

Mitschrieb: Algebraic Groups
SS 20

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Vorwort

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Lecture
from
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Recall: Last time we introduced the **Zariski-Topology** on X .

There, algebraic sets equal closed sets.

We called a set X **irreducible** iff each open subset lies dense in X .

Lemma 1. *For an algebraic set X , the following are equivalent:*

- (1) X is irreducible.
- (2) $k[X] = k[x_1, \dots, x_n]/I(X)$ is a domain.
- (3) $I(X)$ is a prime ideal.

The proof of (2) \iff (3) is a basic algebraic result.

Lemma 2. *An open base for the Zariski-Topology on an algebraic set X is given by sets:*

$$D(f) := \{p \in X \mid f(p) \neq 0\}$$

for each $f \in k[X]$. We call the $D(f)$ **basic open sets**.

Proof. Suppose $U \subseteq X$ is nonempty and open. Set

$$Z := X \setminus U$$

then Z is closed. Thus

$$Z = \{x \in X \mid f(x) = 0 \forall f \in I\}$$

for some ideal $I \subseteq k[X]$. Let $p \in U$, then there is an $f \in Z$ s.t.

$$f(p) \neq 0.$$

Also, $D(f) \cap Z = \emptyset$, thus $p \in D(f) \subseteq U$. □

Lemma 1. It is clear that (2) is equivalent to (3).

(1) is equivalent to

$$\forall \text{ nonempty, open } U_1, U_2 \subset X : U_1 \cap U_2 \neq \emptyset$$

$$\stackrel{\text{Lemma 2}}{\iff} \forall \text{ nonempty, basic open } D(f_1), D(f_2) \subset X : D(f_1) \cap D(f_2) \neq \emptyset$$

Since $D(f_1) \cap D(f_2) = D(f_1 f_2)$, this is equivalent to the statement

$$f_1, f_2 \in k[X] : f_1, f_2 \neq 0 \implies f_1 f_2 \neq 0.$$

Which states that $k[X]$ is a domain. □

Lemma 3. *Let X be an algebraic set. We have bijections*

$$\{\text{closed subsets } Z \subseteq X\} \leftrightarrow \{\text{radical ideals } I \subset k[X]\}$$

and

$$\{\text{irreducible, closed subsets } Z \subseteq X\} \leftrightarrow \{\text{prime ideals } I \subset k[X]\}$$

and

$$\{\text{points of } X\} \leftrightarrow \{\text{maximum ideals } I \subset k[X]\}.$$

Lemma 4 (Primary Decompositions, Atiyah, Macdonald Ch. 4). *For an ideal I we call $P \supseteq I$ a **minimal prime** of I if P is a prime ideal and we have for each prime ideal Q :*

$$P \supseteq Q \supseteq I \implies P = Q.$$

*Any radical ideal I of $k[x_1, \dots, x_n]$ has only finitely many **minimal** primes P_1, \dots, P_r . In particular,*

$$I = \bigcap_{i=1}^n P_i$$

and for each i

$$P_i \not\supseteq \bigcap_{j:j \neq i} P_j.$$

Definition 1. An **(irreducible) component** Z of X is a maximal irreducible closed subset, i.e., an irreducible closed $Z \subseteq X$ s.t. there does not exist an irreducible closed $Y \subset X$ s.t. $Y \supsetneq Z$.

Then, we have the bijection

$$\{\text{irreducible components of } X\} \leftrightarrow \{\text{minimal primes of } I(X)\}.$$

Lemma 5. *Any algebraic set X has finitely many components Z_1, \dots, Z_r . We have*

$$X = Z_1 \cup \dots \cup Z_r$$

and for each i

$$Z_i \not\subset \bigcup_{j:j \neq i} Z_j.$$

Example 1. 1. Let $X = V(x \cdot y) \subset k^2$. Then $X = Z_1 \cup Z_2$ where $Z_1 = V(x)$, $Z_2 = V(y)$.

X is connected, but not irreducible ($D(x)$ does not lie dense in X).

2. Let X be a **finite** algebraic set. It is easy to check that every subset of X is closed:

$$\{p\} = V(x_1 - p_1, \dots, x_n - p_n)$$

for each $p \in X$. Further

$$X = \{p_1\} \cup \dots \cup \{p_r\}.$$

Moreover: Any function $f : X \rightarrow k$ is regular (i.e. given by polynomials).

Lemma 6. *We call an element $e \in k[X]$ **idempotent** iff $e^2 = e$.*

Let X be an algebraic set. Then

$$\begin{aligned} X \text{ connected} &\iff \text{the only idempotents } e \in k[X] \text{ are } 0 \text{ and } 1 \\ &\iff k[X] \not\cong A \times B \text{ for any } k\text{-algebras } A, B. \end{aligned}$$

Lemma 7. *Morphisms of algebraic sets are continuous.*

Proof. Let $\phi : X \rightarrow Y$ be a morphism. It suffices to show that for all closed $Z \subset Y$ that $\phi^{-1}(Z) \subset X$ is closed.

But, if

$$Z = V_Y(S) := \{q \in Y \mid f(q) = 0 \forall f \in S\}$$

for some ideal $S \subset k[Y]$, then

$$\phi^{-1}(Z) = V_X(\phi^*(S)) = \{\phi^*(f) = f \circ \phi \mid f \in S\}.$$

□

Lemma 8. *Isomorphisms of algebraic sets are homeomorphisms. In particular, any isomorphism of algebraic sets $\phi : X \rightarrow X$ permutes the components Z_1, \dots, Z_r of X :*

$$\forall i \exists j : \phi(Z_i) = Z_j.$$

Theorem 1. *Let G be an algebraic group.*

- (i) *There is a unique component G^0 of G with $e \in G^0$.*
- (ii) *Every component Z of G is a coset gG^0 of G for some $g \in Z$.*
- (iii) *G^0 is a normal algebraic subgroup of G .*
- (iv) *G^0 is of finite index, i.e.*

$$[G : G^0] = \#(G/G^0) < \infty.$$

(v) *The irreducible components are also the connected components.*

Proof. Let $G = Z_1 \cup \dots \cup Z_r$ be the decomposition into components. We may assume that $e \in Z_1$.

Recall that $Z_1 \not\subset \bigcup_{j \geq 2} Z_j$. Then, there is an $x \in Z_1 \setminus \bigcup_{j \geq 2} Z_j$. Thus, for all algebraic set isomorphisms $\phi : G \rightarrow G$, we have by some previous lemma that $\phi(x)$ is likewise contained in some unique component of G . For example, we may take ϕ to be

$$\begin{aligned} \phi_g : G &\rightarrow G \\ y &\longmapsto gy \end{aligned}$$

for any $g \in G$. Then, for all $g \in G$, the element $gx = \phi_g(x)$ is contained in only one component of G . Ergo, each $g \in G$ is contained in exactly one component.

- (i) Take $g = e$.
- (iii) G^0 is an algebraic subset, by construction. Denote by $m : G \times G \rightarrow G$ and $i : G \rightarrow G$ the continuous multiplication and inversion map on G . **Why is G^0 a subgroup?** We need to show

$$\begin{aligned} m(G^0 \times G^0) &\subseteq G^0. \\ i(G^0) &\subseteq G^0. \end{aligned}$$

We know that $i(G^0)$ is some component of G , since i is an isomorphism. But it contains the identity e , since $e^{-1} = e$. Therefore, $i(G^0) = G^0$.

