.$$

Idea of Proof. Reduce to the case $H = \mathcal{G}_m$ or $H = \mathcal{G}_a$. They give a commutative diagram:

$$\begin{array}{ccc} \mathcal{G}_m & \longrightarrow & X \\ \downarrow & \nearrow \exists \phi & \\ \mathbb{P}^1 & & \end{array}$$

Now, ϕ is a non-constant orbit map, therefore

$$k(X) \hookrightarrow k(\mathbb{P}^1) \cong k(T).$$

By Lüroth's Theorem⁴, there is a transcendent $T' \in k(T)$ s.t. $k(X) = k(T')$. Ergo

$$X = \mathbb{P}^1. \quad \square$$

Lemma 79 (Fact). *We have*

$$\text{Aut}(\mathbb{P}^1) \cong \text{PGL}_2(k).$$

Proposition 9. *Let G be of semisimple rank one. Then, there is a surjective morphism*

$$\rho : G \rightarrow \text{PGL}_2(k)$$

s.t. $\text{Ker} \rho = R(G)$.

In particular, if G is semisimple, then $\text{Ker} \rho$ is finite, since $R(G)$ is trivial in this case.

⁴Lüroth's Theorem states that each intermediate field $L \supset P \supset K$ of a purely transcendental extension $L \supset K$ of degree 1 is either K or purely transcendental over K .

Proof. By dividing out $R(G)$, we can reduce the proof to the case, in which G is semisimple and of rank 1. In particular, G cannot be solvable, ergo $\dim(G) \geq 3$.

We have seen for unsolvable groups, that $\#W(G, T) \geq 2$. But, since $T \cong \mathcal{G}_m$

$$W(G, T) = N_G(T)/Z_G(T) \hookrightarrow \text{Aut}(T) = \{\text{Id}, t \mapsto t^{-1}\} \cong \mathbb{Z}/2\mathbb{Z}.$$

Ergo, $\#W(G, T) = 2$. We have seen, that we have in this case

$$\dim(G/B) = 1.$$

Ergo, G/B is a projective one-dimensional variety. Ergo, $G/B \cong \mathbb{P}^1$. Now, define

$$\begin{aligned} \rho : G &\longrightarrow \text{Aut}(G/B) \cong \text{Aut}(\mathbb{P}^1) \cong \text{PGL}_2(k) \\ g &\longmapsto [xB \mapsto gx B]. \end{aligned}$$

Clearly,

$$\text{Kern}\rho = \{g \in G \mid gx B = x B \forall x \in G\} = \bigcap_{x \in G} x B x^{-1} = \bigcap \{B \subset G \text{ Borel}\}.$$

Ergo $(\text{Kern}\rho)^0 = R(G) = 1$, since G is semisimple.

It remains to show, that ρ is surjective. Indeed, we have

$$\dim(\rho(G)) \geq \dim(G) - \dim(\text{Kern}\rho) \geq 3.$$

Since $\text{PGL}_2(k)$ is 3-dimensional and connected, we have

$$\rho(G) = \text{PGL}_2(k). \quad \square$$

Proposition 10. *Let G be a reductive algebraic group of semisimple rank one.*

Let $\rho : G \twoheadrightarrow \text{Aut}(G/B) \cong \text{PGL}_2(k)$ be as in the previous proposition. Then, $\ker \rho$ is diagonalizable.

Proof. Let T be a maximal torus in G . It suffices to show that

$$\text{Kern}\rho \subset T.$$

By the above proof

$$2 = \#W(G, T) = \#\mathcal{B}^T.$$

Therefore,

$$\mathcal{B}^T = \{B^+, B^-\}.$$

Since $\text{Kern}\rho \subset \bigcap_{B \subset G \text{ Borel}} B = B^+ \cap B^-$ it suffices to show

$$B^+ \cap B^- = T.$$

Which is equivalent to

$$B_u^+ \cap B_u^- = 1.$$

Since G has a semisimple rank of one, we have $G \neq B^\pm$. Ergo, B^\pm is not nilpotent (otherwise $G = B^\pm$). Thus, B_u^\pm is connected and non-trivial. Thus,

