Notes: Algebraic Groups
SS 20, ETH Z

August 10, 2020

Contents

1 Introduction

Let k be an algebraically closed field.

Definition 1. For $I \subseteq k[X] := k[X_1, \dots, X_n]$, we define its **vanishing set** by

$$V(I) := \{p \in k^n \mid \forall f \in I: \ f(p) = 0\}.$$

A set $S \subset k^n$ is called **algebraic**, if

$$S = V(I)$$

for some $I \subseteq k[X]$.

Example 1. The group $\mathsf{GL}_n(k)$ is not an algebraic subset of $k^{n \times n}$. But, we can identify it with an algebraic subset of $(k^{n\times n})^2$ by

$$\mathsf{GL}_n(k) \cong \left\{(x, y) \in k^{n\times n} \mid xy = 1_n\right\} = V(X \cdot Y - 1_n).$$

Definition 2. Let $\iota : \mathsf{GL}_n(k) \hookrightarrow k^{n \times n 2}$ be the injection

$$A \mapsto (A, A^{-1}).$$

A **linear algebraic group** over k is a subgroup $U \subseteq \mathsf{GL}_n(k)$ s.t. $\iota(k)$ is an algebraic subset of k^{2n^2}.

I.e., a linear algebraic group is a matrix-group which can be defined by polynomials over the entries of a matrix and its inverse.

Example 2. The following groups are linear algebraic groups:

1. The multiplicative group $\mathcal{G}_m(k) := k^\times = k \setminus \{0\} = \mathsf{GL}_1(k)$.

2. The general linear group $\mathsf{GL}_n(k)$.

3. The special linear group

$$\mathsf{SL}_n(k) := \{A \in \mathsf{GL}_n(k) \mid \det(A) = 1\}.$$

4. The orthogonal group

$$\mathcal{O}_n(k) := \left\{A \in \mathsf{GL}_n(k) \mid A^T \cdot A = 1\right\}.$$

5. The special orthogonal group

$$\mathsf{SO}_n(k) := \mathcal{O}_n(k) \cap \mathsf{SL}_n(k).$$

6. The upper triangle-matrix group

$$\left\{ \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ & \ddots & \vdots \\ & & a_{n,n} \end{pmatrix} \mid a_{i,j} \in k \right\} \cap \mathsf{GL}_n(k).$$

7. The normed upper triangle-matrix group

$$\left\{ \begin{pmatrix} 1 & \cdots & a_{1,n} \\ & \ddots & \vdots \\ & & 1 \end{pmatrix} \mid a_{i,j} \in k \right\} \cap \mathsf{GL}_n(k).$$

8. The group of n-th roots of unity

$$\mu_n(k) := \{x \in k \mid x^n = 1\}.$$

9. The additive group $(k, +)$ is not a subgroup of $\mathsf{GL}_n(k)$, but it can be identified with the linear algebraic group

$$\left\{ \begin{pmatrix} 1 & a \\ & 1 \end{pmatrix} \mid a \in k \right\} \subset \mathsf{GL}_2(k)$$

10. For $k = \mathbb{C}$, the unit sphere and the unitary groups are NOT linear algebraic groups.

2 Algebraic Groups and Hopf Algebras

Definition 3. A **morphism** $f : X \to Y$ of algebraic sets $X \subset k^m, Y \subset k^n$ is a map which is coordinatewise described by polnomials.

Definition 4. An **algebraic group** is an algebraic set $G \subset k^n$ together with a fixed element $e \in G$ and morphisms $m : G \times G \to G, i : G \to G$ s.t. (G, m, i, e) is a group.

A **morphism of algebraic groups** is a morphism of algebraic sets that is also a group homomorphism.

Definition 5. Let $V \subset k^n$ be any subset. Then, we define the vanishing ideal of V by

$$I(V) := \{f \in k[x] \mid f(V) = 0\}.$$

Definition 6. For a commutative ring R we define the **radical** of an ideal $I \subseteq R$ by

$$\sqrt{I} := \{r \in R \mid r^m \in I \text{ for some } m \in \mathbb{N}_0\}.$$

R is called **reduced**, if $\sqrt{0} = 0$.

Lemma 1 (Zariskis Lemma)**.** *Let $L \supseteq k$ be fields. If L is finitely generated as a k-algebra, then the extension $L \supseteq k$ is finite, i.e., L is a finitely-generated k-vector space.*

Theorem 1 (Hilberts Nullstellensatz)**.** *For any ideal $I \subseteq k[x]$, we have*

$$I(V(I)) = \sqrt{I}.$$

Proof. It is easy to see that

$$I \subset \sqrt{I} \subset I(V(I)).$$

Now, let $f \in I(V(I))$ and assume – for the sake of contradiction – that $f \notin \sqrt{I}$. Since $\sqrt{I}$ is the intersection of its upper prime ideals, there is a prime ideal $p \supset I$, s.t. $f \notin p$. Now, define the zero divisor-free ring

$$R := (k[x]/p)[f^{-1}].$$

And let $\phi : k[x] \to R$ be the corresponding ring homomorphism.

Let $m \subseteq R$ be a maximal ideal in R. Then, R/m is a field, which contains k and is finitely generated as k-algebra. According to Zariski's lemma, R/m is a finite (ergo algebraic) extension of k. Since k is algebraically closed, we have $R/m = k$. Let $\pi_m : R \to k$ be the corresponding ring homomorphism.

Now, for $x_1, \ldots, x_n$, set

$$t_i := \pi_m(\phi(x_i)).$$

Then, $t = (t_1, \ldots, t_n) \in k^n$. We now have

1. $t \in V(I)$: For each $g \in I$, we have $\phi(g) = 0$. On the other hand

$$g(t) = g(\pi_m \circ \phi(x)) = \pi_m \circ \phi(g) = 0.$$

2. $f(t) \neq 0$: $\phi(f)$ is invertible in R, therefore $\phi(f) \neq 0$ and $\phi(f) \notin m$. Ergo

$$f(t) = \pi_m \circ \phi(f) \neq 0.$$

Ergo, there is a point $t \in V(I)$ s.t. $f(t) \neq 0$. This yields a contradiction, since we assumed $f \in I(V(I))$. $\square$

Definition 7. For an algebraic set $X \subset k^n$, we define its **coordinate ring** by

$$k[X] := k[x_1, \ldots, x_n]/I(X).$$

Lemma 2. *For a morphism $f : X \to Y$ of algebraic sets define the following homomorphism of k-algebras.*

$$\begin{aligned} f^* : k[Y] &\longrightarrow k[X] \\ p &\longmapsto p \circ f. \end{aligned}$$

We have a contravariant functor $\_^*$ from the categories of algebraic sets over k to the category of k-algebras:

$$\begin{aligned} X &\longmapsto k[X] \\ \mathsf{Hom}\,(X, Y) &\longmapsto \mathsf{Hom}_k\,(k[Y], k[X]) \\ f &\longmapsto f^*. \end{aligned}$$

Lemma 3. *We have*

$$k[X \times Y] \cong k[X] \otimes k[Y].$$

Proof.

$$k[X] \otimes k[Y] = k[x]/I(X) \otimes_k k[y]/I(Y) = k[x, y]/I(X) \otimes k[y] + k[x] \otimes I(Y).$$

But

$$V(I(X) \otimes k[y] + k[x] \otimes I(Y)) = V(I(X) \otimes k[y]) \cap V(k[x] \otimes I(Y)) = X \times Y.$$

$\square$

Theorem 2. *Every finitely generated reduced k-algebra A is isomorphic to some $k[X]$ for some algebraic X.*

Proof. Choose some $\pi : k[x_1, \dots, x_n] \twoheadrightarrow A$ and set $X := V(\ker \pi)$. Then $\ker \pi = I(X)$, since π's kernel is radical since A is reduced. $\square$

Corollary 1. *The contravariant functor $\_^* : \mathcal{C}_{\text{algSets}} \to \mathcal{C}_{k-\text{alg.s}}$ gives an antiequivalence of categories.*

Lemma 4. *An algebraic set X is isomorphic to some algebraic subset of Y iff there is an epimorphism $k[Y] \twoheadrightarrow k[X]$.*

Lemma 5. *Let $G \subset k^n$ be an algebraic group. Then, we have maps*

$$\begin{aligned} m : G \times G &\longrightarrow G \\ i : G &\longrightarrow G \\ e : * &\longrightarrow G. \end{aligned}$$

They induce dual maps in the category of k-algebras:

$$\begin{aligned} \Delta := m^* : k[G] &\longrightarrow k[G] \otimes_k k[G] \\ \iota := i^* : k[G] &\longrightarrow k[G] \\ \varepsilon := e^* : k[G] &\longrightarrow k \end{aligned}$$

Definition 8. A **Hopf-algebra** over k is a (reduced?!) k-algebra together with maps $\Delta, \varepsilon, \iota$ as above s.t. the following holds:

$$\begin{gathered} (\Delta \otimes \mathrm{Id})\Delta = (\mathrm{Id} \otimes \Delta)\Delta \\ s^* \circ (\iota \otimes \mathrm{Id})\Delta = s^* \circ (\mathrm{Id} \otimes \iota)\Delta = \varepsilon \\ (\varepsilon \otimes \mathrm{Id})\Delta = (\mathrm{Id} \otimes \varepsilon)\Delta = \mathrm{Id} \end{gathered}$$

where $s : G \to G \times G, g \mapsto (g, g)$ is the diagonal map.

A morphism of Hopf-algebras is a homomorphism of k-algebra $F : A \to B$ s.t.

$$\Delta \circ F = (F \otimes F) \circ \Delta.$$

Theorem 3. *The contravariant functor $\_^*$ gives an anti-equivalence of the categories of algebraic groups and the categories of finitely generated Hopf-algebras over k.*

Example 3. 1. Let $G = \mathcal{G}_a = (k, +)$. Then, $k[G] = k[x]$, since $I(x) = 0$. Then, we have

$$\begin{aligned}\Delta(x) &= x \otimes 1 + 1 \otimes x \\ \iota(x) &= -x \\ \varepsilon(x) &= 0.\end{aligned}$$

2. Let $G = \mathcal{G}_m = \{(a, a^{-1}) \mid a \neq 0\} \cong k^{\times}$. Then, $k[G] = k[x, y]/(xy - 1) = k[x, x^{-1}]$. Then, we have

$$\begin{aligned}\Delta(x) &= x \otimes x \\ \iota(x) &= x^{-1} \\ \varepsilon(x) &= 1.\end{aligned}$$

3. Let $G = \mathsf{GL}_n(k)$. Then, $k[G] = k[x, y]/(xy - 1_n) = k[x_{i,j}, \frac{1}{\det}]$. Then, we have

$$\begin{aligned}\Delta(x_{i,j}) &= \sum_k x_{i,k} \otimes x_{k,j} \\ \Delta(\frac{1}{\det(x)}) &= \frac{1}{\det(x)} \otimes \frac{1}{\det(x)} \\ \iota(x_{i,j}) &= (x^{-1})_{i,j} \\ \varepsilon(x_{i,j}) &= \delta_{i,j}.\end{aligned}$$

2.1 An Aside on the General Group

Let $G = \mathsf{GL}_n(k) = \{(x, y) \mid xy = \mathrm{Id}_n\}$. Since we have

$$x^{-1} = \frac{1}{\det(x)} \cdot \mathrm{adj}(x)$$

where the adjoint $\mathrm{adj}(x)$ can be expressed by polnomials in the entries of x, we have isomorphisms

$$\begin{aligned}k[x, y]/(xy - 1_n) &\longrightarrow k[x, 1/\det(x)] = k[x, t]/(\det(x) \cdot t = 1) \\ (x, y) &\longmapsto (x, \det(y))\end{aligned}$$

and

$$\begin{aligned}k[x, 1/\det(x)] &\longrightarrow k[x, y]/(xy - 1_n) \\ (x, t) &\longmapsto (x, t \cdot \mathrm{adj}(x)).\end{aligned}$$

Lemma 6.

$$k[\mathsf{GL}_n(k)] \cong k[x_{i,j}, \frac{1}{\det(x)}].$$

Lemma 7. *Let V be a finite-dimensional k-vector space. If we choose a basis for V, we get an isomorphism $\mathsf{GL}(V) \cong \mathsf{GL}_n(k) \subset k^{n\times n}$. Hence, $\mathsf{GL}(V)$ is an algebraic group whose structure is up to a unique isomorphism independent of the choice of basis.*

3 Actions

Remark 1. Let $G \curvearrowright M$ be a group action of algebraic sets, then the morphism

$$G \times M \longrightarrow M$$

yields an homomorphism

$$\Delta : k[M] \to k[G] \otimes k[M].$$

This turns $k[M]$ to a **comodule** of the Hopf-Algebra $k[G]$.

Definition 9. Let V be vector space and G an algebraic group. A morphism $r_V : G \to \mathsf{GL}(V)$ of groups is called **representation** of G, if there is a linear map

$$\Delta : V \to V \otimes_k k[G] (= \mathsf{Hom}_{alg}(G, V))$$

s.t. we have for each $v \in V$ and $g \in G$

$$r_V(g) \cdot v = \sum_i v_i \cdot f_i(g)$$

where $\Delta v = \sum_i v_i \otimes f_i$.

That is, V is a comodule for $k[G]$.

A map $\phi : V \to W$ is called **equivariant** for two representations r_V, r_W of G, if

$$\phi(r_V(g)v) = r_W(g)\phi(v)$$

for all g, v.

Example 4. Let $G = \mathsf{GL}_n(k)$, $V = k^n$ and r_V be the canonical representation. For an orthonormal basis $(b_i)_{i=1,\dots,n}$, we for example can set

$$\Delta v = \sum_{i=1}^{n} b_i \otimes f_i$$

where

$$f_i(A) := b_i^T A v.$$

Then, we have

$$r_V(A) \cdot v = A \cdot v = \sum_{i=1}^{n} b_i \cdot b_i^T A v = \Delta(v)(A).$$

Example 5. Let M be a right G-set. Then, G also acts on $k[M]$, therefore we have a map

$$\rho : G \to \mathsf{GL}(k[M])$$

by, for $v \in k[M]$,

$$(\rho(g)v)(m) := v(m.g).$$

Further, we have an algebra morphism

$$\Delta : k[M] \to k[M] \otimes k[G] = k[M \times G]$$

with

$$(\Delta v)(m, g) = v(m.g).$$

With $\Delta v = \sum_i v_i \otimes f_i$

$$\rho(g)v(m) = v(mg) = \Delta v(m, g) = \sum_{i=1} f_i(g)v_i(m).$$

Ergo, ρ is a representation of G.

When $M = G$ with action given by the right translation, then $\rho : G \to \mathsf{GL}(k[G])$ is called the **right regular representation** of G.

Lemma 8. *Let G be an algebraic group and V a finite-dimensional k-vector space. Then $\rho : G \to \mathsf{GL}(V)$ is morphism of algebraic groups iff it is a representation.*

Definition 10. Let G be an algebraic group and V a representation of G. A subspace $W \subset V$ is called **invariant** or **subrepresentation**, if we have $W.G = W$.

Lemma 9. *The following are equivalent:*

1. *W is invariant.*
2. *$\Delta(W) \subseteq W \otimes k[G]$.*

Lemma 10. *Any representation V is a filtered union of its finite-dim. subrepresentations:*

1. *Each $v \in V$ is contained in some fin.-dim. subrep.*
2. *Any two finite-dim. subrep. are contained in some bigger fin.-dim. subrep.*

Theorem 4. *Every algebraic group G is isomorphic to a linear algebraic group.*

Proof. Let $\rho : G \to \mathsf{GL}(k[G])$ be the right regular representation. $k[G]$ is a finitely-generated k-algebra. Then, there is a finite-dim. subrepresentation $V \subseteq k[G]$ s.t. V generates $k[G]$ as k-algebra. Then

$$\phi : G \longrightarrow \mathsf{GL}(V)$$

is morphism of algebraic groups.

Consider the dual map

$$\phi^* : k[\mathsf{GL}(V)] \to k[G].$$

We need to show that ϕ^* is surjective. It is enough to show that $V \subset \mathsf{Img}\phi^*$. Define

$$\begin{aligned} l : V \subset k[G] &\longrightarrow k \\ f &\longmapsto f(e). \end{aligned}$$

Let $f \in V$ and set $a(g) := l(g \cdot f)$ for $g \in \mathsf{GL}(V)$. Then $a \in k[\mathsf{GL}(V)]$ is regular. Further,

$$\phi^*(a)(g) = a(\rho(g)) = l(\rho(g)f) = f(eg) = f(g).$$

Therefore, $f = \phi^*(a) \in \mathsf{Img}(\phi^*)$. Since V generates $k[G]$, the surjectivity of ϕ^* follows. $\square$

Theorem 5. *Let H be an algebraic subgroup of an algebraic group G. There is a finite-dim. representation V of G and a line $L \subset V$ s.t. H is the stabilizer in G of L, i.e.*

$$H = \{g \in G \mid L.g = g\}.$$

Proof. Let V be like in the previous proof. Consider

$$I \hookrightarrow k[G] \twoheadrightarrow k[H].$$

We can now set $L' := V \cap I$. We then have for $g \in G$.

$$I.g \subseteq I \iff g \in H.$$

Now, in general L' is not of dimension one. Set $d = \dim(L')$ and consider the one-dimensional subspace $L := \Lambda^d(L') \subseteq \Lambda^d(V)$. G acts on $\Lambda^d(V)$ in the natural way.

It is clear, that H stabilizes L. For the other direction, let $g \notin H$ and let $e_1, \ldots, e_n$ be a basis of V s.t. $L' = \langle e_1, \ldots, e_d \rangle$. Then,

$$L = \langle e_1 \wedge \ldots \wedge e_d \rangle$$

and, since g does not stabilize L', w.l.o.g. we can assume $e_1.g = e_{d+1}$. Then, we have $g(e_1 \wedge \ldots \wedge e_d) = g(e_1) \wedge \ldots \wedge g(e_d) =: v$. Now, v cannot be zero and it cannot lie L because $e_1.g = e_{d+1}$. Therefore, $g \notin H$ does not stabilize L. $\square$

Theorem 6. *Let H be a normal algebraic subgroup of an algebraic group G. Then, there is a finite-dimensional $\rho : G \to \mathsf{GL}(V)$ s.t. $H = \ker(\rho)$.*

Proof. Let V, L and $\phi : G \to \mathsf{GL}(V)$ be like in the preceding theorem. Set

$$V_H := \{v \in V \mid H.v \subset \langle v \rangle\}.$$

Then, V_H is G-invariant, since

$$h.(g.v) = (hg).v = (gh').v = g.(h'v) = g.(\kappa \cdot v) = \kappa \cdot g.v$$

for all $g \in G, h \in H, v \in V_H$ and fitting $h' \in H, \kappa \in k^\times$. W.l.o.g. we have $V = V_H$. V is not trivial, because $L \subset V$.

Let χ range through all homorphism $H \to k^\times$, then we have

$$V = \otimes_\chi V_\chi$$

where

$$V_\chi = \{v \in V \mid h.v = \chi(h) \cdot v\}.$$

Then each $g \in G$ permutes those eigenspaces by

$$g.V_\chi = V_{\chi(g^{-1}\_g)}.$$

Now, let $W := \bigoplus_\chi \mathsf{End}(V_\chi) \subset \mathsf{End}(V)$. For $g \in G$ and $\lambda \in \mathsf{End}(V)$, define

$$\begin{aligned} \widetilde{\gamma} : G &\longrightarrow \mathsf{GL}(\mathsf{End}(V)) \\ g &\longmapsto \widetilde{\gamma}(g) : [\lambda \mapsto \phi(g) \circ \lambda \circ \phi(g)^{-1}]. \end{aligned}$$

The action $\widetilde{\gamma}(g)$ stabilizes W, since each $\phi(g)$ just permutes the V_χ and $\phi(g)^{-1}$ permutes them back. Therefore, we have a subrepresentation

$$\gamma : G \to \mathsf{GL}(W).$$

We now have to show

$$\ker(\gamma) = H.$$

Since elements of H don't permute V_χ, we have $\gamma(H) = \mathrm{Id}$.

One the other side, let $g \in G$ with $\gamma(g) = \mathrm{Id}$. Then, we can choose the projection $\pi : V \twoheadrightarrow L$ in W and get

$$\phi(g) \circ \pi = \pi \circ \phi(g).$$

Therefore, g leaves each L invariant. But now, we have $g \in H$. □

4 Connected Components

Lemma 11. *Let $I_1, I_2, I_\lambda \subset k[x]$ be ideals, then*

$$V(I_1 \cap I_2) = V(I_1) \cup V(I_2)$$
$$V(\bigcup_\lambda I_\lambda) = \bigcap_\lambda V(I_\lambda).$$

Definition 11. A topological space X is called **connected**, if any of the following equivalent condition holds:

- There is no pair of non-empty closed subsets $Z_1, Z_2 \subseteq X$, s.t. $X = Z_1 \dot\cup Z_2$.
- There is no pair of non-empty open closed subsets $U_1, U_2 \subseteq X$, s.t. $X = U_1 \dot\cup U_2$.
- Each nonempty open subset of X is dense.

Definition 12. A topological space X is called **irreducibel**, if any of the following equivalent condition holds:

- There is no pair of proper closed subsets $Z_1, Z_2 \subseteq X$, s.t. $X = Z_1 \cup Z_2$.
- For each pair $U_1, U_2 \subseteq X$ of non-empty open subsets we have $U_1 \cap U_2 \neq \emptyset$.
- Each nonempty open subset of X is dense.

Example 6. $V(xy)$ is connected but not irreducible.

Lecture from 03.03.2020

Recall: Last time we introduced the **Zariski-Topology** on X.
There, algebraic sets equal closed sets.
We called a set X **irreducible** iff each open subset lies dense in X.

Lemma 12. *For an algebraic set X, the following are equivalent:*

(1) X is irreducible.

(2) $k[X] = k[x_1, \dots, x_n]/I(X)$ is an (integral) domain.

(3) $I(X)$ is a prime ideal.

The proof of (2) $\iff$ (3) is a basic algebraic result.

Lemma 13. *An open base for the Zariski-Topology on an algebraic set X is given by sets:*

$$D(f) := \{p \in X \mid f(p) \neq 0\}$$

for each $f \in k[X]$. We call the $D(f)$ ***basic open sets****.*

Proof. Suppose $U \subseteq X$ is nonempty and open. Set

$$Z := X \setminus U$$

then Z is closed. Thus

$$Z = \{x \in X \mid f(x) = 0 \ \forall f \in I\} = V(I)$$

for some ideal $I \subseteq k[X]$. Let $p \in U$, then there is an $f \in Z$ s.t.

$$f(p) \neq 0.$$

Also, $D(f) \cap Z = 0$, thus $p \in D(f) \subseteq U$. □

Proof: Lemma 1. It is clear that (2) is equivalent to (3).
(1) is equivalent to

$$\forall \text{ nonempty, open } U_1, U_2 \subset X : U_1 \cap U_2 \neq \emptyset$$
$$\overset{\text{Lemma 2}}{\iff} \forall \text{ nonempty, basic open } D(f_1), D(f_2) \subset X : D(f_1) \cap D(f_2) \neq \emptyset$$

Since $D(f_1) \cap D_(f_2) = D(f_1 f_2)$, this is equivalent to the statement

$$\forall f_1, f_2 \in k[X] : \ f_1, f_2 \neq 0 \implies f_1 f_2 \neq 0.$$

Which states that $k[X]$ is a domain. □

Lemma 14. *Let X be an algebraic set. We have bijections*

$$\{\textit{closed subsets } Z \subseteq X\} \leftrightarrow \{ \textit{ radical ideals } I \subset k[X]\}$$

and

$$\{\textit{irreducible, closed subsets } Z \subseteq X\} \leftrightarrow \{ \textit{ prime ideals } I \subset k[X]\}$$

and

$$\{\textit{points of } X\} \leftrightarrow \{ \textit{ maximum ideals } I \subset k[X]\}.$$

Lemma 15 (Primary Decompositions, Atiyah, Macdonald Ch. 4)**.** *For an ideal I we call $P \supseteq I$ a* ***minimal prime*** *of I if P is a prime ideal and we have for each prime ideal Q:*

$$P \supseteq Q \supseteq I \implies P = Q.$$

Any radical ideal I of $k[x_1, \dots, x_n]$ has only finitely many ***minimal*** *primes $P_1, \dots, P_r$. Inparticular,*

$$I = \bigcap_{i=1}^{r} P_i$$

and for each i

$$P_j \not\supseteq \bigcap_{j:j\neq i} P_j.$$

Definition 13. An **(irreducible) component** Z of X is a maximal irreducible closed subset, i.e., an irreducible closed $Z \subseteq X$ s.t. there does not exist an irreducible closed $Y \subset X$ s.t. $Y \supsetneq Z$.

Then, we have the bijection

$$\{\text{irreducible components of } X\} \leftrightarrow \{ \text{ mimimal primes of } I(X)\}.$$

Lemma 16. *Any algebraic set X has finitely many irreducible components $Z_1, \dots, Z_r$. We have*

$$X = Z_1 \cup \ldots \cup Z_r$$

and for each i

$$Z_i \not\subset \bigcup_{j:j\neq i} Z_j.$$

Example 7. 1. Let $X = V(x \cdot y) \subset k^2$. Then $X = Z_1 \cup Z_2$ where $Z_1 = V(x), Z_2 = V(y)$.

X is connected, but not irreducible ($D(x)$ does not lie dense in X).

2. Let X be a **finite** algebraic set. It is easy to check that every subset of X is closed:
$$\{p\} = V(x_1 - p_1, \ldots, x_n - p_n)$$
for each $p \in X$. Further
$$X = \{p_1\} \cup \ldots \cup \{p_r\}.$$
Moreover: Any function $f : X \to k$ is regular (i.e. given by polynomials).

Lemma 17. *We call an element $e \in k[X]$* ***idempotent*** *iff $e^2 = e$.*
Let X be an algebraic set. Then
$$\begin{aligned} X \text{ connected} &\iff \text{the only idempotents } e \in k[X] \text{ are } 0 \text{ and } 1 \\ &\iff k[X] \ncong A \times B \text{ for any } k\text{-algebras } A, B. \end{aligned}$$

Lemma 18. *Morphisms of algebraic sets are continuous.*

Proof. Let $\phi : X \to Y$ be a morphism. It suffices to show that for all closed $Z \subset Y$ that $\phi^{-1}(Z) \subset X$ is closed.

But, if
$$Z = V_Y(S) := \{q \in Y \mid f(q) = 0 \forall f \in S\}$$
for some ideal $S \subset k[Y]$, then
$$\phi^{-1}(Z) = V_X(\phi^*(S)) = \{\phi^*(f) = f \circ \phi \mid f \in S\}.$$
□

Lemma 19. *Isomorphisms of algebraic sets are homeorphisms. In particular, any isomorphism of algebraic sets $\phi : X \to X$ permutes the irreducible components $Z_1, \ldots, Z_r$ of X:*
$$\forall i \exists j : \phi(Z_i) = Z_j.$$

Theorem 7. *Let G be an algebraic group.*

(i) There is a unique irreducible component G^0 of G with $e \in G^0$.

(ii) Every irreducible component Z of G is a coset gG^0 of G for some $g \in Z$.

(iii) G^0 is a normal algebraic subgroup of G.

(iv) G^0 is of finite index, i.e.
$$[G : G^0] = \#\left(G/G^0\right) < \infty.$$

(v) The irreducible components are also the connected components.

Proof. Let $G = Z_1 \cup \ldots \cup Z_r$ be the decomposition into components. We may assume that $e \in Z_1$.

Recall that $Z_1 \not\subset \bigcup_{j \geq 2} Z_j$. Then, there is an $x \in Z_1 \setminus \bigcup_{j \geq 2} Z_j$. Thus, for all algebraic set isomorphisms $\phi : G \to G$, we have by some previous lemma that $\phi(x)$ is likewise contained in some unique component of G. For example, we may take ϕ to be

$$\begin{aligned} \phi_g : G &\to G \\ y &\longmapsto gy \end{aligned}$$

for any $g \in G$. Then, for all $g \in G$, the element $gx = \phi_g(x)$ is contained in only one component of G. Ergo, each $g \in G$ is contained in exactly one component.

(i) Take $g = e$.

(iii) G^0 is an algebraic subset, by construction. Denote by $m : G \times G \to G$ and $i : G \to G$ the continuous multiplication and inversion map on G. **Why is G^0 a subgroup?** We need to show

$$\begin{aligned} m(G^0 \times G^0) &\subseteq G^0. \\ i(G^0) &\subseteq G^0. \end{aligned}$$

We know that $i(G^0)$ is some component of G, since i is an isomorphism. But it contains the identity e, since $e^{-1} = e$. Therefore, $i(G^0) = G^0$.

If $g \in G$, then gG^0 is some component of G. Suppose $g \in G^0$. Then $gG^0 \cap G^0 \supseteq \{g\}$, therefore $gG^0 = G^0$. Ergo, G^0 is closed under multiplication.

