Mitschrieb: Algebraic Groups
SS 20

Akin

April 6, 2020

Vorwort

Contents

Lecture from 03.03.2020

Recall: Last time we introduced the **Zariski-Topology** on X.
There, algebraic sets equal closed sets.
We called a set X **irreducible** iff each open subset lies dense in X.

Lemma 1. *For an algebraic set X, the following are equivalent:*

(1) X is irreducible.

(2) $k[X] = k[x_1, \dots, x_n]/I(X)$ is a domain.

(3) $I(X)$ is a prime ideal.

The proof of (2) $\iff$ (3) is a basic algebraic result.

Lemma 2. *An open base for the Zariski-Topology on an algebraic set X is given by sets:*

$$D(f) := \{p \in X \mid f(p) \neq 0\}$$

for each $f \in k[X]$. We call the $D(f)$ ***basic open sets****.*

Proof. Suppose $U \subseteq X$ is nonempty and open. Set

$$Z := X \setminus U$$

then Z is closed. Thus

$$Z = \{x \in X \mid f(x) = 0 \forall f \in I\}$$

for some ideal $I \subseteq k[X]$. Let $p \in U$, then there is an $f \in Z$ s.t.

$$f(p) \neq 0.$$

Also, $D(f) \cap Z = 0$, thus $p \in D(f) \subseteq U$. $\square$

Lemma 1. It is clear that (2) is equivalent to (3).
(1) is equivalent to

$$\forall \text{ nonempty, open } U_1, U_2 \subset X : U_1 \cap U_2 \neq \emptyset$$
$$\overset{\text{Lemma 2}}{\iff} \forall \text{ nonempty, basic open } D(f_1), D(f_2) \subset X : D(f_1) \cap D(f_2) \neq \emptyset$$

Since $D(f_1) \cap D_(f_2) = D(f_1 f_2)$, this is equivalent to the statement

$$f_1, f_2 \in k[X] : f_1, f_2 \neq 0 \implies f_1 f_2 \neq 0.$$

Which states that $k[X]$ is a domain. $\square$

Lemma 3. *Let X be an algebraic set. We have bijections*

$$\{\textit{closed subsets } Z \subseteq X\} \leftrightarrow \{ \textit{ radical ideals } I \subset k[X]\}$$

and

$$\{\textit{irreducible, closed subsets } Z \subseteq X\} \leftrightarrow \{ \textit{ prime ideals } I \subset k[X]\}$$

and

$$\{\textit{points of } X\} \leftrightarrow \{ \textit{ maximum ideals } I \subset k[X]\}.$$

Lemma 4 (Primary Decompositions, Atiyah, Macdonald Ch. 4)**.** *For an ideal I we call $P \supseteq I$ a* ***minimal prime*** *of I if P is a prime ideal and we have for each prime ideal Q:*

$$P \supseteq Q \supseteq I \implies P = Q.$$

Any radical ideal I of $k[x_1, \dots, x_n]$ has only finitely many ***minimal*** *primes $P_1, \dots, P_r$. Inparticular,*

$$I = \bigcap_{i=1}^{n} P_i$$

and for each i

$$P_j \not\supseteq \bigcap_{j:j \neq i} P_j.$$

Definition 1. An **(irreducible) component** Z of X is a maximal irreducible closed subset, i.e., an irreducible closed $Z \subseteq X$ s.t. there does not exist an irreducible closed $Y \subset X$ s.t. $Y \supsetneq Z$.

Then, we have the bijection

$$\{\text{irreducible components of } X\} \leftrightarrow \{ \text{ mimimal primes of } I(X)\}.$$

Lemma 5. *Any algebraic set X has finitely many components $Z_1, \dots, Z_r$. We have*

$$X = Z_1 \cup \ldots \cup Z_r$$

and for each i

$$Z_i \not\subset \bigcup_{j:j \neq i} Z_j.$$

Example 1.

1. Let $X = V(x \cdot y) \subset k^2$. Then $X = Z_1 \cup Z_2$ where $Z_1 = V(x), Z_2 = V(y)$.

   X is connected, but not irreducible ($D(x)$ does not lie dense in X).

2. Let X be a **finite** algebraic set. It is easy to check that every subset of X is closed:
$$\{p\} = V(x_1 - p_1, \dots, x_n - p_n)$$
for each $p \in X$. Further
$$X = \{p_1\} \cup \dots \cup \{p_r\}.$$
Moreover: Any function $f : X \to k$ is regular (i.e. given by polynomials).

Lemma 6. *We call an element $e \in k[X]$* ***idempotent*** *iff $e^2 = e$.*
Let X be an algebraic set. Then
$$\begin{aligned} X \text{ connected} &\iff \text{the only idempotents } e \in k[X] \text{ are 0 and 1} \\ &\iff k[X] \ncong A \times B \text{ for any } k\text{-algebras } A, B. \end{aligned}$$

Lemma 7. *Morphisms of algebraic sets are continuous.*

Proof. Let $\phi : X \to Y$ be a morphism. It suffices to show that for all closed $Z \subset Y$ that $\phi^{-1}(Z) \subset X$ is closed.

But, if
$$Z = V_Y(S) := \{q \in Y \mid f(q) = 0 \forall f \in S\}$$
for some ideal $S \subset k[Y]$, then
$$\phi^{-1}(Z) = V_X(\phi^*(S)) = \{\phi^*(f) = f \circ \phi \mid f \in S\}.$$
□

Lemma 8. *Isomorphisms of algebraic sets are homeorphisms. In particular, any isomorphism of algebraic sets $\phi : X \to X$ permutes the components $Z_1, \dots, Z_r$ of X:*
$$\forall i \exists j : \phi(Z_i) = Z_j.$$

Theorem 1. *Let G be an algebraic group.*

(i) There is a unique component G^0 of G with $e \in G^0$.

(ii) Every component Z of G is a coset gG^0 of G for some $g \in Z$.

(iii) G^0 is a normal algebraic subgroup of G.

(iv) G^0 is of finite index, i.e.
$$[G : G^0] = \#\left(G/G^0\right) < \infty.$$

(v) *The irreducible components are also the connected components.*

Proof. Let $G = Z_1 \cup \ldots \cup Z_r$ be the decomposition into components. We may assume that $e \in Z_1$.

Recall that $Z_1 \not\subset \bigcup_{j \geq 2} Z_j$. Then, there is an $x \in Z_1 \setminus \bigcup_{j \geq 2} Z_j$. Thus, for all algebraic set isomorphisms $\phi : G \to G$, we have by some previous lemma that $\phi(x)$ is likewise contained in some unique component of G. For example, we may take ϕ to be

$$\begin{aligned} \phi_g : G &\to G \\ y &\longmapsto gy \end{aligned}$$

for any $g \in G$. Then, for all $g \in G$, the element $gx = \phi_g(x)$ is contained in only one component of G. Ergo, each $g \in G$ is contained in exactly one component.

(i) Take $g = e$.

(iii) G^0 is an algebraic subset, by construction. Denote by $m : G \times G \to G$ and $i : G \to G$ the continuous multiplication and inversion map on G. **Why is G^0 a subgroup?** We need to show

$$\begin{aligned} m(G^0 \times G^0) &\subseteq G^0. \\ i(G^0) &\subseteq G^0. \end{aligned}$$

We know that $i(G^0)$ is some component of G, since i is an isomorphism. But it contains the identity e, since $e^{-1} = e$. Therefore, $i(G^0) = G^0$.

If $g \in G$, then gG^0 is some component of G. Suppose $g \in G^0$. Then $gG^0 \cap G^0 \supseteq \{g\}$, therefore $gG^0 = G^0$. Ergo, G^0 is closed under multiplication.

Why is G^0 a normal? If $g \in G$, then gG^0g^{-1} is a component that contains e, therefore $G^0 = gG^0g^{-1}$.

(Alternative proof that $m(G^0 \times G^0) = G^0$: Consider

- any continuous image of an irreducible set is irreducible.
- the closure of any irreducible set is irreducible.

Ergo $\overline{m(G^0 \times G^0)}$ is a closed irreducible set containing e. Ergo, $\overline{m(G^0 \times G^0)} = G^0$).

(ii) Let $Z \subset G$ be a component. Let $g \in Z$. Then $g \in (gG^0 \cap Z)$, so $gG^0 = Z$.

(iv) This follows from some previous lemma.

(v) This is left as a topological exercise. It is true whenever the irreducible components do not intersect.

