Mitschrieb: Algebraic Groups
SS 20

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Vorwort

Contents

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Recall: Last time we introduced the **Zariski-Topology** on X.
There, algebraic sets equal closed sets.
We called a set X **irreducible** iff each open subset lies dense in X.

Lemma 1. *For an algebraic set X, the following are equivalent:*

(1) X is irreducible.

(2) $k[X] = k[x_1, \dots, x_n]/I(X)$ is a domain.

(3) $I(X)$ is a prime ideal.

The proof of (2) $\iff$ (3) is a basic algebraic result.

Lemma 2. *An open base for the Zariski-Topology on an algebraic set X is given by sets:*

$$D(f) := \{p \in X \mid f(p) \neq 0\}$$

for each $f \in k[X]$. We call the $D(f)$ ***basic open sets****.*

Proof. Suppose $U \subseteq X$ is nonempty and open. Set

$$Z := X \setminus U$$

then Z is closed. Thus

$$Z = \{x \in X \mid f(x) = 0 \forall f \in I\}$$

for some ideal $I \subseteq k[X]$. Let $p \in U$, then there is an $f \in Z$ s.t.

$$f(p) \neq 0.$$

Also, $D(f) \cap Z = 0$, thus $p \in D(f) \subseteq U$. □

Lemma 1. It is clear that (2) is equivalent to (3).
(1) is equivalent to

$$\begin{aligned} &\forall \text{ nonempty, open } U_1, U_2 \subset X : U_1 \cap U_2 \neq \emptyset \\ \overset{\text{Lemma 2}}{\iff} &\forall \text{ nonempty, basic open } D(f_1), D(f_2) \subset X : D(f_1) \cap D(f_2) \neq \emptyset \end{aligned}$$

Since $D(f_1) \cap D_(f_2) = D(f_1 f_2)$, this is equivalent to the statement

$$f_1, f_2 \in k[X] : f_1, f_2 \neq 0 \implies f_1 f_2 \neq 0.$$

Which states that $k[X]$ is a domain. □

Lemma 3. *Let X be an algebraic set. We have bijections*

$$\{\textit{closed subsets } Z \subseteq X\} \leftrightarrow \{\textit{ radical ideals } I \subset k[X]\}$$

and

$$\{\textit{irreducible, closed subsets } Z \subseteq X\} \leftrightarrow \{\textit{ prime ideals } I \subset k[X]\}$$

and

$$\{\textit{points of } X\} \leftrightarrow \{\textit{ maximum ideals } I \subset k[X]\}.$$

Lemma 4 (Primary Decompositions, Atiyah, Macdonald Ch. 4)**.** *For an ideal I we call $P \supseteq I$ a **minimal prime** of I if P is a prime ideal and we have for each prime ideal Q:*

$$P \supseteq Q \supseteq I \implies P = Q.$$

*Any radical ideal I of $k[x_1, \ldots, x_n]$ has only finitely many **minimal** primes $P_1, \ldots, P_r$. Inparticular,*

$$I = \bigcap_{i=1}^{n} P_i$$

and for each i

$$P_j \not\supseteq \bigcap_{j:j\neq i} P_j.$$

Definition 1. An **(irreducible) component** Z of X is a maximal irreducible closed subset, i.e., an irreducible closed $Z \subseteq X$ s.t. there does not exist an irreducible closed $Y \subset X$ s.t. $Y \supsetneq Z$.

Then, we have the bijection

$$\{\text{irreducible components of } X\} \leftrightarrow \{\text{ mimimal primes of } I(X)\}.$$

Lemma 5. *Any algebraic set X has finitely many components $Z_1, \ldots, Z_r$. We have*

$$X = Z_1 \cup \ldots \cup Z_r$$

and for each i

$$Z_i \not\subset \bigcup_{j:j\neq i} Z_j.$$

Example 1. 1. Let $X = V(x \cdot y) \subset k^2$. Then $X = Z_1 \cup Z_2$ where $Z_1 = V(x), Z_2 = V(y)$.