Why is G^0 a normal? If $g \in G$, then gG^0g^{-1} is a component that contains e, therefore $G^0 = gG^0g^{-1}$.

(Alternative proof that $m(G^0 \times G^0) = G^0$: Consider

- any continuous image of an irreducible set is irreducible.
- the closure of any irreducible set is irreducible.

Ergo $\overline{m(G^0 \times G^0)}$ is a closed irreducible set containing e. Ergo, $\overline{m(G^0 \times G^0)} = G^0$).

(ii) Let $Z \subset G$ be a component. Let $g \in Z$. Then $g \in (gG^0 \cap Z)$, so $gG^0 = Z$.

(iv) This follows from some previous lemma.

(v) This is left as a topological exercise. It is true whenever the irreducible components do not intersect.

$\square$

It now follows:

$$\{\text{finite algebraic groups}\} \longleftrightarrow \{\text{finite groups}\}$$

where the above arrow is an equivalence of categories.

Example 8.
- Let $G = \{g_1, \ldots, g_r\}$ be a finite algebraic group. Then,

$$G^0 = \{e\}.$$

- Without proofs:

$$G \in \{\mathsf{GL}_n(k), \mathsf{SO}_n(k), \mathsf{SL}_n(k)\} \implies G^0 = G.$$

Further,
$$G = O_n(k) \implies G^0 = \mathsf{SO}_n(k).$$
And if $-1 = 1$ i.e. $\mathsf{char} k = 2$, then $[G : G^0] = 1$. Otherwise $[G : G^0] = 2$.

5 Jordan Decomposition

As usual, $k = \overline{k}$ is an algebraically closed field.

Definition 14. Let V be a finite-dimensional vector space.

An element $x \in \mathsf{End}(V)$ is **semisimple**, if it is diagonalizable, i.e. it has a basis of eigenvectors, or equivalently, if the minimal polynomial of x is square-free.

Then, there is a decomposition $V = \bigoplus_{i=1}^r V_i$ and distinct elements $\lambda_1, \ldots, \lambda_n \in k$ s.t.

$$x|_{V_i} = \lambda_i.$$

If $\dim(V_i) = n_i$, then

$$\text{char polynomial of } x = \prod_{i=1}^r (T_i - \lambda_i)^{n_i} \in k[T]$$

and

$$\text{minimal polynomial of } x = \prod_{i=1}^r (T_i - \lambda_i) \in k[T].$$

(Where the minimal polynomial of x is defined as the least degree monic polynomial $m \in k[T]$ s.t. $m(x) = 0$.)

Remark 2. Let $m(T) \in k[T]$ be the minimal polynomial of $x \in k^{n \times n}$.

The theorem of Cayley and Hamilton states that we have for each $p \in k[T]$:

$$p(x) = 0 \implies m|p.$$

Definition 15. $x \in End(V)$ is **nilpotent** if $x^n = 0$ for some n.

x is **unipotent**, if $x - 1$ is nilpotent.

Lemma 20. *x is nilpotent iff the characteristic polynomial of x is $T^{\dim(V)}$. (Use Cayley-Hamilton for one of the directions).*

Lemma 21. *If x is semisimple and nilpotent, then $x = 0$.*

If x is semisimple and unipotent, then $x = 1$.

Lemma 22. *If x, y are commuting elements, that are semisimple resp. unipotent resp. nilpotent, then so is xy.*

Proof. It is easy to see, that this is true for nilpotent x, y.

Now, let x, y be unipotent and commuting. Then, we have

$$xy - 1 = (x+1)(y-1) + (x-y).$$

Since x, y commute, $(x+1)(y-1)$ must be nilpotent. $(x-y)$ must be nilpotent because the sum of commuting nilpotent elements must be nilpotent. Because everything commutes, also $xy - 1$ as the sum of two commuting, nilpotent elements must be nilpotent.

Now, let $A, B \in k^{n \times n}$ be two diagonalizable and commuting matrices. Let $\lambda_1, \ldots, \lambda_r$ be different eigenvectors of A and let E_i be the corresponding eigenspaces. We then have

$$A \cdot (BE_i) = BAE_i = \lambda_i \cdot BE_i.$$

Ergo, each E_i is invariant under B. Since $B_{|E_i}$ stays diagonalizable, we can simply choose a basis of eigenvectors $b_1, \ldots, b_n \in \bigcup_i E_i$ of B. Since each b_i lies in a E_j, those vectors are also eigenvectors for A. Therefore, $b_1, \ldots, b_n$ is basis of eigenvectors for both matrices. □

Theorem 8 (Goal)**.** *For all algebraic groups G and for all $g \in G$, there exist unique group elements $g_s, g_u \in G$ s.t.*

$$g = g_s g_u = g_u g_s$$

and for all finite-dimensional representations $\rho : G \to \mathsf{GL}(V)$, $\rho(g_s)$ is semisimple and $\rho(g_u)$ is unipotent.

Example 9. If $g = \begin{pmatrix} \lambda & 1 & 0 \\ & \lambda & 1 \\ & & \lambda \end{pmatrix} \in G = \mathsf{GL}_3(k)$, then $g_s = \begin{pmatrix} \lambda & 0 & 0 \\ & \lambda & 0 \\ & & \lambda \end{pmatrix}$, $g_u = \begin{pmatrix} 1 & \lambda^{-1} & 0 \\ & 1 & \lambda^{-1} \\ & & 1 \end{pmatrix}$.

Theorem 9 (Goal Theorem)**.** *Let G an algebraic group. For all $g \in G$ there is exactly one pair $g_s, g_u \in G$ s.t.* Lecture from 09.03.2020

$$g = g_s g_u = g_u g_s$$

and for all finite-dimensional representations $r : G \to \mathsf{GL}_n(V)$, the element $r(g_s)$ resp. $r(g_u)$ is semisimple resp. unipotent.

Last time, we saw:

Lemma 23.
- *If g, h are commuting and semisimple resp. commuting and unipotent then so is gh.*
- *If g is semisimple and unipotent, then $g = 1$.*

Proposition 1. *Let V be a finite-dimensional vector space and $g \in \mathsf{GL}(V)$. There exist unique elements $g_s, g_u \in \mathsf{GL}(V)$ s.t.*

$$g = g_s g_u = g_u g_s$$

and g_s is semisimple and g_u is unipotent.

Moreover, $g_s, g_u \in k[g] = \{\sum_{i=1}^m a_i g^i \mid a_i \in k\} \subseteq \mathsf{End}(V)$.

Proof. **Existence** (Sketch): Say

$$g = \begin{pmatrix} \lambda & 1 & 0 \\ & \lambda & 1 \\ & & \lambda \end{pmatrix}$$

then take

$$g_s = \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{pmatrix}, \quad g_u = \begin{pmatrix} 1 & \lambda^{-1} & 0 \\ & 1 & \lambda^{-1} \\ & & 1 \end{pmatrix}.$$

For $\lambda \in k$, define the **generalized λ-eigenspace** of g by

$$V_\lambda := \{v \in V \mid \exists n \in \mathbb{N}_0 : \ (g - \lambda)^n v = 0\}.$$

Then

$$V = \bigoplus_{\lambda \in k} V_\lambda.$$

Here V_λ = sum of domains of all Jordan blocks with λs on the diagonal. (It follows from the Jordan Decomposition for matrices that such a decomposition exist.)

Let's define $g_s \in \mathsf{GL}(V)$ by

$$g_s|_{V_\lambda} = \lambda \cdot \mathrm{Id}.$$

Note that $gV_\lambda \subset V_\lambda$, hence g commutes with g_s, hence g, g_s commutes with $g_u := gg_s^{-1}$. Then, $g = g_s g_u = g_u g_s$.

Write $\det(T - g) = \prod_\lambda (T-\lambda)^{n(\lambda)}$, $n(\lambda) = \dim(V_\lambda)$. Since the polynomials $T - \lambda$ for $\lambda \in k$ are coprime, the chinese remainder theorem implies that there is a $Q \in k[T]$ s.t.

$$Q \equiv \lambda \mod (T-\lambda)^{n(\lambda)}$$

for each $\lambda \in k$.

We claim that

$$Q(g) = g_s.$$

Indeed, since $gV_\lambda \subseteq V_\lambda$, we have

$$Q(g)V_\lambda \subseteq V_\lambda.$$

So, it suffices to show for all $v \in V_\lambda$

$$Q(g)v = g_s v = \lambda v.$$

Note that, by Cayley-Hamilton,

$$V_\lambda = \left\{ v \in V \mid (g-\lambda)^{n(\lambda)} v = 0. \right\}$$

Write

$$Q = \lambda + R \cdot (T-\lambda)^{n(\lambda)}$$

for some $R \in k[T]$. Since $(g-\lambda)^{n(\lambda)} v = 0$, deduce that $Q(g)v = \lambda v$, as required.

If $P \equiv \lambda^{-1} \mod (T-\lambda)^{n(\lambda)}$, then $P(g) = g_s^{-1}$.

Therefore,

$$g_u = g \cdot P(g)$$

for $T \cdot P(T) \in k[T]$.

Uniqueness: Suppose given some other decomposition

$$g = g_s' g_u' = g_u' g_s'$$

with g_s' semisimple and g_u' unipotent. Then g_s' commutes with g_s' and g_u', hence with g, hence also with any element in $k[g]$. Ergo, g_s' commutes with g_s and g_u. Similarly, g_u' commutes with g_s and g_u.

Consider

$$h := g'_s g_s^{-1} = g'_s g'_u (g'_u)^{-1} g_s^{-1} = g (g'_u)^{-1} g_s^{-1} = g_u (g'_u)^{-1}.$$

Then $h = g'_s g_s^{-1}$ is a product of semisimple elements and $h = g_u (g'_u)^{-1}$ is a product of unipotent elements. By proceeding lemmas, h is semisimple and unipotent, ergo trivial. It follows $g'_s = g_s$ and $g'_u = g_u$. $\square$

Corollary 2. *Let $g \in \mathsf{GL}(V)$, let $W \subset V$ be any g-**invariant subspace**, i.e. $gW \subseteq W$.*

Then, W is g_s-invariant and g_u-invariant.

Proof. This is clear, since g_s and g_u are algebraically generated by g over g. $\square$

Lemma 24. *Let $\phi : V \to W$ be a linear map between finite-dimensional vector spaces.*

Let $\alpha \in \mathsf{GL}(W)$ and $\beta \in \mathsf{GL}(W)$ s.t.

$$\begin{array}{ccc} V & \xrightarrow{\alpha} & V \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ W & \xrightarrow{\beta} & W, \end{array}$$

i.e. $\phi \circ \alpha = \beta \circ \phi$.

Then,

$$\phi \circ \alpha_s = \beta_s \circ \phi,$$
$$\phi \circ \alpha_u = \beta_u \circ \phi.$$

Proof. Write $V = \bigoplus_{\lambda \in k} V_\lambda$, $W = \bigoplus_{\lambda \in k} W_\lambda$ where V_λ are the generalized α-eigenspaces and W_λ are the generalized β-eigenspaces.

We claim that

$$\phi(V_\lambda) \subset W_\lambda.$$

Indeed, let $v \in V_\lambda$, then

$$(\beta - \lambda)^n \phi(v) = \phi((\alpha - \lambda)^n v) = 0.$$

Since $(\alpha - \lambda)^n v = 0$, the claim follows.

Since, $\alpha_s|_{V_\lambda} = \lambda \mathrm{Id}$ and $\beta_s|_{W_\lambda} = \lambda \mathrm{Id}$, deduce that

$$\phi \circ \alpha_s = \beta_s \circ \phi.$$

Indeed, both sides are given on V_λ by $\lambda \cdot \phi$. Thus

$$\begin{aligned}\phi \circ \alpha_u &= \phi \circ \alpha\alpha_s^{-1} \\ &= \beta\beta_s^{-1} \circ \phi \\ &= \beta_u \circ \phi.\end{aligned}$$

□

Lemma 25. *Let $\alpha \in GL(V)$, $\beta \in GL(W)$. Then the **tensor** $\alpha \otimes \beta \in GL(V \otimes W)$ is defined by*

$$(\alpha \otimes \beta)(u \otimes v) = \alpha(u) \otimes \beta(v).$$

Then, we have

$$(\alpha \otimes \beta)_s \stackrel{(1)}{=} \alpha_s \otimes \beta_s$$

$$(\alpha \otimes \beta)_u \stackrel{(2)}{=} \alpha_u \otimes \beta_u.$$

Proof. It suffices to prove (1), since

$$\begin{aligned}(\alpha \otimes \beta)_u &= (\alpha \otimes \beta) \circ (\alpha \otimes \beta)_s^{-1} \\ &\stackrel{(1)}{=} (\alpha \otimes \beta) \circ (\alpha_s \otimes \beta_s)^{-1} \\ &= \alpha\alpha_s^{-1} \otimes \beta\beta_s^{-1} \\ &= \alpha_u^{-1} \otimes \beta_u^{-1}\end{aligned}$$

For (1), consider

$$V = \bigoplus_{\lambda \in k} V_\lambda,$$

$$W = \bigoplus_{\lambda \in k} W_\lambda.$$

It follows

$$V \otimes W = \bigoplus_{\lambda, \mu \in k} V_\lambda \otimes W_\mu.$$

Now,

$$\alpha_s \otimes \beta_s|_{V_\lambda \otimes W_\mu} = \lambda\mu \cdot \mathrm{Id}.$$

Ergo, $\alpha_s \otimes \beta_s$ is semisimple. By Proposition, we reduce to checking that $\alpha_u \otimes \beta_u$ is unipotent. Indeed,

$$\alpha_u \otimes \beta_u - 1 = (\alpha_u - 1) \otimes (\beta_u - 1) + 1 \otimes (\beta_u - 1) + (\alpha_u - 1) \otimes 1$$

is nilpotent (You can also check that $\alpha_u \otimes \beta_u = (\alpha_u \otimes 1) \circ (1 \otimes \beta_u)$ is unipotent.) □

Example 10. Let $1 \in \mathsf{GL}(V)$. Then $1_s = 1$ and $1_u = 1$.

Summary : Let G be an algebraic group. Let $r_V : G \to \mathsf{GL}(V)$ be a finite-dimensional representation. Also, fix $g \in G$.

Let $\lambda_V := r_V(g)_s$ (or $r_V(g)_u$).

We get a family of operators $\lambda_V \in \mathsf{End}(V)$ with the following properties:

(i) if $V = k$ and $r_V(g') = 1$ for all $g' \in G$, then $\lambda_V = 1$.

(ii) for any two representations in V and W, we have

$$\lambda_{V \otimes W} = \lambda_V \otimes \lambda_W.$$

(iii) for all G-equivariant $\phi : V \to W$ we have

$$\phi \circ \lambda_V = \lambda_W \circ \phi.$$

Theorem 10. *Let G be an algebraic group. Let $\lambda_V \in \mathsf{End}(V)$ (i.e. $V = (r_V, V)$ is a finite-dim. representation of G) be a family of operations satisfying (i), (ii), (iii).*

Then, there is exactly one $g \in G$ s.t. $\lambda_V = r_V(g)$ for all V.

Note, that this theorem implies our goal theorem.

Applying the theorem to $\lambda_V = r_V(g)_s$ implies

$$\exists_1 g_s \in G : r_V(g_s) = r_V(g)_s$$

and

$$\exists_1 g_u \in G : r_V(g_u) = r_V(g)_u.$$

Proof of Goal Theorem. There exist unique $g_s, g_u \in G$ s.t.

$$g \overset{(*)}{=} g_u g_s = g_s g_u,$$

Then, $r_V(g) = r_V(g_s) r_V(g_u) = r_V(g_u) r_V(g_s)$.

Since $r_V(g_u)$ is unipotent and $r_V(g_s)$ is semisimple, it follows $r_V(g_u) = r_V(g)_u$ and $r_V(g_s) = r_V(g)_s$.

To deduce $(*)$, take any $r_V : G \hookrightarrow \mathsf{GL}(V)$. We know for each V

$$r_V(g) = r_V(g_s) r_V(g_u) = r_V(g_u) r_V(g_s).$$

□

Proof of Theorem 10. We first extend the assignment

$$V \mapsto \lambda_V$$

to all representations of G.

Say $V = \bigcup_j W_j$ where each W_j is a finite-dimensional G-invariant subspace. Try to define $\lambda_V \in \mathsf{End}(V)$ by

$$\lambda_V|_{W_j} := \lambda_{W_j}.$$

For this to be well-defined, we need to show for each i, j

$$\lambda_{W_i}|_{W_i \cap W_j} \overset{(*)}{=} \lambda_{W_j}|_{W_i \cap W_j}.$$

Proof of $(*)$**:** Apply assumption (iii) to the G-equivariant linear maps

$$W_i \cap W_j \overset{\phi}{\hookrightarrow} W_i,$$
$$W_i \cap W_j \overset{\phi'}{\hookrightarrow} W_j.$$

Then,

$$\begin{aligned} \lambda_{W_i}|_{W_i \cap W_j} &= \lambda_{W_i} \circ \phi \\ &\overset{(iii)}{=} \phi \circ \lambda_{W_i \cap W_j} \\ &= \phi' \circ \lambda_{W_i \cap W_j} \end{aligned}$$

and

$$\lambda_{W_j}|_{W_i \cap W_j} = \lambda_{W_i} \circ \phi' = \phi' \circ \lambda_{W_i \cap W_j}.$$

Recall here that any finite-dimensional G-invariant $W \subset V$ is a representation. $\square$

[0]Not necessarily finite-dimensional, but may be written as a filtered union of finite-dimensional G-invariant subspaces of W.

Let G be an algebraic group. Lecture from 11.03.2020

Easy Exercise : If V_1, V_2 are representations r_1, r_2 of G, then $V_1 \otimes V_2$ is also a representation with

$$r = r_1 \otimes r_2 : G \to \mathsf{GL}(V_1 \otimes V_2)$$

given by

$$r(g)(v_1 \otimes v_2) = (r_1(g)v_1) \otimes (r_2(g)v_2).$$

Proof. Given $\Delta_j : V_j \to V_j \otimes k[G]$, define

$$\Delta : V_1 \otimes V_2 \longrightarrow V_1 \otimes V_2 \otimes k[G]$$

by: if

$$\Delta_1 u = \sum_i u_i \otimes f_i, \qquad \Delta_2 v = \sum_j v_j \otimes h_j,$$

then

$$\Delta(u \otimes v) = \sum_i \sum_j u_i \otimes v_j \otimes f_i h_j.$$

Set $A := k[G]$, then

$$r_A := \text{right regular representation with } r_A(g)f(x) = f(xg).$$

The map

$$\begin{aligned} A \otimes A &\xrightarrow{m} A \\ f_1 \otimes f_2 &\longmapsto f_1 f_2 \end{aligned}$$

defines a morphism of representations

$$(A, r_A) \otimes (A, r_A) \to (A, r_A).$$

Indeed,

$$\begin{aligned} m((r_A \otimes r_A)(g)(f_1 \otimes f_2))(x) &= f_1(xg) f_2(xg), \\ &= f_1 f_2(xg) = r_A(g)(m(f_1 \otimes f_2))(x), \end{aligned}$$

since $f_1(\_g) \otimes f_2(\_g) = (r_A \otimes r_A)(g)(f_1 \otimes f_2)$.

Ergo $m \circ (r_A \otimes r_A)(g) = r_A(g) \circ m$. ∎

Recall: We stated to prove the following theorem

Theorem 11. *Let $\lambda_V \in \mathit{End}(V)$ be given s.t. for all finite-dim. rep.s V of G s.t.:*

(i) $\lambda_k = 1$

(ii) $\lambda_{V\otimes W} = \lambda_V \otimes \lambda_W$

(iii) for all morphisms of rep.s $\phi : V \to W$ we have

$$\phi \circ \lambda_V = \lambda_W \circ \phi.$$

Then, there is exactly one $g \in G$ s.t. $\lambda_V = r_V(g)$ for all V.

Proof. Last time, we saw that any such family $V \mapsto \lambda_V$ extends to **all** rep.s V of G.

Let's note also that, if (V_0, r_0) is any representation of G with trivial action, i.e. $r(g) = 1$ for all g, then $\lambda_{V_0} = 1$. Indeed, let $v \in V_0$. We must check that $\lambda_{V_0} v = v$. Since the action is trivial, any subsapce of V_0 is G-invariant.

Consider the map

$$\begin{aligned} \phi : k &\longrightarrow V_0 \\ \alpha &\longmapsto \alpha v \end{aligned}$$

where $v = \phi(1)$. Then, ϕ is a morphism of rep.s because the action is trivial.

Thus,

$$\lambda_V v = (\lambda_V \circ \phi)(1) \overset{(iii)}{=} (\phi \circ \lambda_k)(1) \overset{(i)}{=} \phi(1) = v.$$

Consider $\lambda_A \in \mathsf{End}(A)$. Then,

$$\lambda_{A\otimes A} = \lambda_A \otimes \lambda_A.$$

It is an easy exercise to see that $m : (A, r_A) \otimes (A, r_A) \to (A, r_A)$ is a morphism of rep.s.

By (iii) it follows, $m \circ (\lambda_A \otimes \lambda_A) = \lambda_A \circ m$, i.e.

$$\lambda_A(f_1 f_2) = \lambda_A(f_1)\lambda_A(f_2)$$

for all $f_1, f_2 \in A$. Thus, λ_A is an algebra morhism (check, using the morphism $k \hookrightarrow A$, that $\lambda_A(1) = 1$).

Thus, $\lambda_A = \phi^*$ for some unique morphism ϕ of algebraic sets $\phi : G \to G$.

We claim that ϕ commutes left multiplication i.e.

$$\phi(hx) = h\phi(x)$$

for all $h, x \in G$. Indeed, let's consider the map

$$\begin{aligned} A &\longrightarrow A \\ f &\longmapsto f(h \cdot \_). \end{aligned}$$

This induces a morphism

$$(A, r_A) \xrightarrow{\psi} (A, r_A).$$

By (ii), $\psi \circ \lambda_A = \lambda_A \circ \psi$.

Since $\lambda_A = \phi^*$, this implies the claim.

Now, set $g := \phi(e)$. Then for all $h \in G$,

$$\phi(h) = \phi(he) = hg.$$

Thus, $\lambda_A = \phi^* = r_A(g)$.

(It remains to show that

$$\lambda_V = r_V(g)$$

for each finite-dim. rep. V.)

Let $V = (V, r)$ be any rep. This induces a map

$$\Delta : V \longrightarrow V \otimes A.$$

If $\Delta v = \sum v_i \otimes f_i$, then

$$hv = \sum f_i(h_i) \otimes v_i.$$

Let

$$\begin{aligned} \varepsilon : V \otimes A &\longrightarrow V \\ v \otimes f &\longmapsto f(1)v. \end{aligned}$$

It follows $\varepsilon \circ \Delta : V \to V$ is the identity map.

Let (V_0, r_0) be the representation of G with $V_0 := V$ and r_0 the trivial action. Then, $\Delta : V \to V_0 \otimes A$ is a morphism of representations.

(Indeed, if $\Delta v = \sum v_i \otimes f_i$, then

$$\Delta(r(h)v) \stackrel{?}{=} (r_0(h) \otimes r_A(h))\Delta v$$

since

$$\begin{aligned}\Delta v &= \sum v_i \otimes f_i \\ \iff \ xv &= \sum f_i(x_i)v_i \ \forall x \in G \\ \iff \ xhv &= \sum f_i(xh)v_i \ \forall x, h \in G.\end{aligned}$$

Since $r(h)v = hv$, it follows

$$\Delta(hv) = \sum v_i \otimes f_i(\cdot h) \implies (?).)$$

We want to show

$$\lambda_V = r_V(g).$$

We have

$$\begin{aligned}\Delta \circ \lambda_V &\overset{(iii)}{=} \lambda_{V_0 \otimes A} \circ \Delta \\ &\overset{(ii)}{=} \lambda_{V_0} \otimes \lambda_A \\ &= 1 \otimes \lambda_A = 1 \otimes r_A(g).\end{aligned}$$

This implies

$$\Delta \circ \lambda_V = (1 \otimes r_A(g)) \circ \Delta$$

but also

$$\Delta \circ r_V(g) = (1 \otimes r_A(g)) \circ \Delta.$$

Because of the injectivity of Δ it now follows

$$\lambda_V = r_V(g).$$

□

Corollary 3. *Let $\phi : G \to H$ be any morphism of algebraic groups. Then, for all $g \in G$*

$$\begin{aligned}\phi(g)_s &= \phi(g_s) \\ \phi(g)_u &= \phi(g_u).\end{aligned}$$

Proof. Let V be any **faithful** representation of H, i.e. $r_V : H \to \mathsf{GL}(V)$ is injective, (for a finite-dim. V).

Then, $r_V \circ \phi$ is a rep. of G. To prove (i), it suffices to show

$$r_V(\phi(g)_s) = r_V(\phi(g_s))$$

since H operates faithfully on V.

We know that

$$r_V(\phi(g)_s) = r_V(\phi(g))_s$$

(characterizing property of h_s for $h \in H$). On the other hand,

$$r_V(\phi(g_s)) = (r_V \circ \phi)(g_s) = r_V(\phi(g))_s.$$

Therefore, claim (i) follows. (ii) works analogously. □

Definition 16. Let $g \in G$ where G is an algebraic group. We call g **semisimple**, if $g = g_s$.

We call g **unipotent**, if $g = g_u$.

Lemma 26. *For $g \in G$, the following are equivalent:*

(i) g is semisimple.

(ii) $r_V(g)$ is semisimple for all finite-dim. rep. V.

(iii) $r_V(g)$ is semisimple for at least one faithful f.d. rep. V of G.

We get an analogous lemma for unipotent group elements.

Proof. We have

$$\begin{aligned}
(i) &\iff g = g_s \\
&\overset{\text{Def. of } g_s \text{ by goal thm.}}{\iff} r_V(g) = r_V(g)_s \forall \text{ f.d. } V \\
&\iff r_V(g) \text{ is semisimple} \\
&\iff (ii) \implies (iii).
\end{aligned}$$

On the other hand,

$$(iii) \implies \exists \text{ faithful f.d. } V \text{ s.t. } r_V(g) = r_V(g)_s = r_V(g_s) \implies g = g_s.$$

□

6 Non-Commutative Algebra

Definition 17. A **ring** R (for now) is unital, associative but not necessarily commutative.

Example 11. The ring of matrices over some field or ring.

Definition 18. A **left ideal** $I \subset R$ is a subset that is an abelian subgroup of $(R, +)$ s.t. $ra \in I$ for all $r \in R, a \in I$.

A **right ideal** $I \subset R$ is a subset that is an abelian subgroup with

$$IR \subset I.$$

A two-sided ideal I is a subset that is a left and a right ideal of R.

It is easy to check that for any homomorphism of rings $\phi : R \to S$, $\mathsf{Kern}\phi$ is a two-sided ideal. Also, if $J \subset R$ is any two-sided ideal, then there exists a unique ring structure on R/J s.t. the projection $R \to R/J$ is a ring homomorphism.

Definition 19. A **left module** M for R is an abelian group equipped with a ring homomorphism

$$R \xrightarrow{\alpha} \mathsf{End}(M)$$

where $\mathsf{End}(M)$ acts on the left of M. We write

$$rm := \alpha(r)m.$$

We have

$$(r_1 r_2)(m) = r_1(r_2(m)).$$

If R **acts** on M by the left, we write

$$R \curvearrowright M.$$

Example 12. $M_n(k) \curvearrowright k^n$ where k^n is the space of column vectors.

If k^n denotes the space of row vectors, we have $k^n \curvearrowleft M_n(k)$.

Definition 20. A **(left) submodule** $N \subset M$ is an algebraic subgroup s.t.

$$RN \subset N.$$

It follows that N is itself is a left module.

Definition 21. A (left) module M of R is **simple** (or irreducible) if it has exactly the two submodules: $0 = \{0\}$ and M.

Definition 22. A ring R is a **division ring** (aka **skew field**) if it satisfies any of the following equivalent requirements:

(i) $R^\times = R \setminus \{0\}$ where[1] $R^\times = \{r \in R \mid \exists a, b \in R : ar = rb = 1.\}$

(ii) R has no nontrivials left or right ideals.

Definition 23. If $R \curvearrowright M$, then we can define

$$\mathsf{End}_R(M) := \{\phi \in \mathsf{End}(M) \mid \phi(rm) = r\phi(m) \ \forall r \in R, m \in M\}.$$

Note, that $\mathsf{End}_R(M)$ is a ring.

Lemma 27 (Schur's Lemma)**.** *If M is simple, then $\mathsf{End}_R(M)$ is a division ring.*

Lemma 28. *Let k be a field. Then, $M_n(k)$ has no nontrivial twosided ideals.*

Theorem 12 (Jacobson Density Theorem (Double Commutant Theorem))**.** *Suppose M is a simple left module which is finitely generated as a right D-module for $D = \mathsf{End}_R(M)$.*

Assume that R acts faithfully on M, i.e. $R \to \mathsf{End}_R(M)$ is injective.