□

It now follows:

$$\{\text{finite algebraic groups}\} \longleftrightarrow \{finitegroups\}$$

where the above arrow is an equivalence of categories.

Example 2.
- Let $G = \{g_1, \ldots, g_r\}$ be a finite algebraic group. Then,
$$G^0 = \{e\}.$$
- Without proofs:
$$G \in \{\mathsf{GL}_n(k), \mathsf{SO}_n(k), \mathsf{SL}_n(k)\} \implies G^0 = G.$$
Further,
$$G = O_n(k) \implies G^0 = \mathsf{SO}_n(k)$$
(but only if $-1 = 1$ i.e. $\mathsf{char} k = 2$. Otherwise $[G : G^0] = 2$.)

0.1 Jordan Decomposition

As usual, $k = \overline{k}$ is an algebraically closed field.

Definition 2. Let V be a finite-dimensional vector space.

An element $x \in \mathsf{End}(V)$ is **semisimple**, if it is diagonalizable, i.e. it has a basis of eigenvectors, or equivalently, if the minimal polynomial of x is square-free.

Then, there is a decomposition $V = \bigoplus_{i=1}^r V_i$ and distinct elements $\lambda_1, \ldots, \lambda_n \in k$ s.t.

$$x|_{V_i} = \lambda_i.$$

If $\dim(V_i) = n_i$, then

$$\text{char polynomial of } x = \prod_{i=1}^{r} (T_i - \lambda_i)_i^n \in k[T]$$

and

$$\text{minimal polynomial of } x = \prod_{i=1}^{r}(T_i - \lambda_i) \in k[T].$$

(Where the minimal polynomial of x is defined as the least degree monic m s.t. $m(x) = 0$ and Cayley-Hamilton $m|c$.)

Definition 3. $x \in End(V)$ is **nilpotent** if $x^n = 0$ for some n. (Equivalent to: characteristic polynomial of x is $T^{\dim(V)}$.)

x is **unipotent**, if $x - 1$ is nilpotent.

Lemma 9. *If x is semisimple and nilpotent, then $x = 0$.*

If x is semisimple and unipotent, then $x = 1$.

Lemma 10. *If x, y are commuting elements, thar are semisimple resp. unipotent or nilpotent, then so is xy.*

Theorem 2 (Goal)**.** *For all algebraic groups G and for all $g \in G$, there exist unique group elements $g_s, g_u \in G$ s.t.*

$$g = g_s g_u = g_u g_s$$

and for all finite-dimensional representations $\rho : G \to \mathsf{GL}(V)$, $\rho(g_s)$ is semisimple and $\rho(g_u)$ is unipotent.

Example 3. If $g = \begin{pmatrix} \lambda & 1 & 0 \\ & \lambda & 1 \\ & & \lambda \end{pmatrix} \in G = \mathsf{GL}_3(k)$, then $g_s = \begin{pmatrix} \lambda & 0 & 0 \\ & \lambda & 0 \\ & & \lambda \end{pmatrix}$, $g_u = \begin{pmatrix} 1 & \lambda^{-1} & 0 \\ & 1 & \lambda^{-1} \\ & & 1 \end{pmatrix}$.

Lecture from 09.03.2020

Theorem 3 (Goal Theorem)**.** *Let G an algebraic group. For all $g \in G$ there is exactly one pair $g_s, g_u \in G$ s.t.*

$$g = g_s g_u = g_u g_s$$

and for all finite-dimensional representations $r : G \to \mathsf{GL}_n(V)$, the element $r(g_s)$ resp. $r(g_u)$ is semisimple resp. unipotent.

Last time, we saw:

Lemma 11.
- *If g, h are commuting and semisimple resp. commuting and unipotent then so is gh.*
- *If g is semisimple and unipotent, then $g = 1$.*

Proposition 1. *Let V be a finite-dimensional vector space and $g \in \mathsf{GL}(V)$. There exist unique elements $g_s, g_u \in \mathsf{GL}(V)$ s.t.*

$$g = g_s g_u = g_u g_s$$

and g_s is semisimple and g_u is unipotent.

Moreover, $g_s, g_u \in k[g] = \{\sum_{i=1}^m a_i g^i \mid a_i \in k\} \subseteq \mathsf{End}(V)$.

Proof. **Existence** (Sketch): Say

$$g = \begin{pmatrix} \lambda & 1 & 0 \\ & \lambda & 1 \\ & & \lambda \end{pmatrix}$$

then take

$$g_s = \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{pmatrix}, \quad g_u = \begin{pmatrix} 1 & \lambda^{-1} & 0 \\ & 1 & \lambda^{-1} \\ & & 1 \end{pmatrix}.$$

For $\lambda \in k$, define the **generalized λ-eigenspace** of g by

$$V_\lambda := \{v \in V \mid \exists n \in \mathbb{N}_0 : \ (g-\lambda)^n v = 0\}.$$

Then

$$V = \bigoplus_{\lambda \in k} V_\lambda.$$

Here V_λ = sum of domains of all Jordan blocks with λs on the diagonal. (It follows from the Jordan Decomposition for matrices that such a decomposition exist.)

Let's define $g_s \in \mathsf{GL}(V)$ by
$$g_s|_{V_\lambda} = \lambda \cdot \mathrm{Id}.$$
Note that $gV_\lambda \subset V_\lambda$, hence g commutes with g_s, hence g, g_s commutes with $g_u := gg_s^{-1}$. Then, $g = g_s g_u = g_u g_s$.

Write $\det(T-g) = \prod_\lambda (T-\lambda)^{n(\lambda)}$, $n(\lambda) = \dim(V_\lambda)$. Since the polynomials $T-\lambda$ for $\lambda \in k$ are coprime, the chinese remainder theorem implies that there is a $Q \in k[T]$ s.t.
$$Q \equiv \lambda \mod (T-\lambda)^{n(\lambda)}$$
for each $\lambda \in k$.

We claim that
$$Q(g) = g_s.$$
Indeed, since $gV_\lambda \subseteq V_\lambda$, we have
$$Q(g)V_\lambda \subseteq V_\lambda.$$
So, it suffices to show for all $v \in V_\lambda$
$$Q(g)v = g_s v = \lambda v.$$
Note that, by Cayley-Hamilton,
$$V_\lambda = \left\{v \in V \mid (g-\lambda)^{n(\lambda)}v = 0.\right\}$$
Write
$$Q = \lambda + R \cdot (T-\lambda)^{n(\lambda)}$$
for some $R \in k[T]$. Since $(g-\lambda)^{n(\lambda)}v = 0$, deduce that $Q(g)v = \lambda v$, as required.

If $Q' \equiv T(T-\lambda)^{n(\lambda)}$, then
$$Q'(g) = g_u.$$
If $Q'' \equiv \lambda^{-1}(T-\lambda)^{n(\lambda)}$, then $Q''(g) = g_s^{-1}$. check corresponding stuff for g_u.

Uniqueness: Suppose given some other decomposition
$$g = g_s' g_u' = g_u' g_s'$$
with g_s' semisimple and g_u' unipotent. Then g_s' commutes with g_s' and g_u', hence with g, hence also with any element in $k[g]$. Ergo, g_s' commutes with g_s and g_u. Similarly, g_u' commutes with g_s and g_u.

Consider

$$h := g'_s g_s^{-1} = g'_s g'_u (g'_u)^{-1} g_s^{-1} = g(g'_u)^{-1} g_s^{-1} = g_u (g'_u)^{-1}.$$

Then $h = g'_s g_s^{-1}$ is a product of semisimple elements and $h = g_u(g'_u)^{-1}$ is a product of unipotent elements. By proceeding lemmas, h is semisimple and unipotent, ergo trivial. It follows $g'_s = g_s$ and $g'_u = g_u$. □

Corollary 1. *Let $g \in \mathsf{GL}(V)$, let $W \subset V$ be any g-**invariant subspace**, i.e. $gW \subseteq W$.*

Then, W is g_s-invariant and g_u-invariant.

Proof. This is clear, since g_s and g_u are algebraically generated by g over g. □

Lemma 12. *Let $\phi : V \to W$ be a linear map between finite-dimensional vector spaces.*

Let $\alpha \in \mathsf{GL}(W)$ and $\beta \in \mathsf{GL}(W)$ s.t.

$$\begin{array}{ccc} V & \xrightarrow{\alpha} & V \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} \\ W & \xrightarrow{\beta} & W, \end{array}$$

i.e. $\phi \circ \alpha = \beta \circ \phi$.