If $g \in G$, then gG^0 is some component of G . Suppose $g \in G^0$. Then $gG^0 \cap G^0 \supseteq \{g\}$, therefore $gG^0 = G^0$. Ergo, G^0 is closed under multiplication.

Why is G^0 a normal? If $g \in G$, then gG^0g^{-1} is a component that contains e , therefore $G^0 = gG^0g^{-1}$.

(Alternative proof that $m(G^0 \times G^0) = G^0$: Consider

- any continuous image of an irreducible set is irreducible.
- the closure of any irreducible set is irreducible.

Ergo $\overline{m(G^0 \times G^0)}$ is a closed irreducible set containing e . Ergo, $\overline{m(G^0 \times G^0)} = G^0$.

- (ii) Let $Z \subset G$ be a component. Let $g \in Z$. Then $g \in (gG^0 \cap Z)$, so $gG^0 = Z$.

- (iv) This follows from some previous lemma.
- (v) This is left as a topological exercise. It is true whenever the irreducible components do not intersect.

□

It now follows:

$$\{\text{finite algebraic groups}\} \longleftrightarrow \{\text{finite groups}\}$$

where the above arrow is an equivalence of categories.

Example 2. • Let $G = \{g_1, \dots, g_r\}$ be a finite algebraic group. Then,

$$G^0 = \{e\}.$$

- Without proofs:

$$G \in \{\mathrm{GL}_n(k), \mathrm{SO}_n(k), \mathrm{SL}_n(k)\} \implies G^0 = G.$$

Further,

$$G = \mathrm{O}_n(k) \implies G^0 = \mathrm{SO}_n(k)$$

(but only if $-1 = 1$ i.e. $\mathrm{char} k = 2$. Otherwise $[G : G^0] = 2$.)

0.1 Jordan Decomposition

As usual, $k = \bar{k}$ is an algebraically closed field.

Definition 2. Let V be a finite-dimensional vector space.

An element $x \in \mathrm{End}(V)$ is **semisimple**, if it is diagonalizable, i.e. it has a basis of eigenvectors, or equivalently, if the minimal polynomial of x is square-free.

Then, there is a decomposition $V = \bigoplus_{i=1}^r V_i$ and distinct elements $\lambda_1, \dots, \lambda_n \in k$ s.t.

$$x|_{V_i} = \lambda_i.$$

If $\dim(V_i) = n_i$, then

$$\text{char polynomial of } x = \prod_{i=1}^r (T_i - \lambda_i)^{n_i} \in k[T]$$

and

$$\text{minimal polynomial of } x = \prod_{i=1}^r (T_i - \lambda_i) \in k[T].$$

(Where the minimal polynomial of x is defined as the least degree monic m s.t. $m(x) = 0$ and Cayley-Hamilton $m|c$.)

Definition 3. $x \in \text{End}(V)$ is **nilpotent** if $x^n = 0$ for some n . (Equivalent to: characteristic polynomial of x is $T^{\dim(V)}$.)

x is **unipotent**, if $x - 1$ is nilpotent.

Lemma 9. *If x is semisimple and nilpotent, then $x = 0$.*

If x is semisimple and unipotent, then $x = 1$.

Lemma 10. *If x, y are commuting elements, then x is semisimple resp. unipotent or nilpotent, then so is xy .*

Theorem 2 (Goal). *For all algebraic groups G and for all $g \in G$, there exist unique group elements $g_s, g_u \in G$ s.t.*

$$g = g_s g_u = g_u g_s$$

and for all finite-dimensional representations $\rho : G \rightarrow \text{GL}(V)$, $\rho(g_s)$ is semisimple and $\rho(g_u)$ is unipotent.

Example 3. If $g = \begin{pmatrix} \lambda & 1 & 0 \\ & \lambda & 1 \\ & & \lambda \end{pmatrix} \in G = \text{GL}_3(k)$, then $g_s = \begin{pmatrix} \lambda & 0 & 0 \\ & \lambda & 0 \\ & & \lambda \end{pmatrix}$, $g_u = \begin{pmatrix} 1 & \lambda^{-1} & 0 \\ & 1 & \lambda^{-1} \\ & & 1 \end{pmatrix}$.

Lecture

from

09.03.2020

Theorem 3 (Goal Theorem). *Let G an algebraic group. For all $g \in G$ there is exactly one pair $g_s, g_u \in G$ s.t.*

$$g = g_s g_u = g_u g_s$$

and for all finite-dimensional representations $r : G \rightarrow GL_n(V)$, the element $r(g_s)$ resp. $r(g_u)$ is semisimple resp. unipotent.

Last time, we saw:

Lemma 11. • *If g, h are commuting and semisimple resp. commuting and unipotent then so is gh .*

• *If g is semisimple and unipotent, then $g = 1$.*

Proposition 1. *Let V be a finite-dimensional vector space and $g \in GL(V)$. There exist unique elements $g_s, g_u \in GL(V)$ s.t.*

$$g = g_s g_u = g_u g_s$$

and g_s is semisimple and g_u is unipotent.

Moreover, $g_s, g_u \in k[g] = \{\sum_{i=1}^m a_i g^i \mid a_i \in k\} \subseteq \text{End}(V)$.

Proof. Existence (Sketch): Say

$$g = \begin{pmatrix} \lambda & 1 & 0 \\ & \lambda & 1 \\ & & \lambda \end{pmatrix}$$

then take

$$g_s = \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{pmatrix}, \quad g_u = \begin{pmatrix} 1 & \lambda^{-1} & 0 \\ & 1 & \lambda^{-1} \\ & & 1 \end{pmatrix}.$$

For $\lambda \in k$, define the **generalized λ -eigenspace** of g by

$$V_\lambda := \{v \in V \mid \exists n \in \mathbb{N}_0 : (g - \lambda)^n v = 0\}.$$

Then

$$V = \bigoplus_{\lambda \in k} V_\lambda.$$

Here V_λ = sum of domains of all Jordan blocks with λ s on the diagonal. (It follows from the Jordan Decomposition for matrices that such a decomposition exist.)

Let's define $g_s \in \text{GL}(V)$ by

$$g_s|_{V_\lambda} = \lambda \cdot \text{Id}.$$

Note that $gV_\lambda \subset V_\lambda$, hence g commutes with g_s , hence g, g_s commutes with $g_u := gg_s^{-1}$. Then, $g = g_s g_u = g_u g_s$.

Write $\det(T - g) = \prod_\lambda (T - \lambda)^{n(\lambda)}$, $n(\lambda) = \dim(V_\lambda)$. Since the polynomials $T - \lambda$ for $\lambda \in k$ are coprime, the chinese remainder theorem implies that there is a $Q \in k[T]$ s.t.

$$Q \equiv \lambda \pmod{(T - \lambda)^{n(\lambda)}}$$

for each $\lambda \in k$.

We claim that

$$Q(g) = g_s.$$

Indeed, since $gV_\lambda \subseteq V_\lambda$, we have

$$Q(g)V_\lambda \subseteq V_\lambda.$$

So, it suffices to show for all $v \in V_\lambda$

$$Q(g)v = g_s v = \lambda v.$$

Note that, by Cayley-Hamilton,

$$V_\lambda = \{v \in V \mid (g - \lambda)^{n(\lambda)} v = 0.\}$$

Write

$$Q = \lambda + R \cdot (T - \lambda)^{n(\lambda)}$$

for some $R \in k[T]$. Since $(g - \lambda)^{n(\lambda)} v = 0$, deduce that $Q(g)v = \lambda v$, as required.