Then, the map $R \to \mathsf{End}_D(M)$ is an isomorphism.

[1] If $ar = rb = 1$, then $a = arb = b$.

Lecture from 16.03.2020 (Corona-Madness started here...)

Recap:

– Basics: definitions, Hopf-algebras, ...

– Jordan decomposition

– Primer on non-commutative algebra

- Jacobson density theorem

– Unipotent groups

– Tori

We had last week

$$\mathsf{End}_D(M) := \{\phi \in \mathsf{End}(M) \mid \phi \circ d = d \circ \phi \ \forall d \in D\}.$$

Let k be an algebraically cloesd field, V a non-trivial finite-dimensional k-vector space and let G be a subgroup of $\mathsf{GL}(V)$ that acts **irreducibly** on V, i.e., V is G-**irreducible**, i.e., the only G-invariant subspaces of V are 0 and V.

Set

$$D := \{d \in \mathsf{End}_k(V) \mid dg = gd \ \forall g \in G\} = \operatorname{span}(G) = \left\{ \sum_{i=1}^{n} c_i g_i \mid c_i \in k, g_i \in G, n \in \mathbb{N}_0 \right\}.$$

Then,

$$D = \mathsf{End}_R(V)$$

where R is the k-subalgebra of $\mathsf{End}(V)$ that is generated by G.

Lemma 29 (Schur's Lemma). *We understand $k \hookrightarrow \mathsf{End}(V)$ as the inclusion of operations which operate by scalar multiplication*

$$k \xrightarrow{\cong} \{\phi : V \to V \mid \phi : v \mapsto t \cdot v \ \text{ for some } t \in k\}.$$

Let V be G-irreducible. Then, we have

$$D \cong k.$$

Proof. Let $d \in D$. Since $V \neq 0$, there is an eigenspace $V_\lambda \neq 0$ for d. Observe that V_λ has to be G-invariant:
if $g \in G$ and $v \in V_\lambda$, then $gv \in V_\lambda$, since

$$dgv = gdv = g(\lambda v) = \lambda gv.$$

Since V_λ is a non-trivial G-invariant subspace and V is irreducible under G, we have

$$V_\lambda = V.$$

Ergo $d = \lambda$ in the sense of $k \hookrightarrow \mathsf{End}(V)$. $\square$

Consequence of the Jacobson Density Theorem: $R = \mathsf{End}_k(V)$, i.e., G generates all linear operations on V, if V is G-irreducible.

We will prove this after a lemma.

Lemma 30. *Let V be G-irreducible.*

Let $n \in \mathbb{N}$. Set

$$V^n := V \oplus V \oplus \ldots \oplus V = V_1 \oplus \ldots \oplus V_n$$

where each $V_i = V$.

Let $v = (v_1, \ldots, v_n) \in V^n$ and set

$$Rv := \{(rv_1, \ldots, rv_n) \mid r \in R\} = \operatorname{span}\{(gv_1, \ldots, gv_n) \mid g \in G\}.$$

Then, $Rv \neq V^n$ iff the v_j are linearly dependent over k.

Consequence: Take $n := \dim(V)$. Let $\{e_1, \ldots, e_n\}$ be a basis of V and set

$$e := (e_1, \ldots, e_n) \in V^n.$$

Since the $(e_i)_i$ are linearly independent, the lemma states that $Re = V^n$.

Now, let $x \in \mathsf{End}_k(V)$. Choose $r \in R$ s.t.

$$re = (xe_1, \ldots, xe_n).$$

Then $re_i = xe_i$ for all i, thus $x = r$. Hence, $R = \mathsf{End}_k(V)$.

Proof. For $v = (v_1, \ldots, v_n) \in V^n$ choose $J \in \{1, \ldots, n\}$ as large as possible with

$$Rv + V_1 + V_2 + \ldots + V_{J-1} =: U \neq V^n.$$

Such an J does exist, since we know that $Rv \neq V^n$.

Then, $V_J \nsubseteq U$, otherwise we may increase J. Also, U is invariant by the diagonal action of G on V^n. Thus, $V_J \cap U \subsetneq V_J$ is a proper G-invariant subspace of the G-irreducible $V_J \cong V$. Therefore, $V_J \cap U = 0$.

On the other hand, by maximality of J, we have

$$U \oplus V_J = V^n.$$

Ergo, the map (composition)

$$V \cong V_J \hookrightarrow V^n \twoheadrightarrow V^n/U$$

is a G-equivariant isomorphism, since U is G-invariant.

Let $z : V^n/U \xrightarrow{\cong} V$ be the inverse isomorphism. Let l be the G-equivariant map given by

$$\begin{tikzcd} V^n \arrow[r, "l"] \arrow[d] & V \\ V^n/U \arrow[ur, "z"] & \end{tikzcd}$$

and let l_j be the G-equivariant maps by restricting l on V_j. Then $l_j \in D \cong k$.

Say $l_j = t_j \in k$. Then,

$$l(w) = t_1 w_1 + \ldots t_n w_n.$$

Since z is an isomorphism, l is nonzero and $(t_1, \ldots, t_n) \neq (0, \ldots, 0)$.

Since $l|_U = 0$, we can deduce for all $u \in U$

$$t_1 u_1 + \ldots + t_n u_n = 0.$$

But $v \in Rv \subseteq U$, so we may conclude – as required – that the $(v_i)_i$ are linearly dependent ($l(v) = 0$). $\square$

7 Unipotent Groups

Let G be a subgroup of $\mathsf{GL}(V)$ where V is a finite-dimensional vector space and k an algebrically closed field.

Definition 24. We say that G is **unipotent** if one of the following equivalent conditions hold for each $g \in G$:

- g is unipotent (i.e. $(g-1)^n = 0$ for some $n \in \mathbb{N}$).
- all eigenvalues of g are 1.
- g is conjugate to $\begin{pmatrix} 1 & \star & \star \\ 0 & \ddots & \star \\ 0 & 0 & 1 \end{pmatrix}$.

Theorem 13. *Any unipotent subgroup of $\mathsf{GL}_n(k)$ is conjugate to a subgroup of*

$$U_n := \left\{ \begin{pmatrix} 1 & \star & \star \\ 0 & \ddots & \star \\ 0 & 0 & 1 \end{pmatrix} \right\} = \left\{ U \in M_n(k) \mid U_{i,j} = \begin{cases} 0, & \text{if } i > j \\ 1, & \text{if } i = j \\ \text{arbitrary}, & \text{otherwise.} \end{cases} \right\}.$$

Definition 25. For two subgroups G, H of some common supergroup, define their **commutator** by

$$[G, H] := \left\langle ghg^{-1}h^{-1} \mid g \in G, h \in H \right\rangle.$$

A group G is called **nilpotent**, if one of its commutators is trivial, i.e. if we set

$$G_0 := G \text{ and } G_{i+1} := [G_i, G],$$

then G is called nilpotent iff there is an $j \in \mathbb{N}$ with $G_j = 1$.

Corollary 4. *Any unipotent subgroup of $\mathsf{GL}(V)$ is nilpotent.*

Definition 26. A group G is called **solvable**, if $G^{(n)} = 1$ for some n where

$$G^{(0)} := G,$$
$$G^{(i+1)} := [G^{(i)}, G^{(i)}].$$

Note that nilpotent groups are solvable, since $G^{(i)} \subset G_i$.

Notation 1. In the following, we will write $G' := [G, G]$.

Definition 27. Let $n := \dim(V)$. A **complete flag** is a maximal strictly increasing chain of subspaces
$$0 = V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_n = V.$$

Any complete flag is of the form
$$V_j := \mathrm{span}\{e_1, \ldots, e_j\}$$
for some basis $e_1, \ldots, e_n$ of V.

Let B be the basis of some flag $0 = V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_n = V$. For $x \in \mathsf{End}(V)$, we have that x is upper-triangle with respect to B iff x leaves each member V_i of the flag invariant, i.e. $xV_i \subseteq V_i$.

Proposition 2 (Key Proposition)**.** *Let G be a unipotent subgroup of $\mathsf{GL}(V)$. Then there is a complete flag $V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_n$ consisting of G-invariant subspaces, i.e., each V_i is G-invariant.*

Proof. Recall, that G is a unipotent subgroup of $\mathsf{GL}_n(V)$. We will give an induction on $n = \dim V$.

If $n = 0$, there is nothing to show.

Let $n \geq 1$. We may assume that V is G-irreducible. Because, if not, there is a G-invariant subspace $0 \neq W \subset V$ s.t. W and V/W have dimension $< n$. Then there exist complete G-invariant flags in W and V/W and the claim – that there is a complete G-invariant flag in V – follows by the induction hypothesis.

By Jacobson Density Theorem, we have
$$R := \mathrm{span}(G) = \mathsf{End}(V) := \mathsf{End}_k(V).$$

Since G is unipotent, we have for each $g \in G$
$$\mathrm{trace}(g) = n.$$

Ergo, for $g, h \in G$
$$\mathrm{trace}(gh) = \mathrm{trace}(h)$$
and
$$\mathrm{trace}((g-1)h) = \mathrm{trace}(gh) - \mathrm{trace}(h) = 0.$$

Since $\mathrm{span}(G) = \mathsf{End}(V)$, it now in particularly follows for all $g \in G, \phi \in \mathsf{End}(V)$
$$\mathrm{trace}((g-1)\phi) = 0.$$

Since the above holds for all $\phi \in \mathsf{End}(V)$, it must hold

$$g - 1 = 0$$

for all $g \in G$ (take for example the elementary matrices $\phi = E_{i,j}$). Ergo, G is trivial. Then, any complete flag is trivially G-invariant. $\square$

Remark 3. This gives the group analogue of Engel's Theorem.

Proof Goal Theorem. Let B be a basis of V s.t. G leaves each subspace in the corresponding flag invariant. Then, G is upper-triangle with respect to this basis.

On the other hand, each $g \in G$ is unipotent, hence its diagonal (i.e. eigenvalues) are all 1. Thus, with respect to B

$$G \subseteq \left\{ \begin{pmatrix} 1 & \star & \star \\ 0 & \ddots & \star \\ 0 & 0 & 1 \end{pmatrix} \right\} = U_n.$$

$\square$

Remark 4. Tori are of the form $(k^\times)^n$. In the case $k = \mathbb{C}$, $(\mathbb{C}^\times)^n$ are the complexification of $U(1)^n$. This equals tori in top. sense.

$$\begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix} \subseteq \mathsf{GL}_2(\mathbb{C})$$

is a non-algebraic unipotent group.

Exercise. (to be discussed next time)

it would have sufficed to prove the Goal theorem in the special case that G is algebraic.

Corollary of Proof: If $G \subset \mathsf{GL}(V)$ (with $V \neq 0$) is unipotent and V is G-irreducible, then $G = 1$, $\dim V = 1$.

Lecture from 18.03.2020

Answer to last Exercise: Recall that the main point was to show that any unipotent subgroup $G \subseteq \mathsf{GL}(V)$ leaves invariant some complete flag $\mathcal{F} = (V_0 \subset V_1 \ldots)$. But by some homework (problem 1), the group

$$\mathsf{GL}(V)_{\mathcal{F}} := \{g \in \mathsf{GL}(V) \mid g\mathcal{F} = \mathcal{F}\}$$

is algebraic.

Proof: If $\mathcal{F}$ is the standard flag with $V_i = \mathrm{span}(e_1, \ldots, e_i)$ for the standard basis $\{e_1, \ldots, e_n\}$, then

$$\mathsf{GL}(V)_{\mathcal{F}} = \{A \in \mathsf{GL}(V) \mid A \text{ is upper-triangle}\}.$$

The condition that A is upper triangle can be realized by polynomials. $\square$

Thus,

$$\begin{aligned} & G \text{ fixes } \mathcal{F} \\ \iff & G \subseteq \mathsf{GL}(V)_{\mathcal{F}} \\ \overset{\mathsf{GL}(V)_{\mathcal{F}} \text{ is algebraic}}{\iff} & \overline{G} \subseteq \mathsf{GL}(V)_{\mathcal{F}} \\ \iff & \overline{G} \text{ fixes } \mathcal{F}. \end{aligned}$$

Now, the Zariski-Closure $\overline{G}$ of any group G is an algebraic group (shown in some homework).

Further, if G is unipotent, then $\overline{G}$ is unipotent.

8 Tori

Definition 28. A **torus** is an algebraic group that is isomorphic to $\mathcal{G}_m^n$ for some $n \in \mathbb{N}_0$ where $\mathcal{G}_m = k^\times = \mathsf{GL}_1(k)$ is the unit group of k.

We think of $\mathcal{G}_m^n \subseteq \mathsf{GL}_n(k)$ as the subgroup of diagonal matrices.

Lemma 31. *Let G be a commutative algebraic group. Then the following are equivalent:*

(i) each $g \in G$ is semisimple.

(ii) for each finite-dimensional representation V of G and for each $g \in G$, the operator $r_V(g)$ is diagonalizable.

(iii) for all finite-dimensional representations V of G, there is a basis of common eigenvectors for $r_V(G)$, i.e. a basis s.t.

$$r_V(G) \subseteq \mathcal{G}_m^n.$$

(iv) G is isomorphic to an algebraic subgroup of a torus.

Proof. We show:

(i) $\iff$ (ii): This follows from the Jordan decomposition and definition of semisimple.

(ii) $\implies$ (iii) : This is homework. Note that any commutative subset S of $\mathsf{GL}(V)$ consisting of semisimple operators may be diagonalized simultaneously.

(iii) $\implies$ (iv) : Take any faithful representation V of G and diagonalize it simultaneously. Then, $G \cong r_V(G) \subseteq \mathcal{G}_m^n$.

(iv) $\implies$ (i) : Any diagonal matrix is semisimple.

□

Definition 29. A commutative algebraic group G is called **diagonalizable**, if it satisfies one of the above equivalent conditions.

Definition 30. A **character** χ of an algebraic group G is an element $\chi \in \mathsf{Hom}_{\text{alg.grp.}}(G, k^\times)$, i.e., a homomorphism $\chi : G \to k^\times$ of algebraic groups.

Notation 2. For an algebraic group G, set $\mathfrak{X}(G) := \mathsf{Hom}_{\text{alg.grp.}}(G, k^\times)$.

Also denote now by $\mathcal{O}(X) := k[T]/I(X)$ the coordinate ring of an algebraic set X (rather than $k[X]$).

Lemma 32. *There is a bijection*

$$\mathfrak{X}(G) = \{\textit{characters } \chi \textit{ of } G\} \longleftrightarrow \{x \in \mathcal{O}(G)^\times \mid \Delta(x) = x \otimes x\}.$$

Proof. Note, that any $x \in O(G)^\times$ can be thought of as a map $x : G \to k^\times \subset k$.

We have

$$\begin{aligned}\mathsf{Hom}_{\text{alg.grp.}}(G, \mathcal{G}_m) &= \{\phi \in \mathsf{Hom}_{\text{alg.sets}}(G, \mathcal{G}_m) \mid \phi(gh) = \phi(g)\phi(h)\ \forall g, h\} \\ &= \{\phi \in \mathsf{Hom}_{k-\text{alg.}}(\mathcal{O}(\mathcal{G}_m), \mathcal{O}(G)) \mid (\phi \otimes \phi) \circ \Delta = \Delta \circ \phi\}.\end{aligned}$$

Recall: $\mathcal{O}(\mathcal{G}_m) \cong k[t, \frac{1}{t}]$ with $\Delta(t) = t \otimes t$.

Thus for any k-algebra A, $\mathsf{Hom}_{k-\text{alg.}}(\mathcal{O}(\mathcal{G}_m), A) \overset{A^\times}{\cong}$ via

$$[t \mapsto a, (t^{-1} \mapsto a^{-1})] \longleftrightarrow a.$$

Thus,

$$\mathsf{Hom}_{\text{alg.grp.}}(G, \mathcal{G}_m) \cong \left\{a \in \mathcal{O}(G)^\times \mid a \otimes a = \Delta(a)\right\}.$$

Therefore, it suffices to test the condition $(\phi \otimes \phi) \circ \Delta = \Delta \circ \phi$ on the generators t, t^{-1} of $\mathcal{O}(\mathcal{G}_m)$. Now, the above isomorphism is given by

$$\phi \mapsto a = \phi(t)$$

which is equivalent or regarding $\chi : G \to \mathcal{G}_m$ as a map $\chi : G \to k$. □

Example 13. Let $G = \mathcal{G}_m$, then $\mathcal{O}(G) = k[t, \frac{1}{t}]$.

Which $x = \sum_{m \in \mathbb{Z}} c_m t^m \in \mathcal{O}(G)$ – with almost all $c_m = 0$, but not all of them – have the property

$$\Delta(x) = x \otimes x?$$

We have

$$\begin{aligned} x \otimes x &= \sum_{m,n \in \mathbb{Z}} c_m c_n t^m \otimes t^n, \\ \Delta(x) &= \sum_{m \in \mathbb{Z}} c_m t^m \otimes t^m. \end{aligned}$$

Those sums equal, if

$$\begin{aligned} &c_m c_n = o \text{ for all } m \neq n, \\ &c_m^2 = c_m \text{ for all m.} \end{aligned}$$

By those conditions, it follows

$$x = t^m.$$

Therefore

$$\mathfrak{X}(G) = \{\chi_m \mid m \in \mathbb{Z}\} \cong \mathbb{Z}$$

with

$$\chi_m(y) = y^m.$$

Example 14. Let $T \cong \mathcal{G}_m^n$ be a torus. Then,

$$\mathfrak{X}(T) = \{\chi_m \mid m \in \mathbb{Z}^n\} \cong \mathbb{Z}^n$$

where $\chi_m(y) = y^m = y_1^{m_1} \cdots y_n^{m_n}$.

Note: For each algebraic group G, $\mathfrak{X}(G)$ is naturally an abelian group:

$$(\chi_1 \cdot \chi_2)(g) := \chi_1(g) \cdot \chi_2(g).$$

Given a morphism of algebraic groups $f : G \to H$, we get a morphism of abelian groups

$$\begin{aligned} f^* : \mathfrak{X}(H) &\longrightarrow \mathfrak{X}(G) \\ \chi &\longmapsto \chi \circ f =: f^*(\chi). \end{aligned}$$

This induces a contravariant functor from the category of algebraic groups to the category of abelian groups.

Lemma 33. *Let G be a diagonalizable algebraic group. Then, $\mathfrak{X}(G)$ is a k-vector space basis for $\mathcal{O}(G)$.*

Example 15. Let $G = \mathcal{G}_m^n$ be a torus. Then, we have the embedding

$$\begin{aligned} \mathfrak{X}(G) &\hookrightarrow \mathcal{O}(G) \\ \chi_{(m_1,\ldots,m_n)} &\longmapsto t^{(m_1,\ldots,m_n)}. \end{aligned}$$

The lemma is obvious in this case: each elment of $\mathcal{O}(G) = k[t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1}]$ can be written uniquely as a linear combination of monomials.

Proof. (i) $\mathfrak{X}(G)$ spans $\mathcal{O}(G)$:

Choose an embedding $G \subset \mathcal{G}_m^n$ of algebraic groups. Then, by restriction, we get

$$\mathcal{O}(\mathcal{G}_m^n) \twoheadrightarrow \mathcal{O}(G).$$

Since the $\chi_m, m \in \mathbb{Z}^n$, span $\mathcal{O}(\mathcal{G}_m^n)$, their images $\chi_m|_G \in \mathfrak{X}(G)$ span $\mathcal{O}(G)$.

(ii) $\mathfrak{X}(G)$ is linearly independent:
Suppose otherwise and let $\phi_1, \dots, \phi_m$ be a linearly dependent subset of $\mathfrak{X}(G)$ with $m \geq 1$ chosen minimally, with $c_1, \dots, c_m \in k^\times$ s.t.

$$\sum_{i=1}^{m} c_i \phi_i = 0.$$

We distinguish the following cases:

$m = 1$: In this case, we have $\phi_1 = 0$, but $\phi_1(1) = 1$, a contradiction.

$m > 1$: We can assume $\phi_1 \neq \phi_2$, so there is an $h \in G$ s.t. $\phi_1(h) \neq \phi_2(h)$. Then,

$$\phi_1(h) \sum_{i=1}^{m} c_i \phi_i = 0,$$

but also for all $h, g \in G$

$$\sum_{i=1}^{m} c_i \phi_i(hg) = \sum_{i=1}^{m} c_i \phi_i(h) \phi_i(g) = 0.$$

This implies

$$\sum_{i=1}^{m} c_i \phi_i(h) \phi = 0.$$

Ergo

$$\sum_{i=1}^{m} c_j(\phi_i(h) - \phi_1(h))\phi_i = \sum_{i=2}^{m} c_j(\phi_i(h) - \phi_1(h))\phi_i = 0.$$

Now, $\phi_i(h) - \phi_1(h)$ is zero if $i = 1$ and non-zero, if $i = 2$. Therefore, this yields a shorter linear dependency for the elements

$$\phi_2, \dots, \phi_m,$$

which contradicts our requirement.

$\square$

Definition 31. Let M be an abelian group. The **group algebra** on M is the k-algebra $k[M]$ (not a coordinate ring!) defined as follows:

$$\begin{aligned} k[M] &:= \text{the } k\text{-vectorspace with basis } M \\ &:= \left\{ \sum_{m \in M} c_m \cdot m \mid c_m \in k, \text{ almost all } c_m = 0 \right\}, \end{aligned}$$

where the multiplication on $k[M]$ extends that on M:

$$(\sum_{m\in M} c_m m)(\sum_{n\in M} d_n n) = \sum_{m,n\in M} c_m d_n mn.$$

Corollary 5. *For a diagonalizable G, we have*

$$\mathcal{O}(G) \cong k[\mathfrak{X}(G)].$$

Fact: For an abelian group M, there is exactly one Hopf algebra structure on $k[M]$ given by $\Delta(m) = m \otimes m$ for all $m \in M$.

With this definition, the above isomorphism is one of Hopf algebras.

Lemma 34. *If G, H are diagonalizable algebraic groups, then*

$$\mathit{Hom}_{\text{alg.grp.s}}(G,H) \xrightarrow{f\mapsto f^*} \mathit{Hom}_{\text{grp.}s}(\mathfrak{X}(H), \mathfrak{X}(G))$$

is a bijection.

Proof.

$$\begin{aligned}\mathsf{Hom}\,(G,H) \cong & \mathsf{Hom}_{\text{Hopf}-\text{alg.}}(\mathcal{O}(H), \mathcal{O}(G)) \\ & \cong \{\phi \in \mathsf{Hom}_{k-\text{alg.}}(\mathcal{O}(H), \mathcal{O}(G)) \mid (\phi\otimes\phi)\circ\Delta = \Delta\circ\phi\}.\end{aligned}$$

Since $\mathsf{Hom}_{k-\text{alg.}}(\mathcal{O}(H), \mathcal{O}(G)) \cong \mathsf{Hom}\,(k[\mathfrak{X}(H)], k[\mathfrak{X}(G)])$, this reduces to the following lemma:

Lemma 35. *Let $M_1, M2$ be two abelian groups. Then*

$$\begin{aligned}\mathit{Hom}\,(M_1, M_2) &\xrightarrow{\cong} \mathit{Hom}_{\text{Hopf}-\text{alg.}}(k[M_1], k[M_2]) \\ \phi &\longmapsto \left[\sum c_m m \mapsto \sum c_m \phi(m)\right].\end{aligned}$$

Proof. We have to show that

$$M = \left\{x \in K[M]^\times \mid \Delta(x) = x \otimes x\right\}.$$

Then, by this, it follows for each $\phi \in \mathsf{Hom}_{\text{Hopf}-\text{alg.}}(k[M_1], k[M_2])$,

$$\phi(M_1) \subseteq M_2.$$

Ergo, $\phi|_{M_1} \in \mathsf{Hom}\left(M_1, M_2\right)$. Therefore, the surjectivity of the claimed bijection is shown. The injectivity is clear, since M generates $k[M]$ as a k-algebra.

To show

$$M = \left\{x \in K[M]^\times \mid \Delta(x) = x \otimes x\right\},$$

let

$$\begin{aligned} x &= \sum c_m m \in K[M]^\times \\ \Delta(x) &= \sum c_m m \otimes m \\ x \otimes x &= \sum c_m c_n m \otimes n. \end{aligned}$$

If $\Delta(x) = x \otimes x$, then it follows

$$x = m$$

for some $m \in M$.

$\square$

$\square$

Recall: We have seen that for diagonalizable algebraic groups G, H Lecture from 25.03.2020

$$\mathsf{Hom}\,(G, H) \cong \mathsf{Hom}\,(\mathfrak{X}(H), \mathfrak{X}(G))\,.$$

If G is diagonalizable, then

$$\mathcal{O}(G) \cong k[\mathfrak{X}(G)].$$

Theorem 14. *The functor*

$$\begin{aligned} G &\longrightarrow \mathfrak{X}(G) \\ f &\longmapsto f^* \end{aligned}$$

defines an equivalence of categories:

$\{$*diagonalizable alg. groups*$\} \cong \{$*finite-dim. abelian groups with no* $\mathrm{char}(k)$*-torsion*$\}$.

This amounts to the bijection above between Hom-spaces and the following lemma.

Lemma 36. *(i) Let G be a diagonalizable alg. group. Then, $\mathfrak{X}(G)$ is a finitely generated abelian group with no* $\mathrm{char}(k)$*-torsion.*

(ii) Let Γ be a finitely generated abelian group with no $\mathrm{char}(k)$*-torsion. Then, there is a diagonalizable algebraic group G s.t. $\mathfrak{X}(G) \cong \Gamma$.*

Proof. We will use the following facts:

- Let $n \in \mathbb{N}$. Then, $t^n - 1$ is square-free in $k[t]$ iff the ideal $(t^n - 1)$ is radical in $k[t]$ iff $t^n - 1$ has not repetitive root iff either $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) = p > 0$ and $p \not| n$.

  (Proof: Galois Theory, seperable/inseperable extensions.)

- Let $M := \mathbb{Z}/n\mathbb{Z}$. Then, the k-group-algebra generated by M

  $$k[M] \cong k[t]/(t^n - 1)$$

  is reduced iff either $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) = p > 0, p \not| n$.

- If M_1, M_2 are abelian groups, then we have the following isomorphism of Hopf algebras

$$\begin{aligned} k[M_1] \otimes_k k[M_2] &\xrightarrow{\cong} k[M_1 \oplus M_2] \\ m_1 \otimes m_2 &\longmapsto m_1 m_2 \end{aligned}$$

where $M_1 \oplus M_2 \cong M_1 \times M_2$.

(i) Embed $G \hookrightarrow T := \mathcal{G}_m^n$ for some n. Then, we have a surjection $\mathbb{Z}^n \cong \mathfrak{X}(T) \twoheadrightarrow \Xi(G)$. Ergo, $\mathfrak{X}(G)$ is finitely generated.

Suppose $\mathrm{char}(k) = p > 0$. Let $\chi \in \mathfrak{X}(G)$ with $\chi^p = 1$. Then, for all $g \in G$, $\chi^p(g) = \chi(g^p) = 1$. The unit group $k^\times$ has not p-torsion, therefore $G \hookrightarrow T = (k^\times)^n$ has also no p-torsion. Therefore, the frobenius $g \mapsto g^p$ is an isomorphism on G. Therefore, $\chi = 1$ is a trivial character. Ergo $\mathfrak{X}(G)$ has no p-torsion.

(ii) Let Γ be a finitely generated abelian group with no $\mathrm{char}(k)$-torsion. Then,

$$\Gamma \cong \mathbb{Z}^r \oplus \mathbb{Z}/n_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/n_l\mathbb{Z}$$

where $\mathrm{char}(k) \nmid n_1, \ldots, n_l$. We may reduce to the cases:

(a) $\Gamma = \mathbb{Z}$: take $G = \mathcal{G}_m$, then $\Xi(G) \cong \mathbb{Z} \cong \Gamma$.

(b) $\Gamma = \mathbb{Z}/n\mathbb{Z}$ with $\mathrm{char}(k) =: p \nmid n$:
take $G := \mu_n := \{y \in k^\times \mid y^n = 1\}$. Then, since $p \nmid n$, $(t^n - 1)$ is radical. So,

$$\mathcal{O}(\mu_n) \overset{\text{Nullstellensatz}}{=} k[t]/(t^n - 1) \overset{\text{as Hopf algebras}}{\cong} k[\Gamma]$$

where t gets mapped to the generator of Γ.

$\square$

Corollary 6. *We have the bijection*

$$\{\textit{tori}\} \cong \{\textit{ finitely generated free abelian groups}(\cong \mathbb{Z}^n)\}.$$

Remark 5.

$$\{\text{algebraic group schemes}/k\} \overset{\text{not necessarily natural}}{\cong} \{\text{ f.g. Hopf algebras}\}.$$

by

$$G \mapsto \mathcal{O}(G)$$

and

$$\{\text{diagonalizable algebraic group schemes}/k\} \cong \{\text{ f.g. abelian groups}\}.$$

by

$$G \mapsto \mathfrak{X}(G).$$

Where μ_p in the left hand term gets mapped to $\mathcal{O}(\mu_p) = k[t]/(t^p - 1)$ with $p = \mathrm{char} k$.