Then,

$$\phi \circ \alpha_s = \beta_s \circ \phi,$$
$$\phi \circ \alpha_u = \beta_u \circ \phi.$$

Proof. Write $V = \bigoplus_{\lambda \in k} V_\lambda$, $W = \bigoplus_{\lambda \in k} W_\lambda$ where V_λ are the generalized α-eigenspaces and W_λ are the generalized β-eigenspaces.

We claim that

$$\phi(V_\lambda) \subset W_\lambda.$$

Indeed, let $v \in V_\lambda$, then

$$(\beta - \lambda)^n \phi(v) = \phi((\alpha - \lambda)^n v) = 0.$$

Since $(\alpha - \lambda)^n v = 0$, the claim follows.

Since, $\alpha_s|_{V_\lambda} = \lambda \mathrm{Id}$ and $\beta_s|_{W_\lambda} = \lambda \mathrm{Id}$, deduce that

$$\phi \circ \alpha_s = \beta_s \circ \phi.$$

Indeed, both sides are given on V_λ by $\lambda \cdot \phi$. Thus

$$\begin{aligned}\phi \circ \alpha_u &= \phi \circ \alpha\alpha_s^{-1}\\ &= \beta\beta_s^{-1} \circ \phi\\ &= \beta_u \circ \phi.\end{aligned}$$

□

Lemma 13. *Let $\alpha \in GL(V)$, $\beta \in GL(W)$. Then the* ***tensor*** *$\alpha \otimes \beta \in GL(V \otimes W)$ is defined by*

$$(\alpha \otimes \beta)(u \otimes v) = \alpha(u) \otimes \beta(v).$$

Then, we have

$$(\alpha \otimes \beta)_s \overset{(1)}{=} \alpha_s \otimes \beta_s$$

$$(\alpha \otimes \beta)_u \overset{(2)}{=} \alpha_u \otimes \beta_u.$$

Proof. It suffices to prove (1), since

$$\begin{aligned}(\alpha \otimes \beta)_u &= (\alpha \otimes \beta) \circ (\alpha \otimes \beta)_s^{-1}\\ &\overset{(1)}{=} (\alpha \otimes \beta) \circ (\alpha_s \otimes \beta_s)^{-1}\\ &= \alpha\alpha_s^{-1} \otimes \beta\beta_s^{-1}\\ &= \alpha_u^{-1} \otimes \beta_u^{-1}\end{aligned}$$

For (1), consider

$$V = \bigoplus_{\lambda \in k} V_\lambda,$$
$$W = \bigoplus_{\lambda \in k} W_\lambda.$$

It follows

$$V \otimes W = \bigoplus_{\lambda, \mu \in k} V_\lambda \otimes W_\mu.$$

Now,

$$\alpha_s \otimes \beta_s|_{V_\lambda \otimes W_\mu} = \lambda\mu \cdot \mathrm{Id}.$$

Ergo, $\alpha_s \otimes \beta_s$ is semisimple. By Proposition, we reduce to checking that $\alpha_u \otimes \beta_u$ is unipotent. Indeed,

$$\alpha_u \otimes \beta_u - 1 = (\alpha_u - 1) \otimes (\beta_u - 1) + 1 \otimes (\beta_u - 1) + (\alpha_u - 1) \otimes 1$$

is nilpotent (You can also check that $\alpha_u \otimes \beta_u = (\alpha_u \otimes 1) \circ (1 \otimes \beta_u)$ is unipotent.) □

Example 4. Let $1 \in \mathsf{GL}(V)$. Then $1_s = 1$ and $1_u = 1$.

Summary : Let G be an algebraic group. Let $r_V : G \to \mathsf{GL}(V)$ be a finite-dimensional representation. Also, fix $g \in G$.

Let $\lambda_V := r_V(g)_s$ (or $r_V(g)_u$).

We get a family of operators $\lambda_V \in \mathsf{End}(V)$ with the following properties:

(i) if $V = k$ and $r_V(g') = 1$ for all $g' \in G$, then $\lambda_V = 1$.

(ii) for any two representations in V and W, we have

$$\lambda_{V\otimes W} = \lambda_V \otimes \lambda_W.$$

(iii) for all G-equivariant $\phi : V \to W$ we have

$$\phi \circ \lambda_V = \lambda_W \circ \phi.$$

Theorem 4. *Let G be an algebraic group. Let $\lambda_V \in \mathsf{End}(V)$ (i.e. $V = (r_V, V)$ is a finite-dim. representation of G) be a family of operations satisfying (i), (ii), (iii).*

Then, there is exactly one $g \in G$ s.t. $\lambda_V = r_V(g)$ for all V.

Note, that this theorem implies our goal theorem.

Applying the theorem to $\lambda_V = r_V(g_s)$ implies

$$\exists g_s \in G : r_V(g_s) = r_V(g)_s$$

and

$$\exists g_u \in G : r_V(g_u) = r_V(g)_u.$$

Proof of Goal Theorem. There exist unique $g_s, g_u \in G$ s.t.

$$g \overset{(*)}{=} g_u g_s = g_s g_u,$$

Then, $r_V(g) = r_V(g_s)r_V(g_u) = r_V(g_u)r_V(g_s)$.

Since $r_V(g_u)$ is unipotent and $r_V(g_s)$ is semisimple, it follows $r_V(g_u) = r_V(g)_u$ and $r_V(g_s) = r_V(g)_s$.

To deduce $(*)$, take any $r_V : G \hookrightarrow \mathsf{GL}(V)$. We know for each V

$$r_V(g) = r_V(g_s)r_V(g_u) = r_V(g_u)r_V(g_s).$$

□

Proof of Theorem. We first extend the assignment

$$V \mapsto \lambda_V$$

to all representations of G.

Say $V = \bigcup_j W_j$ where each W_j is a finite-dimensional G-invariant subspace. Try to define $\lambda_V \in \mathsf{End}(V)$ by

$$\lambda_V|_{W_j} := \lambda_{W_j}.$$

For this to be well-defined, we need to show for each i, j

$$\lambda_{W_i}|_{W_i \cap W_j} \stackrel{(*)}{=} \lambda_{W_j}|_{W_i \cap W_j}.$$

Proof of $(*)$**:** Apply assumption (iii) to the G-equivariant linear maps

$$W_i \cap W_j \stackrel{\phi}{\hookrightarrow} W_i,$$
$$W_i \cap W_j \stackrel{\phi'}{\hookrightarrow} W_j.$$

Then,

$$\begin{aligned} \lambda_{W_i}|_{W_i \cap W_j} &= \lambda_{W_i} \circ \phi \\ &\stackrel{(iii)}{=} \phi \circ \lambda_{W_i \cap W_j} \\ &= \phi' \circ \lambda_{W_i \cap W_j} \end{aligned}$$

and

$$\lambda_{W_j}|_{W_i \cap W_j} = \lambda_{W_i} \circ \phi' = \phi' \circ \lambda_{W_i \cap W_j}.$$

Recall here that any finite-dimensional G-invariant $W \subset V$ is a representation. $\square$

[0]Not necessarily finite-dimensional, but may be written as a filtered union of finite-dimensional G-invariant subspaces of W.

Lecture from 11.03.2020

Let G be an algebraic group.

Easy Exercise : If V_1, V_2 are representations r_1, r_2 of G, then $V_1 \otimes V_2$ is also a representation with

$$r = r_1 \otimes r_2 : G \to \mathsf{GL}(V_1 \otimes V_2)$$

given by

$$r(g)(v_1 \otimes v_2) = (r_1(g)v_1) \otimes (r_2(g)v_2).$$

Proof. Given $\Delta_j : V_j \to V_j \otimes k[G]$, define

$$\Delta : V_1 \otimes V_2 \longrightarrow V_1 \otimes V_2 \otimes k[G]$$

by: if

$$\Delta_1 u = \sum u_i \otimes f_i, \quad \Delta_2 v = \sum v_j \otimes h_j,$$

then

$$\Delta(u \otimes v) \sum \sum u_i \otimes v_j \otimes f_i h_j.$$

Set $A := k[G]$, then

$$r_A := \text{right regular representation with } r_A(g)f(x) = f(xg).$$

The map

$$\begin{aligned} A \otimes A &\xrightarrow{m} A \\ f_1 \otimes f_2 &\longmapsto f_1 f_2 \end{aligned}$$

defines a morphism of representations

$$(A, r_A) \otimes (A, r_A) \to (A, r_A).$$

Indeed,

$$\begin{aligned} m((r_A \otimes r_A)(g)(f_1 \otimes f_2))(x) &= f_1(xg) f_2(xg), \\ &= f_1 f_2(xg) = r_A(g)(m(f_1 \otimes f_2))(x), \end{aligned}$$

since $f_1(\_g) \otimes f_2(\_g) = (r_A \otimes r_A)(g)(f_1 \otimes f_2)$.