If $Q' \equiv T(T - \lambda)^{n(\lambda)}$, then

$$Q'(g) = g_u.$$

If $Q'' \equiv \lambda^{-1}(T - \lambda)^{n(\lambda)}$, then $Q''(g) = g_s^{-1}$. check corresponding stuff for g_u .

Uniqueness: Suppose given some other decomposition

$$g = g'_s g'_u = g'_u g'_s$$

with g'_s semisimple and g'_u unipotent. Then g'_s commutes with g'_s and g'_u , hence with g , hence also with any element in $k[g]$. Ergo, g'_s commutes with g_s and g_u . Similarly, g'_u commutes with g_s and g_u .

Consider

$$h := g'_s g_s^{-1} = g'_s g'_u (g'_u)^{-1} g_s^{-1} = g(g'_u)^{-1} g_s^{-1} = g_u (g'_u)^{-1}.$$

Then $h = g'_s g_s^{-1}$ is a product of semisimple elements and $h = g_u (g'_u)^{-1}$ is a product of unipotent elements. By proceeding lemmas, h is semisimple and unipotent, ergo trivial. It follows $g'_s = g_s$ and $g'_u = g_u$. \square

Corollary 1. *Let $g \in GL(V)$, let $W \subset V$ be any g -invariant subspace, i.e. $gW \subseteq W$.*

Then, W is g_s -invariant and g_u -invariant.

Proof. This is clear, since g_s and g_u are algebraically generated by g over g . \square

Lemma 12. *Let $\phi : V \rightarrow W$ be a linear map between finite-dimensional vector spaces.*

Let $\alpha \in GL(W)$ and $\beta \in GL(W)$ s.t.

$$\begin{array}{ccc} V & \xrightarrow{\alpha} & V \\ \downarrow \phi & & \downarrow \phi \\ W & \xrightarrow{\beta} & W, \end{array}$$

i.e. $\phi \circ \alpha = \beta \circ \phi$.

Then,

$$\begin{aligned} \phi \circ \alpha_s &= \beta_s \circ \phi, \\ \phi \circ \alpha_u &= \beta_u \circ \phi. \end{aligned}$$

Proof. Write $V = \bigoplus_{\lambda \in k} V_\lambda$, $W = \bigoplus_{\lambda \in k} W_\lambda$ where V_λ are the generalized α -eigenspaces and W_λ are the generalized β -eigenspaces.

We claim that

$$\phi(V_\lambda) \subset W_\lambda.$$

Indeed, let $v \in V_\lambda$, then

$$(\beta - \lambda)^n \phi(v) = \phi((\alpha - \lambda)^n v) = 0.$$

Since $(\alpha - \lambda)^n v = 0$, the claim follows.

Since, $\alpha_s|_{V_\lambda} = \lambda \text{Id}$ and $\beta_s|_{W_\lambda} = \lambda \text{Id}$, deduce that

$$\phi \circ \alpha_s = \beta_s \circ \phi.$$

Indeed, both sides are given on V_λ by $\lambda \cdot \phi$. Thus

$$\begin{aligned}\phi \circ \alpha_u &= \phi \circ \alpha \alpha_s^{-1} \\ &= \beta \beta_s^{-1} \circ \phi \\ &= \beta_u \circ \phi.\end{aligned}$$

□

Lemma 13. *Let $\alpha \in GL(V)$, $\beta \in GL(W)$. Then the **tensor** $\alpha \otimes \beta \in GL(V \otimes W)$ is defined by*

$$(\alpha \otimes \beta)(u \otimes v) = \alpha(u) \otimes \beta(v).$$

Then, we have

$$\begin{aligned}(\alpha \otimes \beta)_s &\stackrel{(1)}{=} \alpha_s \otimes \beta_s \\ (\alpha \otimes \beta)_u &\stackrel{(2)}{=} \alpha_u \otimes \beta_u.\end{aligned}$$

Proof. It suffices to prove (1), since

$$\begin{aligned}(\alpha \otimes \beta)_u &= (\alpha \otimes \beta) \circ (\alpha \otimes \beta)_s^{-1} \\ &\stackrel{(1)}{=} (\alpha \otimes \beta) \circ (\alpha_s \otimes \beta_s)^{-1} \\ &= \alpha \alpha_s^{-1} \otimes \beta \beta_s^{-1} \\ &= \alpha_u^{-1} \otimes \beta_u^{-1}\end{aligned}$$

For (1), consider

$$\begin{aligned}V &= \bigoplus_{\lambda \in k} V_\lambda, \\ W &= \bigoplus_{\lambda \in k} W_\lambda.\end{aligned}$$

It follows

$$V \otimes W = \bigoplus_{\lambda, \mu \in k} V_\lambda \otimes W_\mu.$$

Now,

$$\alpha_s \otimes \beta_s|_{V_\lambda \otimes W_\mu} = \lambda \mu \cdot \text{Id}.$$

Ergo, $\alpha_s \otimes \beta_s$ is semisimple. By Proposition, we reduce to checking that $\alpha_u \otimes \beta_u$ is unipotent. Indeed,

$$\alpha_u \otimes \beta_u - 1 = (\alpha_u - 1) \otimes (\beta_u - 1) + 1 \otimes (\beta_u - 1) + (\alpha_u - 1) \otimes 1$$

is nilpotent (You can also check that $\alpha_u \otimes \beta_u = (\alpha_u \otimes 1) \circ (1 \otimes \beta_u)$ is unipotent.) □

Example 4. Let $1 \in GL(V)$. Then $1_s = 1$ and $1_u = 1$.

Summary : Let G be an algebraic group. Let $r_V : G \rightarrow \mathrm{GL}(V)$ be a finite-dimensional representation. Also, fix $g \in G$.

Let $\lambda_V := r_V(g)_s$ (or $r_V(g)_u$).

We get a family of operators $\lambda_V \in \mathrm{End}(V)$ with the following properties:

- (i) if $V = k$ and $r_V(g') = 1$ for all $g' \in G$, then $\lambda_V = 1$.
- (ii) for any two representations in V and W , we have

$$\lambda_{V \otimes W} = \lambda_V \otimes \lambda_W.$$

- (iii) for all G -equivariant $\phi : V \rightarrow W$ we have

$$\phi \circ \lambda_V = \lambda_W \circ \phi.$$

Theorem 4. *Let G be an algebraic group. Let $\lambda_V \in \mathrm{End}(V)$ (i.e. $V = (r_V, V)$ is a finite-dim. representation of G) be a family of operations satisfying (i), (ii), (iii).*

Then, there is exactly one $g \in G$ s.t. $\lambda_V = r_V(g)$ for all V .

Note, that this theorem implies our goal theorem.

Applying the theorem to $\lambda_V = r_V(g_s)$ implies

$$\exists g_s \in G : r_V(g_s) = r_V(g)_s$$

and

$$\exists g_u \in G : r_V(g_u) = r_V(g)_u.$$

Proof of Goal Theorem. There exist unique $g_s, g_u \in G$ s.t.

$$g \stackrel{(*)}{=} g_u g_s = g_s g_u,$$

Then, $r_V(g) = r_V(g_s)r_V(g_u) = r_V(g_u)r_V(g_s)$.

Since $r_V(g_u)$ is unipotent and $r_V(g_s)$ is semisimple, it follows $r_V(g_u) = r_V(g)_u$ and $r_V(g_s) = r_V(g)_s$.