$$\dim(B_u^\pm) \geq 1.$$

Also,

$$(\text{Kern}\rho)^o \cap B_u^\pm = R(G) \cap B_u^\pm \subset R_u(G) = 1$$

since G is reductive. So, $\text{Kern}(\rho|_{B_u^\pm})$ is finite. And therefore

$$\dim(\rho(B_u^\pm)) \geq 1.$$

But $\rho(B_u^\pm)$ is a unipotent subgroup of $PGL_2(k)$. Therefore

$$\dim(\rho(B_u^\pm)) = 1.$$

Since $\text{Kern}(\rho|_{B_u^\pm})$ is finite, we have

$$\dim B_u^\pm = 1.$$

Ergo, $B_u^\pm \cong \mathcal{G}_a$.

In characteristic zero, we proved that T acts on B_u^\pm by conjugation. We have that the composition

$$T \longrightarrow \text{GL}(\mathcal{G}_a) \cong \mathcal{G}_m$$

is nontrivial, since B^\pm is not nilpotent.

We still want to show

$$B_u^+ \cap B_u^- = 1.$$

Assume, for the sake of contradiction, that we have a non-trivial $x \in B_u^+ \cap B_u^-$. Then, $T.x = \{txt^{-1} \mid t \in T\}$ lies dense in $B_u^+ \cap B_u^-$, in fact

$$T.x \cong \mathcal{G}_a \setminus \{0\}.$$

Since B_u^\pm is one-dimensional, it follows $B_u^+ = B_u^-$. Therefore,

$$B^+ = B^-.$$

This is a contradiction to $\#W(G, T) = 2$. □

Corollary 26 (Eventual Corollary). *Any connected reductive group G of semisimple rank one is isomorphic to one of the following:*

$$\begin{aligned} &SL_2(k) \times T \\ &PGL_2(k) \times T \\ &GL_2(k) \times T \end{aligned}$$

for some torus T .

19.9 Isogenies

Definition 66. A morphism $\phi : G_1 \rightarrow G_2$ of connected algebraic groups is an **isogeny**, if ϕ is surjective and $\text{Kern}\phi$ is finite.

An isogeny is called **multiplicative**, if moreover $\text{Kern}\phi$ is diagonalizable.

Thus, the last result said:

If G is a semisimple connected group of rank one, then there is a multiplicative isogeny $G \twoheadrightarrow \text{PGL}_2(k)$. (Which implies $G \cong \text{PGL}_2(k)$ or $G \cong \text{SL}_2(k)$.)

Definition 67. G is **simply connected**, if every multiplicative isogeny $\tilde{G} \twoheadrightarrow G$ is an isomorphism.

Remark 25 (Facts). • If G is semisimple, then there is a simply connected semisimple G^{sc} and a multiplicative isogeny $G^{sc} \twoheadrightarrow G$ s.t. $G^{sc} \twoheadrightarrow G$ is initial in the category of all multiplicative isogenies $\tilde{G} \twoheadrightarrow G$.

- $\text{SL}_2(k) \twoheadrightarrow \text{PGL}_2(k)$ is simply connected.

20 Root Data

20.1 More on Reductive Groups

Proposition 11. *Let G be a connected, reductive algebraic group. Then,*

$$R(G) = Z(G)^\circ.$$

Proof. $Z(G)^\circ$ is connected, normal and commutative, ergo solvable. Ergo $Z(G)^\circ$ is contained in $R(G)$.

Since $R(G)$ is a normal subgroup of the connected group G , we have

$$G = N_G(R(G)) = N_G(R(G))^\circ.$$

Since $R(G)$ is a torus in a connected group G , we have

$$N_G(R(G))^\circ = Z_G(R(G))^\circ.$$

$G = Z_G(R(G))^\circ$ implies $R(G) \subset Z(G)$. □

Proposition 12. *Let G be a connected, reductive algebraic group. Then,*

$$R(G) \cap [G, G]$$

is finite.