X is connected, but not irreducible ($D(x)$ does not lie dense in X).

2. Let X be a **finite** algebraic set. It is easy to check that every subset of X is closed:
$$\{p\} = V(x_1 - p_1, \ldots, x_n - p_n)$$
for each $p \in X$. Further
$$X = \{p_1\} \cup \ldots \cup \{p_r\}.$$
Moreover: Any function $f : X \to k$ is regular (i.e. given by polynomials).

Lemma 6. *We call an element $e \in k[X]$* ***idempotent*** *iff $e^2 = e$.*
Let X be an algebraic set. Then
$$\begin{aligned} X \textit{ connected} &\iff \textit{ the only idempotents } e \in k[X] \textit{ are 0 and 1} \\ &\iff k[X] \ncong A \times B \textit{ for any } k\textit{-algebras } A, B. \end{aligned}$$

Lemma 7. *Morphisms of algebraic sets are continuous.*

Proof. Let $\phi : X \to Y$ be a morphism. It suffices to show that for all closed $Z \subset Y$ that $\phi^{-1}(Z) \subset X$ is closed.

But, if
$$Z = V_Y(S) := \{q \in Y \mid f(q) = 0 \forall f \in S\}$$
for some ideal $S \subset k[Y]$, then
$$\phi^{-1}(Z) = V_X(\phi^*(S)) = \{\phi^*(f) = f \circ \phi \mid f \in S\}.$$
□

Lemma 8. *Isomorphisms of algebraic sets are homeorphisms. In particular, any isomorphism of algebraic sets $\phi : X \to X$ permutes the components $Z_1, \ldots, Z_r$ of X:*
$$\forall i \exists j : \phi(Z_i) = Z_j.$$

Theorem 1. *Let G be an algebraic group.*

(i) There is a unique component G^0 of G with $e \in G^0$.

(ii) Every component Z of G is a coset gG^0 of G for some $g \in Z$.

(iii) G^0 is a normal algebraic subgroup of G.

(iv) G^0 is of finite index, i.e.
$$[G : G^0] = \#\left(G/G^0\right) < \infty.$$

(v) *The irreducible components are also the connected components.*

Proof. Let $G = Z_1 \cup \ldots \cup Z_r$ be the decomposition into components. We may assume that $e \in Z_1$.

Recall that $Z_1 \not\subset \bigcup_{j\geq 2} Z_j$. Then, there is an $x \in Z_1 \setminus \bigcup_{j\geq 2} Z_j$. Thus, for all algebraic set isomorphisms $\phi : G \to G$, we have by some previous lemma that $\phi(x)$ is likewise contained in some unique component of G. For example, we may take ϕ to be

$$\begin{aligned} \phi_g : G &\to G \\ y &\longmapsto gy \end{aligned}$$

for any $g \in G$. Then, for all $g \in G$, the element $gx = \phi_g(x)$ is contained in only one component of G. Ergo, each $g \in G$ is contained in exactly one component.

(i) Take $g = e$.

(iii) G^0 is an algebraic subset, by construction. Denote by $m : G \times G \to G$ and $i : G \to G$ the continuous multiplication and inversion map on G. **Why is G^0 a subgroup?** We need to show

$$\begin{aligned} m(G^0 \times G^0) &\subseteq G^0. \\ i(G^0) &\subseteq G^0. \end{aligned}$$

We know that $i(G^0)$ is some component of G, since i is an isomorphism. But it contains the identity e, since $e^{-1} = e$. Therefore, $i(G^0) = G^0$.

If $g \in G$, then gG^0 is some component of G. Suppose $g \in G^0$. Then $gG^0 \cap G^0 \supseteq \{g\}$, therefore $gG^0 = G^0$. Ergo, G^0 is closed under multiplication.

Why is G^0 a normal? If $g \in G$, then gG^0g^{-1} is a component that contains e, therefore $G^0 = gG^0g^{-1}$.