9 Trigonalization

We say a representation $r : G \to \mathsf{GL}(V)$ of a group G on a finite-dimensional k-vectorspace V is **trigonalizable** if it admits a basis with respect to which $r(V)$ is upper-triangular:

$$r(G) \subseteq \left\{ \begin{pmatrix} * & \dots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & * \end{pmatrix} \right\}$$

Definition 32. We call a subgroup $G \subseteq \mathsf{GL}(V)$ **trigonalizable**, if the identity representation is.

Lemma 37. *Let G be an algebraic group. The following are equivalent:*

(i) Every finite-dimensional representation $r : G \to \mathsf{GL}(V)$ is trigonalizable.

(ii) Every irreducible representation of G is 1-dimensional.

(iii) G is isomorphic to an algebraic subgroup of

$$B_n := \left\{ \begin{pmatrix} * & \dots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & * \end{pmatrix} \right\} \subseteq \mathsf{GL}_n(k).$$

(iv) There is a normal unipotent algebraic subgroup U of G s.t. G/U is diagonalizable.

Proof. We prove as follows:

(i) $\implies$ (ii): Let V be an irreducible representation. Then, $V \neq 0$. Choose a basis $e_1, \dots, e_n$ of V s.t.

$$r(G) \subseteq B_n.$$

Then, $r(G)e_1 \subseteq ke_1$, so $V_0 := ke_1$ is G-invariant. Ergo $V = V_0$ is 1-dimensional.

(ii) $\implies$ (i): Let V be a f.d. representation. We show by induction on $\dim(V)$ that $r : G \to \mathsf{GL}(V)$ is trigonalizable:

In the cases $\dim(V) = 0, 1$, there is nothing to show.

In the case $\dim(V) \geq 2$, assume that V is not irreducible. Then, there is a G-invariant V_0 with $0 \neq V_0 \neq V$.

By the induction hypothesis, V_0 and V/V_0 are trigonalizable. Ergo, V is trigonalizable.

(**Recall:** we used this criterion above in the proof that unipotent groups are trigonalizable by showing that every ??? of each G is trivial.)

(i) $\implies$ (iii): Choose a faithful representation V of G. Then, $G \cong r(G)$. Since r is trigonalizable, there is a basis of V s.t.

$$r(G) \subseteq B_n \subseteq \mathsf{GL}_n(k).$$

(iii) $\implies$ (ii): Suppose $G \subseteq B_n \subseteq \mathsf{GL}_n(k)$. Set

$$\begin{aligned} A_n &:= \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & * \end{pmatrix} \right\} \subseteq \mathsf{GL}_n(k), \\ U_n &:= \left\{ \begin{pmatrix} 1 & \dots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & 1 \end{pmatrix} \right\} \subseteq \mathsf{GL}_n(k) \text{ normal algebraic subgroup of } B_n, \\ U &:= G \cap U_n \text{ normal unipotent algebraic subgroup of } G. \end{aligned}$$

Let V be an irreducible representation of G, then V is not zero. Consider the subspace of V fixed by U

$$V^U := \{v \in V \mid r(u)v = v \forall u \in U\}.$$

Then, we get a representation

$$r|_U : U \longrightarrow \mathsf{GL}(V).$$

Then, $r(U)$ is a unipotent algebraic group of $\mathsf{GL}(V)$. Ergo,

$$r(U) \subseteq \left\{ \begin{pmatrix} 1 & \dots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Ergo, $V^U \neq 0$. Since U is normal in G, the subspace V^U of V is G-invariant: if $v \in V^U, g \in G$, then for all $u \in U$ we have

$$r(u)r(g)v = r(g)r(g^{-1}ug)v = r(g)v$$

since $v \in V^U$. Ergo $r(g)v \in V^U$.

Since V is irreducible, $V = V^U$, i.e. U acts trivially on V. Ergo, r descends to a representation of the group G/U.

But $G/U \hookrightarrow B_n/U_n \cong A_n$. Therefore, G/U and $r(G)$ are commutative. Moreover, for all $g \in G$, $r(g) \in \mathsf{GL}(V)$ is semisimple:

if $g = g_s g_u$, then $g_u \in U$, because U_n is the group of unipotent elements of B_n.

Hence, $r(g) = r(g_s)r(g_u) = r_(g_s)$ is semisimple.

It follows that $r(G)$ is commutative and consists of semisimple elements. By some HW: $r(G)$ is trigonalizable. It is easy to show now that V is one-dimensional. (Since V is irreducible and ke_1 is G-invariant.)

□

Definition 33. G is **trigonalizable**, if it satisfies one of the above equivalent conditions.

Later, we will see, that if G is connected, then being trigonalizable implies being solvable.

10 Commutative Groups

Let G be an algebraic group. Denote by G_s resp. G_u the subsets of semisimple resp. unipotent elements of G.

Then, G_u is always algebraical i.e. closed: if $G \hookrightarrow \mathsf{GL}_n(k)$, then $G_u = \{g \mid (g-1)^n = 0\}$. G_u does not need to be closed under multiplication (for example, take $G = \mathsf{SL}_2(k)$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$).

G_s needs not to be algebraic: for example, take $G = \mathsf{SL}_2(k)$ and if G_s were algebraic, then

$$\left\{\lambda \in k^\times \mid \begin{pmatrix} \lambda & 1 \\ 0 & \lambda^{-1} \end{pmatrix} \in G_s\right\} = \left\{\lambda \mid \lambda \neq \lambda^{-1}\right\}$$

but the last set is not algebraic. Also, G_s does not need to be a group.

We have the a surjective map of sets

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$$\begin{aligned} G_s \times G_u &\longrightarrow G \\ (g_1, g_2) &\longmapsto g_1 g_2. \end{aligned}$$

Example 16 (Non-Example)**.** Take generic $g \in G_s, h \in G_u$ for $G = \mathsf{SL}_2(k)$. Then, g, h do not commute and we have

$$((gh)_s, (gh_u)) \neq (g, h)$$

because Jordan components don't commute.

Theorem 15. *Let G be a commutative algebraic group. Then:*

(i) G_s, G_u are closed subgroups and the multiplicative map $G_s \times G_u \to G$ is an isomorphism of algebraic groups.

(ii) G is trigonalizable. Moreover, for each finite dimensional representation $r : G \to \mathsf{GL}(V)$ there is a basis s.t.

$$r(G_s) \subseteq \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & * \end{pmatrix} \right\} \qquad r(G_u) \subseteq \left\{ \begin{pmatrix} 1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

(iii) G_s is diagonalizable.

Proof. (ii) Let V be any irreducible representation of G. We have seen that commuting semisimple operators may be simultanously diagonalizable, then

$$V = \bigoplus_{\chi : G_s \to \mathcal{G}_m} V_\chi$$

where
$$V_\chi = \{ v \in V \mid r(h)v = \chi(h)v \ \forall h \in G_s \}.$$

Since G is commutative, each subspace V_χ is G-invariant ($r(h)r(g)v = r(g)r(h)v = r(g)\chi(h)v = \chi(h)r(g)v$).

Since V is irreducible, we must have $V = V_\chi$ for some χ.

Recall that $G \cong G_s \times G_u$ as abstract groups. We have seen that $r(G_s) \subseteq \mathcal{G}_m^n$. We proved a while ago that any unipotent group, such as G_u, is trigonalizable. Ergo, V is trigonalizable. Since V is irreducible, we have $\dim V = 1$.

If we apply the same argument without assuming that V is irreducible, then we see that V is the coproduct of V_χ's as above and that each V_χ admits a basis s.t.

$$r(G_s)|_{V_\chi} \subseteq \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & * \end{pmatrix} \right\} \qquad r(G_u)|_{V_\chi} \subseteq \left\{ \begin{pmatrix} 1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

This yields the same conclusion for V.

(i) We have to show that G_s and G_u are closed and $j : G_s \times G_u \to G$ is an isomorphism of groups. Take any faithful representation

$$G \xrightarrow{\cong, r} r(G) \subseteq \mathsf{GL}(V)$$

and apply (ii). Then we have

$$r(G) \subseteq \left\{ \begin{pmatrix} * & * & * \\ 0 & \ddots & * \\ 0 & 0 & * \end{pmatrix} \right\} =: B$$

$$B_u = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & 1 \end{pmatrix} \right\},$$

$$r(G_s) \subseteq \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & * \end{pmatrix} \right\} =: A.$$

In fact, $r(G_s) = r(G) \cap A$, because if $g \in G$ with $r(g) \in A$, then $r(g)$ is semisimple, so $g \in G_s$.

Therefore, G_s is closed in G. Ergo, G_s and G_u are closed subgroups.

Then, the map j is a morphism of algebraic groups.

We need to show that j^{-1} is a morphism of algebraic groups. For this, it suffices to verify that the projection $G \to G_s$ is a morphism. But this map is given under r by the morphism:

$$t := \begin{pmatrix} a_1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & a_n \end{pmatrix} \longmapsto \begin{pmatrix} a_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_n \end{pmatrix} =: t_s.$$

This suffices because if $g = g_s g_u$, then $g_u = g_s^{-1} g$, so if the map $g \mapsto g_s$ is a morphism, so is $g \mapsto g g_s^{-1} = g_u$, hence so is $g \mapsto (g_s, g_u)$.

(iii) We have seen that G_s is a closed subgroup. Hence G_s is a commutative algebraic group where elements are semisimple. Ergo, G_s is diagonalizable.

□

11 Connected Solvable Groups

Theorem 16 (Lie-Kolchin)**.** *Let G be a connected solvable algebraic group. Then G is trigonalizable.*

(By comparison, recall that we have seen so far that, if G is commutative or unipotent, then G is trigonalizable.) We can reformulate this theorem as: Any connected solvable subgroup of $\mathsf{GL}(V)$ stabilizes some complete flag $\mathcal{F} = (V_0 \subsetneq \ldots \subsetneq V_n)$.

Generalization (Borel's Fixed Point Theorem): Any connected algebraic group G acting on a projective variety X has a fixed point in X.

We get a relation between complete flags and projective varieties.

Proof. Induct on the number n s.t. $G^{(n)} = 1$.

For $n = 0$, there is nothing to show.

If $n = 1$, $(G, G) = 1$, then G is commutative, ergo trigonalizable.

Let $n \geq 2$. Then, we have $G' := (G, G) \neq 1$. We will show the following lemma:

□

Lemma 38. *Let $G \subseteq \mathsf{GL}(V)$ be a subgroup.*

If G is connected, then the group G' with the induced topology is connected ($\iff$ the Zariski Closure of G' is connected).

Proof. We have the following facts:

- An increasing union of connected spaces is connected.
- A continuous image of a connected space is connected.

We have

$$\begin{aligned} G' &= \left\langle (g, h) := ghg^{-1}h^{-1} \mid g, h \in G \right\rangle \\ &= \bigcup_{j \geq 0} \bigcup_{g_1, h_1, \ldots, g_j, h_j \in G} \{(g_1, h_1) \cdots (g_j, h_j)\}. \end{aligned}$$

Since

$$\bigcup_{g_1,h_1,\ldots,g_j,h_j\in G}\{(g_1,h_1)\cdots(g_j,h_j)\} = \mathsf{Img}\phi_j$$

for some continuous map $\phi_j : G^{2j} \to G$, the claim follows. Ergo, G' is connected. $\square$

Remark 6. It is equivalent to show that $(*)$ any subgroup G of $\mathsf{GL}(V)$ – s.t. G is connected and solvable – is trigonalizable in $\mathsf{GL}(V)$.

Indeed, the theorem implies $(*)$: the Zariski closure of G is a connected algebraic group that is solvable (which extends by continuity). If $Zcl(G)$ is trigonalizable, then also G is trigonalizable.

On the other hand: $(*)$ implies the theorem, since if G is given as in the theorem, apply $(*)$ to $r(G) \subseteq \mathsf{GL}(V)$.

Proof of Theorem. If $G^{(n)} = 1$, then $(G')^{(n-1)} = G^{(n)} = 1$. By induction, we may assume that G' satisfies the following:

For all finite dimensional representations $r : G \to \mathsf{GL}(V)$, $r(G')$ is trigonalizable.

Our aim is to show that any irreducible representation V of G has dimension 1.

The induction hypothesis implies that $r(G')$ is trigonalizable. In particular, there exists an eigenspace $V_\chi \subseteq V$ for G' for some character $\chi : G' \to k^\times$. Since G' is normal in G we know that G acts from the left on

$$\{\text{eigenspaces } V_\chi \text{ in } V \text{ for } G'\}.$$

Ergo, $\bigoplus_{\chi:G'\to k^\times} V_\chi$ is G-invariant. Since V is G-irreducible, we have

$$V = \bigoplus_{\chi:G'\to k^\times} V_\chi = \bigoplus_{\chi\in\mathfrak{X}'} V_\chi$$

for some finite subset $\mathfrak{X}' = \{\chi \mid V_\chi \neq 0\}$ of $\mathsf{Hom}\,(G', \mathcal{G}_m)$, since V is finite dimensional.

Claim: Let $h \in G'$. Then, the map

$$\begin{aligned} G &\longrightarrow \mathsf{GL}(V) \\ g &\longmapsto r(ghg^{-1}) \end{aligned}$$

has a finite image.

Proof. Denote by $\chi \mapsto \chi^g$ the action of $g \in G$ in $\mathsf{Hom}(G', \mathcal{G}_m)$ given by $\chi^g(h) := \chi(ghg^{-1})$. This is an action, since G' is normal.

Note, that $\mathfrak{X}' \subseteq \mathsf{Hom}(G', \mathcal{G}_m)$ is a finite subset.

Also note, that the action $\chi \mapsto \chi^g$ descends to an action $G \curvearrowright \mathfrak{X}'$.

Now, let $\mathfrak{X}' = \{\chi_1, \dots, \chi_r\}$. The matrix $r(h)$ is totally determined by the values $\chi_1(h), \dots, \chi_r(h)$. Then, the element $r(ghg^{-1})$ is totally determined by the values $\chi_1^g(h), \dots, \chi_r^g(h)$. It follows

$$\#\left\{r(ghg^{-1}) \mid g \in G\right\} \leq r!.$$

□

The following lemma is easy to show:

Lemma 39. *Let G be an algebraic set. Then, G is connected iff for each finite algebraic set X, and for each morphism $f : G \to X$ of algebraic sets, we have that f is constant.*

Claim with the Lemma implies that the map $g \mapsto t(ghg^{-1})$ is constant. This implies that $r(ghg^{-1}) = r(h)$ for all $g \in G, h \in G'$. Ergo, G stabilizes each eigenspace V_χ for G'. Ergo, $V = V_{\chi_0}$, since V is irreducible. □

Lemma 40. *Let G be any group with a finite dimensional representation $r : G \to \mathsf{GL}(V)$. Then, the subspaces V_χ for $\chi \in \mathsf{Hom}(G, k^\times)$ are linearly independent, i.e., the map*

$$\oplus V_\chi \longrightarrow V$$

is injective.

Proof. The spaces V_χ are G-invariant. Suppose, there exist distinct $\chi_1, \dots, \chi_n$ of non-zero $v_j \in V_{\chi_j}$ s.t. $\sum_j v_j = 0$.

We may assume that n, the number of v_j, is minimal. W.l.o.g., $n \geq 2$.

Choose $g \in G$ s.t. $\chi_1(g) \neq \chi_2(g)$. Use that $0 = g \sum_j v_j = \sum_j g v_j$ and take the linear combination as in the proof of linear indpendence of characters to contradict the minimality of n.

($g - \chi_1(g)$ is not zero, but reduces $\sum_j v_j$ by one summand.) □

Finishing Proof of Theorem. Since $G' = \langle ghg^{-1}h^{-1} \mid g, h \in G\rangle$, so $\det(r(G')) = 1$. Lecture from 01.04.2020

On the other hand, for each $g \in G'$, we have

$$r(g) = \begin{pmatrix} \chi_0(g) & & \\ & \ddots & \\ & & \chi_0(g) \end{pmatrix}$$

since $V = V_{\chi_0}$. This implies

$$1 = \det(r(g)) = \chi_0(g)^d.$$

Ergo, χ_0 defines a morphism

$$\chi_0 : G' \longrightarrow \mu_d \subseteq \mathcal{G}_m.$$

But G' is connected and μ_d is finite. Since χ_0 is a morphism, χ_0 must be constant, ergo the trivial character.

As a consequence, we get $r(G') = 1$ on $V = V_{\chi_0}$.

Lemma 41. *Let G be an algebraic group, $r : G \to \mathsf{GL}(V)$ a representation. $v \in V$ shall be a simultaneous non-zero eigenvector for $r(G)$.*

Then, for each $g \in G$, there is a value $\chi(g) \in k^\times$ s.t.

$$r(g)v =: \chi(g)v.$$

Then, the mapping $\chi : G \to \mathcal{G}_m$ is a morphism of algebraic groups.

Therefore, r descends to a representation of the commutative group

$$\overline{r} : G/G' \longrightarrow \mathsf{GL}(V).$$

Ergo, $r(G/G') = r(G)$ is commutative and therefore trigonalizable (because of irreducibility).

□

Example 17 (Non-Example)**.**

- Take $G = D_4 \hookrightarrow \mathsf{GL}_2(\mathbb{C})$ which is solvable and has an irreducible and faithful representation over $\mathbb{C}^2$.

- Consider the solvable group

$$G = \left\langle \begin{pmatrix} \pm 1 & \\ & \pm 1 \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \right\rangle$$

  which is a finite subgroup of $\mathsf{GL}_2(\mathbb{C})$, s.t. $\mathbb{C}^2$ defines an irreducible representation of G.

Lemma 42 (Form of Schur's Lemma)**.** *Let S be any commutative subset of $\mathsf{GL}(V)$ for a finite-dimensional $0 \neq V$ over an algebrically closed field k. Let V be S-irreducible.*

Then, $\dim V = 1$.

Proof. There is nothing to show if S is empty.

Let $s \in S$ and denote by $V_\lambda \subseteq V$ the λ-eigenspace for s. Then, since S is commutative, V_λ is S-invariant. Therefore, $V = V_\lambda$ for one $\lambda \in k^\times$.

Thus, every $s \in S$ acts by scaling, therefore every subspace of V is S-invariant. Since V is invariant, we get $\dim V = 1$. □

Corollary 7. *Let G be a connected algebraic group. Then, G is solvable iff G is trigonalizable.*

Proposition 3. *If G is trigonalizable, then G_u is a normal algebraic subgroup.*

Proof. We have

$$G \hookrightarrow B := \left\{ \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ & & * \end{pmatrix} \right\} \subseteq \mathsf{GL}_n(k).$$

B has the normal subgroup $U := \left\{ \begin{pmatrix} 1 & \cdots & * \\ & \ddots & \vdots \\ & & 1 \end{pmatrix} \right\}$ and we have $G_u = G \cap U$. Now, U is the kernel of the multiplicative morphism

$$\begin{pmatrix} a_1 & \cdots & * \\ & \ddots & \vdots \\ & & a_n \end{pmatrix} \longmapsto \begin{pmatrix} a_1 & \\ & a_n \end{pmatrix}.$$

□

Corollary 8. *If G is connected and solvable, then G_u is a normal algebraic subgroup.*

12 Semisimple Elements of nilpotent Groups

Theorem 17. *Let G be a connected nilpotent algebraic group. Then, we have*

$$G_s \subseteq Z(G)$$

*where $Z(G)$ denotes the **center** of G, i.e.*

$$Z(G) = \{g \in G \mid \forall h \in G : \; gh = hg\}.$$

Theorem 18 (Lie-algebraic Analogue)**.** *Let V be a finite-dimensional vectorspace. Let $\mathfrak{g}$ be the Lie-Subalgebra of $\mathsf{End}(V)$, i.e. $\mathfrak{g}$ is a subspace s.t. we have for each $x, y \in \mathfrak{g}$*

$$[x, y] := xy - yx \in \mathfrak{g}.$$

Assume that $\mathfrak{g}$ is nilpotent, i.e. there is an $n \in \mathbb{N}_0$ s.t.

$$[x_1, [x_2, [\dots, [x_{n-1}, x_n]]]] = 0$$

for all $x_1, \dots, x_n \in \mathfrak{g}$.

*Then, any semisimple (semisimple in $\mathsf{End}(V)$ that is) $x \in \mathfrak{g}$ is **central** in $\mathfrak{g}$, i.e. $[x, y] = 0$ for each $y \in \mathfrak{g}$.*

Remark 7. The Lie-algebraic Analogue implies the general theorem if – for example – $k = \mathbb{C}$.

Proof. Let $g \in G_s$. We want to show $Z_G(g) = G$.

Fact from the theory of Lie-Algebras: For the Lie-Algebra $\mathrm{Lie} Z_G(g)$ we have

$$\mathrm{Lie} Z_G(g) = \ker(\mathsf{Ad}(g))$$

where Ad is the map

$$\begin{aligned} \mathsf{Ad} : G &\longrightarrow \mathsf{GL}(\mathfrak{g}) \\ x &\longmapsto gxg^{-1}. \end{aligned}$$

Since G is connected, it suffices to verify

$$\ker(\mathsf{Ad}(g)) = \mathfrak{g}$$

i.e. $\mathsf{Ad}(g) = 1$.

Since g is semisimple, we have for suitable basis

$$g = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$$

with $a_j \in \mathbb{C}^\times$. This is $\exp(x)$ for a suitable diagonal matrix $x \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} \in$ $\mathsf{GL}_n(\mathbb{C})$.

Fact: We may assume that $x \in \mathfrak{g} := \mathrm{Lie}(G)$.

Since G is nilpotent, it can be shown that $\mathfrak{g}$ is nilpotent.

By the theorem, x is central in $\mathfrak{g}$. By the properties of exp we have

$$\mathsf{Ad}(g) = \exp(\mathrm{ad}(g)) = 1$$

ergo $\mathrm{ad}(x) = 0$ where $\mathrm{ad} : \mathfrak{g} \to \mathfrak{g}$ is defined by

$$\mathrm{ad}(x) \cdot y := [x, y].$$

□

Proof. If $\mathfrak{g}$ is nilpotent, then $\mathrm{ad}(x) \in \mathsf{End}(\mathfrak{g})$ is nilpotent.

Since x is semisimple, $\mathrm{ad}(x)$ is semisimple, because $\mathrm{ad}(x)$ is the restriction to $\mathfrak{g}$ of the map

$$\begin{aligned} \mathsf{End}(V) &\longrightarrow \mathsf{End}(V) \\ y &\longmapsto [x, y] \end{aligned}$$

and, if $e_1, \ldots, e_n$ are a basis of eigenvectors for x, then $E_{i,j}$ is a basis of eigenvectors for ℓ.

So, $\mathrm{ad}(x)$ is nilpotent and semisimple, therefore $\mathrm{ad}(x) = 0$. □

Proof Theorem. Let G be a connected nilpotent algebraic group, $G \overset{\mathsf{GL}}{\hookrightarrow} (V)$.

Let $g \in G_s$, we want to show that $g \in Z(G)$.

Assume otherwise, then we have a $h \in G$ s.t. $(g, h) = ghg^{-1}h^{-1} \neq 1$.

Since G is connected and nilpotent (ergo solvable), we know by Lie-Kolchin that G stabilizes some complete flag $V_0 \subset \ldots \subset V_n$.

We have $g|_{V_i}, h|_{V_i} \in \mathsf{GL}(V_i)$. They commute, if $i = 0$, but not if $i = n$.

So, there is an i s.t. $g|_{V_i}, h|_{V_i}$ commute but $g|_{V_{i+1}}, h|_{V_{i+1}}$ don't commute. W.l.o.g. $V = V_{i+1}, g = g|_{V_{i+1}}, h = h|_{V_{i+1}}$. Set $a := g|_{V_i}, b := h|_{V_i} \in \mathsf{GL}(V_i)$. a will be semisimple, since g is.

Since g is semisimple, there is an eigenvector $v \in V_{i+1}$ for g s.t.

$$V_{i+1} = V_i \oplus \langle v \rangle .$$

We have an isomorphism of vector spaces

$$\mathsf{End}(V_{i+1}) \cong \mathsf{End}(V_i) \oplus \mathsf{Hom}\left(\langle v \rangle, V_i\right) \oplus \mathsf{Hom}\left(V_i, \langle v \rangle\right) \oplus \mathsf{End}(\langle v \rangle)$$

with

$$\mathsf{End}(\langle v \rangle) \cong k \text{ and } \mathsf{Hom}\left(\langle v \rangle, V_i\right) \cong V_i.$$

So, we can write $g|_{V_{i+1}}, h|_{V_{i+1}}$ write as

$$g = \begin{pmatrix} a & \\ & * \in k \end{pmatrix} \text{ and } h = \begin{pmatrix} b & c \in V_i \\ & * \end{pmatrix}.$$

We may replace g, h with scalar multiples to reduce to the case that $* = 1$. Then, So, we can write $g|_{V_{i+1}}, h|_{V_{i+1}}$ write as

$$g = \begin{pmatrix} a & \\ & 1 \end{pmatrix} \text{ and } h = \begin{pmatrix} b & c \\ & 1 \end{pmatrix}.$$

Then,

$$h \neq ghg^{-1} = \begin{pmatrix} b & ac \\ & 1 \end{pmatrix}.$$

Ergo, $c \neq ac$, i.e. $c \notin \ker(a-1)$. Define

$$h_1 := h^{-1}ghg^{-1} = \begin{pmatrix} 1 & b^{-1}(a-1)c \\ & 1 \end{pmatrix}.$$

We claim that h_1 does not commute with g. This claim implies the theorem, since we can iterate the claim to obtain elements h_i by $h_{i+1} := h_i^{-1} g h_i g^{-1}$. Then, h_i does not commute with g. But G is nilpotent, therefore $h_i = 1$ for some large enough i.

We can prove the claim as follows: By some calculation as for h and g, we see, that h_1 and g don't commute iff $b^{-1}(a-1)c \notin \ker(a-1)$. This is equivalent to

$$\begin{aligned}
&\iff (a-1)b^{-1}(a-1)c \neq 0 \\
&\iff b^{-1}(a-1)^2 c \neq 0 \\
&\iff (a-1)^2 c \neq 0 \\
&\iff c \notin \ker((a-1)^2).
\end{aligned}$$

But a being semisimple implies $a-1$ being semisimple, therefore

$$\ker((a-1)^2) = \ker(a-1).$$

So h_1, g don't commute iff $c \in \ker(a-1)$ iff h, g don't commute. $\square$

13 Algebraic Geometry

Lecture from 06.04.2020

13.1 Projective Algebraic Sets

Let V be a finite-dimensional vector space. Then $\mathcal{G}_m = k^\times$ acts on V by scalar multiplication. $\{0\}$ is a $\mathcal{G}_m$-invariant subspace of V. We are interested on the orbits of $\mathcal{G}_m$ on $V \setminus \{0\}$.

Define the **projective space** over V by

$$\mathbb{P}V := \mathcal{G}_m \backslash (V - 0) = (V - 0)/\sim \cong \{\text{lines in } V\}$$

where for $a, b \in V - 0$ we set

$$a \sim b :\Longleftrightarrow \exists \lambda \in k^\times : \lambda a = b.$$

If $V = k^{n+1}$, we denote the n-dimensional projective space by $\mathbb{P}^n := \mathbb{P}V$.

Given $a = (a_0, a_1 \ldots, a_n) \in k^{n+1} - 0$, we denote the $\sim$-class of a by

$$[a] = [a_0, \ldots a_n] \in \mathbb{P}^n.$$

Define S to be the graded algebra of polynomials in k

$$S := k[x_0, \ldots, x_n] = \bigoplus_{d \geq 0} S_d$$

where each S_d is the space of homogenous polynomials of degree d, i.e.

$$S_d = \bigoplus_{i_1, \ldots, i_d \in \{0, \ldots, n\}} k \cdot x_{i_1} \cdots x_{i_d}.$$

We identify k with the space of constant polynomials $S_0 \subseteq S$.

We have

$$S_d = \left\{ f \in S \mid f(\lambda X) = \lambda^d f(X) \ \forall \lambda \in k^\times \right\}.$$

Given $f \in S_d$, the set

$$\left\{ a \in k^{n+1} \mid f(a) = 0 \right\}$$

is $\mathcal{G}_m$-invariant. In other words, given $a \in \mathbb{P}^n$ and $f \in S^d$, it is well-defined to state $f(a) = 0$ and $f(a) \neq 0$.