Ergo $m \circ (r_A \otimes r_A)(g) = r_A(g) \circ m$. $\square$

Recall: We stated the following theorem

Theorem 5. *Let $\lambda_V \in \mathsf{End}(V)$ be given s.t. for all finite-dim. rep.s V of G s.t.:*

(i) $\lambda_k = 1$

(ii) $\lambda_{V\otimes W} = \lambda_V \otimes \lambda_W$

(iii) for all morphisms of rep.s $\phi : V \to W$ we have

$$\phi \circ \lambda_V = \lambda_W \circ \phi.$$

Then, there is exactly one $g \in G$ s.t. $\lambda_V = r_V(g)$ for all V.

Proof. Last time, we saw that any such family $V \mapsto \lambda_V$ extends to **all** rep.s V of G.

Let's note also that, if (V_0, r_0) is any representation of G with trivial action, i.e. $r(g) = 1$ for all g, then $\lambda_{V_0} = 1$. Indeed, let $v \in V_0$. We must check that $\lambda_{V_0} v = v$. Since the action is trivial, any subsapce of V_0 is G-invariant.

Consider the map

$$\begin{aligned} \phi : k &\longrightarrow V_0 \\ \alpha &\longmapsto \alpha v \end{aligned}$$

where $v = \phi(1)$. Then, ϕ is a morphism of rep.s because the action is trivial.

Thus,

$$\lambda_V v = (\lambda_V \circ \phi)(1) \overset{(iii)}{=} (\phi \circ \lambda_k)(1) \overset{(i)}{=} \phi(1) = v.$$

Consider $\lambda_A \in \mathsf{End}(A)$. Then,

$$\lambda_{A\otimes A} = \lambda_A \otimes \lambda_A.$$

It is an easy exercise to see that $m : (A, r_A) \otimes (A, r_A) \to (A, r_A)$ is a morphism of rep.s.

By (iii) it follows, $m \circ (\lambda_A \otimes \lambda_A) = \lambda_A \circ m$, i.e.

$$\lambda_A(f_1 f_2) = \lambda_A(f_1)\lambda_A(f_2)$$

for all $f_1, f_2 \in A$. Thus, λ_A is an algebra morhism (check, using the morphism $k \hookrightarrow A$, that $\lambda_A(1) = 1$).

Thus, $\lambda_A = \phi^*$ for some unique morphism ϕ of algebraic sets $\phi : G \to G$.

We claim that ϕ commutes left multiplication i.e.

$$\phi(hx) = h\phi(x)$$

for all $h, x \in G$. Indeed, let's consider the map

$$\begin{aligned} A &\longrightarrow A \\ f &\longmapsto f(h \cdot \_). \end{aligned}$$

This induces a morphism

$$(A, r_A) \xrightarrow{\psi} (A, r_A).$$

By (ii), $\psi \circ \lambda_A = \lambda_A \circ \psi$.

Since $\lambda_A = \phi^*$, this implies the claim.

Now, set $g := \phi(e)$. Then for all $h \in G$,

$$\phi(h) = \phi(he) = hg.$$

Thus, $\lambda_A = \phi^* = r_A(g)$.

(It remains to show that

$$\lambda_V = r_V(g)$$

for each finite-dim. rep. V.)

Let $V = (V, r)$ be any rep. This induces a map

$$\Delta : V \longrightarrow V \otimes A.$$

If $\Delta v = \sum v_i \otimes f_i$, then

$$hv = \sum f_i(h_i) \otimes v_i.$$

Let

$$\begin{aligned} \varepsilon : V \otimes A &\longrightarrow V \\ v \otimes f &\longmapsto f(1)v. \end{aligned}$$

It follows $\varepsilon \circ \Delta : V \to V$ is the identity map.

Let (V_0, r_0) be the representation of G with $V_0 := V$ and r_0 the trivial action. Then, $\Delta : V \to V_0 \otimes A$ is a morphism of representations.

(Indeed, if $\Delta v = \sum v_i \otimes f_i$, then

$$\Delta(r(h)v) \stackrel{?}{=} (r_0(h) \otimes r_A(h))\Delta v$$

since

$$\begin{aligned} \Delta v &= \sum v_i \otimes f_i \\ \iff xv &= \sum f_i(x_i) v_i \ \forall x \in G \\ \iff xhv &= \sum f_i(xh) v_i \ \forall x, h \in G. \end{aligned}$$

Since $r(h)v = hv$, it follows

$$\Delta(hv) = \sum v_i \otimes f_i(\cdot h) \implies (?).)$$

We want to show

$$\lambda_V = r_V(g).$$

We have

$$\begin{aligned} \Delta \circ \lambda_V &\overset{(iii)}{=} \lambda_{V_0 \otimes A} \circ \Delta \\ &\overset{(ii)}{=} \lambda_{V_0} \otimes \lambda_A \\ &= 1 \otimes \lambda_A = 1 \otimes r_A(g). \end{aligned}$$

This implies

$$\Delta \circ \lambda_V = (1 \otimes r_A(g)) \circ \Delta$$

but also

$$\Delta \circ r_V(g) = (1 \otimes r_A(g)) \circ \Delta.$$

Because of the injectivity of Δ it now follows

$$\lambda_V = r_V(g).$$

$\square$

Corollary 2. *Let $\phi : G \to H$ be any morphism of algebraic groups. Then, for all $g \in G$*

$$\begin{aligned} \phi(g)_s &= \phi(g_s) \\ \phi(g)_u &= \phi(g_u). \end{aligned}$$

Proof. Let V be any **faithful** representation of H, i.e. $r_V : H \to \mathsf{GL}(V)$ is injective, (for a finite-dim. V).

Then, $r_V \circ \phi$ is a rep. of G. To prove (i), it suffices to show

$$r_V(\phi(g)_s) = r_V(\phi(g_s))$$

since H operates faithfully on V.

We know that

$$r_V(\phi(g)_s) = r_V(\phi(g))_s$$

(characterizing property of h_s for $h \in H$). On the other hand,

$$r_V(\phi(g_s)) = (r_V \circ \phi)(g_s) = r_V(\phi(g))_s.$$

Therefore, claim (i) follows. (ii) works analogously. □

Definition 4. Let $g \in G$ where G is an algebraic group. We call g **semisimple**, if $g = g_s$.

We call g **unipotent**, if $g = g_u$.

Lemma 14. *For $g \in G$, the following are equivalent:*

(i) g is semisimple.

(ii) $r_V(g)$ is semisimple for all finite-dim. rep. V.

(iii) $r_V(g)$ is semisimple for at least one faithful f.d. rep. V of G.

We get an analogous lemma for unipotent group elements.

Proof. We have

$$\begin{aligned}
(i) &\iff g = g_s \\
&\overset{\text{Def. of } g_s \text{ by goal thm.}}{\iff} r_V(g) = r_V(g)_s \forall \text{ f.d. } V \\
&\iff r_V(g) \text{ is semisimple} \\
&\iff (ii) \implies (iii).
\end{aligned}$$

On the other hand,

$$(iii) \implies \exists \text{ faithful f.d. } V \text{ s.t. } r_V(g) = r_V(g)_s = r_V(g_s) \implies g = g_s.$$

□

0.2 Non-Commutative Algebra

Definition 5. A **ring** R (for now) is unital, associative but not necessarily commutative.

Example 5. The ring of matrices over some field or ring.

Definition 6. A **left ideal** $I \subset R$ is a subset that is an abelian subgroup of $(R, +)$ s.t.$ra \in I$ for all $r \in R, a \in I$.

A **right ideal** $I \subset R$ is a subset that is an abelian subgroup with

$$IR \subset I.$$

A two-sided ideal I is a subset that is a left and a right ideal of R.

It is easy to check that for any homomorphism of rings $\phi : R \to S$, $\mathsf{Kern}\phi$ is a two-sided ideal. Also, if $J \subset R$ is any two-sided ideal, then there exists a unique ring structure on R/J s.t. the projection $R \to R/J$ is a ring homomorphism.