To deduce (*), take any $r_V : G \hookrightarrow \mathrm{GL}(V)$. We know for each V

$$r_V(g) = r_V(g_s)r_V(g_u) = r_V(g_u)r_V(g_s).$$

□

Proof of Theorem. We first extend the assignment

$$V \mapsto \lambda_V$$

to all representations of G .

Say $V = \bigcup_j W_j$ where each W_j is a finite-dimensional G -invariant subspace. Try to define $\lambda_V \in \text{End}(V)$ by

$$\lambda_V|_{W_j} := \lambda_{W_j}.$$

For this to be well-defined, we need to show for each i, j

$$\lambda_{W_i}|_{W_i \cap W_j} \stackrel{(*)}{=} \lambda_{W_j}|_{W_i \cap W_j}.$$

Proof of (*): Apply assumption (iii) to the G -equivariant linear maps

$$W_i \cap W_j \xrightarrow{\phi} W_i,$$

$$W_i \cap W_j \xrightarrow{\phi'} W_j.$$

Then,

$$\begin{aligned} \lambda_{W_i}|_{W_i \cap W_j} &= \lambda_{W_i} \circ \phi \\ &\stackrel{(iii)}{=} \phi \circ \lambda_{W_i \cap W_j} \\ &= \phi' \circ \lambda_{W_i \cap W_j} \end{aligned}$$

and

$$\lambda_{W_j}|_{W_i \cap W_j} = \lambda_{W_j} \circ \phi' = \phi' \circ \lambda_{W_i \cap W_j}.$$

Recall here that any finite-dimensional G -invariant $W \subset V$ is a representation. \square

⁰Not necessarily finite-dimensional, but may be written as a filtered union of finite-dimensional G -invariant subspaces of W .

Lecture
from
11.03.2020

Let G be an algebraic group.

Easy Exercise : If V_1, V_2 are representations r_1, r_2 of G , then $V_1 \otimes V_2$ is also a representation with

$$r = r_1 \otimes r_2 : G \rightarrow \mathbf{GL}(V_1 \otimes V_2)$$

given by

$$r(g)(v_1 \otimes v_2) = (r_1(g)v_1) \otimes (r_2(g)v_2).$$

Proof. Given $\Delta_j : V_j \rightarrow V_j \otimes k[G]$, define

$$\Delta : V_1 \otimes V_2 \longrightarrow V_1 \otimes V_2 \otimes k[G]$$

by: if

$$\Delta_1 u = \sum u_i \otimes f_i, \quad \Delta_2 v = \sum v_j \otimes h_j,$$

then

$$\Delta(u \otimes v) \sum \sum u_i \otimes v_j \otimes f_i h_j.$$

Set $A := k[G]$, then

$$r_A := \text{right regular representation with } r_A(g)f(x) = f(xg).$$

The map

$$\begin{aligned} A \otimes A &\xrightarrow{m} A \\ f_1 \otimes f_2 &\longmapsto f_1 f_2 \end{aligned}$$

defines a morphism of representations

$$(A, r_A) \otimes (A, r_A) \rightarrow (A, r_A).$$

Indeed,

$$\begin{aligned} m((r_A \otimes r_A)(g)(f_1 \otimes f_2))(x) &= f_1(xg)f_2(xg), \\ &= f_1 f_2(xg) = r_A(g)(m(f_1 \otimes f_2))(x), \end{aligned}$$

since $f_1(_g) \otimes f_2(_g) = (r_A \otimes r_A)(g)(f_1 \otimes f_2)$.

Ergo $m \circ (r_A \otimes r_A)(g) = r_A(g) \circ m$. □

Recall: We stated the following theorem

Theorem 5. Let $\lambda_V \in \text{End}(V)$ be given s.t. for all finite-dim. rep.s V of G s.t.:

- (i) $\lambda_k = 1$
- (ii) $\lambda_{V \otimes W} = \lambda_V \otimes \lambda_W$
- (iii) for all morphisms of rep.s $\phi : V \rightarrow W$ we have

$$\phi \circ \lambda_V = \lambda_W \circ \phi.$$

Then, there is exactly one $g \in G$ s.t. $\lambda_V = r_V(g)$ for all V .

Proof. Last time, we saw that any such family $V \mapsto \lambda_V$ extends to **all** rep.s V of G .

Let's note also that, if (V_0, r_0) is any representation of G with trivial action, i.e. $r(g) = 1$ for all g , then $\lambda_{V_0} = 1$. Indeed, let $v \in V_0$. We must check that $\lambda_{V_0}v = v$. Since the action is trivial, any subspace of V_0 is G -invariant.

Consider the map

$$\begin{aligned} \phi : k &\longrightarrow V_0 \\ \alpha &\longmapsto \alpha v \end{aligned}$$

where $v = \phi(1)$. Then, ϕ is a morphism of rep.s because the action is trivial.

Thus,

$$\lambda_V v = (\lambda_V \circ \phi)(1) \stackrel{(iii)}{=} (\phi \circ \lambda_k)(1) \stackrel{(i)}{=} \phi(1) = v.$$

Consider $\lambda_A \in \text{End}(A)$. Then,

$$\lambda_{A \otimes A} = \lambda_A \otimes \lambda_A.$$

It is an easy exercise to see that $m : (A, r_A) \otimes (A, r_A) \rightarrow (A, r_A)$ is a morphism of rep.s.

By (iii) it follows, $m \circ (\lambda_A \otimes \lambda_A) = \lambda_A \circ m$, i.e.

$$\lambda_A(f_1 f_2) = \lambda_A(f_1) \lambda_A(f_2)$$

for all $f_1, f_2 \in A$. Thus, λ_A is an algebra morphism (check, using the morphism $k \hookrightarrow A$, that $\lambda_A(1) = 1$).

Thus, $\lambda_A = \phi^*$ for some unique morphism ϕ of algebraic sets $\phi : G \rightarrow G$.

We claim that ϕ commutes left multiplication i.e.

$$\phi(hx) = h\phi(x)$$

for all $h, x \in G$. Indeed, let's consider the map

$$\begin{aligned} A &\longrightarrow A \\ f &\longmapsto f(h \cdot _). \end{aligned}$$

This induces a morphism

$$(A, r_A) \xrightarrow{\psi} (A, r_A).$$

By (ii), $\psi \circ \lambda_A = \lambda_A \circ \psi$.

Since $\lambda_A = \phi^*$, this implies the claim.

Now, set $g := \phi(e)$. Then for all $h \in G$,

$$\phi(h) = \phi(he) = hg.$$

Thus, $\lambda_A = \phi^* = r_A(g)$.

(It remains to show that

$$\lambda_V = r_V(g)$$

for each finite-dim. rep. V .)

Let $V = (V, r)$ be any rep. This induces a map

$$\Delta : V \longrightarrow V \otimes A.$$

If $\Delta v = \sum v_i \otimes f_i$, then

$$hv = \sum f_i(h) \otimes v_i.$$

Let

$$\begin{aligned} \varepsilon : V \otimes A &\longrightarrow V \\ v \otimes f &\longmapsto f(1)v. \end{aligned}$$

It follows $\varepsilon \circ \Delta : V \rightarrow V$ is the identity map.

Let (V_0, r_0) be the representation of G with $V_0 := V$ and r_0 the trivial action. Then, $\Delta : V \rightarrow V_0 \otimes A$ is a morphism of representations.