Definition 34. A **projective** algebraic subset $X \subseteq \mathbb{P}^n$ is a set of the form

$$X = V(\Sigma) := V_{\mathbb{P}^n}(\Sigma)$$

where Σ is a collection of homogenous elements of S, where

$$V_{\mathbb{P}^n}(\Sigma) := \{a \in \mathbb{P}^n \mid f(a) = 0 \ \forall f \in \Sigma\}.$$

Facts:

- Hilbert's basis theorem states

  $$V(\Sigma) = V(f_1, \dots, f_m)$$

  for some finite collection $f_1, \dots, f_m \in \Sigma$.

- It is useful to extend the meaning of "$f(a) = 0$" for $a \in \mathbb{P}^n$ to *general* elements $f \in S$ by requiring that $f(a') = 0$ for each $a' \in [a]$.

  If we write $f = \sum_{d \geq 0} f_d$, $f_d \in S_d$, then we have

  $$f(a) = 0 \iff f_d(a) = 0 \ \forall d \geq 0.$$

  Therefore, we can extend the definition of $V(\Sigma)$ to any $\Sigma \subseteq S$.

- We have $V(\Sigma) = V((\Sigma))$ where (Σ) is the ideal generated by some finite subset of Σ.

- We call an ideal $I \subseteq S$ **homogenous** if it is the direct sum of its d-homogeneous components, i.e.

  $$I = \sum_{d \geq 0} I_d$$

  where $I_d = \{f \in I \mid f \text{ is homogenous of degree } d\}$.

  I is homogeneous iff it is generated by homogeneous elements.

- We have the following *Nullstellensatz*:

  For any $X \subseteq \mathbb{P}^n$, set $I(X)$ to be the ideal generated by all homogeneous polynomials of S vanishing on X.

  Let $I \subseteq S$ be a *homogeneous* ideal which is *not equal* to $(x_0, \dots, x_n)$. Then, we have

  $$I(V_{\mathbb{P}^n}(I)) = \sqrt{I}.$$

  Example 18 (Anti-example)**.** The second property is necessary:

  Set $I = (x_0, \dots, x_n)$. Then $V_{k^{n+1}}(I) = 0$. Therefore, $V_{\mathbb{P}^n}(I) = \emptyset$. However,

  $$I(V_{\mathbb{P}^n}(I)) = S.$$

- The above point induces a bijection between algebraic subsets of $\mathbb{P}^n$ and radical ideals $I \subset S$ which are not $(x_0, \dots, x_n)$.

For $i = 0, \ldots, n$, set $D(x_i) := \{a \in \mathbb{P}^n \mid a_i \neq 0\}$. $D(x_i)$ is an open set homeomorphic to k^n by mapping

$$\phi_i : a \longmapsto (\frac{a_0}{a_i}, \ldots, \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i}, \ldots, \frac{a_n}{a_i}).$$

The $D(x_i)$ cover $\mathbb{P}^n = \bigcup_i D(x_i)$.

Given a projective algebraic subset $X \subset \mathbb{P}^n$, define $X^{(i)} \subset k^n$ by

$$X^{(i)} := \phi_i(X \cap D(x_i)).$$

If $X = V_{\mathbb{P}^n}(I)$, then

$$X^{(i)} = V_{k^n}(I^{(i)})$$

where

$$I^{(i)} := \left\{ f^{(i)} \mid f \in I \right\}$$

where $f^{(i)}(t_1, \ldots, t_n) := f(t_1, \ldots, t_{i-1}, 1, t_i, \ldots, t_n)$. Thus, $X^{(i)}$ is an algebraic subset of k^n.

Definition 35. The **Zariski topology** on $\mathbb{P}^n$ is defined by setting the set of closed sets to be the set of projective algebraic sets.

Facts:

- $D(x_i)$ is open in $\mathbb{P}^n$, since $D(x_i) = \mathbb{P}^n - V(x_i)$.
- The bijections $D(x_i) \cong k^n$ are homeomorphims.

Definition 36. A **quasi-projective** algebraic set Y is an open subset of a projective algebraic set $X \subseteq \mathbb{P}^n$.

Example 19. Any algebraic set in k^n is quasi-projective.

Definition 37. A **quasi-projective variety** is defined as an irreducible quasi-projective algebraic set.

Lemma 43 (Products)**.** *Define the* ***Segre-embedding*** *by*

$$\begin{aligned} S^{n,m} : \mathbb{P}^n \times \mathbb{P}^m &\hookrightarrow \mathbb{P}^{nm+n+m} \\ (a, b) &\longmapsto [(a_i b_j)_{i,j=0,\ldots,n}]. \end{aligned}$$

We have:

1. *$S^{n,m}$ is injective.*

2. *$S^{n,m}$ has a closed image.*

3. *$k^n \times k^m \cong D(z_{00}) \cap S^{n,m}(\mathbb{P}^n \times \mathbb{P}^m) = S^{n,m}(D(x_0) \times D(y_0))$.*

Definition 38. For quasi-projective algebraic sets $X \subset \mathbb{P}^n, Y \subset \mathbb{P}^m$, we define their product by

$$X \times Y := S^{n,m}(X, Y) \subseteq \mathbb{P}^{nm+n+m}.$$

Then, $X \times Y$ is a quasi-projective algebraic subset of $\mathbb{P}^{nm+n+m}$.

13.2 Flag Varieties

Definition 39. We define the **Grassmannian manifold** by

$$G(n,d) := \{W \subset k^n \mid W \text{ is a } d\text{-dimensional subvectorspace}\}.$$

Then, we have the **Plücker-embedding** by

$$\begin{aligned} P_d : G(n,d) &\longrightarrow \mathbb{P}\left(\bigwedge^d k^n\right) = \mathbb{P}^{\binom{n}{d}} \\ W &\longmapsto [w_1 \wedge w_d] \end{aligned}$$

where $w_1, \ldots, w_d$ is a basis of W.

Lemma 44. *P_d is injective and has a closed image.*
Therefore, we can see $G(n,d)$ as a projective algebraic set.

Definition 40. Let V be a finite-dimensional vector space of dimension n. Set

$$\mathrm{Gr}_d(V) := \{d\text{-dim. subspaces of} V\} \cong G(n,d).$$

Define further the **flag manifold** to be

$$\mathrm{Flag}(V) := \{\text{complete flags } \mathcal{F} = (0 = V_0 \subseteq V_1 \subseteq \ldots \subseteq V_n = V)\}.$$

Then, we have a map

$$\begin{aligned} P_v : \mathrm{Flag}(V) &\longrightarrow \mathrm{Gr}_0(V) \times \ldots \times \mathrm{Gr}_n(V) \\ \mathcal{F} &\longmapsto (V_0, \ldots, V_n). \end{aligned}$$

Lemma 45. *P_V has a closed image and is injective.*
Thus, we can see $\mathrm{Flag}(V)$ as a projective algebraic set.

Lemma 46. *$\mathrm{Gr}_d(V)$ and $\mathrm{Flag}(V)$ are both irreducible, hence projetive alg. varieties.*
*$\mathrm{Flag}(V)$ is called the **variety of complete flags**.*

13.3 Local Rings and Function Fields

Definition 41. An **affine variety** is an irredudible algebraic subset of k^n.

Definition 42. If X is an affine variety, then the coordinate ring $\mathcal{O}(X)$ is a domain. Define the **function field** of X by

$$k(X) := \mathrm{Frac}(\mathcal{O}(X)) := \left\{ \frac{a}{b} \mid a, b \in \mathcal{O}(X), b \neq 0 \right\}.$$

Definition 43. Let $p \in X$. We define the **local ring** of $\mathcal{O}(X)$ at p by

$$\mathcal{O}_{X,p} := \left\{ \frac{a}{b} \mid a \in O(X), 0 \neq b \in O(X), b(p) \neq 0 \right\} \subset k(X).$$

We have an **evaluation** map

$$\begin{aligned} \mathsf{eval}_p : \mathcal{O}_{X,p} &\longrightarrow k \\ \frac{a}{b} &\longmapsto \frac{a(p)}{b(p)}. \end{aligned}$$

Lemma 47. *Let X be an affine variety. Then*

$$\mathcal{O}(X) = \bigcap_{p \in X} \mathcal{O}_{X,p}.$$

Definition 44. Let $X \subset \mathbb{P}^n$ be a projective variety. Denote by $I_{\mathbb{P}}(X)$ its homogenous vanishing ideal.

Define its **function field** by

$$k(X) := R/M,$$

where

$$\begin{aligned} R &:= \left\{ \frac{f}{g} \mid f, g \in k[x_0, \dots, x_n] \text{ homogen.}, \deg f = \deg g, g \notin I_{\mathbb{P}}(X) \right\}, \\ M &:= \left\{ \frac{f}{g} \in R \mid f \in I_{\mathbb{P}}(X) \right\}. \end{aligned}$$

Lemma 48. *M is a maximal ideal in R and R/M is a field.*

Lemma 49. *If X is a projective variety, then $X^{(i)} \subset k^n$ is an affine variety.*

If $X^{(i)} \neq \emptyset$, then

$$k(X) \cong k(X^{(i)}).$$

Definition 45. Let X be a projective variety. For $p \in X$, we define its **local ring** at p by

$$\mathcal{O}_{X,p} := \left\{ \frac{f}{g} \in k(X) \mid g(p) \neq 0 \right\} \subset k(X).$$

Lemma 50. *For a projective variety X, we have:*

1. *For $p \in X^{(i)}$: $\mathcal{O}_{X,p} \cong \mathcal{O}_{X^{(i)},p}$.*
2. *For $p \in X^{(i)} \cap X^{(j)}$: $\mathcal{O}_{X^{(j)},p} \cong \mathcal{O}_{X^{(i)},p}$.*

Definition 46. If $X \subset \mathbb{P}^n$ is quasi-projective variety, there is a minimal projective variety $\overline{X} \subset \mathbb{P}^n$ which contains X as an open subset.

Then, we can set

$$k(X) := k(\overline{X})$$
$$\mathcal{O}_{X,p} := \mathcal{O}_{\overline{X},p}.$$

13.4 Regular Functions and Morphisms

Definition 47. Let X be quasi-projective variety. Let $U \subseteq X$ be open. Then, we define the **ring of regular functions** on U by

$$\mathcal{O}(U) := \bigcap_{P \in U} \mathcal{O}_{X,P} \subseteq k(X).$$

Definition 48. Let X, Y be two quasi-projective varieties. A map $f : X \to Y$ is called a **morphism**, if f is continuous and we have

$$f^*\mathcal{O}(U) := \{h \circ f \mid h \in \mathcal{O}(U)\} \subseteq \mathcal{O}(f^{-1}(U)).$$

Remark 8. Let X, Y be affine varieties and $f : X \to Y$ be a map. Then we have

$$f^*\mathcal{O}(U) \subseteq \mathcal{O}(f^{-1}(U))$$

iff f is given by polynomials.

Lemma 51. *Let X be a quasi-projective variety and let $p \in X$.*

Then there is an open neighborhood U of p in X s.t. U is isomorphic (as quasi-projetive varieties) to an affine variety $U' \subset k^n$.

Proof. Let Y be a projective variety s.t. X lies open in Y. By replacing $X \hookrightarrow Y$ with $X^{(i)} \hookrightarrow Y^{(i)}$, we may assume that X is an open subset of an affine variety Y in k^n.

Since the sets $D(f)$ give an open basis of k^n, there is a $f \in O(Y)$ s.t.

$$p \in D_Y(f) := \{y \in Y \mid f(y) \neq 0\} \subset X.$$

Now, $D_Y(f)$ is affine, because the map

$$\begin{aligned} D_Y(f) &\longrightarrow \big\{(q, r) \in k^{n+1} \mid q \in Y, f(q)r = 1\big\} \\ q &\longmapsto (q, \frac{1}{f(q)}) \end{aligned}$$

is an isomorphism of quasi-projective varieties. $\square$

13.5 Dimensions

Definition 49. Let X be a quasi-projective variety. We define its **dimension** as the transcendency degree of its function field, i.e.

$$\dim(X) := \text{tr.-deg}_k(k(X)).$$

Remark 9. If X is affine, then

$$\begin{aligned}
\dim(X) &= \text{tr.-deg}_k(k(X)) \\
&= \dim_{\text{Krull}}(\mathcal{O}(X)) \\
&= \sup \{n \in \mathbb{N}_0 \mid P_0 \subsetneq P_1 \subsetneq \dots P_n, P_i \text{ prime in } \mathcal{O}(X)\} \\
&= \sup \{n \in \mathbb{N}_0 \mid Z_n \subsetneq \dots Z_0, Z_i \text{ closed, irreducible in } X\}
\end{aligned}$$

Remark 10. If $U \subset X$ is open, then $k(U) = k(X)$ and $\dim(U) = \dim(X)$.

Lemma 52. *Let $\phi : X \to Y$ be a surjective morphism of quasi-projective varieties. Then,*

$$\dim X \geq \dim Y.$$

Proof. ϕ induces an inclusion

$$\begin{aligned}
\phi^* : k(Y) &\hookrightarrow k(X) \\
[(U_i, \alpha_i)_i] &\longmapsto [(\phi^{-1}(U_i), \alpha_i \circ \phi)].
\end{aligned}$$

This map is indeed injective, since ϕ is surjective. Therefore, the claim follows. $\square$

Lemma 53. *Let X be a quasi-projective variety and Y a proper, closed subvariety. Then,*

$$\dim(Y) < \dim(X).$$

Proof. By going from X to its closure $\widetilde{X}$ and from there to $\widetilde{X}^{(i)}$, we can assume that X is affine.

Then, $I_X(Y)$ is a non-trivial prime ideal in $\mathcal{O}(X)$. Therefore, we have

$$\dim_{\text{Krull}}(\mathcal{O}(Y)) = \dim_{\text{Krull}}(\mathcal{O}(X)/I_X(Y)) \leq \dim_{\text{Krull}}(\mathcal{O}(X)),$$

since A is a finitely generated k-algebra and a domain. $\square$

Lemma 54. *Let X be an affine variety and $f \in \mathcal{O}(X)$ be non-zero.*

Then, the set

$$V_X(f) := \{p \in X \mid f(p) = 0\}$$

is a proper, closed subset of X and we can decompose it into irreducible components

$$V_X(f) = Z_1 \cup \ldots \cup Z_l.$$

For each of those Z_i, we have

$$\dim(Z_i) = \dim(X) - 1.$$

Proof. The Z_i correspond to minimal prime ideals P_i in $\mathcal{O}(X)$ which contain (f). Since, they are minimal, we have

$$\text{height}(P_i) = 1.$$

$\square$

Lemma 55. *Let X be an quasi-projective algebraic set. Then – as in the affine case – we may write*

$$X = Z_1 \cup \ldots \cup Z_l$$

*where each $Z_i \subseteq X$ is an **irreducible component**, i.e. a maximal closed irreducible subset.*

We then define

$$\dim(X) := \max_i \dim(Z_j).$$

Lemma 56. *Let $\phi : X \overset{Y}{\to}$ be a morphism of quasi-projective varieties. Further, let ϕ be **dominant**, i.e., $\mathsf{Img}\phi$ is dense.*

Then, for all $p \in \mathsf{Img}(\phi)$, we have the following for the fiber of ϕ along p:

$$\dim(\phi^{-1}(p)) \geq \dim(X) - \dim(Y).$$

13.6 Images of Morphisms

Lemma 57. *Let Y be a quasi-projective set. Then for each $p \in Y$, there is an open, affine neighborhood $Y_0 \subset Y$ which contains Y.*

Proof. Denote by $\overline{Y}$ the algebraic closure of Y in $\mathbb{P}^n$. Assume that $p_i \neq 0$. Then, the affine sets $Y^{(i)} = Y \cap D(x_i)$ and $\overline{Y}^{(i)} = \overline{Y} \cap D(x_i)$ lie dense in Y and $\overline{Y}$.

Now, $Y^{(i)}$ is open in $\overline{Y}^{(i)}$. Since the $D_{\overline{Y}^{(i)}}(f)$, $f \in \mathcal{O}(\overline{Y}^{(i)})$, give a basis of the topology of $\overline{Y}^{(i)}$, there is an $f \in \mathcal{O}(\overline{Y}^{(i)})$ s.t.

$$p \in D_{\overline{Y}^{(i)}}(f) \subseteq Y^{(i)} \subset Y.$$

The neighborhood $D_{\overline{Y}^{(i)}}(f)$ is, in particular, affine. $\square$

Lemma 58. *Let Y be a quasi-projective algebraic set. Then the diagonal*

$$\Delta Y := \{(y, y) \mid y \in Y\}$$

is closed in $Y \times Y$.

Proof. If we cover Y by affine open subsets, then, we can reduce the claim to the case, where Y is affine, i.e. closed in k^n.

Then, $\Delta Y = (Y \times Y) \cap \Delta k^n \subset k^n \times k^n$. Since $Y \times Y$ is algebraic, it suffices to show that Δk^n is algebraic. And, indeed,

$$\Delta k^n = \{(x, y) \mid x - y = 0\}.$$

$\square$

Theorem 19 (Thm2)**.** *Let X be a projective variety and Y be a quasi-projective variety. Then, the projection*

$$\pi_Y : X \times Y \longrightarrow Y$$

*is **closed**, i.e. $\pi_Y(Z)$ is closed for each $Z \subseteq X \times Y$ closed.*

Proof. Since $X \hookrightarrow \mathbb{P}^n$ is a closed map (since X is closed in $\mathbb{P}^n$), it suffices to show the claim for $X = \mathbb{P}^n$.

Actually, at this point, we are done, since $\mathbb{P}^n$ with the Zariski-topology is topologically quasi-compact. $\square$

Theorem 20 (Thm1)**.** *Let X be a projective variety and Y be a quasi-projective variety. Then, for each morphism $\phi : X \to Y$, the image $\phi(X)$ is closed in Y.*

Proof. First, we show that

$$\Gamma := \{(x,y) \in X \times Y \mid \phi(x) = y\}$$

is closed in $X \times Y$. In fact, we have

$$\Gamma = (\phi \times 1)^{-1}(\Delta Y)$$

where $\Delta Y \subseteq Y \times Y$ is closed.

Now, we can consider the chain

$$X \xrightarrow{\mathrm{Id} \times \phi} X \times Y \xrightarrow{\pi_Y} Y.$$

We have $\phi(X) = \pi_Y(\Gamma)$. Since π_Y and Γ are closed, the claim follows. □

Example 20.

1. The condition that X is a *projective* variety is necessary. Consider

   $$\pi_x : \{(x,y) \mid xy = 1\} \longrightarrow k.$$

   The image $k^\times = \pi_x(\{(x,y) \mid xy = 1\})$ is not closed in k.

2. Let $Y = k \subset_o \mathbb{P}^1$. Then any morphism $\phi : X \to k$ is constant.

   This is, because $\phi(X)$ must be closed in $\mathbb{P}^1$, ergo a finite set. Now, this finite set cannot contain multiple elements. Otherwise, X would not be irreducible.

Corollary 9. *Let X be a projective variety and Y be an affine variety. Then, any morphism $X \to Y$ is constant.*

Proof. We have the chain

$$X \longrightarrow Y \hookrightarrow k^m \xrightarrow{\pi_i} k.$$

For each π_i this chain must be constant. □

Theorem 21 (Thm3)**.** *Let $\phi : X \to Y$ be a morphism of quasi-projective varieties. Assume that ϕ is **dominant**, i.e. $\phi(X)$ is dense in Y.*

Then, $\phi(X)$ contains a nonempty open (hence dense) subset of Y.

Proof. Postponed... □

Corollary 10. *Let $\phi : G \to H$ be a morphism of algebraic groups. Then, $\phi(G)$ is closed.*

Proof. Since G can be reduced to finite many irreducible components and since $\phi(G) = \bigcup_i \phi(g_i)\phi(G^o)$, it suffices to show the claim in the case where $G = G^o$ is irreducible.

Set $Y = \overline{\phi(G)}$. Y is irreducible and closed. Further, Y is a subgroup of H.

We are finished, if we can show $\phi(G) = \overline{\phi(G)}$.

By the previous theorem, $\phi(G)$ contains a nonempty open subset U of Y, hence $\phi(G)$ is dense in Y. Now, assume there are any $h \in Y - \phi(G)$. The map $y \mapsto hy$ is an isomorphism, hence $h\phi(G)$ lies dense in Y. Ergo

$$\phi(G) \cap (h\phi(G)) \neq \emptyset.$$

Take $u_1, u_2 \in \phi(G)$ s.t.

$$u_1 = hu_2.$$

Then, it follow $h = u_1 u_2^{-1} \in \phi(G)$. A contradiction. □

13.7 Borel's Fixed Point Theorem (special case)

Theorem 22. *Let G be a connected solvable algebraic subgroup of $\mathsf{GL}(V)$, where V is a finite-dimensional non-trivial vector space.*

Then, G acts algebraically on $\mathbb{P}(V)$.

Let $X \subseteq$ be a non-empty, closed G-stable subset. Then, G has a fixed point in X.

Proof. We prove this by an induction on $n = \dim(V)$:

- $n = 1$:
  In this case, $\mathbb{P}(V)$ contains only one element.

- $n = 2$:
  We have $\mathbb{P}V \cong \mathbb{P}^1$. If $X = \mathbb{P}(V)$, then there is a complete invariant flag $0 \subset \langle v \rangle \subset V$ which is G-stable.

  Then, $[v]$ is fixed by G.

  If X is finite, let $x \in X$. Then, $G.x$ is a connected subset of X, hence $G.x = \{x\}$.

- $n \geq 3$:
  Take again a complete G-stable flag $0 \subset \langle v \rangle \subset \ldots \subset V$. If $[v] \in X$, we are done.

  Otherwise, consider the morphism

  $$\phi : X \longrightarrow \mathbb{P}(V/\langle v \rangle).$$

  Since $\langle v \rangle$ is G-invariant, G acts on $\mathbb{P}(V/\langle v \rangle)$ and ϕ is G-equivariant.

  The image $\phi(X)$ is closed by a theorem in the preceeding subsection. By the induction hypothesis, there is a fixed point $[w + \langle v \rangle] \in \phi(X) \subseteq \mathbb{P}(V/\langle v \rangle)$.

  In particular, $[w + \langle v \rangle]$ has a preimage $[w]$ in X. Consider the subset

  $$W := \langle w, v \rangle \subseteq V.$$

  W is G-stable and we have $\mathbb{P}W \cap X \neq \emptyset$. Since $\mathbb{P}W \cap X$ is closed in $\mathbb{P}W \cong \mathbb{P}^2$, it follows from a previous case that there is a G-fixed point in $\mathbb{P}W \cap X$.

$\square$

13.8 Orbits

Definition 50. Let G be an algebraic group and Y a quasi-projective variety.

An **action** $G \curvearrowright Y$ is an action described by a morhism[2]

$$\phi : G \times Y \longrightarrow Y.$$

Lemma 59. *Let G be an algebraic group which acts on a quasi-projective algebraic set Y. For an orbit $O \subset Y$, we have that O is open in $\overline{O}$.*

Proof. Let G_i be an irreducible component of G. For a point $p \in O$, the map

$$\begin{aligned} G_i &\longrightarrow \overline{G_i.p} \\ g &\longmapsto g.p \end{aligned}$$

is dominant. Ergo, $G_i.p$ contains a nonempty open subset of $\overline{G_i.p}$. Ergo, the set $O = G.p$ contains a nonempty open subset U of $\overline{O} = \overline{G.p}$.

Now, for $q \in O$, there is some isomorphism $g \in G$ s.t. $q \in q.U$. Ergo, O is open. $\square$

Definition 51. If O is a G-orbit in a quasi-projective variety Y, we can consider it to be a quasi-projective set. Therefore, the notion of the dimension of an orbit O is well-defined.

Lemma 60 (Minimal Orbit Lemma)**.** *Let G be an algebraic group. Let Y be a quasi-projective variety s.t. Y is projective or affine.*

Let O be a G-orbit in Y s.t. the dimension of O is minimal among all G-orbits in Y.

Then, O is closed.

Proof. Since the action of an element of G does not change the dimension of a quasi-projective set, we can reduce the claim to the case that G is connected.

Then, O is irreducible. Further $\overline{O}$ is reduced and, because of the previous lemma, $\overline{O} - O$ is closed. It is easy to see, that G operates on $\overline{O} - O$.

Let Z be an irreducible component of $\overline{O} - O$. Since Z is a proper closed subset of $\overline{O}$, we have

$$\dim(Z) < \dim(\overline{O}) = \dim(O).$$

Since O is dimensionally minimal, we must have $Z = \emptyset$. Ergo, $O = \overline{O}$. $\square$

Corollary 11. *Let G be an algebraic group. Let Y be a quasi-projective variety s.t. Y is projective or affine.*

Then G has a closed orbit in Y.

---

[2] If G is connected, ϕ shall be a morphism of quasi-projective varieties. Otherwise, we just require that $G^o \times Y \to Y$ is a morphism of quasi-projective varieties.

13.9 Borel's Fixed Point Theorem (General Case)

Theorem 23. *Let G be a connected solvable algebraic group which acts on a projective variety X.*

Then, there exists a G-fixed point in X.

Proof. Since orbits of minimal dimensions are closed, we can replace X by a G-orbit. That is, we can assume that G acts transitively on X.

For $p \in X$, the G-stabilizer set

$$\mathrm{Stab}_G(p) = \{g \in G \mid g.p = p\}$$

is a closed subgroup in G, since it is the preimage of p under the continuous map $g \mapsto g.p$.

We showed earlier, that there exist a finite-dimensional representation $\rho : G \to \mathsf{GL}(V)$ with a one-dimensional subspace $L \subset V$ s.t.

$$G_p = \{g \in G \mid gL = L\}.$$

Let $q = [L] \in \mathbb{P}(V)$. Then G operates on $\mathbb{P}V$ and

$$G_q := \mathrm{Stab}_G(q) = \{g \in G \mid g.q = q\} = G_p.$$

Now, define

$$\begin{aligned} Y &:= G.q \subset \mathbb{P}V \\ Z &:= G.(p, q) \subset X \times \mathbb{P}V. \end{aligned}$$

Y and Z are quasi-projective varieties, since G is connected. We then have a G-equivariant diagram of quasi-projective varieties:

$$X \Longleftarrow Z \xrightarrow{\pi} Y$$

via

$$X \Longleftarrow X \times \mathbb{P}(V) \xrightarrow{\pi} \mathbb{P}(V).$$

Since X is projective, π is closed. Since $G_p = G_q$, the maps are bijective. Since all maps are bijective an G-equivariant, we need only to show that Y has a fixed point.

Since π is closed, the existence of a G-fixed point Y follows by the closedness of Z, because of Borel's special fixed point theorem.

The closedness of Z in $X \times \mathbb{P}(V)$ follows, if we can show that Z is an orbit of minimal dimension in $X \times \mathbb{P}(V)$. Indeed, we have.

- If $O \subset X \times \mathbb{P}(V)$ is a G-orbit, the projection $O \to X$ is G-equivariant and surjective, because X is a G-orbit. Since X, Y are quasi-procetive varieties, it then follows
$$\dim(X) \leq \dim(O).$$
- The map $X \to Z$ is bijective, hence
$$\dim(*) \geq \dim(Z) - \dim(X).$$
Ergo
$$\dim(Z) \leq \dim(X).$$

$\square$

13.10 Generic Openness

Proposition 4. *Let $\phi : X \to Y$ be a dominant morphism of quasi-projective varieties.*

Then, there is an open nonempty set $U \subset X$, s.t., $\phi_{|U}$ is open, that is, it maps open sets to open sets.

Corollary 12. *Let G be a connected algebraic group.*

Then, $[G, G] = \langle aba^{-1}b^{-1} \mid a, b \in G \rangle$ is a closed subgroup of G.

Proof. For $a, b \in G$, set

$$[a, b] = aba^{-1}b^{-1}.$$

For $n \geq 0$, define

$$\begin{aligned} \phi_n : G^{2n} &\longrightarrow G \\ (a_1, b_1, \ldots, a_n, b_n) &\longmapsto [a_1, b_1] \cdots [a_n, b_n]. \end{aligned}$$

Let $Z_n := \overline{\mathsf{Img}\phi_n}$. Then, we have an ascending chain

$$Z_1 \subseteq Z_2 \subseteq \ldots$$

Each Z_n is closed and irreducible, because G^{2n} is connected.

Then, at some point the chains of Z_i's must become stationary, because $\dim(G) < \infty$, because $\mathcal{O}(G)$ is a fintely generated k-algebra.

Let N be s.t.

$$Z_N = Z_{N+1} = \ldots.$$

Since $[G, G] = \bigcup_n \mathsf{Img}(\phi_n)$, we have then

$$Z_N = \bigcup_n Z_n = \overline{[G, G]}.$$

Since $\phi_n : G^{2n} \to Z_n$ is dominant and since G^{2n} and Z_n are quasi-projective varieties, $\mathsf{Img}\phi_n = [G, G]$ contains a nonempty open subset U.