Definition 7. A **left module** M for R is an abelian group equipped with a ring homomorphism

$$R \xrightarrow{\alpha} \mathsf{End}(M)$$

where $\mathsf{End}(M)$ acts on the left of M. We write

$$rm := \alpha(r)m.$$

We have

$$(r_1 r_2)(m) = r_1(r_2(m)).$$

If R **acts** on M by the right, we write

$$R \curvearrowright M.$$

Example 6. $M_n(k) \curvearrowright k^n$ where k^n is the space of column vectors.

If k^n denotes the space of row vectors, we have $k^n \curvearrowleft M_n(k)$.

Definition 8. A **(left) submodule** $N \subset M$ is an algebraic subgroup s.t.

$$RN \subset N.$$

It follows that N is itself is a left module.

Definition 9. A (left) module M of R is **simple** (or irreducible) if it has exactly the two submodules: $0 = \{0\}$ and M.

Definition 10. A ring R is a **division ring** if it satisfies any of the following equivalent requirements:

(i) $R^\times = R \setminus \{0\}$ where $R^\times = \{r \in R \mid \exists a, b \in R : ar = rb = 1.\}$

(ii) R has no nontrivials left or right ideals.

Definition 11. If $R \curvearrowright M$, then we can define

$$\mathsf{End}_R(M) := \{\phi \in \mathsf{End}(M) \mid \phi(rm) = r\phi(m) \ \forall r \in R, m \in M\}.$$

Note, that $\mathsf{End}_R(M)$ is a ring.

Lemma 15 (Schur's Lemma)**.** *If M is simple, then $\mathsf{End}_R(M)$ is a division ring.*

Lemma 16. *Let k be a field. Then, $M_n(k)$ has no nontrivial twosided ideals.*

Theorem 6 (Jacobson Density Theorem (Double Commutant Theorem))**.** *Suppose M is a simple left module which is finitely generated as a right D-module for $D = \mathsf{End}_R(M)$.*

Assume that R acts faithfully on M, i.e. $R \to \mathsf{End}_R(M)$ is injective.

Then, the map $R \to \mathsf{End}_D(M)$ is an isomorphism.

[0]If $ar = rb = 1$, then $a = arb = b$.

Lecture from 16.03.2020 (Corona-Madness started here...)

Recap:

– Basics: definitions, Hopf-algebras, ...

– Jordan decomposition

– Primer on non-commutative algebra

  • Jacobson density theorem

– Unipotent groups

– Tori

0.2.1 Jacobson Density Theorem

We had last week

$$\mathsf{End}_D(M) := \{\phi \in \mathsf{End}(M) \mid \phi \circ d = d \circ \phi \forall d \in D\}.$$

Let k be an algebraically cloesd field, V a non-trivial finite-dimensional k-vector space and let G be a subgroup of $\mathsf{GL}(V)$ that acts **irreducibly** on V, i.e., V is **G-irreducible**, i.e., the only G-invariant subspaces of V are 0 and V.

Set

$$D := \{d \in \mathsf{End}_k(V) \mid dg = gd \forall g \in G\} = \operatorname{span}(G) = \left\{ \sum_{i=1}^{n} c_i g_i \mid c_i \in k, g_i \in G, n \in \mathbb{N}_0 \right\}.$$

Then,

$$D = \mathsf{End}_R(V)$$

where R is the k-subalgebra of $\mathsf{End}(V)$ that is generated by G.

Lemma 17 (Schur's Lemma)**.** *We understand $k \overset{End}{\hookrightarrow} (V)$ as the inclusion of operations which operate by scalar multiplication*

$$k \xrightarrow{\cong} \{\phi : V \to V \mid \phi : v \mapsto t \cdot v \quad \textit{for some } t \in k\}.$$

Then, we have

$$D \cong k.$$

Proof. Let $d \in D$. Since $V \neq 0$, there is an eigenspace $V_\lambda \neq 0$ for d. Observe that V_λ has to be G-invariant:
if $g \in G$ and $v \in V_\lambda$, then $gv \in V_\lambda$, since

$$dgv = gdv = g(\lambda v) = \lambda gv.$$

Since V_λ is a non-trivial G-invariant subspace and V is irreducible under G, we have

$$V_\lambda = V.$$

Ergo $d = \lambda$ in the sense of $k \hookrightarrow \mathsf{End}(V)$. □

Consequence of the Jacobson Density Theorem: $R = \mathsf{End}_k(V)$, i.e., G generates all linear operations on V, if V is G-irreducible.

We will prove this after a lemma.

Lemma 18. *Let $n \in \mathbb{N}$. Set*

$$V^n := V \oplus V \oplus \ldots \oplus V = V_1 \oplus \ldots \oplus V_n$$

where each $V_i = V$.

Let $v = (v_1, \ldots, v_n) \in V^n$ and set

$$Rv := \{(rv_1, \ldots, rv_n) \mid r \in R\} = \operatorname{span}\{(gv_1, \ldots, gv_n) \mid g \in G\}.$$

Then, $Rv \neq V^n$ iff the v_j are linearly dependent over k.

Consequence: Take $n := \dim(V)$. Let $\{e_1, \ldots, e_n\}$ be a basisi of V and set

$$e := (e_1, \ldots, e_n) \in V^n.$$

Since the $(e_i)_i$ are linearly independent, the lemma states that $Re = V^n$.

Now, let $x \in \mathsf{End}_k(V)$. Choose $r \in R$ s.t.

$$re = (xe_1, \ldots, xe_n).$$

Then $re_i = xe_i$ for all i, thus $x = r$. Hence, $R = \mathsf{End}_k(V)$.

Proof. Choose $J \in \{1, \ldots, n\}$ as large as possible with

$$Rv + V_1 + V_2 + \ldots + V_{J-1} =: U \neq V^n$$

. Such an J does exist, since we know that $Rv \neq V^n$.

Then, $V_J \nsubseteq U$, otherwise we may increase J. Also, U is invariant by the diagonal action of G on V^n. Thus, $V_J \cap U \subseteq V_J$ is a proper G-invariant subspace of the G-irreducible $V_J \cong V$. Therefore, $V_J \cap U = 0$.

On the other hand, by maximality of J, we have

$$U \oplus V_J = V^n.$$

Ergo, the map (composition)

$$V \cong V_J \hookrightarrow V^n \twoheadrightarrow V^n/U$$

is a G-equivariant isomorphism, since U is G-invariant.

Let $z : V^n/U \xrightarrow{\cong} V$ be the inverse isomorphism. Let l be the G-equivariant map given by

$$\begin{tikzcd} V^n \arrow[r, "l"] \arrow[d] & V \\ V^n/U \arrow[ur, "z"] & \end{tikzcd}$$

and let l_j be the G-equivariant maps by restricting l on V_j. Then $l_j \in D \cong k$.

Say $l_j = t_j \in k$. Then,

$$l(w) = t_1 w_1 + \ldots t_n w_n.$$

Since z is an isomorphism, l is nonzero and $(t_1, \ldots, t_n) \neq (0, \ldots, 0)$.

Since $l|_U = 0$, we can deduce for all $u \in U$

$$t_1 u_1 + \ldots + t_n u_n = 0.$$

But $v \in Rv \subseteq U$, so we may conclude – as required – that the $(v_i)_i$ are linearly dependent ($l(v) = 0$). $\square$

0.2.2 Unipotent Groups

Let G be a subgroup of $\mathsf{GL}(V)$ where V is a finite-dimensional vector space and k an algebrically closed field.

Definition 12. We say that G is **unipotent** if one of the following equivalent conditions hold:

- each $g \in G$ is unipotent (i.e. $(g-1)^n = 0$ for some $n \in \mathbb{N}$).
- all eigenvalues of g are 1.
- g is conjugate to $\begin{pmatrix} 1 & \star & \star \\ 0 & \ddots & \star \\ 0 & 0 & 1 \end{pmatrix}$.

Theorem 7. *Any unipotent subgroup of $\mathsf{GL}_n(k)$ is conjugate to a subgroup of*

$$U_n := \left\{ \begin{pmatrix} 1 & \star & \star \\ 0 & \ddots & \star \\ 0 & 0 & 1 \end{pmatrix} \right\} = \left\{ U \in M_n(k) \mid U_{i,j} = \begin{matrix} 0, & \text{if } i > j \\ 1, & \text{if } i = j \\ \text{arbitrary}, & \text{otherwise.} \end{matrix} \right\}.$$

Definition 13. For two subgroups G, H of some common supergroup, define their **commutator** by

$$[G,H] := \left\langle ghg^{-1}h^{-1} \mid g \in G, h \in H \right\rangle.$$

A group G is called **nilpotent**, if one of its commutators is trivial, i.e. if we set

$$G_0 := G \text{ and } G_{i+1} := [G_i, G],$$

then G is called nilpotent iff there is an $j \in \mathbb{N}$ with $G_j = 1$.