(Indeed, if $\Delta v = \sum v_i \otimes f_i$, then

$$\Delta(r(h)v) \stackrel{?}{=} (r_0(h) \otimes r_A(h))\Delta v$$

since

$$\begin{aligned}\Delta v &= \sum v_i \otimes f_i \\ \iff xv &= \sum f_i(x_i)v_i \quad \forall x \in G \\ \iff xhv &= \sum f_i(xh)v_i \quad \forall x, h \in G.\end{aligned}$$

Since $r(h)v = hv$, it follows

$$\Delta(hv) = \sum v_i \otimes f_i(\cdot h) \implies (?).$$

We want to show

$$\lambda_V = r_V(g).$$

We have

$$\begin{aligned}\Delta \circ \lambda_V &\stackrel{(iii)}{=} \lambda_{V_0 \otimes A} \circ \Delta \\ &\stackrel{(ii)}{=} \lambda_{V_0} \otimes \lambda_A \\ &= 1 \otimes \lambda_A = 1 \otimes r_A(g).\end{aligned}$$

This implies

$$\Delta \circ \lambda_V = (1 \otimes r_A(g)) \circ \Delta$$

but also

$$\Delta \circ r_V(g) = (1 \otimes r_A(g)) \circ \Delta.$$

Because of the injectivity of Δ it now follows

$$\lambda_V = r_V(g).$$

□

Corollary 2. *Let $\phi : G \rightarrow H$ be any morphism of algebraic groups. Then, for all $g \in G$*

$$\begin{aligned}\phi(g)_s &= \phi(g_s) \\ \phi(g)_u &= \phi(g_u).\end{aligned}$$

Proof. Let V be any **faithful** representation of H , i.e. $r_V : H \rightarrow \text{GL}(V)$ is injective, (for a finite-dim. V).

Then, $r_V \circ \phi$ is a rep. of G . To prove (i), it suffices to show

$$r_V(\phi(g)_s) = r_V(\phi(g_s))$$

since H operates faithfully on V .

We know that

$$r_V(\phi(g)_s) = r_V(\phi(g))_s$$

(characterizing property of h_s for $h \in H$). On the other hand,

$$r_V(\phi(g_s)) = (r_V \circ \phi)(g_s) = r_V(\phi(g))_s.$$

Therefore, claim (i) follows. (ii) works analogously. \square

Definition 4. Let $g \in G$ where G is an algebraic group. We call g **semisimple**, if $g = g_s$.

We call g **unipotent**, if $g = g_u$.

Lemma 14. For $g \in G$, the following are equivalent:

- (i) g is semisimple.
- (ii) $r_V(g)$ is semisimple for all finite-dim. rep. V .
- (iii) $r_V(g)$ is semisimple for at least one faithful f.d. rep. V of G .

We get an analogous lemma for unipotent group elements.

Proof. We have

$$\begin{aligned}
 (i) & \iff g = g_s \\
 & \stackrel{\text{Def. of } g_s \text{ by goal thm.}}{\iff} r_V(g) = r_V(g)_s \forall \text{ f.d. } V \\
 & \iff r_V(g) \text{ is semisimple} \\
 & \iff (ii) \implies (iii).
 \end{aligned}$$

On the other hand,

$$(iii) \implies \exists \text{ faithful f.d. } V \text{ s.t. } r_V(g) = r_V(g)_s = r_V(g_s) \implies g = g_s.$$

\square

0.2 Non-Commutative Algebra

Definition 5. A **ring** R (for now) is unital, associative but not necessarily commutative.

Example 5. The ring of matrices over some field or ring.

Definition 6. A **left ideal** $I \subset R$ is a subset that is an abelian subgroup of $(R, +)$ s.t. $ra \in I$ for all $r \in R, a \in I$.

A **right ideal** $I \subset R$ is a subset that is an abelian subgroup with

$$IR \subset I.$$

A two-sided ideal I is a subset that is a left and a right ideal of R .

It is easy to check that for any homomorphism of rings $\phi : R \rightarrow S$, $\text{Kern}\phi$ is a two-sided ideal. Also, if $J \subset R$ is any two-sided ideal, then there exists a unique ring structure on R/J s.t. the projection $R \rightarrow R/J$ is a ring homomorphism.

Definition 7. A **left module** M for R is an abelian group equipped with a ring homomorphism

$$R \xrightarrow{\alpha} \text{End}(M)$$

where $\text{End}(M)$ acts on the left of M . We write

$$rm := \alpha(r)m.$$

We have

$$(r_1 r_2)(m) = r_1(r_2(m)).$$

If R **acts** on M by the right, we write

$$R \curvearrowright M.$$

Example 6. $M_n(k) \curvearrowright k^n$ where k^n is the space of column vectors.

If k^n denotes the space of row vectors, we have $k^n \curvearrowleft M_n(k)$.

Definition 8. A **(left) submodule** $N \subset M$ is an algebraic subgroup s.t.

$$RN \subset N.$$

It follows that N is itself is a left module.

Definition 9. A (left) module M of R is **simple** (or irreducible) if it has exactly the two submodules: $0 = \{0\}$ and M .

Definition 10. A ring R is a **division ring** if it satisfies any of the following equivalent requirements:

- (i) $R^\times = R \setminus \{0\}$ where $R^\times = \{r \in R \mid \exists a, b \in R : ar = rb = 1.\}$
- (ii) R has no nontrivial left or right ideals.

Definition 11. If $R \curvearrowright M$, then we can define

$$\text{End}_R(M) := \{\phi \in \text{End}(M) \mid \phi(rm) = r\phi(m) \forall r \in R, m \in M\}.$$

Note, that $\text{End}_R(M)$ is a ring.

Lemma 15 (Schur's Lemma). *If M is simple, then $\text{End}_R(M)$ is a division ring.*

Lemma 16. *Let k be a field. Then, $M_n(k)$ has no nontrivial two-sided ideals.*

Theorem 6 (Jacobson Density Theorem (Double Commutant Theorem)). *Suppose M is a simple left module which is finitely generated as a right D -module for $D = \text{End}_R(M)$.*

Assume that R acts faithfully on M , i.e. $R \rightarrow \text{End}_R(M)$ is injective.

Then, the map $R \rightarrow \text{End}_D(M)$ is an isomorphism.

⁰If $ar = rb = 1$, then $a = arb = b$.

Lecture
from
16.03.2020
(Corona-
Madness
started
here...)

Recap:

- Basics: definitions, Hopf-algebras, ...
- Jordan decomposition
- Primer on non-commutative algebra
 - Jacobson density theorem
- Unipotent groups
- Tori

0.2.1 Jacobson Density Theorem

We had last week

$$\text{End}_D(M) := \{\phi \in \text{End}(M) \mid \phi \circ d = d \circ \phi \forall d \in D\}.$$

Let k be an algebraically closed field, V a non-trivial finite-dimensional k -vector space and let G be a subgroup of $\text{GL}(V)$ that acts **irreducibly** on V , i.e., V is **G -irreducible**, i.e., the only G -invariant subspaces of V are 0 and V .

Set

$$D := \{d \in \text{End}_k(V) \mid dg = gd \forall g \in G\} = \text{span}(G) = \left\{ \sum_{i=1}^n c_i g_i \mid c_i \in k, g_i \in G, n \in \mathbb{N}_0 \right\}.$$

Then,

$$D = \text{End}_R(V)$$

where R is the k -subalgebra of $\text{End}(V)$ that is generated by G .