Proof. Take a faithful representation $G \hookrightarrow \mathrm{GL}(V)$. $R(G)$ is a torus in G , therefore we can decompose V into eigenspaces of $R(G)$:

$$V = \bigoplus_{\chi} V_{\chi}$$

where $\chi \in \mathfrak{X}(R(G))$ and

$$V_{\chi} = \{v \in V \mid h.v = \chi(h)v \ \forall h \in R(G)\}.$$

Since $R(G)$ is normal in G , G acts on each V_{χ} . Consider the representations

$$\rho_{\chi} : G \longrightarrow \mathrm{GL}(V_{\chi}).$$

It is easy to see that

$$\rho_{\chi}([G, G]) \subseteq \mathrm{SL}(V_{\chi})$$

and

$$\rho_\chi(R(G)) \subseteq Z_{\mathrm{GL}(V_\chi)}(\mathrm{GL}(V_\chi)) = k^\times.$$

Ergo,

$$\rho_\chi(R(G) \cap [G, G]) \subseteq \mu_{\dim(V_\chi)}.$$

And, therefore,

$$\#([G, G] \cap R(G)) \leq \prod_{\chi: V_\chi \neq 0} \dim(V_\chi). \quad \square$$

Example 32. Let $G = \mathrm{GL}_n(k)$. Then

$$R(G) = k^\times \cdot 1_n.$$

$$[G, G] = \mathrm{SL}_n.$$

Ergo,

$$R(G) \cap [G, G] \cong \mu_n.$$

Proposition 13. *Let G be a connected, reductive algebraic group. Then, $[G, G]$ is semisimple, i.e. $R([G, G]) = 1$.*

Proof. If B' is Borel group in $[G, G]$, then gBg^{-1} stays a Borel group in $[G, G]$ for each $g \in G$. Therefore, $R([G, G])$ is normal in G .

Now, take a Borel subgroup B of G s.t.

$$R([G, G]) \subset B.$$

Then, we have

$$R([G, G]) \subset \bigcap_{g \in G} gBg^{-1} = R(G).$$

So,

$$R([G, G]) \subset R(G) \cap [G, G].$$

Since $R([G, G])$ is finite and connected, it is trivial. Ergo, $[G, G]$ is semisimple. \square

Proposition 14. *Let G be a connected, reductive algebraic group.*

Let $S \subset G$ be a torus. Then, $Z_G(S)$ is reductive.

If T is a maximal torus, then $Z_G(T) = T$.

Proof. Since S is a torus, $Z_G(S)$ is connected. Note that:

- (i) every Borel subgroup of $Z_G(S)$ is contained in some Borel subgroup of G .

(ii) for each Borel $B \subset G$ which contains S , the intersection

$$Z_B(S) = Z_G(S) \cap B$$

is a Borel subgroup of $Z_G(S)$.

(iii) From the above, it follows

$$R(Z_G(S)) = \bigcap_{S \subset B \subset G \text{ Borel}} Z_B(S) \subset \bigcap_{S \subset B \subset G \text{ Borel}} B.$$

(iv) Since $R_u(G)$ is connected, we have

$$R_u(G) = \left(\bigcap_{B \subset G \text{ Borel}} B \right)_u^o.$$

(v) One can show for any maximal torus T

$$R_u(G) = \left(\bigcap_{T \subset B \subset G \text{ Borel}} B \right)_u^o.$$

Now, it follows

$$R_u(Z_G(S)) \subset \left(\bigcap_{S \subset B \subset G \text{ Borel}} B \right)_u^o \subset R_u(G) = 1.$$

For the second part: T is central in $Z_G(T)$ and a maximal torus in $Z_G(T)$. Therefore, $Z_G(T)$ must be nilpotent. Now, $W(Z_G(T), T) = 1$, ergo $Z_G(T)$ only has one Borel subgroup $R(Z_G(T))$. But $R(Z_G(T))$ must be a torus, hence

$$R(Z_G(T)) = T. \quad \square$$

20.2 Root Data – Definition

Definition 68. A **root datum** is a tuple

$$\Psi = (X, X^\vee, R, R^\vee)$$

where X, X^\vee are finitely generated free \mathbb{Z} -modules equipped with a pairing

$$\begin{aligned} X \times X^\vee &\longrightarrow \mathbb{Z} \\ (x, \xi) &\longmapsto \langle x \mid \xi \rangle \end{aligned}$$

which is **perfect**, i.e. $\langle \cdot \mid \cdot \rangle$ induces isomorphism $X \cong \operatorname{Hom}_{\mathbb{Z}}(X^\vee, \mathbb{Z})$ and $X^\vee \cong \operatorname{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$.