(Alternative proof that $m(G^0 \times G^0) = G^0$: Consider

- any continuous image of an irreducible set is irreducible.
- the closure of any irreducible set is irreducible.

Ergo $\overline{m(G^0 \times G^0)}$ is a closed irreducible set containing e. Ergo, $\overline{m(G^0 \times G^0)} = G^0$).

(ii) Let $Z \subset G$ be a component. Let $g \in Z$. Then $g \in (gG^0 \cap Z)$, so $gG^0 = Z$.

(iv) This follows from some previous lemma.

(v) This is left as a topological exercise. It is true whenever the irreducible components do not intersect.

□

It now follows:

$$\{\text{finite algebraic groups}\} \longleftrightarrow \{finitegroups\}$$

where the above arrow is an equivalence of categories.

Example 2.

- Let $G = \{g_1, \ldots, g_r\}$ be a finite algebraic group. Then,

$$G^0 = \{e\}.$$

- Without proofs:

$$G \in \{\mathsf{GL}_n(k), \mathsf{SO}_n(k), \mathsf{SL}_n(k)\} \implies G^0 = G.$$

  Further,

$$G = O_n(k) \implies G^0 = \mathsf{SO}_n(k)$$

  (but only if $-1 = 1$ i.e. $\mathsf{char}k = 2$. Otherwise $[G : G^0] = 2$.)

0.1 Jordan Decomposition

As usual, $k = \overline{k}$ is an algebraically closed field.

Definition 2. Let V be a finite-dimensional vector space.

An element $x \in \mathsf{End}(V)$ is **semisimple**, if it is diagonalizable, i.e. it has a basis of eigenvectors, or equivalently, if the minimal polynomial of x is square-free.

Then, there is a decomposition $V = \bigoplus_{i=1}^r V_i$ and distinct elements $\lambda_1, \ldots, \lambda_n \in k$ s.t.

$$x|_{V_i} = \lambda_i.$$

If $\dim(V_i) = n_i$, then

$$\text{char polynomial of} x = \prod_{i=1}^r (T_i - \lambda_i)_i^n \in k[T]$$

and

$$\text{minimal polynomial of} x = \prod_{i=1}^{r}(T_i - \lambda_i) \in k[T].$$

(Where the minimal polynomial of x is defined as the least degree monic m s.t. $m(x) = 0$ and Cayley-Hamilton $m|c$.)

Definition 3. $x \in End(V)$ is **nilpotent** if $x^n = 0$ for some n. (Equivalent to: characteristic polynomial of x is $T^{\dim(V)}$.)

x is **unipotent**, if $x - 1$ is nilpotent.

Lemma 9. *If x is semisimple and nilpotent, then $x = 0$.*

If x is semisimple and unipotent, then $x = 1$.

Lemma 10. *If x, y are commuting elements, thar are semisimple resp. unipotent or nilpotent, then so is xy.*

Theorem 2 (Goal)**.** *For all algebraic groups G and for all $g \in G$, there exist unique group elements $g_s, g_u \in G$ s.t.*

$$g = g_s g_u = g_u g_s$$

and for all finite-dimensional representations $\rho : G \to \mathit{GL}(V)$, $\rho(g_s)$ is semisimple and $\rho(g_u)$ is unipotent.

Example 3. If $g = \begin{pmatrix} \lambda & 1 & 0 \\ & \lambda & 1 \\ & & \lambda \end{pmatrix} \in G = \mathsf{GL}_3(k)$, then $g_s = \begin{pmatrix} \lambda & 0 & 0 \\ & \lambda & 0 \\ & & \lambda \end{pmatrix}$, $g_u = \begin{pmatrix} 1 & \lambda^{-1} & 0 \\ & 1 & \lambda^{-1} \\ & & 1 \end{pmatrix}$.

$$\begin{tikzcd} \gamma \arrow[r, hook] \arrow[dr, "s_i:="'] & \pi^* E = \bigoplus L_i \arrow[d, "Proj."] \\ & L_i \end{tikzcd}$$