Lemma 17 (Schur's Lemma). *We understand $k \xrightarrow{\text{End}} (V)$ as the inclusion of operations which operate by scalar multiplication*

$$k \xrightarrow{\cong} \{\phi : V \rightarrow V \mid \phi : v \mapsto t \cdot v \text{ for some } t \in k\}.$$

Then, we have

$$D \cong k.$$

Proof. Let $d \in D$. Since $V \neq 0$, there is an eigenspace $V_\lambda \neq 0$ for d . Observe that V_λ has to be G -invariant:

if $g \in G$ and $v \in V_\lambda$, then $gv \in V_\lambda$, since

$$dgv = gdv = g(\lambda v) = \lambda gv.$$

Since V_λ is a non-trivial G -invariant subspace and V is irreducible under G , we have

$$V_\lambda = V.$$

Ergo $d = \lambda$ in the sense of $k \hookrightarrow \text{End}(V)$. □

Consequence of the Jacobson Density Theorem: $R = \text{End}_k(V)$, i.e., G generates all linear operations on V , if V is G -irreducible.

We will prove this after a lemma.

Lemma 18. *Let $n \in \mathbb{N}$. Set*

$$V^n := V \oplus V \oplus \dots \oplus V = V_1 \oplus \dots \oplus V_n$$

where each $V_i = V$.

Let $v = (v_1, \dots, v_n) \in V^n$ and set

$$Rv := \{(rv_1, \dots, rv_n) \mid r \in R\} = \text{span}\{(gv_1, \dots, gv_n) \mid g \in G\}.$$

Then, $Rv \neq V^n$ iff the v_j are linearly dependent over k .

Consequence: Take $n := \dim(V)$. Let $\{e_1, \dots, e_n\}$ be a basis of V and set

$$e := (e_1, \dots, e_n) \in V^n.$$

Since the $(e_i)_i$ are linearly independent, the lemma states that $Re = V^n$.

Now, let $x \in \text{End}_k(V)$. Choose $r \in R$ s.t.

$$re = (xe_1, \dots, xe_n).$$

Then $re_i = xe_i$ for all i , thus $x = r$. Hence, $R = \text{End}_k(V)$.

Proof. Choose $J \in \{1, \dots, n\}$ as large as possible with

$$Rv + V_1 + V_2 + \dots + V_{J-1} =: U \neq V^n$$

. Such an J does exist, since we know that $Rv \neq V^n$.

Then, $V_J \not\subseteq U$, otherwise we may increase J . Also, U is invariant by the diagonal action of G on V^n . Thus, $V_J \cap U \subseteq V_J$ is a proper G -invariant subspace of the G -irreducible $V_J \cong V$. Therefore, $V_J \cap U = 0$.

On the other hand, by maximality of J , we have

$$U \oplus V_J = V^n.$$

Ergo, the map (composition)

$$V \cong V_J \hookrightarrow V^n \twoheadrightarrow V^n/U$$

is a G -equivariant isomorphism, since U is G -invariant.

Let $z : V^n/U \xrightarrow{\cong} V$ be the inverse isomorphism. Let l be the G -equivariant map given by

$$\begin{array}{ccc} V^n & \xrightarrow{l} & V \\ \downarrow & \nearrow z & \\ V^n/U & & \end{array}$$

and let l_j be the G -equivariant maps by restricting l on V_j . Then $l_j \in D \cong k$.

Say $l_j = t_j \in k$. Then,

$$l(w) = t_1 w_1 + \dots t_n w_n.$$

Since z is an isomorphism, l is nonzero and $(t_1, \dots, t_n) \neq (0, \dots, 0)$.

Since $l|_U = 0$, we can deduce for all $u \in U$

$$t_1 u_1 + \dots + t_n u_n = 0.$$

But $v \in Rv \subseteq U$, so we may conclude – as required – that the $(v_i)_i$ are linearly dependent ($l(v) = 0$). \square

0.2.2 Unipotent Groups

Let G be a subgroup of $\mathrm{GL}(V)$ where V is a finite-dimensional vector space and k an algebraically closed field.

Definition 12. We say that G is **unipotent** if one of the following equivalent conditions hold:

- each $g \in G$ is unipotent (i.e. $(g - 1)^n = 0$ for some $n \in \mathbb{N}$).
- all eigenvalues of g are 1.
- g is conjugate to $\begin{pmatrix} 1 & \star & \star \\ 0 & \ddots & \star \\ 0 & 0 & 1 \end{pmatrix}$.

Theorem 7. Any unipotent subgroup of $\mathrm{GL}_n(k)$ is conjugate to a subgroup of

$$U_n := \left\{ \begin{pmatrix} 1 & \star & \star \\ 0 & \ddots & \star \\ 0 & 0 & 1 \end{pmatrix} \right\} = \left\{ U \in M_n(k) \mid U_{i,j} = \begin{cases} 0, & \text{if } i > j \\ 1, & \text{if } i = j \\ \text{arbitrary,} & \text{otherwise.} \end{cases} \right\}.$$

Definition 13. For two subgroups G, H of some common supergroup, define their **commutator** by

$$[G, H] := \langle ghg^{-1}h^{-1} \mid g \in G, h \in H \rangle.$$

A group G is called **nilpotent**, if one of its commutators is trivial, i.e. if we set

$$G_0 := G \text{ and } G_{i+1} := [G_i, G],$$

then G is called nilpotent iff there is an $j \in \mathbb{N}$ with $G_j = 1$.

Corollary 3. Any unipotent subgroup of $\mathrm{GL}(V)$ is nilpotent.

Definition 14. A group G is called **solvable**, if $G^{(n)} = 1$ for some n where

$$\begin{aligned} G^{(0)} &:= G, \\ G^{(i+1)} &:= [G^{(i)}, G^{(i)}]. \end{aligned}$$

Notation 1. In the following, we will write $G' := [G, G]$.

Definition 15. Let $n := \dim(V)$. A **complete flag** is a maximal strictly increasing chain of subspaces

$$0 = V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_n = V.$$

Any complete flag is of the form

$$V_j := \text{span}\{e_1, \dots, e_j\}$$

for some basis e_1, \dots, e_n of V .

Let B be the basis of some flag $0 = V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_n = V$. For $x \in \text{End}(V)$, we have that x is upper-triangular with respect to B iff x leaves each member V_i of the flag invariant, i.e. $xV_i \subseteq V_i$.

Proposition 2 (Key Proposition). *Let G be a unipotent subgroup of $\text{GL}(V)$. Then there is a complete flag $V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_n$ consisting of G -invariant subspaces, i.e., each V_i is G -invariant.*

Proof. Recall, that G is a unipotent subgroup of $\text{GL}_n(V)$. We will give an induction on $n = \dim V$.

If $n = 0$, there is nothing to show.

Let $n \geq 1$. We may assume that V is G -irreducible. Because, if not, there is a G -invariant subspace $0 \neq W \subset V$ s.t. W and V/W have dimension $< n$. Then there exist complete G -invariant flags in W and V/W and the claim – that there is a complete G -invariant flag in V – follows by the induction hypothesis.

By Jacobson Density Theorem, we have

$$R := \text{span}(G) = \text{End}(V) := \text{End}_k(V).$$

Since G is unipotent, we have for each $g \in G$

$$\text{trace}(g) = n.$$

Ergo, for $g, h \in G$

$$\text{trace}(gh) = \text{trace}(h)$$

and

$$\text{trace}((g - 1)h) = \text{trace}(gh) - \text{trace}(h) = 0.$$

Since $\text{span}(G) = \text{End}(V)$, it now in particular follows for all $g \in G, \phi \in \text{End}(V)$

$$\text{trace}((g - 1)\phi) = 0.$$

Since the above holds for all $\phi \in \text{End}(V)$, it must hold

$$g - 1 = 0$$

for all $g \in G$ (take for example the elementary matrices $\phi = E_{i,j}$). Ergo, G is trivial. Then, any complete flag is trivially G -invariant. \square

Remark 1. This gives the group analogue of Engel's Theorem.