Now, let $h \in \overline{[G, G]}$. Then, $hU \cap U \neq \emptyset$, since both are nonempty open and $\overline{[G, G]}$ is irreducible. Therefore, we have $u_1, u_2 \in U \subseteq [G, G]$ with

$$hu_1 = u_2.$$

Ergo, h lies in $[G, G]$. $\square$

14 Homogenous Spaces

Definition 52. Let G be a connected algebraic group. A homogenous space for G is a quasi-projective variety X equipped with a **transitive** action $G \curvearrowright X$.

Let G be now disconnected. Then, we only demand that X is a finite union of irreducible components. Still G needs to act transitively on X.

A morphism of G-homogenous spaces is a **morphism** of quasi-projective varieties/sets which is G-equivariant.

Corollary 13. *If $\phi : X \to Y$ is a morphism of G-homogenous spaces, then ϕ is an open map.*

Proof. It suffices, if we show this statement for an irreducible X. Note, that ϕ must be surjective, ergo dominant.

By a previous proposition, X must contain an open nonempty subset U s.t. $\phi_{|U}$ is an open map. Since G acts transitively on X, we can cover X with such open sets gU. □

Proposition 5. *Let G be an algebraic group and H a closed subgroup.*

Then, there is a homogenous space X for G and a point $p \in X$ s.t.

$$H = \mathrm{Stab}_G(p)$$

and the map

$$\begin{aligned} G/H &\longrightarrow X \\ gH &\longmapsto g.p \end{aligned}$$

is a bijection.

Proof. There is a faithful representation $\rho : G \to \mathsf{GL}(V)$ with V finite-dimensional s.t. there is a one-dimensional subspace $L \subset V$ with

$$H = \{g \in G \mid gL = L\} .$$

Set $p := [L] \in \mathbb{P}(V)$. Then, we can set

$$X := G.p.$$

Then, X is an orbit of G, ergo a quasi-projective set/variety. □

14.1 Quotients

Definition 53. A (left) **quotient** of an algebraic group G by a closed group H is a pair (X, ρ) s.t.

(1) X is a quasi-projective variety.

(2) $\rho : G \to X$ is a morphism with

$$\rho(hg) = \rho(g)$$

for all $h \in H, g \in G$.

Further, we demand that a quotient is **initial** in the category of all objects satisfying the above conditions. I.e. for each pair (X', ρ') there must be a unique morphism ϕ s.t. the following diagram commutes:

$$\begin{tikzcd} G \arrow[d, "\rho"] \arrow[dr, "\rho'"] & \\ X \arrow[r, dashed, "\phi"] & X' \end{tikzcd}$$

Remark 11. Set theoretically, we just have $X = G/H$.

Lemma 61. *Let (X, ρ) satisfy conditions (1) and (2) from the above definition. Suppose further*

(i) $\{$fibers of $\rho\} = \{$left H-cosets of $G\}$,

(ii) X is a G-homogenous space and ρ is G-equivariant,

(iii) for each open $U \subset X$ the pullback map

$$\begin{aligned} \rho^* : \mathcal{O}(U) &\longrightarrow \mathcal{O}(\rho^{-1}(U)) \\ f &\longmapsto f \circ \rho \end{aligned}$$

defines an isomorphism

$$\mathcal{O}(U) \cong \left\{ f \in \mathcal{O}(\rho^{-1}U) \mid f(Hg) = f(g) \right\} =: \mathcal{O}(\rho^{-1}(U))^H.$$

Then, (X, ρ) is a quotient of G by H.

Proof. We have to show that (X, ρ) is initial. Let (X', ρ') be another object satisfying (1), (2). Because of (i), we have a unique settheoretic map $\phi : X \to X'$ s.t. the diagram

$$\begin{tikzcd} G \arrow[d, "\rho"] \arrow[dr, "\rho'"] & \\ X \arrow[r, dashed, "\phi"] & X' \end{tikzcd}$$

commutes. We need to check that ϕ is a morphism:

- ϕ is continuous, since ρ' is continuous and ρ is open (since X is a G-homogenous space). Therefore, $\phi = \rho' \circ \rho^{-1}$ is continuous.
- Let $U' \subset X'$ be open. We need to show
$$\phi^* \mathcal{O}(U') \subseteq \mathcal{O}(\phi^{-1} U').$$
Let $f \in \mathcal{O}(U')$ and set $U := \phi^{-1} U'$. Since ρ' is a morphism, we have
$$\rho'^*(f) \in \mathcal{O}(\rho'^{-1} U').$$
Because of (iii), we have
$$\mathcal{O}(U) \cong \mathcal{O}(\rho^{-1} U)^H.$$
Therefore, it suffices to show
$$\rho'^*(f) \in \mathcal{O}(\rho^{-1} U)^H.$$
And, indeed
$$f \circ \rho'(hg) = f \circ \rho'(g)$$
for $g \in G, h \in H$.

$\square$

Lemma 62. *Suppose* $\operatorname{char} k = 0$. *Any injective morphism of quasi-projective varieties with dense image is **birational**, i.e., induces, via pullback an isomorphism*

$$k(X) \cong k(Y).$$

Theorem 24. *Let G be an algebraic group with a closed subgroup H.*

- *A quotient (X, ρ) exists and X is a homogenous space for G s.t. $H = \operatorname{Stab}_G(p)$ for some $p \in X$.*

- *If* $\operatorname{char}(k) = 0$*, then each* G*-homogenous space* X *together with a point* $p \in X$ *s.t.* $H = \operatorname{Stab}_G(p)$ *gives a quotient of* G *by* H*, where* $\rho(g) = g.p$*.*

Proof. We only prove the theorem for the case $\operatorname{char} k = 0$. We construct X as in a previous proposition, i.e. $X = G.p$ for a point $p \in \mathbb{P}(V)$ s.t. $H = \operatorname{Stab}_G(p)$.

It is then clear, that conditions (i) and (ii) of the previous lemma are met. We only need to show

$$\rho^*\mathcal{O}(U) = \mathcal{O}(\rho^{-1}U)^H.$$

Naturally, $\rho^*\mathcal{O}(U)$ is contained in $\mathcal{O}(\rho^{-1}U)^H$.

Let $f \in \mathcal{O}(\rho^{-1}U)^H$. W.l.o.g., we can assume that U is affine. Consider the diagram

$$\begin{tikzcd}
\rho^{-1}U \arrow[r, "f"] \arrow[d, "\rho"] & k \\
U \arrow[ur, dashed, "g"] &
\end{tikzcd}$$

$g := f \circ \rho^{-1}$ is well-defined, because f is H-invariant. We need to show, that g is regular, i.e. $g \in \mathcal{O}(\rho^{-1}U)$.

We can blow up the diagram as follows:

$$\begin{tikzcd}
U \times k \arrow[dr, "\pi_2"] & & V \supset \mathsf{Img}(\rho \times f) \arrow[ll, hook, "\text{open}"] \arrow[dl, "\pi_2"] \arrow[dd, "p"] \\
& k & \\
\rho^{-1}U \arrow[uu, "\rho\times f"] \arrow[ur, "f"] \arrow[rr, "\rho"] & & U \arrow[ul, "g"']
\end{tikzcd}$$

Then V is a quasi-projective variety and p is dominant and injective, hence birational. Therefore, we have

$$k(U) \cong k(V).$$

Since X is homogenous space for G, it is smooth. On a smooth quasi-projective variety, every rational function that fails to be regular must have a pole.

In particular, we do have $\pi_2 \in \mathcal{O}(V)$ and therefore

$$g = p^*(\pi_2) \in \mathcal{O}(U).$$

$\square$

Example 21 (Non-Example)**.** The proof of the theorem does not hold, if $\mathrm{char}(k) = p > 0$.

Consider,

$$\begin{aligned} G &:= \mathcal{G}_a \\ H &:= 1 \\ V &= k^2 \end{aligned}$$

and

$$\begin{aligned} G &\longrightarrow \mathsf{GL}(V) \\ x &\longmapsto \begin{pmatrix} 1 & x^{p^n} \\ & 1 \end{pmatrix} \end{aligned}$$

for some $n \in \mathbb{N}_0$.

For $q = [1, 0] \in \mathbb{P}(V)$, we have

$$X := G.q = \left\{ [1, x^{p^n}] \mid x \in k \right\} \cong k.$$

Define ρ by by

$$\begin{aligned} \rho : G &\longrightarrow X \\ g &\longmapsto g.q. \end{aligned}$$

Then, (ρ, X) fulfills the conditions of the above theorem, but it is NOT a quotient for $n \geq 1$.

Indeed, for $n_1 \geq n_2$, we have non-isomorphic maps

$$\begin{aligned} X_{n_2} &\longrightarrow X_{n_1} \\ x &\longmapsto x^{p^{n_1 - n_2}}. \end{aligned}$$

15 Borel and Parabolic Groups

Let G be a connected algebraic group.

Definition 54. A subgroup $B \subset G$ is called **Borel**, if B is maximal among all connected solvable closed subgroups.

Since $\dim(G) < \infty$, Borel subgroups exist.

Definition 55. A subgroup $P \subset G$ is called **parabolic**, if the quasi-projective variety G/P is **projective**, i.e. closed in $\mathbb{P}^n$.

Lemma 63. *Let G connected, P parabolic, B Borel. Then, P contains some conjugate of B.*

Proof. B acts on the projective variety G/P. According to Borel's fixed point theorem, there is a fixed point $gP \in G/P$ s.t.

$$bgP = gP$$

for each $b \in B$. Ergo

$$g^{-1}bg \in P$$

for each $b \in B$. $\square$

Theorem 25. *Let G be connected.*

Any two Borel subgroups are conjugate.

Proof. Take a faithful representation $G \hookrightarrow \mathsf{GL}(V)$ with a finite-dimensional V. Let $\mathcal{F} = \mathrm{Flag}(V)$ be the flag variety of V.

Choose $F \in \mathcal{F}$ s.t. the orbit $G.F$ has a minimal dimension. Then, $G.F$ is closed, hence projective. If we set

$$H := \mathrm{Stab}_G(F),$$

then H is parabolic. Therefore, each Borel group B has a conjugate in H. Since B is connected, its conjugate is contained in an irreducible component H^o of the neutral element.

Since H is solvable[3], H^o is a connected, solvable, closed subrgoup. Ergo H^o is the conjugate of B. $\square$

Proposition 6. *Let G be connected. Then, each Borel group is parabolic.*

[3] Why is H solvable?

Proof. Let B be a Borel subgroup of G.

Take a representation $G \to \mathsf{GL}(V)$ with a finite-dimensional V s.t. there is a one-dimensional $L \subseteq V$ s.t.

$$B = \{g \in G \mid gL = L\}.$$

B acts on V/L. Since B is connected and solvable there must be a complete B-invariant flag $\overline{F}$ in V/L. We can lift $\overline{F}$ to a complete flag $F = (L = V_1 \subset \ldots \subset V_n)$ of V. Then, it is easy to see

$$B = \mathrm{Stab}_G(F).$$

Choose $F' \in \mathrm{Flag}(V)$ s.t. the orbit $G.F'$ has a minimal dimension. Then, $G.F'$ is closed, hence projective. If we set

$$H := \mathrm{Stab}_G(F'),$$

we have (by conjugating)

$$B = H^o.$$

Consider the map

$$G/B = G/H^o \twoheadrightarrow G/H.$$

This map has finite fibers, because $[H : B] < \infty$. Ergo

$$\dim(G/B) \leq \dim(G/H).$$

Ergo, G/B is of minimal dimension, hence closed. Hence, B is parabolic. $\square$

Corollary 14. *Let P be an algebraic subgroup of a connected algebraic group G. Then, P is parabolic iff it contains a Borel group.*

Proof. The direction to the right is known.

Let P contain a Borel group B. Consider the maps

$$G/B \twoheadrightarrow G/P \hookrightarrow \mathbb{P}^n.$$

Since B is parabolic, G/B is closed. Therefore, the morphism $G/B \to \mathbb{P}^n$ has a closed image. But its image is exactly G/P. Ergo, P is parabolic. $\square$

Corollary 15. *Let B be an algebraic subgroup of a connected algebraic group G. Then, B is Borel iff it is a minimal parabolic subgroup.*

Example 22. If $G = \mathsf{GL}_n(k)$, then

$$B = \left\{ \begin{pmatrix} * & \dots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & * \end{pmatrix} \right\}$$

is a Borel group.

Let $n = n1 + \ldots + n_r$ and set

$$P_{(n_1,\ldots,n_r)} := \left\{ \begin{pmatrix} \mathsf{GL}_{n_1}(k) & * & * \\ 0 & \ddots & * \\ 0 & 0 & \mathsf{GL}_{n_r}(k) \end{pmatrix} \right\}.$$

Each $P_{(n_1,\ldots,n_r)}$ is closed, since it is the stabilizer of an incomplete flag.

In fact, each parabolic group is conjugate to one of those $P_{(n_1,\ldots,n_r)}$.

If $P \neq G$ is parabolic, P is called a **proper** parabolic subgroup.

Example 23.

- $G = \mathsf{SL}_n(k)$: In this case parabolic groups are like in the above case, but inside of $\mathsf{SL}_n(k)$.

- $G = \mathsf{SO}_n(k)$: Then, we can embed G in $\mathsf{GL}(V)$. Let $\langle \cdot \mid \cdot \rangle$ be (any?) symmetric bilinear form.

  A subspace $W \subset V$ is called **isotopic** iff $\langle \cdot \mid \cdot \rangle_{|W \times W} \equiv 0$.

  Then, we have the equivalence

  $$\{\text{Borel Group } B \subset G\} \Leftrightarrow \{\text{maximal isotropic flags } \mathcal{F} \text{ in } V\}.$$

- $G = \mathsf{SP}_{2n}$: The symplectic group is defined by

  $$\mathsf{SP}_{2n} := \left\{ A \in \mathsf{GL}_{2n}(k) \mid A^T \cdot \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \cdot A = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \right\}.$$

  Embed again G in $\mathsf{GL}(V)$.

  Let $\langle \cdot \mid \cdot \rangle$ be a **symplectic** form on V, i.e., $\langle \cdot \mid \cdot \rangle$ is bilinear, alternating ($\langle v \mid v \rangle = 0$) and nonsingular, i.e. $\langle v \mid \_ \rangle \equiv 0 \iff v = 0$.

  Then, again, we have the equivalence

  $$\{\text{Borel Group } B \subset G\} \Leftrightarrow \{\text{maximal isotropic flags } \mathcal{F} \text{ in } V\}.$$

Further, we can take a basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ of V with

$$\begin{aligned} \langle e_i \mid e_j \rangle &= \langle f_i \mid f_j \rangle = 0 \\ \langle e_i \mid f_j \rangle &= \delta_{i,j}. \end{aligned}$$

Then, one can for example set

$$V_j = \operatorname{span}\{e_1, \ldots, e_j\}$$

to get a flag $V_0 \subset V_1 \subset \ldots$.

Vice versa, one can convert each maximal isotropic flag to such a symplectic basis.

15.1 Radicals

Let G be a connected algebraic group.

Definition 56. The **radical** $R(G)$ of G is defined as the intersection of all Borel subgroups of G i.e.

$$R(G) := \bigcap_{B \subset G \text{ Borel}} B.$$

The **unipotent radical** is defined by

$$R_u(G) := R(G)_u = \{\text{unipotent elements of } R(G)\}.$$

Lemma 64. *Let G be a connected algebraic group.*

$R(G)$ is the largest connected solvable normal algebraic subgroup of G.

Proof. It is clear that $R(G)$ is connected, solvable, normal and algebraic.

We need to show that each connected solvable normal algebraic subgroup H of G is contained in $R(G)$.

Clearly, H is contained in one Borel group B. Since H is normal, we have for each $g \in G$

$$H = gHg^{-1} \subset gBg^{-1}.$$

Since gBg^{-1} is a Borel group and all Borel groups are conjugated, it follows H is contained in each Borel group, ergo it is contained in $R(G)$. □

Definition 57. We call G **semisimple** iff $R(G) = 1$.

We call G **reductive** iff $R_u(G) = 1$ (iff $R(G)$ is a torus).

Example 24.

- Let $n \geq 1$ and $G = \mathsf{GL}_n(k)$. G is reductive, but not semisimple:

  G has two Borel groups:

  $$B = \left\{ \begin{pmatrix} * & * \\ & * \end{pmatrix} \right\} \qquad B' = \left\{ \begin{pmatrix} * & \\ * & * \end{pmatrix} \right\}.$$

  Ergo, we have for the radical

  $$R(G) \subset B \cap B' = \left\{ \begin{pmatrix} * & \\ & * \end{pmatrix} \right\} =: T.$$

  But, now we have

  $$\left\{ t \in T \mid gtg^{-1} \in T \ \forall g \in G \right\} = k^\times.$$

  Ergo,

  $$R(G) = k^\times.$$

Let $G = \mathsf{SL}_n(k)$. G is semisimple and reductive:

As above, one can compute

$$Z = G \cap \left\{ \begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{pmatrix} \right\}.$$

However, Z is not connected. In particular

$$R(G) = Z^o = 1.$$

$G = \mathcal{G}_m^n$ is a torus: It is easy to see that $R(G) = G$ in this case.

G is solvable (and connected): Trivially, we have then $R(G) = G$.

G is unipotent: In this case, we know that G is solvable. Further, we even have $R_u(G) = G$.

If G is SO_n or SP_{2n}, then $R(G) = R_u(G) = 1$.

16 Reductivity

Let G be a connected algebraic group which acts on an affine variety X.

Definition 58. A **quotient** of X by G is a pair (Y, ρ) s.t.

1. Y is an affine variety
2. and $\rho : X \to Y$ is a morphism which is constant on G-orbits.

Further, we demand that a quotient is initial in the category of all objects which fulfill the above conditions. I.e.

$$\begin{tikzcd} X \arrow[d, "\rho"] \arrow[dr, "\rho'"] & \\ Y \arrow[r, dashed, "\exists_1 \phi"] & Y' \end{tikzcd}$$

Remark 12.

- Such quotients need not to exist.
- Even when such quotients exist, they don't need to describe orbits. I.e., $G \backslash X$ must not be related to Y.

Example 25. Consider the action of $G = \mathcal{G}_m$ on $X = k^1$. This action has to orbits: the open orbit $k \setminus \{0\}$ and the closed orbit $\{0\}$.
Then the quotient of X by G is given by $(Y, \rho) = (\{0\}, x \mapsto 0)$.
Note, if $f : X \to k$ is regular and constant on G-orbits, then f is constant on X, because $k \setminus \{0\}$ lies dense in k.

Definition 59. Let G be a connected algebraic group.
We call G **geometrically reductive** if we have for each finite-dimensional representation V of G:

$$\forall v \in V^G \; \exists f : V \to k : \; f \text{ is a homogenous } G\text{-invariant polynomial s.t. } f(v) \neq 0$$

where

$$V^G = \{v \in V \mid g.v = v \; \forall g \in G\}.$$

Remark 13. G is geometrically reductive iff for each affine X on which G operates and for each pair of closed G-invariant disjoint subsets $W_1, W_2 \subset X$ there is an $f \in \mathcal{O}(X)^G$ s.t.

$$f_{|W_1} \equiv 1,$$
$$f_{|W_2} \equiv 0.$$

Is this easy to see? I only see the backwards direction (take $X = V, W_1 = 0, W_2 = v$).

Theorem 26. *Let G be a connected algebraic group.*

Then, G is reductive iff G is geometrically reductive.

Theorem 27. *Let G be a connected algebraic group which is geometrically reductive and acts on an affine set X.*

Then, there is a quotient (Y, ρ) of X by G.

Moreover, ρ induces a bijection

$$\{\textit{closed } G\textit{-orbits in } X\} \Longleftrightarrow Y.$$

Definition 60. Let G be a connected algebraic group.

We call G **linearly reductive** if we have for each finite-dimensional representation V of G:

$$\forall v \in V^G \setminus \{0\}\ \exists f : V \to k : \ f \text{ is a linear } G\text{-invariant polynomial s.t. } f(v) \neq 0.$$

Remark 14. Naturally, linear reductivity implies geometrical reductivity. The converse does hold iff $\mathrm{char} k = 0$.

Remark 15. $\mathsf{GL}_n(k)$ is linear reductive.

Remark 16. G is linear reductive iff every finite-dimensional representation V of G is completely **reducible**, i.e.

$$V = \bigoplus_i V_i$$

where each V_i is irreducible.

17 Union of Borel Subgroups

Theorem 28. *Let G be a connected algebraic group. Then,*

$$G = \bigcup_{B \text{ Borel}} B.$$

Because of Jordan Decomposition, it is clear that the theorem holds for $\mathsf{GL}_n(k)$. We will prove it only for the case $k = \mathbb{C}$.

Lemma 65. *Let k be any (not necessarily algebraically closed) field. Let B be some Borel subgroup.*

Then, $X := \bigcup_{g \in G} gBg^{-1}$ is closed in G.

Proof. Our intuition is as follows:

gBg^{-1} only depends on $gB \in G/B$. Since B is Borel, ergo parabolic, G/B is projective, ergo somewhat 'compact'. Then, $X = \bigcup_{g \in G} gBg^{-1}$ is a union of 'compactly-many' closed sets.

Now, the actual proof works as follows: We want to use that $G/B \times G \to G$ is a closed map. Consider the chain

$$G \times B \overset{\phi(g,b)=(g,gbg^{-1})}{\longrightarrow} G \times G \longrightarrow G/B \times G \longrightarrow G.$$

X is the image of the composition $(g, b) \mapsto gbg^{-1}$. It therefore suffices to show that the image of

$$\pi \times \mathrm{Id} : G \times G \longrightarrow G/B \times G$$

is closed.

Set

$$Y := (\pi \times \mathrm{Id})(\phi(G \times B)).$$

If we can show, that $(\pi \times \mathrm{Id})^{-1}(Y)$ is closed, then Y is closed, because $\pi \times \mathrm{Id}$ is, as a morphism of homogenous spaces, open. However, we have

$$(\pi \times \mathrm{Id})^{-1}(Y) = \mathsf{Img}\phi.$$

Now, $\mathsf{Img}\phi$ is closed, since morphisms of algebraic groups have closed images. ∎

Lemma 66. *Let $k = \mathbb{C}$.*

Then, $X = \bigcup_{g \in G} gBg^{-1}$ is dense in G.

Proof idea. We want to show $\overleftarrow{X} = G$.

Since G is connected, it would suffice to show that X contains an Euclidean neighborhood of $1 \in G$.

Let $\mathfrak{g} := \mathrm{Lie}(G)$ be the Lie-algebra of G. A Borel-subalgebra $\mathfrak{b} \subset \mathfrak{g}$ is a maximal solvable subalgebra.

Then, one can show, that for each Borel-subalgebra $\mathfrak{b} \subset \mathfrak{g}$ there is a Borel-subgroup $B \subset G$ s.t. $\mathfrak{b} = \mathrm{Lie}(B)$.

Is easy to see, that each $x \in \mathfrak{g}$ is contained in some Borel-subalgebra, since $\mathbb{C} \cdot x$ is a solvable subalgebra.

With the two above facts, it follows that X contains a small euclidean neighborhood of 1. $\square$

18 Splitting Solvable Groups

Let B be a connected solvable algebraic group. (Then, B is trigonalizable.)

Then, $U := B_u$ is a unipotent normal algebraic subgroup (since $U = R_u(B)$, since $B = R(B)$).

Lemma 67. *The group B/U is a torus.*

Proof. We have an injective morphism

$$B \hookrightarrow \left\{ \begin{pmatrix} * & \dots & * \\ & \ddots & \vdots \\ & & * \end{pmatrix} \right\}$$

with

$$U = B \cap \left\{ \begin{pmatrix} 1 & \dots & * \\ & \ddots & \vdots \\ & & 1 \end{pmatrix} \right\}.$$

Therefore, we get an injection

$$B/U \hookrightarrow \left\{ \begin{pmatrix} * & \dots & * \\ & \ddots & \vdots \\ & & * \end{pmatrix} \right\} / \left\{ \begin{pmatrix} 1 & \dots & * \\ & \ddots & \vdots \\ & & 1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \right\}.$$

Ergo, B/U is diagonalizable. Since B is connected, B/U is connected, too. It follows that B/U is a torus. □

Theorem 29. *Let B be a connected solvable algebraic group.*

Then, there is a torus $T \subset B$ s.t. the composition

$$T \hookrightarrow B \twoheadrightarrow B/U$$

is an isomorphism.

Before, we can prove the therenm we need some lemmata:

Lemma 68. *Suppose* $\operatorname{char} k = 0$. *Let T be a torus.*

Then, there is an $s \in T$ s.t.

$$\overline{\langle s \rangle} = T.$$

s is called the ***generator*** *of T.*

Remark 17. The lemma does not hold, if char$k > 0$.

Proof. Recall that we have the following correspondence:

$$\{\text{tori}\} \overset{T \mapsto \mathfrak{X}(T)}{\longleftrightarrow} \{\text{f.g. free } \mathbb{Z}\text{-modules}\}.$$

and in particular for each torus T:

$$\begin{aligned} \{\text{alg. subgroups } H \text{ of } T\} &\longleftrightarrow \{\text{submodules } \Gamma \text{ of } \mathfrak{X}(T)\} \\ H &\longmapsto \left\{\chi \in \mathfrak{X}(T) \mid \chi \in \mathfrak{X}(T) : \ \chi_{|H} \equiv 1\right\} \\ \{t \in T \mid \chi(t) = 1 \ \forall \chi \in \Gamma\} &\longleftarrow \Gamma. \end{aligned}$$

So, we have $\overline{\langle s \rangle}$ iff

$$\chi(s) \neq 1$$

for each $1 \neq \chi \in \mathfrak{X}(T)$.

W.l.o.g. $T = (k^\times)^n$. Then

$$\mathfrak{X}(T) = \{\chi_m \mid m \in \mathbb{Z}^n\}$$

with

$$\chi_m(t_1, \ldots, t_n) = t_1^{m_1} \ldots t_n^{m_n}.$$

We can then pick

$$s = (2, 3, 5, 7, \ldots).$$

$\square$

Lemma 69. *If* char$k = 0$, *then any bijective morphism of algebraic groups is an isomorphism of algebraic groups.*

Remark 18. This does not need to hold for non-zero charateristic. If char$k = p$, then

$$\begin{aligned} k &\longrightarrow k \\ x &\longmapsto x^p \end{aligned}$$

is bijective without being isomorphic.

Proof of Theorem. We only show the theorem in case char$k = 0$.

Let B be a connected solvable algebraic group.

We need to show, that there is a torus $T \subset B$ s.t. the composition

$$T \hookrightarrow B \twoheadrightarrow B/U$$

is an isomorphism where
$$U = B_u.$$

We know, that B/U is a torus. Take $s' \in B/U$ s.t.
$$\overline{\langle s' \rangle} = B/U.$$
Take a preimage $g \in B$ s.t. $\pi(g) = s'$.

We can decompose $g = su$ into a semisimple and a unipotent element. We then have
$$\phi(g) = \phi(s) \cdot \phi(u) = \phi(s),$$
since $\phi(u)$ must be unipotent, ergo trivial.

Set
$$T = \overline{\langle s \rangle}.$$
Since s is semisimple, T must be diagonalizable. Ergo
$$T \cap U = 1.$$
Ergo, the chain
$$T \hookrightarrow B \twoheadrightarrow B/U$$
must be bijective, hence an isomorphism, since char$k = 0$. □

The theorem gives the structure of a semidirect product of algebraic groups:
$$B = U \rtimes T$$
(where $T \curvearrowright U$ by conjugation.)

Definition 61. Let G_1, G_2 be algebraic groups. Let G_2 act algebraically on G_1 via $b : G_2 \to \mathsf{Aut}\,(G_1)$ s.t. the map
$$\begin{aligned} G_2 \times G_1 &\longrightarrow G_1 \\ (g_2, g_1) &\longmapsto b(g_2)(g_1) \end{aligned}$$
is a morphism.

Their semidirect group $G_1 \rtimes_b G_2$ is an algebraic group which is:

- set-theoretically $G_1 \times G_2$,
- group-theoretically the semidirect product $G_1 \rtimes_b G_2$. I.e. multiplication works by
$$(g_1, g_2) \cdot (h_1, h_2) = (g_1 \cdot b(g_2)(h_1), g_2 h_2).$$

Remark 19. Even if we are given an algebraic group G with closed subgroups G_1, G_2 s.t.

$$G = G_1 \rtimes G_2$$

as abstract groups, it does not need to be the case that

$$G_1 \rtimes G_2 \longrightarrow G$$

is an isomorphism. (However, it is the case, if $\mathrm{char} k = 0$.)

18.1 An Aside

Let G be an algebraic group and H a normal algebraic subgroup.

Then, G/H is a quasi-projective variety equipped with a G-action. Ergo, we have an algebraic group structure on G/H.

Theorem 30. *G/H is an affine algebraic group.*

Proof. We need to show that G/H is affine.