R and R^\vee are to be finite subsets $R \subset X, R^\vee \subset X^\vee$ with a bijective map

$$\begin{aligned} R &\longrightarrow R^\vee \\ \alpha &\longmapsto \alpha^\vee. \end{aligned}$$

The (X, X^\vee, R, R^\vee) shall meet the following axioms:

(i) For each $\alpha \in R$, we have

$$\langle \alpha \mid \alpha^\vee \rangle = 2.$$

(ii) For each $\alpha \in R$, define the maps

$$\begin{aligned} S_\alpha : X &\longrightarrow X \\ x &\longmapsto x - \langle x \mid \alpha^\vee \rangle \alpha \\ S_{\alpha^\vee} : X^\vee &\longrightarrow X^\vee \\ \xi &\longmapsto \xi - \langle \alpha \mid \xi \rangle \alpha^\vee. \end{aligned}$$

These maps satisfy for each $\alpha \in R$

$$S_\alpha(R) \subseteq R \text{ and } S_{\alpha^\vee}(R^\vee) \subset R^\vee.$$

(iii) We call Ψ **reduced**, if additionally the following is fulfilled:

If $\alpha, c\alpha \in R$, for some $c \in \mathbb{Q}$, then $c = \pm 1$.

Elements $\alpha \in R$ are called **roots**, while corresponding elements $\alpha^\vee \in R^\vee$ are called **coroots**.

Remark 26. Let $\Psi = (X, X^\vee, R, R^\vee)$ be a root datum:

1. For $\alpha \in R$, we have

$$S_\alpha(\alpha) = -\alpha.$$

2. In particular, we have for $\alpha \in R$

$$S_\alpha^2 = \text{Id}_R.$$

3. If $\Psi = (X, X^\vee, R, R^\vee)$ is a root datum, then so is $\Psi^\vee = (X^\vee, X, R^\vee, R)$.

4. If we have for $\alpha, \beta \in R$

$$\langle _ \mid \alpha^\vee \rangle \equiv \langle _ \mid \beta^\vee \rangle,$$

then $\alpha = \beta$.

Lemma 80. *In the definition of a root datum, it would suffice to demand that*

$$\begin{aligned} R &\longrightarrow R^\vee \\ \alpha &\longmapsto \alpha^\vee \end{aligned}$$

is only surjective.

Proof. Let $\alpha, \beta \in R$ s.t.

$$\alpha^\vee = \beta^\vee.$$

If α, β are linearly dependent, they must be equal, because of

$$\langle \alpha \mid \alpha^\vee \rangle = 2 = \langle \beta \mid \alpha^\vee \rangle.$$

Assume, therefore, that they are linearly independent. Let V be the \mathbb{Z} -module spanned by the basis (α, β) . Regarding this basis, the action of S_α and S_β can be represented by the following matrices:

$$\begin{aligned} S_\alpha &\hat{=} \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix} \\ S_\beta &\hat{=} \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix}. \end{aligned}$$

It is now easy to show that

$$\begin{aligned} (S_\alpha \circ S_\beta)^2 &\hat{=} \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}, \\ (S_\alpha \circ S_\beta)^n &\hat{=} \begin{pmatrix} 2n+1 & 2n \\ -2n & 1-2n \end{pmatrix}. \end{aligned}$$

But R must be closed under the action of S_α and S_β . Since R must be finite, α, β cannot be linearly independent. \square

Definition 69. The **Weyl group** $W(\Psi)$ of a root datum Ψ is the subgroup of $\text{Aut}_\mathbb{Z}(X) \cong \text{GL}_n(\mathbb{Z})$ which is generated by

$$\{S_\alpha \mid \alpha \in R\}.$$

Our goal is to construct for each connected reductive group G with a maximal torus T a root datum

$$\Phi(G, T) = \Phi(G)$$

s.t. the Weyl groups $W(G, T)$ and $W(\Phi(G))$ are isomorphic (in a canonical way).