Proof Goal Theorem. Let B be a basis of V s.t. G leaves each subspace in the corresponding flag invariant. Then, G is upper-triangle with respect to this basis.

On the other hand, each $g \in G$ is unipotent, hence its diagonal (i.e. eigenvalues) are all 1. Thus, with respect to B

$$G \subseteq \left\{ \begin{pmatrix} 1 & \star & \star \\ 0 & \ddots & \star \\ 0 & 0 & 1 \end{pmatrix} \right\} = U_n.$$

□

Remark 2. Tori are of the form $(k^\times)^n$. In the case $k = \mathbb{C}$, $(\mathbb{C}^\times)^n$ are the complexification of $U(1)^n$. This equals tori in top. sense.

$$\begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix} \subseteq \mathrm{GL}_2(\mathbb{C})$$

is a non-algebraic unipotent group.

Exercise. (to be discussed next time)

it would have sufficed to prove the Goal theorem in the special case that G is algebraic.

Corollary of Proof: If $G \subset \mathrm{GL}(V)$ (with $V \neq 0$) is unipotent and acts irreducibly (?), then $G = 1$, $\dim V = 1$.

Answer to last Exercise: Recall that the main point was to show that any unipotent subgroup $G \subseteq \mathrm{GL}(V)$ leaves invariant some complete flag $\mathcal{F} = (V_0 \subset V_1 \dots)$. But by some homework (problem 1), the group

$$\mathrm{GL}(V)_{\mathcal{F}} := \{g \in \mathrm{GL}(V) \mid g\mathcal{F} = \mathcal{F}\}$$

is algebraic.

Proof: If \mathcal{F} is the standard flag with $V_i = \mathrm{span}(e_1, \dots, e_i)$ for the standard basis $\{e_1, \dots, e_n\}$, then

$$\mathrm{GL}(V)_{\mathcal{F}} = \{A \in \mathrm{GL}(V) \mid A \text{ is upper-triangle}\}.$$

The condition that A is upper triangle can be realized by polynomials. □

Thus,

$$\begin{aligned} G \text{ fixes } \mathcal{F} \\ \iff G \subseteq \mathrm{GL}(V)_{\mathcal{F}} \\ \mathrm{GL}(V)_{\mathcal{F}} \text{ is algebraic} \iff \overline{G} \subseteq \mathrm{GL}(V)_{\mathcal{F}} \\ \iff \overline{G} \text{ fixes } \mathcal{F}. \end{aligned}$$

Now, the Zariski-Closure \overline{G} of any group G is an algebraic group (shown in some homework).

Further, if G is unipotent, then \overline{G} is unipotent.

0.3 Tori

Definition 16. A **torus** is an algebraic group that is isomorphic to \mathcal{G}_m^n for some $n \in \mathbb{N}_0$ where $\mathcal{G}_m = k^\times = \mathrm{GL}_1(k)$ is the unit group of k .

We think of $\mathcal{G}_m^n \subseteq \mathrm{GL}_n(k)$ as the subgroup of diagonal matrices.

Lemma 19. Let G be a commutative algebraic group. Then the following are equivalent:

- (i) each $g \in G$ is semisimple.
- (ii) for each finite-dimensional representation V of G and for each $g \in G$, the operator $r_V(g)$ is diagonalizable.

(iii) for all finite-dimensional representations V of G , there is a basis of common eigenvectors for $r_V(G)$, i.e. a basis s.t.

$$r_V(G) \subseteq \mathcal{G}_m^n.$$

(iv) G is isomorphic to an algebraic subgroup of a torus.

~~(i) Proof~~ (ii): This follows from the Jordan decomposition and definition of semisimple.

(ii) \implies (iii) : This is homework. Note that any commutative subset S of $\mathbf{GL}(V)$ consisting of semisimple operators may be diagonalized simultaneously.

(iii) \implies (iv) : Take any faithful representation V of G and diagonalize it simultaneously. Then, $G \cong r_V(G) \subseteq \mathcal{G}_m^n$.

(iv) \implies (i) : Any diagonal matrix is semisimple.

□

Definition 17. A commutative algebraic group G is called **diagonalizable**, if it satisfies one of the above equivalent conditions.

Definition 18. A character χ of any algebraic group F is an element $\chi \in \mathbf{Hom}_{\text{alg.grp.}}(F, k^\times)$, i.e., a homomorphism $\chi : F \rightarrow k^\times$ of algebraic groups.

Notation 2. For an algebraic group G , set $\Xi(G) := \mathbf{Hom}_{\text{alg.grp.}}(G, k^\times)$.

Also denote now by $\mathcal{O}(X) := k[T]/I(X)$ the coordinate ring of an algebraic set X (rather than $k[X]$).

Lemma 20. *There is a bijection*

$$\Xi(G) = \{\text{characters } \chi \text{ of } G\} \longleftrightarrow \{x \in \mathcal{O}(G)^\times \mid \Delta(x) = x \otimes x\}.$$

Proof. Note, that any $x \in \mathcal{O}(G)^\times$ can be thought of as a map $x : G \rightarrow k^\times \subset k$.

We have

$$\begin{aligned} \mathbf{Hom}_{\text{alg.grp.}}(G, \mathcal{G}_m) &= \{\phi \in \mathbf{Hom}_{\text{alg.sets}}(G, \mathcal{G}_m) \mid \phi(gh) = \phi(g)\phi(h) \ \forall g, h\} \\ &= \{\phi \in \mathbf{Hom}_{k\text{-alg.}}(\mathcal{O}(\mathcal{G}_m), \mathcal{O}(G)) \mid (\phi \otimes \phi) \circ \Delta = \Delta \circ \phi\}. \end{aligned}$$

Recall: $\mathcal{O}(\mathcal{G}_m) \cong k[t, \frac{1}{t}]$ with $\Delta(t) = t \otimes t$.

Thus for any k -algebra A , $\text{Hom}_{k\text{-alg.}}(\mathcal{O}(\mathcal{G}_m), A) \cong^{A^\times}$ via

$$[t \mapsto a, (t^{-1} \mapsto a^{-1})] \longleftrightarrow a.$$

Thus,

$$\text{Hom}_{\text{alg.grp.}}(G, \mathcal{G}_m) \cong \{a \in \mathcal{O}(G)^\times \mid a \otimes a = \Delta(a)\}.$$

Therefore, it suffices to test the condition $(\phi \otimes \phi) \circ \Delta = \Delta \circ \phi$ on the generators t, t^{-1} of $\mathcal{O}(\mathcal{G}_m)$. Now, the above isomorphism is given by

$$\phi \mapsto a = \phi(t)$$

which is equivalent or regarding $\chi : G \rightarrow \mathcal{G}_m$ as a map $\chi : G \rightarrow k$. \square

Example 7. Let $G = \mathcal{G}_m$, then $\mathcal{O}(G) = k[t, \frac{1}{t}]$. Which $x = \sum_{m \in \mathbb{Z}} c_m t^m \in \mathcal{O}(G)$, almost all $c_m = 0$, but not all of them, have the property

$$\Delta(x) = x \otimes x.$$

We have

$$\begin{aligned} x \otimes x &= \sum_{m, n \in \mathbb{Z}} c_m c_n t^m \otimes t^n, \\ \Delta(x) &= \sum_{m \in \mathbb{Z}} c_m t^m \otimes t^m. \end{aligned}$$

Those sums equal, if

$$\begin{aligned} c_m c_n &= 0 \text{ for all } m \neq n, \\ c_m^2 &= c_m \text{ for all } m. \end{aligned}$$

By those conditions, it follows

$$x = t^m.$$

Therefore

$$\Xi(G) = \{\chi_m \mid m \in \mathbb{Z}\} \cong \mathbb{Z}$$

with

$$\chi_m(y) = y^m.$$

Example 8. Let $T \cong \mathcal{G}_m^n$ be a torus. Then,

$$\Xi(T) = \{\chi_m \mid m \in \mathbb{Z}^n\} \cong \mathbb{Z}^n$$

where $\chi_m(y) = y^m = y_1^{m_1} \cdots y_n^{m_n}$.