We showed in a lemma long before, that there is a finite-dimensional representation V, ρ s.t.

$$H = \ker \rho.$$

Therefore, we can simply set

$$G/H := \mathsf{Img}(\rho) \subset \mathsf{GL}(V)$$

which is closed as ρ is a morphism of algebraic groups. $\square$

18.2 Semisimple Elements of Solvable Groups

Theorem 31. *Let $B = U \rtimes T$ as before be a solvable connected algebraic group. Let $s \in B$ be semisimple. Then s is conjugated to one element in T.*

Corollary 16. *Let G be a connected algebraic group. Then, every semisimple element of G is contained in some torus.*

Proof. Let $s \in G$ be semisimple and choose a Borel group $B \subset G$ which contains s. B is of the form $U \rtimes T$, ergo $s \in b^{-1}Tb$ for some $b \in B$. □

Lemma 70. *Suppose* $\mathrm{char} k = 0$.

(i) Let $g \in \mathsf{GL}_n(k)$ be unipotent and set $G(g) := \overline{\langle g \rangle}$. Then, we have the following isomorphism of algebraic groups

$$G(g) = \left\{ g^t \mid t \in k \right\} \cong k$$

where

$$g^t := \exp(t \cdot \log(g))$$
$$-\log(1 - X) = \sum_{n=1}^{\infty} \frac{X^n}{n}$$
$$\exp(Y) = \sum_{k=0}^{\infty} \frac{Y^k}{k!}.$$

(ii) Any unipotent algebraic group is connected. (This does not hold if $\mathrm{char} k > 0$.*)*

(iii) Any unipotent commutative algebraic group is isomorphic to some vector space.

Proof. (i) We will not prove this, but the idea is that $\mathbb{Z}$ is dense in k.

(ii) Let $g, h \in G$ be unipotent. Then the subgroups $G(g), G(h)$ are connected and share a common point (e), ergo g, h are contained in the same component.

(iii) Since all elements commute log gives an isomorphism into an additive group, on which k acts.

□

Proof of Theorem. We only prove the theorem in case $\mathrm{char} k = 0$.
Let $s \in B = U \rtimes T$ be semisimple. Since $\mathrm{char} k = 0$, U is connected.
We induct on $\dim(U)$:

- $\dim(U) = 0$: In this case $U = 1$ and $s \in G = T$.
- $\dim(U) = 1$: This is the crucial case.

  Write
  $$s = ut$$
  with $u \in U$ and $t \in T$.

  If u and t commute, then ut is a Jordan decomposition and we have $u = 1$, ergo $s \in T$.

  Assume therefore, that u, t don't commute. We claim:

  Claim: For each $h \in sU = Us$, we have for the B-conjugacy class $C(h) = \{ghg^{-1} \mid g \in B\}$
  $$C(h) = sU.$$
  The claim implies the theorem, because we then have
  $$t = su \in sU = C(s)$$
  ergo $t = gsg^{-1}$.

  Proof of Claim:

  - First note, that B acts by conjugation on $Us = sU$. This is because G/U is commutative and U is normal. In fact, we have for $g \in B, u \in U$
    $$gsug^{-1} = s \cdot (s^{-1}gsug^{-1}) = s \cdot (s^{-1}gsg^{-1}) \cdot u'.$$
    Now, $(s^{-1}gsg^{-1})$ must lie in U because G/U is commutative.
  - Since $\dim(U) = 1$, we have
    $$U = \left\{v^k \mid k \in K\right\} \cong k.$$
  - $h \in sU$ does not commute with u, since – otherwise – s, t would commute with u.

    Ergo, $h \neq u^{-1}hu$, which means $C(h) \supseteq \{h, u^{-1}hu\}$ contains at least two different elements.

- Note, that $C(h)$ is a B-orbit and therefore connected and **locally closed** (that is a closed subset of an open subset of G). Since G/U is commutative, we have
$$C(h) \subset sU = hU \cong k.$$

Now, the only connected, locally closed subset of k are singletons and complements of finite sets.

Since $C(h)$ is not a singleton, we have

$$C(h) = sU - \Sigma$$

for a finite set Σ.

We claim that Σ is empty. Note, that B acts by conjugation on sU and $C(h)$, ergo also on Σ. If we pick $h' \in \Sigma \subset sU$, then $C(h')$ must be finite, connected and contain two different elements. This is a contradiction.

- $\dim(U) \geq 2$:
We want to reduce this case to the case $\dim(U) = 1$. We need therefore, to show a lemma:

Lemma 71. *Let* $B = U \rtimes T$ *as above and suppose again* $\mathrm{char} k = 0$.

Then, there is an algebraic subgroup $V \subset U$ *s.t.* V *is normal in* B *and*

$$\dim(U/V) = 1.$$

Proof. U is nilpotent, since it is unipotent. Consider the chain

$$U = U_0 \supset U_1 \supset \ldots \supset U_n \supset 1$$

where
$$U_{i+1} := [U_i, U].$$

Since U is normal in B, each U_i is also normal in B. In particular, B acts on each U_i by conjugation.

Now, U/U_1 is unipotent and commutative, hence isomorphic to a vector space.

Further, T acts on U/U_1 by conjugation. Note, that is diagonalizable, ergo reductive. Therefore, U/U_1 must be completely reducible and we can decompose it
$$U/U_1 = \bigoplus_j V_j.$$

Since T is diagonalizable and each V_j is T-invariant, each V_j must be one-dimensional. Set

$$\overline{V} := \bigoplus_{j \geq 2} V_j.$$

And now set

$$V := \pi^{-1}(\overline{V}) = \left\{ u \in U \mid uU_1 \in \overline{V} \right\}.$$

Then, we have

$$U/V = (U/U_1)/(V/U_1) = (U/U_1)/\overline{V} \cong V_1 \cong k.$$

V is normal in U, since U_1 is normal in U and T acts on V and $\overline{V}$ by conjugation.

□

Let $s \in B$ be semisimple and $\dim(U) \geq 2$. Choose $V \subset U$ s.t. $\dim(U/V) = 1$ and V is normal in B. Set

$$\begin{aligned} B' &:= B/V \\ U' &:= U/V. \end{aligned}$$

Then, B' is a connected algebraic group with

$$\begin{aligned} (B')_u &= U' \\ B'/U \cong B/U &= T \\ B' &= U' \rtimes T. \end{aligned}$$

Since $\dim(U') = 1$, we know that $\pi_V(s) \in B$ is contained in a conjugacy class of T. Let $s' \in B$ be the conjugate of $s \in B$ s.t. $\pi_V(s') \in T$. Then,

$$s' \in TV.$$

But TV is a connected solvable algebraic group and we have

$$TV \cong V \rtimes T \subset U \times T.$$

Since $(TV)_u = V$ and $\dim(V) = \dim(U) - 1$, the induction hypothesis does also hold in TV. Ergo, s' is conjugated to some element in T, as we wanted.

□

19 All About Tori

19.1 Maximal Tori

Definition 62. A **maximal torus** $T \subset G$ is a torus that is not contained in any larger torus.

Since G has a finite dimension, it always has at least one maximal torus.

Example 26. If $G = \mathsf{GL}_n(k)$, then $k^\times$ is its maximal torus.

Lemma 72. *Let G be a connected algebraic group. Let $s \in G$ be semisimple. Then, there is a maximal torus in G s.t.*

$$C(s) \cap T \neq \emptyset$$

where $C(s) = \{gsg^{-1} \mid g \in G\}$.

Proof. Choose a Borel group $B \subset G$ s.t. $s \in B$. Then, we can decompose

$$B \cong U \rtimes T.$$

T is a torus. By the previous theorem, we know that a conjugate of s is contained in T. The claim follows, if we enlarge T to a maximal torus in G. □

Theorem 32. *Let G be a connected algebraic group with a maximal torus T. Let $s \in G$ be semisimple. Then,*

$$C(s) \cap T \neq \emptyset.$$

Proof. We only show the theorem in case $\mathrm{char} k = 0$.

Since T is connected and solvable, it is contained in some Borel group B.

Choose further a Borel group $B' = U \rtimes S$ s.t. $s \in B'$ is conjugate to some element $s' \in S$.

Now, B' is conjugate to B. If we choose a generator $t \in T$ s.t.

$$\overline{\langle t \rangle} = T,$$

then t is conjugated to some element $t' \in S$. The torus $T' = \overline{\langle t' \rangle}$ generated by t' is again maximal, therefore

$$T' = S.$$

Since $s' \in T'$, the claim follows. □

Corollary 17. *Let G be a connected algebraic group.*

(i) Any two maximal tori are conjugate.

(ii) For each torus S and each maximal torus T exists a $g \in G$ s.t.

$$gSg^{-1} \subset T.$$

Proof. (i) follows, if we can show (ii).

Let S be torus with a generator s. Then there is a $g \in G$ s.t.

$$gsg^{-1} \in T.$$

Ergo

$$gSg^{-1} = \overline{\langle gsg^{-1} \rangle} \subset T.$$

□

Corollary 18. *Let s be a central semisimple element of G (i.e. s commutates with each other element). Then s is contained in every maximal torus. In other words*

$$Z(G)_s \subset \bigcap_{T \text{ max. torus}} T.$$

Proof. This is clear, since a conjugate of s must be contained in each maximal torus and s commutes with each element. □

Corollary 19. *Let T be a torus in a connected group G. Then, T is maximal iff its dimension is maximal among all dimensions of tori in G.*

19.2 Centralizers of Tori

Lemma 73. *Let G be a connected algebraic group. Let $S \subset T$ be a torus. Let $g \in G$ be a semisimple element which commutes with each element of S.*

Then, $S \cup \{g\}$ is contained in some torus of G.

Proof. Set $H := Z_G(g)^o$. Then, H is a connected algebraic groups that contains S. Then,

$$g \in Z(H)_s \subset \bigcap_{T \text{ maximal tori in } H} T.$$

In particular, there must be some maximal torus of H which contains S. □

Theorem 33. *Let G be a connected algebraic group. Let $S \subset T$ be a torus.*

Then, $Z_G(S)$ is connected.

Proof. We assume char$k = 0$.

Let $g \in Z_G(s)$. Decompose $g = g_s g_u$. Then, we need too show the claim in case:

(i) g is semisimple:
By the previous lemma, there is a torus $T \subset Z_G(S)^o$ which contains S and g.

(ii) g is unipotent:
Since k has characteristic zero, the group

$$\overline{\langle g \rangle} = g^k \cong \begin{cases} k, & g \neq 1 \\ 1, & g = 1 \end{cases}$$

is connected.

□

19.3 Low Dimensional Groups

Lemma 74. *Let G be a connected algebraic group with a Borel subgroup B.*
If B is nilpotent, then G is solvable i.e. $B = G$.

Proof. We induct on $\dim(B)$:

- $\dim(B) = 0$:
  In this case, we have $B = 1$. Then, $G = G/B$ must be projective and connected. Therefore, we must have

  $$\mathcal{O}(G) = \mathcal{O}(\mathbb{P}^n)/I(G) = k$$

  since $\mathcal{O}(\mathbb{P}^n) = k$. On the other side, G affine. Therefore, we have

  $$G = 1.$$

  Or: $G = \bigcup_{g \in G} gBg^{-1}$, since G is connected. Since $B = 1$, it follows $G = 1$.

- $\dim(B) \geq 1$:
  Since B is nilpotent, we have a descending chain

  $$B = B_0 \supsetneq \ldots \supsetneq B_n \supsetneq 1$$

  where

  $$B_{i+1} = [B, B_i].$$

  Note, that each B_i is connected, since B is connected. Let $Z(B) = \{b \in B \mid \forall g \in B: \ gb = bg\}$ be the center of B and let $Z := Z(B)^o$ be the component of the neutral element.
  Then, we have

  $$B_n \subset Z.$$

  Ergo, Z is not the trivial subgroup.

  We want to show

  $$Z \subset Z(G).$$

  Let $z \in Z$ and consider the morphism

  $$\begin{aligned} \phi : G/B &\longrightarrow G \\ gB &\longmapsto gzg^{-1}. \end{aligned}$$

ϕ is well-defined, because $z \in Z(B)$. Since ϕ is a morphism from a projective variety to an affine variety, ϕ must be constant. Thus,

$$Z \subset Z(G).$$

In particular, Z is normal in G. We now get an inclusion of quotient groups

$$B/Z \hookrightarrow G/Z.$$

It is clear that

$$\dim(B/Z) < \dim(B).$$

Further, B/Z is parabolic, since

$$(G/Z)/(B/Z) = G/B$$

is projective. Ergo, B/Z is Borel. By the induction hypothesis, we get

$$G/Z = B/Z.$$

Ergo, $B = G$.

□

Theorem 34. *Let G be connected with* $\dim(G) \leq 2$.
Then, G is solvable.

Example 27 (Non-Example)**.** The condition $\dim(G) \leq 2$ is necessary. Consider e.g. $G = \mathsf{SL}_2(k)$ which has a dimension of 3.

Proof. Let $B \subset G$ be a Borel subgroup. We want to show

$$B = G.$$

Assume otherwise. Then, B is of dimension 1. The key here is, that Borel groups of dimension 1 are nilpotent.

Decompose $B = U \rtimes T$, then we have:

- $U \neq 1$: Then, $\dim(T) = \dim(B/U) = 0$, ergo $T = 1$. Hence

$$B = U.$$

  Since unipotent groups are nilpotent, B is nilpotent.

- $U = 1$: In this case, we have
$$B = T.$$
Now B as a torus is commutative, ergo nilpotent.

Now, the above lemma states
$$B = G$$
since B is nilpotent. Ergo, G is solvable. □

Corollary 20. *Let G be connected with* $\dim(G) = 1$. *Then, G is commutative.*

Proof. Because of the theorem, G is solvable. Therefore, $[G, G]$ is a closed proper subgroup of G. Hence, $\dim([G, G]) = 0$. Since $[G, G]$ is connected, it follows $[G, G] = 1$. □

Remark 20. If G is commutative, it decomposes nicely into semisimple and unipotent elements
$$G = G_s \times G_u.$$
So, if $\dim(G) = 1$ and if G is connected, then $G = G_s \cong \mathcal{G}_m$ is a torus, or $G = G_u \cong \mathcal{G}_a$ is unipotent.

Further, we can consider
$$G = \left\{ \begin{pmatrix} * & * \\ & 1 \end{pmatrix} \right\}.$$
G is connected and of dimension 2. It decomposes
$$G = G_s \times G_u$$
into two groups of dimension 1.

19.4 Characterizing Nilpotent Groups via maximal Tori

Lemma 75. *Let T be a diagonaliable algebraic group. Then,*

$$T = \overline{\bigcup_{n \geq 1} T[n]}$$

where

$$T[n] := \{t \in T \mid t^n = 1\}.$$

Proof. If the claim is true for any T_1, T_2, then it is also true for $T_1 \times T_2$.

Therefore, we can reduce the claim to the cases where T is finite or $T = \mathcal{G}_m$.

A finite T is contained in some $T[n]$.

Let $T = \mathcal{G}_m$. Then, we have

$$\mathcal{O}(T) = k[x, \frac{1}{x}].$$

We need to show for each $f \in \mathcal{O}(T)$:

$$f(\alpha) = 0 \ \forall \alpha \in k, \in \mathbb{N}_0 \text{ s.t. } \alpha^n = 1 \Longrightarrow f = 0.$$

So, let $f \in \mathcal{O}(T)$. By multiplying with a large enough x^r, we can assume $f \in k[x]$. If $f \neq 0$, then f has finitely many roots. However the set

$$\{\alpha \in k \mid \exists n \in \mathbb{N}_0 : \ \alpha^n = 1\}$$

has infinitely many elements. Indeed we have

$$\# \{\alpha \in k \mid \alpha^n = 1\} = n,$$

if $\text{char} k = 0$ or if $\gcd(\text{char} k, n) = 1$. □

Remark 21. If $\text{char} k = 0$ and if U is unipotent, then

$$U[n] = 1$$

for all n.

Theorem 35. *Let G be a connected algebraic group. Then, the following are equivalent:*

(i) G is nilpotent.

(ii) Each maximal torus T of G satisfies $T \subset Z(G)$.

(iii) G has a unique maximal torus.

Proof. We show:

(i) $\implies$ (ii): We have shown, if G is nilpotent and connected, then each semisimple element is central.

(ii) $\implies$ (iii): Any two maximal tori are conjugated.

(iii) + (ii) $\implies$ (i): Since each semisimple element is contained in some torus, we have that each semisimple element is central.

Let $B = U \rtimes T' \subset G$ be a Borel subgroup. Since $T' \subset T$ is central in G, we have $B = U \times T'$. So, B is nilpotent. We showed in this case that

$$G = B.$$

(iii) $\implies$ (ii): Let T be the maximal torus of G. We have to show that T is central.

Since T is unique, it must be normal in G. Then, G acts via conjugation on T and each $T[n]$. Therefore, we get a morphism for each $t \in T[n]$

$$\begin{aligned} G &\longrightarrow T[n] \\ g &\longmapsto gtg^{-1}. \end{aligned}$$

Since G is connected, this morphism must be constant, ergo trivial. Ergo, each $T[n]$ is central in G. Since

$$T = \overline{\bigcup_{n \in \mathbb{N}} T[n]},$$

T must be central in G.

$\square$

19.5 Weyl Groups

Definition 63. Let G be a connected algebraic group with a torus T. Define the **normalizer** of T in G by

$$N_G(T) := \{g \in G \mid gT = Tg\}.$$

Then, the centralizer $Z_G(T)$ is a normal subgroup in $N_G(T)$. Define the **Weyl group** of T as the quotient

$$W(G,T) := N_G(T)/Z_G(T).$$

If T is maximal, then the Weyl group $W(G,T)$ is up to conjugation independent of T.

Proposition 7. *(i)* $\#W < \infty$.

(ii) For each tori $S \subset G$, we have

$$N_G(S)^o = Z_G(S)^o.$$

Proof. It is clear, that (ii) implies (i).

Let $S \subset G$ be a torus. We want to show

$$N_G(S)^o \subset Z_G(S).$$

Note, that $N_G(S)^o$ acts by conjugation on S. Now, it is clear that

$$S = \overline{\bigcup_n S[n]}$$

where $S[n]$ denotes the subgroup of n-th roots of unity. $N_G(S)^o$ acts on each $S[n]$. As before, this gives for each $s \in S[n]$ a group morphism $\phi_s : N_G(S)^o \to S[n]$. Since $S[n]$ is finite and $N_G(S)^o$ is connected, ϕ_s must be trivial. Ergo

$$N_G(S)^o \subset Z_G(S).$$

□

Remark 22. In general, $W(G,T)$ acts on T by conjugation and induces an inclusion

$$W(G,T) \hookrightarrow \mathsf{Aut}(T).$$

Example 28. Let $G = \mathsf{GL}_n(k)$ with the maximal torus

$$T = \left\{ \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \right\}.$$

Denote by $S(n)$ the group of all permutation matrices of G.

Then, we have

$$\begin{aligned} Z_G(T) &= T \\ N_G(T) &= T \cdot S(n) \\ W(G,T) &\cong S(n). \end{aligned}$$

19.6 Normalizers of Borel Subgroups

Lemma 76. *Let G be a connected algebraic group, $B \subset G$ a Borel subgroup and $S \subset B$ any torus.*

Then, $Z_B(S)$ is a Borel subgroup of $Z_G(S)$.

Proof. We showed before, that $Z_G(S)$ is connected, if G is connected. Set

$$U := B_u.$$

- We claim
$$Z_G(S)B = \{g \in G \mid \forall s \in S : \ g^{-1}sg \in sU\} =: A.$$
It is easy to see, that
$$Z_G(S) \subset A.$$
For $b \in B$, we have
$$b^{-1}sb \in sU,$$
since B/U is commutative.

  Now, let $g \in A$. Then,
$$g^{-1}Sg \subset SU \subset B.$$
One can extend S to a maximal torus T of B. Then,
$$B = U \rtimes T \supset SU = U \rtimes S.$$
Since S is closed in T, SU is closed in B. Further, $g^{-1}Sg$ and S are maximal tori in SU. Then, there is a $b \in B$ s.t.
$$b(g^{-1}Sg)b^{-1} = S.$$
Set
$$z := gb^{-1}.$$
We need to show, that z lies in $Z_G(S)$.

  Since B/U is commutative, we have for each $s \in S$
$$z^{-1}sz = b(g^{-1}sg)b^{-1} \in g^{-1}sgU = sU,$$
since $g \in A$. Now, we have for each $s \in S$
$$z^{-1}sz \in sU \cap S = \{s\}.$$
Ergo, $z \in Z_G(S)$.

- We showed that

$$Z_G(S)B = \left\{g \in G \mid \forall s \in S : \; g^{-1}sg \in sU\right\} = \{g \in G \mid \forall s \in S : [g,s] \in U\}\,.$$

Then, $Z_G(S)B$ is closed. Since

$$\pi : G \twoheadrightarrow G/B$$

is an open and surjective map, it is easy to see that

$$Z_G(S)/Z_B(S) \cong \pi(Z_G(S))$$

is closed. Since $Z_B(S)$ is closed, $Z_B(S)$ is a parabolic subgroup of $Z_G(S)$. Since $Z_B(S)$ is contained in B, it is solvable, hence a Borel subgroup.

□

Example 29.

$$\begin{aligned}
G &= \mathsf{GL}_5(k) \\
B &= \left\{ \begin{pmatrix} * & * & * & * & * \\ & * & * & * & * \\ & & * & * & * \\ & & & * & * \\ & & & & * \end{pmatrix} \right\} \\
S &= \left\{ \begin{pmatrix} t_1 & & & & \\ & t_1 & & & \\ & & t_1 & & \\ & & & t_2 & \\ & & & & t_2 \end{pmatrix} \mid t_1, t_2 \in k^\times \right\} \\
Z_G(S) &= \left\{ \begin{pmatrix} * & * & * & & \\ * & * & * & & \\ * & * & * & & \\ & & & * & * \\ & & & * & * \end{pmatrix} \right\} = \mathsf{GL}_3(k) \times \mathsf{GL}_2(k) \\
Z_G(S) &= \left\{ \begin{pmatrix} * & * & * & & \\ & * & * & & \\ & & * & & \\ & & & * & * \\ & & & & * \end{pmatrix} \right\}.
\end{aligned}$$

Theorem 36. *Let G be a connected algebraic group with a Borel subgroup $B \subset G$. Then,*

$$N_G(B) = B.$$

Proof. We induct on $\dim(G)$:

- $\dim(G) = 0$: In this case, we have $G = 1$.
- $\dim(G) = 1$: We have seen, that in this case G is commutative.
- $\dim(G) \geq 2$: Let T be a maximal torus in B. Let $x \in N_G(B)$.

  Then, xTx^{-1} is again a maximal torus in B. Since all maximal tori in B are related via B-conjugation, there is $b \in B$ s.t.

  $$xTx^{-1} = bTb^{-1}.$$

  We therefore replace x by $b^{-1}x$ to achieve

  $$xTx^{-1} = T.$$

  Now, consider the map

  $$\begin{aligned} \rho : T &\longrightarrow T \\ t &\longmapsto txt^{-1}x^{-1}. \end{aligned}$$

  We distinguish two cases:

  - ρ is not surjective:
    Since all tori are irreducible, we have then

    $$\begin{aligned} \dim(\mathsf{Img}(\rho)) &< T \\ \dim(\mathsf{Kern}(\rho)^o) &> 0. \end{aligned}$$

    If we set $S := \mathsf{Kern}(\rho)^o$, then S is a non-trivial torus in T.
    Since $S \subset \mathsf{Kern}(\rho)$, x centralizes S and normalizes B. Hence, x normalizes $Z_B(S)$.
    Because of the previous lemma, $Z_B(S)$ is Borel subgroup of $Z_G(S)$. If $Z_G(S) \neq G$, then the indution hypothesis implies

    $$x \in N_{Z_G(S)}(Z_B(S)) = Z_B(S) \subset B.$$

Otherwise, if $Z_G(S) = G$, then B/S is a Borel subgroup of G/S. So the induction hypothesis implies

$$xS \in N_{G/S}(B/S) = B/S,$$

ergo $x \in B$.

– ρ is surjective:
Then,

$$T = \mathsf{Img}\rho \subset [N_G(B), N_G(B)].$$

Set $H := N_G(B)$. We have to show

$$H = B.$$

Choose a finite-dimensional representation

$$G \hookrightarrow \mathsf{GL}(V)$$

and a line $L \subset V$ s.t.

$$H = \{g \in G \mid gL = L\}.$$

Then, we have a morphism of algebraic groups

$$\gamma : H \longrightarrow \mathsf{GL}(L) = \mathcal{G}_m(k).$$

Since the right side is a torus, we have

$$\gamma_{|H_u} \equiv 1$$
$$\gamma_{|[H,H]} \equiv 1.$$

Ergo, $\gamma(T) = 1$ and, since $B = B_u \rtimes T$, $\gamma(B) = 1$.
Choose a non-zero element $v \in L$ and consider

$$\begin{aligned} \phi : G/B &\longrightarrow V \\ gB &\longmapsto gv. \end{aligned}$$

Since G/B is a projective variety, while V is an affine variety, ϕ must be constant. Therefore, we have for each $g \in G$

$$gv \in L.$$

Ergo, $G = H$ and B is normal in G. But, now

$$G = \bigcup_{g \in G} gBg^{-1} = B.$$

Ergo $H = B$. $\square$

Corollary 21. *We have a bijection:*

$$\begin{aligned} G/B &\longrightarrow \{\textit{Borel Subgroups of } G\} \\ gB &\longmapsto gBg^{-1}. \end{aligned}$$

19.7 Borel Subgroups Containing a Given Torus

Let G be a connected algebraic group with a maximal torus T. Set

$$\mathcal{B}^T := \{B \subset G \text{ Borel} \mid T \subset B\}.$$

Then, $N_G(T)$ acts on $\mathcal{B}^T$ by conjugation.

Example 30. Let $G = \mathsf{GL}_2(k)$ with $T = \left\{\begin{pmatrix} * & \\ & * \end{pmatrix}\right\}$. Then,

$$\mathcal{B}^T = \left\{\begin{pmatrix} * & * \\ & * \end{pmatrix}, \begin{pmatrix} * & \\ * & * \end{pmatrix}\right\}.$$

Lemma 77. *The action of $Z_G(T)$ on $\mathcal{B}^T$ by conjugation is trivial.*

Equivalently (since $B = N_G(B)$), $Z_G(T) \subset B$ for each $B \in B^T$.

Proof. We know, that $Z_G(T)$ is connected, since T is a torus. Further, since $T \subset Z_G(T)$ is central and a maximal torus, is must be the unique maximal torus in $Z_G(T)$. We showed before, that this is equivalent to $Z_G(T)$ being nilpotent. Thus, $Z_G(T)$ is contained in some Borel group $B_0 \in \mathcal{B}^T$.

Let $B \in \mathcal{B}^T$ and choose $g \in G$ s.t.

$$B = gB_0g^{-1}.$$

Since maximal tori in B are B-conjugated, we can choose $g \in G$ s.t. $g \in N_G(T)$. (Otherwise, we can replace g bx bg s.t. $bgTg^{-1}b^{-1} = T$.)

One can show that

$$g \in N_G(T) \implies g \in N_G(Z_G(T)).$$

Thus

$$g^{-1}Z_G(T)g = Z_G(T) \subset B_0$$

which implies

$$Z_G(T) \subset gB_0g^{-1} = B.$$

□

Corollary 22. *The action $N_G(T) \curvearrowright \mathcal{B}^T$ induces am action by the Weyl group $W(G,T) = N_G(T)/Z_G(T)$ on $\mathcal{B}^T$.*

Corollary 23. *In the proof, we could see that $N_G(T)$ and $W(G,T)$ act transitively on $\mathcal{B}^T$.*

Corollary 24.

$$\#\mathcal{B}^T \leq \#W < \infty.$$

Theorem 37. *W acts **simply-transitively** on $\mathcal{B}^T$, i.e., for each $B_1, B_2 \in \mathcal{B}^T$ there is exactly* one *$g \in G$ s.t.*

$$gB_1g^{-1} = B_2.$$

In particular,

$$\#\mathcal{B}^T = \#W.$$

Proof. Let $B \in \mathcal{B}^T$. We need to show

$$N_G(T) \cap N_G(B) \subset Z_G(T).$$

Note, that

$$N_G(T) \cap N_G(B) = N_G(T) \cap B = N_B(T).$$

Set $U := B_u$, then $B = U \rtimes T$.

Choose $b \in N_B(T)$ with $b = ut$, $u \in U, t \in T$. Then,

$$T = bTb^{-1} = uTu^{-1}.$$

Since $t \in Z_G(T)$, it suffices to show that $u \in Z_G(T)$.