Corollary 3. *Any unipotent subgroup of $\mathsf{GL}(V)$ is nilpotent.*

Definition 14. A group G is called **solvable**, if $G^{(n)} = 1$ for some n where

$$\begin{aligned} G^{(0)} &:= G, \\ G^{(i+1)} &:= [G^{(i)}, G^{(i)}]. \end{aligned}$$

Notation 1. In the following, we will write $G' := [G,G]$.

Definition 15. Let $n := \dim(V)$. A **complete flag** is a maximal strictly increasing chain of subspaces

$$0 = V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_n = V.$$

Any complete flag is of the form

$$V_j := \operatorname{span}\{e_1, \dots, e_j\}$$

for some basis $e_1, \dots, e_n$ of V.

Let B be the basis of some flag $0 = V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_n = V$. For $x \in \mathsf{End}(V)$, we have that x is upper-triangle with respect to B iff x leaves each member V_i of the flag invariant, i.e. $xV_i \subseteq V_i$.

Proposition 2 (Key Proposition)**.** *Let G be a unipotent subgroup of $\mathsf{GL}(V)$. Then there is a complete flag $V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_n$ consisting of G-invariant subspaces, i.e., each V_i is G-invariant.*

Proof. Recall, that G is a unipotent subgroup of $\mathsf{GL}_n(V)$. We will give an induction on $n = \dim V$.

If $n = 0$, there is nothing to show.

Let $n \geq 1$. We may assume that V is G-irreducible. Because, if not, there is a G-invariant subspace $0 \neq W \subset V$ s.t. W and V/W have dimension $< n$. Then there exist complete G-invariant flags in W and V/W and the claim – that there is a complete G-invariant flag in V – follows by the induction hypothesis.

By Jacobson Density Theorem, we have

$$R := \operatorname{span}(G) = \mathsf{End}(V) := \mathsf{End}_k(V).$$

Since G is unipotent, we have for each $g \in G$

$$\operatorname{trace}(g) = n.$$

Ergo, for $g, h \in G$

$$\operatorname{trace}(gh) = \operatorname{trace}(h)$$

and

$$\operatorname{trace}((g-1)h) = \operatorname{trace}(gh) - \operatorname{trace}(h) = 0.$$

Since $\operatorname{span}(G) = \mathsf{End}(V)$, it now in particularly follows for all $g \in G, \phi \in \mathsf{End}(V)$

$$\operatorname{trace}((g-1)\phi) = 0.$$

Since the above holds for all $\phi \in \mathsf{End}(V)$, it must hold

$$g - 1 = 0$$

for all $g \in G$ (take for example the elementary matrices $\phi = E_{i,j}$). Ergo, G is trivial. Then, any complete flag is trivially G-invariant. $\square$

Remark 1. This gives the group analogue of Engel's Theorem.

Proof Goal Theorem. Let B be a basis of V s.t. G leaves each subspace in the corresponding flag invariant. Then, G is upper-triangle with respect to this basis.

On the other hand, each $g \in G$ us unipotent, hence its diagonal (i.e. eigenvalues) are all 1. Thus, with respect to B

$$G \subseteq \left\{ \begin{pmatrix} 1 & \star & \star \\ 0 & \ddots & \star \\ 0 & 0 & 1 \end{pmatrix} \right\} = U_n.$$

□

Remark 2. Tori are of the form $(k^\times)^n$. In the case $k = \mathbb{C}$, $(\mathbb{C}^\times)^n$ are the complexification of $U(1)^n$. This equals tori in top. sense.

$$\begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix} \subseteq \mathsf{GL}_2(\mathbb{C})$$

is a non-algebraic unipotent group.

Exercise. (to be discussed next time)

it would have sufficed to prove the Goal theorem in the special case that G is algebraic.

Corollary of Proof: If $G \subset \mathsf{GL}(V)$ (with $V \neq 0$) is unipotent and acts irreducibly (?), then $G = 1$, $\dim V = 1$.

Lecture from 18.03.2020

Answert to last Exercise: Recall that the main point was to show that any unipotent subgroup $G \subseteq \mathsf{GL}(V)$ leaves invariant some complete flag $\mathcal{F} = (V_0 \subset V_1 \ldots)$. But by some homework (problem 1), the group

$$\mathsf{GL}(V)_{\mathcal{F}} := \{g \in \mathsf{GL}(V) \mid g\mathcal{F} = \mathcal{F}\}$$

is algebraic.

Proof: If $\mathcal{F}$ is the standard flag with $V_i = \mathrm{span}(e_1, \ldots, e_i)$ for the standard basis $\{e_1, \ldots, e_n\}$, then

$$\mathsf{GL}(V)_{\mathcal{F}} = \{A \in \mathsf{GL}(V) \mid A \text{ is upper-triangle}\}.$$

The condition that A is upper triangle can be realized by polynomials. □

Thus,

$$\begin{aligned} & G \text{ fixes } \mathcal{F} \\ \iff & G \subseteq \mathsf{GL}(V)_{\mathcal{F}} \\ \overset{\mathsf{GL}(V)_{\mathcal{F}} \text{ is algebraic}}{\iff} & \overline{G} \subseteq \mathsf{GL}(V)_{\mathcal{F}} \\ \iff & \overline{G} \text{ fixes } \mathcal{F}. \end{aligned}$$

Now, the Zariski-Closure $\overline{G}$ of any group G is an algebraic group (shown in some homework).

Further, if G is unipotent, then $\overline{G}$ is unipotent.

0.3 Tori

Definition 16. A **torus** is an algebraic group that is isomorphic to $\mathcal{G}_m^n$ for some $n \in \mathbb{N}_0$ where $\mathcal{G}_m = k^\times = \mathsf{GL}_1(k)$ is the unit group of k.

We think of $\mathcal{G}_m^n \subseteq \mathsf{GL}_n(k)$ as the subgroup of diagonal matrices.

Lemma 19. *Let G be a commutative algebraic group. Then the following are equivalent:*

(i) each $g \in G$ is semisimple.

(ii) for each finite-dimensional representation V of G and for each $g \in G$, the operator $r_V(g)$ is diagonalizable.

(iii) for all finite-dimensional representations V of G, there is a basis of common eigenvectors for $r_V(G)$, i.e. a basis s.t.

$$r_V(G) \subseteq \mathcal{G}_m^n.$$

(iv) G is isomorphic to an algebraic subgroup of a torus.

Proof. (i) $\implies$ (ii): This follows from the Jordan decomposition and definition of semisimple.

(ii) $\implies$ (iii) : This is homework. Note that any commutative subset S of $\mathsf{GL}(V)$ consisting of semisimple operators may be diagonalized simultaneously.

(iii) $\implies$ (iv) : Take any faithful representation V of G and diagonalize it simultaneously. Then, $G \cong r_V(G) \subseteq \mathcal{G}_m^n$.

(iv) $\implies$ (i) : Any diagonal matrix is semisimple.

□

Definition 17. A commutative algebraic group G is called **diagonalizable**, if it satisfies one of the above equivalent conditions.

Definition 18. A character χ of any algebraic group F is an element $\chi \in \mathsf{Hom}_{\text{alg.grp.}}(G, k^\times)$, i.e., a homomorphism $\chi : G \to k^\times$ of algebraic groups.

Notation 2. For an algebraic group G, set $\Xi(G) := \mathsf{Hom}_{\text{alg.grp.}}(G, k^\times)$.

Also denote now by $\mathcal{O}(X) := k[T]/I(X)$ the coordinate ring of an algebraic set X (rather than $k[X]$).

Lemma 20. *There is a bijection*

$$\Xi(G) = \{\textit{characters } \chi \textit{ of } G\} \longleftrightarrow \{x \in \mathcal{O}(G)^\times \mid \Delta(x) = x \otimes x\}.$$

Proof. Note, that any $x \in O(G)^\times$ can be thought of as a map $x : G \to k^\times \subset k$.