Theorem 38 (Facts). *Suppose, we were given a notion of morphism of root data at this point.*

1. $\Phi(G) \cong \Phi(G')$ iff $G \cong G'$.
2. Every root datum is isomorphic to some $\Phi(G)$ for a reductive connected group G .

Remark 27. • Obviously, the notion of root data is independent of k .

- Root data also classify compact connected Lie groups.
- Root data refine the less precise notion of root systems (which classify semisimple Lie algebras and simply connected semisimple algebraic groups).
- Every root system is a finite direct sum of the simple root systems

$$(A_n)_{n \geq 1}, (B_n)_{n \geq 2}, (C_n), (D_n), E_6, E_7, E_8, F_4, G_2.$$

20.3 Lie Algebras

Lemma 81 (Fact). *Let G be an algebraic group. Take a faithful representation*

$$G \hookrightarrow \mathrm{GL}(V).$$

Set

$$I := I(G) \subset \mathcal{O}(\mathrm{GL}(V)).$$

*Consider the nilpotent element ε in $\mathcal{O}(\mathrm{GL}(V))[\varepsilon]/(\varepsilon^2)$ and define the **Lie algebra** of G by*

$$\begin{aligned} \mathfrak{g} := \mathrm{Lie}(G) &:= \{x \in \mathrm{End}(V) \mid \forall f \in I : f(1 + \varepsilon x) = 0 \pmod{(\varepsilon^2)}\} \\ &= \{x \in \mathrm{End}(V) \mid \forall f \in I : f(1 + \varepsilon x) \in (\varepsilon^2)\} \\ &= \left\{ x \in \mathrm{End}(V) \mid \forall f \in I : \frac{d}{dt} \left(\frac{f(1 + tx)}{t} \right) \Big|_{t=0} = 0. \right\}. \end{aligned}$$

Then, \mathfrak{g} is a k -vector space of dimension

$$\dim_k(\mathfrak{g}) = \dim(G).$$

Idea of Proof. The proof boils down to show that G is smooth at 1.

However, each variety is generically smooth. Therefore, G is smooth at some points. Since G acts on itself via isomorphisms, 1 must look locally identically to one of G 's smooth points. Ergo, G is smooth everywhere. \square

Example 33. • $\mathrm{Lie}(\mathrm{GL}(V)) = \mathrm{End}(V)$ because $I(G) = 0$.

•

$$\begin{aligned} \mathrm{Lie}(\mathcal{O}_n(k)) &= \{x \in M_n(k) \mid (1 + \varepsilon x)(1 + \varepsilon x^T) = 1 + (\varepsilon^2)\} \\ &= \{x \in M_n(k) \mid x^T = -x\}. \end{aligned}$$

Definition 70. To each algebraic group we can attach the **adjoint representation**

$$\begin{aligned} \mathrm{Ad} : G &\longrightarrow \mathrm{GL}(\mathfrak{g}) \\ g &\longmapsto [x \mapsto gxg^{-1}]. \end{aligned}$$

If T is a maximal torus in G , we can restrict the adjoint representation

$$\mathrm{Ad} : T \longrightarrow \mathrm{GL}(\mathfrak{g}).$$

Given any representation $\rho : T \rightarrow \mathrm{GL}(V)$, we may decompose V

$$V = \bigoplus_{\chi \in \mathfrak{X}(T)} V^\chi$$

where

$$V^\chi = \{v \in V \mid \forall t \in T : t.v = \chi(t)v\}.$$

Therefore,

$$\mathfrak{g} = \bigoplus_{\chi \in \mathfrak{X}(T)} \mathfrak{g}^\chi = \mathfrak{g}^o \oplus \bigoplus_{0 \neq \alpha \in \mathfrak{X}(T)} \mathfrak{g}^\alpha$$

where

$$\mathfrak{g}^\alpha = \{x \in \mathfrak{g} \mid \forall t \in T : txt^{-1} = \alpha(t)x\}$$

and

$$\mathfrak{g}^o = \{x \in \mathfrak{g} \mid \forall t \in T : txt^{-1} = x\}.$$

Example 34. Let $G \subset \mathrm{GL}_n(k)$ with the torus $T = \mathcal{G}_m^n$. Then,

$$\mathfrak{g} = M_n(k) = \mathfrak{g}^o \oplus \left(\bigoplus_{0 \neq \alpha} \mathfrak{g}^\alpha \right)$$

where

$$\mathfrak{g}^o = \left\{ \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \right\}.$$