Let $t \in T$ and set $t' = utu^{-1} \in T$. Since, we have an isomorphism

$$T \hookrightarrow B \twoheadrightarrow B/U$$

and B/U is commutative, t and t' must be equal in T. Ergo, $u \in Z_G(T)$. $\square$

Corollary 25. *Since $N_B(T) \subset Z_G(T)$ we have for each Borel group B and maximal torus T of G*

$$W(G, T) = 1.$$

In particular,

$$\mathcal{B}^T = \{B\}.$$

Proposition 8. *Let G be a connected non-solvable algebraic group (this implies* $\dim G \geq 3$*). Let B be a Borel subgroup with a maximal torus T. Then,*

$$\#W(G, T) \geq 2.$$

Moreover,

$$\#W = 2 \iff \dim(G/B) = 1.$$

Sketch of Proof. We have a bijection

$$\{\text{Borel subgroups in } G\} \longleftrightarrow G/B.$$

This can be restricted to a bijection

$$\mathcal{B}^T \longleftrightarrow \{gB \in G/B \mid TgB = gB\}.$$

Idea: Show that T acts non-trivally on G/B, since G is non-solvable, and deduce that it has at least two different fixed points:

$$\begin{tikzcd}
\mathcal{G}_m \arrow[r, hook] \arrow[d, hook] & T \arrow[r, "\curvearrowright"] & G/B \\
\mathbb{P}^1 \arrow[urr] & \ni 0, \infty &
\end{tikzcd}$$

One can show, that 0 and ∞ are mapped to different fixed points of G/B. $\square$

19.8 Groups of Semisimple Rank One

Definition 64. Let G be an algebraic groug. The **rank** of G is defined by

$$\operatorname{rank}(G) := \dim(T)$$

for any maximal torus T.

Remark 23. If G is connected and of rank zero, then G is unipotent.

Definition 65. The **semisimple rank** of G is defined by

$$\text{ss-rank}(G) := \operatorname{rank}(G/R(G)).$$

(Note, that $G/R(G)$ is semisimple.)

Example 31.

- If T is a torus, then $\text{ss-rank}(T) = 0$.
- Let $Z = Z_{\mathsf{GL}_n(k)}(\mathsf{GL}_n(k))$ be the centralizer of $\mathsf{GL}_n(k)$. Then, $Z \cong k$. For a matrix group $G \subset \mathsf{GL}_n(k)$, set

  $$PG := G/(G \cap Z).$$

  Then, PG acts on $\mathbb{P}^n$.

  We now have

G	ss-rank(G)	rank(G)
SL_2	1	1
$P\mathsf{GL}_2$	1	1
GL_2	1	2
GL_n	$n-1$	n

- Consider

  $$G = \left\{ \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \right\} \subset \mathsf{GL}_3(k).$$

  Then,

  $$\operatorname{rank}(G) = 3,$$
  $$\text{ss-rank}(G) = 1.$$

Remark 24. Let G be a connected algebraic group. Then, G is of semisimple rank zero iff G is Borel (because $R(G)$ is the connected component of 1 in the intersection of all Borel sungroups in G).

Lemma 78 (Fact). *We call X a **curve**, if X is a one-dimensional variety.*

Let X be a smooth projective curve X that admits a nontrivial action by a nontrivial connected algebraic group H. Then, we have

$$X \cong \mathbb{P}^1.$$

Idea of Proof. Reduce to the case $H = \mathcal{G}_m$ or $H = \mathcal{G}_a$. They give a commutative diagramm:

$$\begin{tikzcd} \mathcal{G}_m \arrow[r] \arrow[d, hook] & X \\ \mathbb{P}^1 \arrow[ur, "\exists\phi"'] & \end{tikzcd}$$

Now, ϕ is a non-constant orbit map, therefore

$$k(X) \hookrightarrow k(\mathbb{P}^1) \cong k(T).$$

By Lüroth's Theorem[4], there is a transcendent $T' \in k(T)$ s.t. $k(X) = k(T')$. Ergo

$$X = \mathbb{P}^1. \qquad \square$$

Lemma 79 (Fact). *We have*

$$\mathsf{Aut}\left(\mathbb{P}^1\right) \cong P\mathsf{GL}_2(k).$$

Proposition 9. *Let G be of semisimple rank one. Then, there is a surjective morphism*

$$\rho : G \to P\mathsf{GL}_2(k)$$

s.t. $\mathsf{Kern}\rho^o = R(G)$.

In particular, if G is semisimple, then $\mathsf{Kern}\rho$ is finite, since $R(G)$ is trivial in this case.

[4] Lüroth's Theorem states that each intermediate field $L \supset P \supset K$ of a purely transcendental extension $L \supset K$ of degree 1 is either K or purely transcendental over K.

Proof. By dividing out $R(G)$, we can reduce the proof to the case, in which G is semisimple and of rank 1. In particular, G cannot be solvable, ergo $\dim(G) \geq 3$.

We have seen for unsolvable groups, that $\# W(G,T) \geq 2$. But, since $T \cong \mathcal{G}_m$

$$W(G,T) = N_G(T)/Z_G(T) \hookrightarrow \mathsf{Aut}\,(T) = \{\mathrm{Id}, t \mapsto t^{-1}\} \cong \mathbb{Z}/2\mathbb{Z}.$$

Ergo, $\# W(G,T) = 2$. We have seen, that we have in this case

$$\dim(G/B) = 1.$$

Ergo, G/B is a projective one-dimensional variety. Ergo, $G/B \cong \mathbb{P}^1$. Now, define

$$\begin{aligned} \rho : G &\longrightarrow \mathsf{Aut}\,(G/B) \cong \mathsf{Aut}\,\left(\mathbb{P}^1\right) \cong P\mathsf{GL}_2(k) \\ g &\longmapsto [xB \mapsto gxB]. \end{aligned}$$

Clearly,

$$\mathsf{Kern}\rho = \{g \in G \mid gxB = xB \forall x \in G\} = \bigcap_{x \in G} xBx^{-1} = \bigcap \{B \subset G \text{ Borel}\}.$$

Ergo $(\mathsf{Kern}\rho)^0 = R(G) = 1$, since G is semisimple.

It remains to show, that ρ is surjective. Indeed, we have

$$\dim(\rho(G)) \geq \dim(G) - \dim(\mathsf{Kern}\rho) \geq 3.$$

Since $P\mathsf{GL}_2(k)$ is 3-dimensional and connected, we have

$$\rho(G) = P\mathsf{GL}_2(k).$$

□

Proposition 10. *Let G be a reductive algebraic group of semisimple rank one.*

Let $\rho : G \twoheadrightarrow \mathsf{Aut}\,(G/B) \cong P\mathsf{GL}_2(k)$ be as in the previous proposition. Then, $\ker \rho$ *is diagonalizable.*

Proof. Let T be a maximal torus in G. It suffices to show that

$$\mathsf{Kern}\rho \subset T.$$

By the above proof

$$2 = \# W(G,T) = \# \mathcal{B}^T.$$

Therefore,

$$\mathcal{B}^T = \left\{B^+, B^-\right\}.$$

Since $\mathsf{Kern}\rho \subset \bigcap_{B\subset G \text{ Borel}} B = B^+ \cap B^-$ it suffices to show

$$B^+ \cap B^- = T.$$

Which is equivalent to

$$B_u^+ \cap B_u^- = 1.$$

Since G has a semisimple rank of one, we have $G \neq B^\pm$. Ergo, $B^\pm$ is not nilpotent (otherwise $G = B^\pm$). Thus, $B_u^\pm$ is connected and non-trivial. Thus,

$$\dim(B_u^\pm) \geq 1.$$

Also,

$$(\mathsf{Kern}\rho)^o \cap B_u^\pm = R(G) \cap B_u^\pm \subset R_u(G) = 1$$

since G is reductive. So, $\mathsf{Kern}(\rho_{|B_u^\pm})$ is finite. And therefore

$$\dim(\rho(B_u^\pm)) \geq 1.$$

But $\rho(B_u^\pm)$ is a unipotent subgroup of $P\mathsf{GL}_2(k)$. Therefore

$$\dim(\rho(B_u^\pm)) = 1.$$

Since $\mathsf{Kern}(\rho_{|B_u^\pm})$ is finite, we have

$$\dim B_u^\pm = 1.$$

Ergo, $B_u^\pm \cong \mathcal{G}_a$.

In characteristic zero, we proved that T acts on $B_u^\pm$ by conjugation. We have that the composition

$$T \longrightarrow \mathsf{GL}(\mathcal{G}_a) \cong \mathcal{G}_m$$

is nontrivial, since $B^\pm$ is not nilpotent.

We still want to show

$$B_u^+ \cap B_u^- = 1.$$

Assume, for the sake of contradiction, that we have a non-trivial $x \in B_u^+ \cap B_u^-$. Then, $T.x = \{txt^{-1} \mid t \in T\}$ lies dense in $B_u^+ \cap B_u^-$, in fact

$$T.x \cong \mathcal{G}_a \setminus \{0\}.$$

Since $B_u^\pm$ is one-dimensional, it follows $B_u^+ = B_u^-$. Therefore,

$$B^+ = B^-.$$

This is a contradiction to $\#W(G,T) = 2$. $\square$

Corollary 26 (Eventual Corollary)**.** *Any connected reductive group G of semisimple rank one is isomorphic to one of the following:*

$$\begin{aligned} \mathsf{SL}_2(k) \times T \\ P\mathsf{GL}_2(k) \times T \\ \mathsf{GL}_2(k) \times T \end{aligned}$$

for some torus T.

19.9 Isogenies

Definition 66. A morphism $\phi : G_1 \to G_2$ of connected algebraic groups is an **isogeny**, if ϕ is surjective and $\mathsf{Kern}\phi$ is finite.

An isogeny is called **multiplicative**, if moreover $\mathsf{Kern}\phi$ is diagonalizable.

Thus, the last result said:

If G is a semisimple connected group of rank one, then there is a multiplicative isogeny $G \twoheadrightarrow P\mathsf{GL}_2(k)$. (Which implies $G \cong P\mathsf{GL}_2(k)$ or $G \cong \mathsf{SL}_2 k$.)

Definition 67. G is **simply connected**, if every multiplicative isogeny $\widetilde{G} \twoheadrightarrow G$ is an isomorphism.

Remark 25 (Facts).

- If G is semisimple, then there is a simply connected semisimple G^{sc} and a multiplicative isogeny $G^{sc} \twoheadrightarrow G$ s.t. $G^{sc} \twoheadrightarrow G$ is initial in the category of all multiplicative isogenies $\widetilde{G} \twoheadrightarrow G$.
- $\mathsf{SL}_2(k) \twoheadrightarrow P\mathsf{GL}_2(k)$ is simply connected.

20 Root Data

20.1 More on Reductive Groups

Proposition 11. *Let G be a connected, reductive algebraic group. Then,*

$$R(G) = Z(G)^o.$$

Proof. $Z(G)^o$ is connected, normal and commutative, ergo solvable. Ergo $Z(G)^o$ is contained in $R(G)$.

Since $R(G)$ is a normal subgroup of the connected group G, we have

$$G = N_G(R(G)) = N_G(R(G))^o.$$

Since $R(G)$ is a torus in a connected group G, we have

$$N_G(R(G))^o = Z_G(R(G))^o.$$

$G = Z_G(R(G))^o$ implies $R(G) \subset Z(G)$. □

Proposition 12. *Let G be a connected, reductive algebraic group. Then,*

$$R(G) \cap [G, G]$$

is finite.

Proof. Take a faithful representation $G \hookrightarrow \mathsf{GL}(V)$. $R(G)$ is a torus in G, therefore we can decompose V into eigenspaces of $R(G)$:

$$V = \bigoplus_{\chi} V_\chi$$

where $\chi \in \mathfrak{X}(R(G))$ and

$$V_\chi = \{v \in V \mid h.v = \chi(h)v \ \forall h \in R(G)\}.$$

Since $R(G)$ is normal in G, G acts on each V_χ. Consider the representations

$$\rho_\chi : G \longrightarrow \mathsf{GL}(V_\chi).$$

It is easy to see that

$$\rho_\chi([G, G]) \subseteq \mathsf{SL}(V_\chi)$$

and

$$\rho_\chi(R(G)) \subseteq Z_{\mathsf{GL}(V_\chi)}(\mathsf{GL}(V_\chi)) = k^\times.$$

Ergo,

$$\rho_\chi(R(G) \cap [G,G]) \subseteq \mu_{\dim(V_\chi)}.$$

And, therefore,

$$\#([G,G] \cap R(G)) \leq \prod_{\chi : V_\chi \neq 0} \dim(V_\chi). \qquad \square$$

Example 32. Let $G = \mathsf{GL}_n(k)$. Then

$$R(G) = k^\times \cdot 1_n.$$

$$[G,G] = \mathsf{SL}_n.$$

Ergo,

$$R(G) \cap [G,G] \cong \mu_n.$$

Proposition 13. *Let G be a connected, reductive algebraic group. Then, $[G,G]$ is semisimple, i.e. $R([G,G]) = 1$.*

Proof. If B' is Borel group in $[G,G]$, then gBg^{-1} stays a Borel group in $[G,G]$ for each $g \in G$. Therefore, $R([G,G])$ is normal in G.

Now, take a Borel subgroup B of G s.t.

$$R([G,G]) \subset B.$$

Then, we have

$$R([G,G]) \subset \bigcap_{g \in G} gBg^{-1} = R(G).$$

So,

$$R([G,G]) \subset R(G) \cap [G,G].$$

Since $R([G,G])$ is finite and connected, it is trivial. Ergo, $[G,G]$ is semisimple. $\square$

Proposition 14. *Let G be a connected, reductive algebraic group.*

Let $S \subset G$ be a torus. Then, $Z_G(S)$ is reductive.

If T is a maximal torus, then $Z_G(T) = T$.

Proof. Since S is a torus, $Z_G(S)$ is connected. Note that:

(i) every Borel subgroup of $Z_G(S)$ is contained in some Borel subgroup of G.

(ii) for each Borel $B \subset G$ which contains S, the intersection

$$Z_B(S) = Z_G(S) \cap B$$

is a Borel subgroup of $Z_G(S)$.

(iii) From the above, it follows

$$R(Z_G(S)) = \bigcap_{S \subset B \subset G \text{ Borel}} Z_B(S) \subset \bigcap_{S \subset B \subset G \text{ Borel}} B.$$

(iv) Since $R_u(G)$ is connected, we have

$$R_u(G) = \left(\bigcap_{B \subset G \text{ Borel}} B \right)_u^o.$$

(v) One can show for any maximal torus T

$$R_u(G) = \left(\bigcap_{T \subset B \subset G \text{ Borel}} B \right)_u^o.$$

Now, it follows

$$R_u(Z_G(S)) \subset \left(\bigcap_{S \subset B \subset G \text{ Borel}} B \right)_u^o \subset R_u(G) = 1.$$

For the second part: T is central in $Z_G(T)$ and a maximal torus in $Z_G(T)$. Therefore, $Z_G(T)$ must be nilpotent. Now, $W(Z_G(T), T) = 1$, ergo $Z_G(T)$ only has one Borel subgroup $R(Z_G(T))$. But $R(Z_G(T))$ must be a torus, hence

$$R(Z_G(T)) = T. \qquad \square$$

20.2 Root Data – Definition

Definition 68. A **root datum** is a tuple

$$\Psi = (X, X^\vee, R, R^\vee)$$

where $X, X^\vee$ are finitely generated free $\mathbb{Z}$-modules equipped with a pairing

$$\begin{aligned} X \times X^\vee &\longrightarrow \mathbb{Z} \\ (x, \xi) &\longmapsto \langle x \mid \xi \rangle \end{aligned}$$

which is **perfect**, i.e. $\langle \cdot \mid \cdot \rangle$ induces isomorphism $X \cong \mathsf{Hom}_{\mathbb{Z}}(X^\vee, \mathbb{Z})$ and $X^\vee \cong \mathsf{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$.

R and $R^\vee$ are to be finite subsets $R \subset X, R^\vee \subset X^\subset$ with a bijective map

$$\begin{aligned} R &\longrightarrow R^\vee \\ \alpha &\longmapsto \alpha^\vee. \end{aligned}$$

The $(X, X^\vee, R, R^\vee)$ shall meet the following axioms:

(i) For each $\alpha \in R$, we have
$$\langle \alpha \mid \alpha^\vee \rangle = 2.$$

(ii) For each $\alpha \in R$, define the maps
$$\begin{aligned} S_\alpha : X &\longrightarrow X \\ x &\longmapsto x - \langle x \mid \alpha^\vee \rangle \alpha \\ S_{\alpha^\vee} : X^\vee &\longrightarrow X^\vee \\ \xi &\longmapsto \xi - \langle \alpha \mid \xi \rangle \alpha^\vee. \end{aligned}$$

These maps satisfy for each $\alpha \in R$
$$S_\alpha(R) \subseteq R \text{ and } S_{\alpha^\vee}(R^\vee) \subset R^\vee.$$

(iii) We call Ψ **reduced**, if additionally the following is fulfilled:

If $\alpha, c\alpha \in R$, for some $c \in \mathbb{Q}$, then $c = \pm 1$.

Elements $\alpha \in R$ are called **roots**, while corresponding elements $\alpha^\vee \in R^\vee$ are called **coroots**.

Remark 26. Let $\Psi = (X, X^\vee, R, R^\vee)$ be a root datum:

1. For $\alpha \in R$, we have
$$S_\alpha(\alpha) = -\alpha.$$
2. In particular, we have for $\alpha \in R$
$$S_\alpha^2 = \mathrm{Id}_R.$$
3. If $\Psi = (X, X^\vee, R, R^\vee)$ is a root datum, then so is $\Psi^\vee = (X^\vee, X, R^\vee, R)$.
4. If we have for $\alpha, \beta \in R$
$$\langle \_ \mid \alpha^\vee \rangle \equiv \langle \_ \mid \beta^\vee \rangle \,,$$
then $\alpha = \beta$.

Lemma 80. *In the definition of a root datum, it would suffice to demand that*
$$\begin{aligned} R &\longrightarrow R^\vee \\ \alpha &\longmapsto \alpha^\vee \end{aligned}$$
is only surjective.

Proof. Let $\alpha, \beta \in R$ s.t.
$$\alpha^\vee = \beta^\vee.$$
If α, β are linearly dependent, they must be equal, because of
$$\langle \alpha \mid \alpha^\vee \rangle = 2 = \langle \beta \mid \alpha^\vee \rangle \,.$$
Assume, therefore, that they are linearly independent. Let V be the $\mathbb{Z}$-module spanned by the basis (α, β). Regarding this basis, the action of S_α and S_β can be represented by the following matrices:
$$\begin{aligned} S_\alpha &\hat{=} \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix} \\ S_\beta &\hat{=} \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix}. \end{aligned}$$
It is now easy to show that
$$\begin{aligned} (S_\alpha \circ S_\beta)^2 &\hat{=} \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}, \\ (S_\alpha \circ S_\beta)^n &\hat{=} \begin{pmatrix} 2n+1 & 2n \\ -2n & 1-2n \end{pmatrix}. \end{aligned}$$
But R must be closed under the action of S_α and S_β. Since R must be finite, α, β cannot be linearly independent. $\square$

Definition 69. The **Weyl group** $W(\Psi)$ of a root datum Ψ is the subgroup of $\mathsf{Aut}_{(}(X)) \cong \mathsf{GL}_n(\mathbb{Z})$ which is generated by

$$\{S_\alpha \mid \alpha \in R\}.$$

Our goal is to construct for each connected reductive group G with a maximal torus T a root datum

$$\Phi(G,T) = \Phi(G)$$

s.t. the Weyl groups $W(G,T)$ and $W(\Phi(G))$ are isomorphic (in a canonical way).

Theorem 38 (Facts)**.** *Suppose, we were given a notion of morphism of root data at this point.*

1. $\Phi(G) \cong \Phi(G')$ *iff* $G \cong G'$.

2. *Every root datum is isomorphic to some* $\Phi(G)$ *for a reductive connected group* G.

Remark 27.

- Obviously, the notion of root data is independent of k.

- Root data also classify compact connected Lie groups.

- Root data refine the less precise notion of root systems (which classify semisimple Lie algebras and simply connected semisimple algebraic groups).

- Every root system is a finite direct sum of the simple root systems

$$(A_n)_{n\geq 1}, (B_n)_{n\geq 2}, (C_n), (D_n), E_6, E_7, E_8, F_4, G_2.$$

20.3 Lie Algebras

Lemma 81 (Fact)**.** *Let G be an algebraic group. Take a faithful representation*

$$G \hookrightarrow \mathsf{GL}(V).$$

Set

$$I := I(G) \subset \mathcal{O}(\mathsf{GL}(V)).$$

*Consider the nilpotent element ε in $\mathcal{O}(\mathsf{GL}(V))[\varepsilon]/(\varepsilon^2)$ and define the **Lie algebra** of G by*

$$\begin{aligned}\mathfrak{g} := \mathrm{Lie}(G) &:= \left\{x \in \mathsf{End}(V) \mid \forall f \in I:\ f(1+\varepsilon x) = 0 \mod (\varepsilon^2)\right\} \\ &= \left\{x \in \mathsf{End}(V) \mid \forall f \in I:\ f(1+\varepsilon x) \in (\varepsilon^2)\right\} \\ &= \left\{x \in \mathsf{End}(V) \mid \forall f \in I:\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{f(1+tx)}{t}\right)_{|t=0} = 0.\right\}.\end{aligned}$$

Then, $\mathfrak{g}$ is a k-vector space of dimension

$$\dim_k(\mathfrak{g}) = \dim(G).$$

Idea of Proof. The proof boils down to show that G is smooth at 1.

However, each variety is generically smooth. Therefore, G is smooth at some points. Since G acts on itself via isomorphisms, 1 must look locally identically to one of G's smooth points. Ergo, G is smooth everywhere. $\square$

Example 33.

- $\mathrm{Lie}(\mathsf{GL}(V)) = \mathsf{End}(V)$ because $I(G) = 0$.
-
$$\begin{aligned}\mathrm{Lie}(\mathcal{O}_n(k)) &= \left\{x \in M_n(k) \mid (1+\varepsilon x)(1+\varepsilon x^T) = 1 + (\varepsilon^2)\right\} \\ &= \left\{x \in M_n(k) \mid x^T = -x\right\}.\end{aligned}$$

Definition 70. To each algebraic group we can attach the **adjoint representation**

$$\begin{aligned}\mathsf{Ad} : G &\longrightarrow \mathsf{GL}(\mathfrak{g}) \\ g &\longmapsto [x \mapsto gxg^{-1}].\end{aligned}$$

If T is a maximal torus in G, we can restrict the adjoint representation

$$\mathsf{Ad} : T \longrightarrow \mathsf{GL}(\mathfrak{g}).$$

Given any representation $\rho : T \to \mathsf{GL}(V)$, we may decompose V

$$V = \bigoplus_{\chi \in \mathfrak{X}(T)} V^{\chi}$$

where

$$V^{\chi} = \{v \in V \mid \forall t \in T : \ t.v = \chi(t)v\}.$$

Therefore,

$$\mathfrak{g} = \bigoplus_{\chi \in \mathfrak{X}(T)} \mathfrak{g}^{\chi} = g^{o} \oplus \bigoplus_{0 \neq \alpha \in \mathfrak{X}(T)} \mathfrak{g}^{\alpha}$$

where

$$\mathfrak{g}^{\alpha} = \left\{x \in \mathfrak{g} \mid \forall t \in T : \ txt^{-1} = \alpha(t)x\right\}$$

and

$$\mathfrak{g}^{o} = \left\{x \in \mathfrak{g} \mid \forall t \in T : \ txt^{-1} = x\right\}.$$

Example 34. Let $G \subset \mathsf{GL}_n(k)$ with the torus $T = \mathcal{G}_m^n$. Then,

$$\mathfrak{g} = M_n(k) = \mathfrak{g}^{o} \oplus \left(\bigoplus_{0 \neq \alpha} \mathfrak{g}^{\alpha} \right)$$

where

$$\mathfrak{g}^{o} = \left\{ \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \right\}.$$

For $i = 1, \ldots, n$ we define $\chi_i \in \mathfrak{X}(T)$ as follows:

$$\chi_i(\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}) = t_i.$$

Then, we have for each $i \neq j$

$$\mathfrak{g}^{\chi_i/\chi_j} = kE_{i,j}$$

and for each other $\chi \in \mathfrak{X}(T)$

$$\mathfrak{g}^{\chi} = 0.$$

Therefore

$$\mathfrak{g} = \mathfrak{t} \oplus \left(\bigoplus_{i \neq j} kE_{i,j} \right)$$

where

$$\mathfrak{t} = \mathrm{Lie}(T) = \mathfrak{g}^{o}.$$

20.4 Root Data – Construction

Let G be a connected reductive group with a torus T. We want to construct a corresponding root datum

$$\Psi = (X, X^\vee, R, R^\vee).$$

- Set X to be the **character lattice**

$$X = \mathfrak{X}(T) = \mathsf{Hom}\left(T, \mathcal{G}_m\right).$$

- Set $X^\vee$ to be the **cocharacter lattice**

$$X^\vee = \mathsf{Hom}\left(\mathcal{G}_m, T\right).$$

- If we are given $x \in X$ and $\xi \in X^\vee$, we define

$$\langle x \mid \xi \rangle := m \in \mathbb{Z}$$

  s.t.

$$\begin{aligned} x \circ \xi : \mathcal{G}_m &\longrightarrow \mathcal{G}_m \\ t &\longmapsto t^m. \end{aligned}$$

  (Recall, $\mathfrak{X}(\mathcal{G}_m) = \mathbb{Z}$.)

- $$R := \left\{0 \neq \alpha \in \mathfrak{X}(T) \mid \mathfrak{g}^\alpha \neq 0\right\}.$$

- Let $g \in \mathcal{G}_m$ be s.t. $\mathcal{G}_m = \overline{\langle g \rangle}$. For $x \in R$, we need to choose $t \in T$ s.t.

$$x(t) = g^2$$

  and, if we set

$$x^\vee = [g^n \mapsto t^n]$$

  the other axioms of a root datum are fulfilled.

Example 35.
- $G = \mathsf{SL}_2(k)$:
  In G we have the maximal torus

$$T = \left\{\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}\right\}.$$

Consider the character

$$\lambda : T \longrightarrow \mathcal{G}_m$$
$$\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \longmapsto t.$$

Set

$$R = \{\alpha, -\alpha\}$$

with

$$\alpha = 2\lambda : \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \longmapsto t^2$$
$$-\alpha : \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \longmapsto t^{-2}.$$

Then, we have

$$\mathfrak{g}^{\alpha} = \left\{ \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \right\}$$
$$\mathfrak{g}^{-\alpha} = \left\{ \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix} \right\},$$

since

$$\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} 0 & t^2 \\ 0 & 0 \end{pmatrix}.$$

Define

$$\alpha^{\vee} := \left[t \mapsto \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \right]$$
$$(-\alpha)^{\vee} := -\alpha^{\vee} \left[t \mapsto \begin{pmatrix} t^{-1} & \\ & t \end{pmatrix} \right].$$

Then, we have

$$\langle \alpha \mid \alpha^{\vee} \rangle = 2.$$

- $G = P\mathsf{GL}_2(k)$:
  We have the maximal torus

$$T = \left\{ \begin{pmatrix} t & \\ & 1 \end{pmatrix} \right\}.$$

Set

$$R = \{\alpha, -\alpha\}$$

with

$$\alpha \begin{pmatrix} t & \\ & 1 \end{pmatrix} := t.$$

Define $\alpha^\vee$ by

$$\alpha^\vee(t) := \begin{pmatrix} t^2 & \\ & 1 \end{pmatrix} = \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}.$$

We have

$$\begin{pmatrix} t & \\ & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} t & \\ & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}.$$

Now, let G be reductive of semisimple rank one. Then:

- $G/R(G) \cong P\mathsf{GL}_2(k)$ with $R(G) = 1$ or $R(G) = \mu_n$.
- $[G, G] \cong \mathsf{SL}_2(k)$ is semisimple of semisimple rank 1.
- $\{\text{roots for } G/R(G)\} = \{\text{roots for } G\}$.
- If α is a root of G, then it is also a root of $[G, G]$. Hence, define

$$\alpha^\vee : \mathcal{G}_m \to [G, G] \hookrightarrow G.$$

Let G be reductive. Take a root $\alpha \in \mathfrak{X}(T)$ and define the torus of codimension 1 in T

$$S_\alpha := \mathsf{Kern}(\alpha)^o.$$

Then,

$$G_\alpha := Z_G(S_\alpha)$$

is connected and reductive. Then,

$$S_\alpha \subset Z_G(G_\alpha)$$

and

$$S_\alpha \subset R(G_\alpha).$$

Ergo, T/S_α is a maximal torus of rank 1 in $G_\alpha/R(G_\alpha)$, while $G_\alpha/R(G_\alpha)$ is semisimple.

Ergo, G_α is reductive of semisimple rank 1.

Example 36. Let $G = \mathsf{GL}_3$ with the maximal torus $T = \left\{ \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix} \right\}$.

Let $\alpha = \chi_1/\chi_2$. Then,

$$S_\alpha = \left\{ \begin{pmatrix} x & & \\ & x & \\ & & y \end{pmatrix} \right\}.$$