We have

$$\begin{aligned}\mathsf{Hom}_{\text{alg.grp.}}(G, \mathcal{G}_m) &= \{\phi \in \mathsf{Hom}_{\text{alg.sets}}(G, \mathcal{G}_m) \mid \phi(gh) = \phi(g)\phi(h)\ \forall g, h\} \\ &= \{\phi \in \mathsf{Hom}_{k-\text{alg.}}(\mathcal{O}(\mathcal{G}_m), \mathcal{O}(G)) \mid (\phi \otimes \phi) \circ \Delta = \Delta \circ \phi\}.\end{aligned}$$

Recall: $\mathcal{O}(\mathcal{G}_m) \cong k[t, \frac{1}{t}]$ with $\Delta(t) = t \otimes t$.

Thus for any k-algebra A, $\mathsf{Hom}_{k-\text{alg.}}\left(\mathcal{O}(\mathcal{G}_m), A\right) \overset{A^\times}{\cong}$ via

$$[t \mapsto a, (t^{-1} \mapsto a^{-1})] \longleftrightarrow a.$$

Thus,

$$\mathsf{Hom}_{\text{alg.grp.}}\left(G, \mathcal{G}_m\right) \cong \left\{a \in \mathcal{O}(G)^\times \mid a \otimes a = \Delta(a)\right\}.$$

Therefore, it suffices to test the condition $(\phi \otimes \phi) \circ \Delta = \Delta \circ \phi$ on the generators t, t^{-1} of $\mathcal{O}(\mathcal{G}_m)$. Now, the above isomorphism is given by

$$\phi \mapsto a = \phi(t)$$

which is equivalent or regarding $\chi : G \to \mathcal{G}_m$ as a map $\chi : G \to k$. $\square$

Example 7. Let $G = \mathcal{G}_m$, then $\mathcal{O}(G) = k[t, \frac{1}{t}]$. Which $x = \sum_{m \in \mathbb{Z}} c_m t^m \in \mathcal{O}(G)$, almost all $c_m = 0$, but not all of them, have the property

$$\Delta(x) = x \otimes x.$$

We have

$$\begin{aligned} x \otimes x &= \sum_{m,n \in \mathbb{Z}} c_m c_n t^m \otimes t^n, \\ \Delta(x) &= \sum_{m \in \mathbb{Z}} c_m t^m \otimes t^m. \end{aligned}$$

Those sums equal, if

$$\begin{aligned} c_m c_n &= o \text{ for all } m \neq n, \\ c_m^2 &= c_m \text{ for all m.} \end{aligned}$$

By those conditions, it follows

$$x = t^m.$$

Therefore

$$\Xi(G) = \{\chi_m \mid m \in \mathbb{Z}\} \cong \mathbb{Z}$$

with

$$\chi_m(y) = y^m.$$

Example 8. Let $T \cong \mathcal{G}_m^n$ be a torus. Then,

$$\Xi(T) = \{\chi_m \mid m \in \mathbb{Z}^n\} \cong \mathbb{Z}^n$$

where $\chi_m(y) = y^m = y_1^{m_1} \cdots y_n^{m_n}$.

Note: For each algebraic group G, $\Xi(G)$ is naturally an abelian group:

$$(\chi_1 \cdot \chi_2)(g) := \chi_1(g) \cdot \chi_2(g).$$

Given a morphism of algebraic groups $f : G \to H$, we get a morphism of abelian groups

$$\begin{aligned} f^* : \Xi(H) &\longrightarrow \Xi(G) \\ \chi &\longmapsto \chi \circ f =: f^*(\chi). \end{aligned}$$

This induces a contravariant functor from the category of algebraic groups to the category of abelian groups.

Lemma 21. *Let G be a diagonalizable algebraic group. Then, $\Xi(G)$ is a k-basis for $\mathcal{O}(G)$.*

Example 9. Let $G = \mathcal{G}_m^n$ be a torus. Then, we have the embedding

$$\begin{aligned} \Xi(G) &\hookrightarrow \mathcal{O}(G) \\ \chi_m &\longmapsto t^m. \end{aligned}$$

The lemma is obvious in this case: each elment of $\mathcal{O}(G) = k[t_1, \dots, t_m, t_1^{-1}, \dots, t_m^{-1}]$ can be written uniquely as a linear combination of monomials.

Proof. (i) $\Xi(G)$ spans $\mathcal{O}(G)$:
Choose an embedding $G \subset \mathcal{G}_m^n$ of algebraic groups. Then, by restriction, we get

$$\mathcal{O}(\mathcal{G}_m^n) \twoheadrightarrow \mathcal{O}(G).$$

Since the $\chi_m, m \in \mathbb{Z}^n$, span $\mathcal{O}(\mathcal{G}_m^n)$, their images $\chi_m|_G \in \Xi(G)$ span $\mathcal{O}(G)$.

(ii) $\Xi(G)$ is linearly independent:
Suppose otherwise and let $\phi_1, \dots, \phi_m$ be a linearly dependent subset of $\Xi(G)$ with $m \geq 1$ chosen minimally, with $c_1, \dots, c_m \in k^\times$ s.t.

$$\sum_{i=1}^m c_i \phi_i = 0.$$

We distinguish the following cases:

$m = 1$: In this case, we have $\phi_1 = 0$, but $\phi_1(1) = 1$, a contradiction.

$m > 1$: We can assume $\phi_1 \neq \phi_2$, so there is an $h \in G$ s.t. $\phi_1(h) \neq \phi_2(h)$. Then,

$$\phi_1(h) \sum_{i=1}^{m} c_i \phi_i = 0,$$

but also for all $h, g \in G$

$$\sum_{i=1}^{m} c_i \phi_i(hg) = \sum_{i=1}^{m} c_i \phi_i(h) \phi_i(g) = 0.$$

This implies

$$\sum_{i=1}^{m} c_i \phi_i(h) \phi = 0.$$

Ergo

$$\sum_{i=1}^{m} c_j(\phi_i(h) - \phi_1(h))\phi_i = \sum_{i=2}^{m} c_j(\phi_i(h) - \phi_1(h))\phi_i = 0.$$

Now, $\phi_i(h) - \phi_1(h)$ is zero if $i = 1$ and non-zero, if $i = 2$. Therefore, this yields a shorter linear dependency for the elements

$$\phi_2, \ldots, \phi_m,$$

which contradicts our requirement.

$\square$

Definition 19. Let M be an abelian group. The **group algebra** on M is the k-algebra $k[M]$ (not a coordinate ring!) defined as follows:

$$\begin{aligned} k[M] &:= \text{the } k\text{-vectorspace with basis } M \\ &:= \left\{ \sum_{m \in M} c_m \cdot m \mid c_m \in k, \text{ almost all } c_m = 0 \right\}, \end{aligned}$$

where the multiplication on $k[M]$ extends that on M:

$$(\sum_{m \in M} c_m m)(\sum_{n \in M} d_n n) = \sum_{m,n \in M} c_m d_n mn.$$

Corollary 4. *For a diagonalizable G, we have*

$$\mathcal{O}(G) \cong k[\Xi(G)].$$

Fact: For an abelian group M, there is exactly one Hopf algebra structure on $k[M]$ given by $\Delta(m) = m \otimes m$ for all $m \in M$.

With this definition, the above isomorphism is one of Hopf algebras.

Lemma 22. *If G, H are diagonalizable algebraic groups, then*

$$\mathsf{Hom}_{\text{alg.grp.s}}(G, H) \xrightarrow{f \mapsto f^*} \mathsf{Hom}_{\text{grp.}s}(\Xi(H), \Xi(G))$$

is a bijection.

Proof.

$$\begin{aligned} \mathsf{Hom}(G, H) &\cong \mathsf{Hom}_{\text{Hopf-alg.}}(\mathcal{O}(H), \mathcal{O}(G)) \\ &\cong \left\{ \phi \in \mathsf{Hom}_{k-\text{alg.}}(\mathcal{O}(H), \mathcal{O}(G)) \mid (\phi \otimes \phi) \circ \Delta = \Delta \circ \phi \right\}. \end{aligned}$$

Since $\mathsf{Hom}_{k-\text{alg.}}(\mathcal{O}(H), \mathcal{O}(G)) \cong \mathsf{Hom}(k[\Xi(H)], k[\Xi(G)])$, this reduces to the following lemma:

Lemma 23. *Let $M_1, M2$ be two abelian groups. Then*

$$\begin{aligned} \mathsf{Hom}(M_1, M_2) &\xrightarrow{\cong} \mathsf{Hom}_{\text{Hopf-alg.}}(k[M_1], k[M_2]) \\ \phi &\longmapsto \left[\sum c_m m \mapsto \sum c_m \phi(m) \right]. \end{aligned}$$

Proof. We have to show that

$$M = \left\{ x \in K[M]^\times \mid \Delta(x) = x \otimes x \right\}.$$

Then, by this, it follows for each $\phi \in \mathsf{Hom}_{\text{Hopf-alg.}}(k[M_1], k[M_2])$,

$$\phi(M_1) \subseteq M_2.$$

Ergo, $\phi|_{M_1} \in \mathsf{Hom}(M_1, M_2)$. Therefore, the surjectivity of the claimed bijection is shown. The injectivity is clear, since M generates $k[M]$ as a k-algebra.