For $i = 1, \dots, n$ we define $\chi_i \in \mathfrak{X}(T)$ as follows:

$$\chi_i \left(\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \right) = t_i.$$

Then, we have for each $i \neq j$

$$\mathfrak{g}^{\chi_i/\chi_j} = kE_{i,j}$$

and for each other $\chi \in \mathfrak{X}(T)$

$$\mathfrak{g}^\chi = 0.$$

Therefore

$$\mathfrak{g} = \mathfrak{t} \oplus \left(\bigoplus_{i \neq j} kE_{i,j} \right)$$

where

$$\mathfrak{t} = \mathrm{Lie}(T) = \mathfrak{g}^o.$$

20.4 Root Data – Construction

Let G be a connected reductive group with a torus T . We want to construct a corresponding root datum

$$\Psi = (X, X^\vee, R, R^\vee).$$

- Set X to be the **character lattice**

$$X = \mathfrak{X}(T) = \operatorname{Hom}(T, \mathcal{G}_m).$$

- Set X^\vee to be the **cocharacter lattice**

$$X^\vee = \operatorname{Hom}(\mathcal{G}_m, T).$$

- If we are given $x \in X$ and $\xi \in X^\vee$, we define

$$\langle x \mid \xi \rangle := m \in \mathbb{Z}$$

s.t.

$$\begin{aligned} x \circ \xi : \mathcal{G}_m &\longrightarrow \mathcal{G}_m \\ t &\longmapsto t^m. \end{aligned}$$

(Recall, $\mathfrak{X}(\mathcal{G}_m) = \mathbb{Z}$.)

•

$$R := \{0 \neq \alpha \in \mathfrak{X}(T) \mid \mathfrak{g}^\alpha \neq 0\}.$$

- Let $g \in \mathcal{G}_m$ be s.t. $\mathcal{G}_m = \overline{\langle g \rangle}$. For $x \in R$, we need to choose $t \in T$ s.t.

$$x(t) = g^2$$

and, if we set

$$x^\vee = [g^n \mapsto t^n]$$

the other axioms of a root datum are fulfilled.

Example 35. • $G = \operatorname{SL}_2(k)$:

In G we have the maximal torus

$$T = \left\{ \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \right\}.$$

Consider the character

$$\begin{aligned}\lambda : T &\longrightarrow \mathcal{G}_m \\ \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} &\longmapsto t.\end{aligned}$$

Set

$$R = \{\alpha, -\alpha\}$$

with

$$\begin{aligned}\alpha = 2\lambda : \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} &\longmapsto t^2 \\ -\alpha : \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} &\longmapsto t^{-2}.\end{aligned}$$

Then, we have

$$\begin{aligned}\mathfrak{g}^\alpha &= \left\{ \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \right\} \\ \mathfrak{g}^{-\alpha} &= \left\{ \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix} \right\},\end{aligned}$$

since

$$\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} 0 & t^2 \\ 0 & 0 \end{pmatrix}.$$

Define

$$\begin{aligned}\alpha^\vee &:= \left[t \mapsto \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \right] \\ (-\alpha)^\vee &:= -\alpha^\vee \left[t \mapsto \begin{pmatrix} t^{-1} & \\ & t \end{pmatrix} \right].\end{aligned}$$

Then, we have

$$\langle \alpha \mid \alpha^\vee \rangle = 2.$$

- $G = \mathrm{PGL}_2(k)$:

We have the maximal torus

$$T = \left\{ \begin{pmatrix} t & \\ & 1 \end{pmatrix} \right\}.$$

Set

$$R = \{\alpha, -\alpha\}$$

with

$$\alpha \begin{pmatrix} t & \\ & 1 \end{pmatrix} := t.$$

Define α^\vee by

$$\alpha^\vee(t) := \begin{pmatrix} t^2 & \\ & 1 \end{pmatrix} = \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}.$$

We have

$$\begin{pmatrix} t & \\ & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} t & \\ & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}.$$