Note: For each algebraic group G , $\Xi(G)$ is naturally an abelian group:

$$(\chi_1 \cdot \chi_2)(g) := \chi_1(g) \cdot \chi_2(g).$$

Given a morphism of algebraic groups $f : G \rightarrow H$, we get a morphism of abelian groups

$$\begin{aligned} f^* : \Xi(H) &\longrightarrow \Xi(G) \\ \chi &\longmapsto \chi \circ f =: f^*(\chi). \end{aligned}$$

This induces a contravariant functor from the category of algebraic groups to the category of abelian groups.

Lemma 21. *Let G be a diagonalizable algebraic group. Then, $\Xi(G)$ is a k -basis for $\mathcal{O}(G)$.*

Example 9. Let $G = \mathcal{G}_m^n$ be a torus. Then, we have the embedding

$$\begin{aligned} \Xi(G) &\hookrightarrow \mathcal{O}(G) \\ \chi_m &\longmapsto t^m. \end{aligned}$$

The lemma is obvious in this case: each element of $\mathcal{O}(G) = k[t_1, \dots, t_m, t_1^{-1}, \dots, t_m^{-1}]$ can be written uniquely as a linear combination of monomials.

Proof. (i) $\Xi(G)$ spans $\mathcal{O}(G)$:

Choose an embedding $G \subset \mathcal{G}_m^n$ of algebraic groups. Then, by restriction, we get

$$\mathcal{O}(\mathcal{G}_m^n) \twoheadrightarrow \mathcal{O}(G).$$

Since the $\chi_m, m \in \mathbb{Z}^n$, span $\mathcal{O}(\mathcal{G}_m^n)$, their images $\chi_m|_G \in \Xi(G)$ span $\mathcal{O}(G)$.

(ii) $\Xi(G)$ is linearly independent:

Suppose otherwise and let ϕ_1, \dots, ϕ_m be a linearly dependent subset of $\Xi(G)$ with $m \geq 1$ chosen minimally, with $c_1, \dots, c_m \in k^\times$ s.t.

$$\sum_{i=1}^m c_i \phi_i = 0.$$

We distinguish the following cases:

$m = 1$: In this case, we have $\phi_1 = 0$, but $\phi_1(1) = 1$, a contradiction.

$m > 1$: We can assume $\phi_1 \neq \phi_2$, so there is an $h \in G$ s.t. $\phi_1(h) \neq \phi_2(h)$. Then,

$$\phi_1(h) \sum_{i=1}^m c_i \phi_i = 0,$$

but also for all $h, g \in G$

$$\sum_{i=1}^m c_i \phi_i(hg) = \sum_{i=1}^m c_i \phi_i(h) \phi_i(g) = 0.$$

This implies

$$\sum_{i=1}^m c_i \phi_i(h) \phi = 0.$$

Ergo

$$\sum_{i=1}^m c_j (\phi_i(h) - \phi_1(h)) \phi_i = \sum_{i=2}^m c_j (\phi_i(h) - \phi_1(h)) \phi_i = 0.$$

Now, $\phi_i(h) - \phi_1(h)$ is zero if $i = 1$ and non-zero, if $i = 2$. Therefore, this yields a shorter linear dependency for the elements

$$\phi_2, \dots, \phi_m,$$

which contradicts our requirement. □

Definition 19. Let M be an abelian group. The **group algebra** on M is the k -algebra $k[M]$ (not a coordinate ring!) defined as follows:

$$\begin{aligned} k[M] &:= \text{the } k\text{-vectorspace with basis } M \\ &:= \left\{ \sum_{m \in M} c_m \cdot m \mid c_m \in k, \text{ almost all } c_m = 0 \right\}, \end{aligned}$$

where the multiplication on $k[M]$ extends that on M :

$$\left(\sum_{m \in M} c_m m \right) \left(\sum_{n \in M} d_n n \right) = \sum_{m, n \in M} c_m d_n mn.$$

Corollary 4. For a diagonalizable G , we have

$$\mathcal{O}(G) \cong k[\Xi(G)].$$

Fact: For an abelian group M , there is exactly one Hopf algebra structure on $k[M]$ given by $\Delta(m) = m \otimes m$ for all $m \in M$.

With this definition, the above isomorphism is one of Hopf algebras.

Lemma 22. *If G, H are diagonalizable algebraic groups, then*

$$\text{Hom}_{\text{alg.grp.s}}(G, H) \xrightarrow{f \mapsto f^*} \text{Hom}_{\text{grp.s}}(\Xi(H), \Xi(G))$$

is a bijection.

Proof.

$$\begin{aligned} \text{Hom}(G, H) &\cong \text{Hom}_{\text{Hopf-alg.}}(\mathcal{O}(H), \mathcal{O}(G)) \\ &\cong \{\phi \in \text{Hom}_{k\text{-alg.}}(\mathcal{O}(H), \mathcal{O}(G)) \mid (\phi \otimes \phi) \circ \Delta = \Delta \circ \phi\}. \end{aligned}$$

Since $\text{Hom}_{k\text{-alg.}}(\mathcal{O}(H), \mathcal{O}(G)) \cong \text{Hom}(k[\Xi(H)], k[\Xi(G)])$, this reduces to the following lemma:

Lemma 23. *Let M_1, M_2 be two abelian groups. Then*

$$\begin{aligned} \text{Hom}(M_1, M_2) &\xrightarrow{\cong} \text{Hom}_{\text{Hopf-alg.}}(k[M_1], k[M_2]) \\ \phi &\longmapsto \left[\sum c_m m \mapsto \sum c_m \phi(m) \right]. \end{aligned}$$

Proof. We have to show that

$$M = \{x \in K[M]^\times \mid \Delta(x) = x \otimes x\}.$$

Then, by this, it follows for each $\phi \in \text{Hom}_{\text{Hopf-alg.}}(k[M_1], k[M_2])$,

$$\phi(M_1) \subseteq M_2.$$

Ergo, $\phi|_{M_1} \in \text{Hom}(M_1, M_2)$. Therefore, the surjectivity of the claimed bijection is shown. The injectivity is clear, since M generates $k[M]$ as a k -algebra.

To show

$$M = \{x \in K[M]^\times \mid \Delta(x) = x \otimes x\},$$

let

$$\begin{aligned} x &= \sum c_m m \in K[M]^\times \\ \Delta(x) &= \sum c_m m \otimes m \\ x \otimes x &= \sum c_m c_n m \otimes n. \end{aligned}$$

If $\Delta(x) = x \otimes x$, then it follows

$$x = m$$

for some $m \in M$.

□

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