Now, let G be reductive of semisimple rank one. Then:

- $G/R(G) \cong PGL_2(k)$ with $R(G) = 1$ or $R(G) = \mu_n$.
- $[G, G] \cong SL_2(k)$ is semisimple of semisimple rank 1.
- $\{\text{roots for } G/R(G)\} = \{\text{roots for } G\}$.
- If α is a root of G , then it is also a root of $[G, G]$. Hence, define

$$\alpha^\vee : \mathcal{G}_m \rightarrow [G, G] \hookrightarrow G.$$

Let G be reductive. Take a root $\alpha \in \mathfrak{X}(T)$ and define the torus of codimension 1 in T

$$S_\alpha := \text{Kern}(\alpha)^o.$$

Then,

$$G_\alpha := Z_G(S_\alpha)$$

is connected and reductive. Then,

$$S_\alpha \subset Z_G(G_\alpha)$$

and

$$S_\alpha \subset R(G_\alpha).$$

Ergo, T/S_α is a maximal torus of rank 1 in $G_\alpha/R(G_\alpha)$, while $G_\alpha/R(G_\alpha)$ is semisimple.

Ergo, G_α is reductive of semisimple rank 1.

Example 36. Let $G = \mathrm{GL}_3$ with the maximal torus $T = \left\{ \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix} \right\}$.

Let $\alpha = \chi_1/\chi_2$. Then,

$$S_\alpha = \left\{ \begin{pmatrix} x & & \\ & x & \\ & & y \end{pmatrix} \right\}.$$

Index

- g -invariant subspace, 25
- (irreducible) component, 17
- (left) submodule, 34
- G -irreducible, 36
- action, 82
- acts, 34
- adjoint representation, 139
- affine variety, 73
- algebraic, 4
- algebraic group, 6
- basic open sets, 16
- birational, 88
- Borel, 91
- center, 64
- central, 64
- character, 43
- character lattice, 141
- closed, 78
- cocharacter lattice, 141
- commutator, 39
- comodule, 11
- complete flag, 40
- connected, 15
- coordinate ring, 7
- coroots, 136
- curve, 128
- diagonalizable, 43
- dimension, 76
- division ring, 35
- dominant, 77, 79
- equivariant, 11
- evaluation, 73
- faithful, 32
- flag manifold, 72
- function field, 73
- generalized λ -eigenspace, 23
- generator, 101
- geometrically reductive, 97
- Grassmannian manifold, 72
- group algebra, 46
- homogenous, 69
- Hopf-algebra, 8
- idempotent, 18
- initial, 87
- invariant, 12
- irreducibel, 15
- irreducible, 16
- irreducible component, 77
- irreducibly, 36
- isogeny, 132
- isotopic, 93
- left ideal, 34
- left module, 34
- Lie algebra, 139
- linear algebraic group, 4
- linearly reductive, 98
- local ring, 73, 74
- locally closed, 107
- maximal torus, 109
- minimal prime, 17
- morphism, 6, 75, 86
- morphism of algebraic groups, 6

multiplicative, 132
 nilpotent, 21, 39
 normalizer, 117
 parabolic, 91
 perfect, 136
 Plücker-embedding, 72
 projective, 68, 91
 projective space, 68
 proper, 93
 quasi-projective, 70
 quasi-projective variety, 70
 quotient, 87, 97
 radical, 6, 95
 rank, 127
 reduced, 6, 136
 reducible, 98
 reductive, 95
 representation, 11
 right ideal, 34
 right regular representation, 12
 ring, 34
 ring of regular functions, 75
 root datum, 136
 roots, 136
 Segre-embedding, 70
 semisimple, 21, 33, 95
 semisimple rank, 127
 simple, 35
 simply connected, 132
 simply-transitively, 125
 skew field, 35
 solvable, 39
 subrepresentation, 12
 symplectic, 93
 tensor, 26
 torus, 43
 transitive, 86
 trigonalizable, 52, 54
 unipotent, 21, 33, 39
 unipotent radical, 95
 vanishing set, 4
 variety of complete flags, 72
 Weyl group, 117, 138
 Zariski topology, 70
 Zariski-Topology, 16