To show

$$M = \left\{ x \in K[M]^\times \mid \Delta(x) = x \otimes x \right\},$$

let

$$\begin{aligned} x &= \sum c_m m \in K[M]^\times \\ \Delta(x) &= \sum c_m m \otimes m \\ x \otimes x &= \sum c_m c_n m \otimes n. \end{aligned}$$

If $\Delta(x) = x \otimes x$, then it follows

$$x = m$$

for some $m \in M$.

$\square$

$\square$

Lecture from 25.03.2020

Recall: We have seen that for diagonalizable algebraic groups G, H

$$\mathsf{Hom}\,(G,H) \cong \mathsf{Hom}\,(\Xi(H),\Xi(G))\,.$$

If G is diagonalizable, then

$$\mathcal{O}(G) \cong k[\Xi(G)].$$

Theorem 8. *The functor*

$$\begin{aligned} G &\longrightarrow \Xi(G) \\ f &\longmapsto f^* \end{aligned}$$

defines an equivalence of categories:

$\{$*diagonalizable alg. groups*$\} \cong \{$*finite-dim. abelian groups with no* $\mathrm{char}(k)$*-torsion*$\}$.

This amounts to the bijection above between Hom-spaces and the following lemma.

Lemma 24. *(i) Let G be a diagonalizable alg. group. Then, $\Xi(G)$ is a finitely generated abelian group with no* $\mathrm{char}(k)$*-torsion.*

(ii) Let Γ be a finitely generated abelian group with no $\mathrm{char}(k)$*-torsion. Then, there is a diagonalizable algebraic group G s.t. $\Xi(G) \cong \Gamma$.*

Proof. We will use the following facts:

- Let $n \in \mathbb{N}$. Then, $t^n - 1$ is square-free in $k[t]$ iff the ideal $(t^n - 1)$ is radical in $k[t]$ iff $t^n - 1$ has not repetitive root iff either $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) = p > 0$ and $p \not| n$.

  (Proof: Galois Theory, seperable/inseperable extensions.)

- Let $M := \mathbb{Z}/n\mathbb{Z}$. Then, the k-group-algebra generated by M

  $$k[M] \cong k[t]/(t^n - 1)$$

  is reduced iff either $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) = p > 0, p \not| n$.

- If M_1, M_2 are abelian groups, then we have the following isomorphism of Hopf algebras

$$\begin{aligned} k[M_1] \otimes_k k[M_2] &\xrightarrow{\cong} k[M_1 \oplus M_2] \\ m_1 \otimes m_2 &\longmapsto m_1 m_2 \end{aligned}$$

where $M_1 \oplus M_2 \cong M_1 \times M_2$.

(i) Embed $G \hookrightarrow T := \mathcal{G}_m^n$ for some n. Then, we have a surjection $\mathbb{Z}^n \cong \Xi(T) \twoheadrightarrow \Xi(G)$. Ergo, $\Xi(G)$ is finitely generated.

Suppose $\mathrm{char}(k) = p > 0$. Let $\chi \in \Xi(G)$ with $\chi^p = 1$. Then, for all $g \in G$, $\chi^p(g) = \chi(g^p) = 1$. The unit group $k^\times$ has not p-torsion, therefore $G \hookrightarrow T = (k^\times)^n$ has also no p-torsion. Therefore, the frobenius $g \mapsto g^p$ is an isomorphism on G. Therefore, $\chi = 1$ is a trivial character. Ergo $\Xi(G)$ has no p-torsion.

(ii) Let Γ be a finitely generated abelian group with no $\mathrm{char}(k)$-torsion. Then,

$$\Gamma \cong \mathbb{Z}^r \oplus \mathbb{Z}/n_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/n_l\mathbb{Z}$$

where $\mathrm{char}(k) \not| n_1, \ldots, n_l$. We may reduce to the cases:

(a) $\Gamma = \mathbb{Z}$: take $G = \mathcal{G}_m$, then $\Xi(G) \cong \mathbb{Z} \cong \Gamma$.

(b) $\Gamma = \mathbb{Z}/n\mathbb{Z}$ with $\mathrm{char}(k) =: p \not| n$:
take $G := \mu_n := \{y \in k^\times \mid y^n = 1\}$. Then, since $p \not| n$, $(t^n - 1)$ is radical. So,

$$\mathcal{O}(\mu_n) \overset{Nullstellensatz}{=} k[t]/(t^n - 1) \overset{asHopfalgebras}{\cong} k[\Gamma]$$

where t gets mapped to the generator of Γ.

□

Corollary 5. *We have the bijection*

$$\{\textit{tori}\} \cong \{\textit{ finitely generated free abelian groups}(\cong \mathbb{Z}^n)\}.$$

Remark 3.

$$\{\text{algebraic group schemes}/k\} \overset{\text{not necessarily natural}}{\cong} \{\text{ f.g. Hopf algebras}\}.$$

by

$$G \mapsto \mathcal{O}(G)$$

and

$$\{\text{diagonalizable algebraic group schemes}/k\} \cong \{\text{ f.g. abelian groups}\}.$$

by

$$G \mapsto \Xi(G).$$

Where μ_p in the left hand term gets mapped to $\mathcal{O}(\mu_p) = k[t]/(t^p - 1)$ with $p = \mathrm{char} k$.

0.3.1 Trigonalization

We say a representation $r : G \to \mathsf{GL}(V)$ of a group G on a finite-dimensional k-vectorspace V is **trigonalizable** if it admits a basis with respect to which $r(V)$ is upper-triangular:

$$r(G) \subseteq \left\{ \begin{pmatrix} * & \dots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & * \end{pmatrix} \right\}$$

Definition 20. We call a subgroup $G \subseteq \mathsf{GL}(V)$ **trigonalizable**, if the identity representation is.

Lemma 25. *Let G be an algebraic group. The following are equivalent:*

(i) Every finite-dimensional representation $r : G \to \mathsf{GL}(V)$ is trigonalizable.

(ii) Every irreducible representation of G is 1-dimensional.

(iii) G is isomorphic to an algebraic subgroup of

$$B_n := \left\{ \begin{pmatrix} * & \dots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & * \end{pmatrix} \right\} \subseteq \mathsf{GL}_n(k).$$

(iv) There is a normal unipotent algebraic subgroup U of G s.t. G/U is diagonalizable.

Proof. We prove as follows:

(i) $\implies$ (ii): Let V be an irreducible representation. Then, $V \neq 0$. Choose a basis $e_1, \dots, e_n$ of V s.t.

$$r(G) \subseteq B_n.$$

Then, $r(G)e_1 \subseteq ke_1$, so $V_0 := ke_1$ is G-invariant. Ergo $V = V_0$ is 1-dimensional.

(ii) $\Longrightarrow$ (i): Let V be a f.d. representation. We show by induction on $\dim(V)$ that $r : G \to \mathsf{GL}(V)$ is trigonalizable:

In the cases $\dim(V) = 0, 1$, there is nothing to show.

In the case $\dim(V) \geq 2$, assume that V is not irreducible. Then, there is a G-invariant V_0 with $0 \neq V_0 \neq V$.

By the induction hypothesis, V_0 and V/V_0 are trigonalizable. Ergo, V is trigonalizable.

(**Recall:** we used this criterion above in the proof that unipotent groups are trigonalizable by showing that every ??? of each G is trivial.)

(i) $\Longrightarrow$ (iii): Choose a faithful representation V of G. Then, $G \cong r(G)$. Since r is trigonalizable, there is a basis of V s.t.

$$r(G) \subseteq B_n \subseteq \mathsf{GL}_n(k).$$

(iii) $\Longrightarrow$ (ii): Suppose $G \subseteq B_n \subseteq \mathsf{GL}_n(k)$. Set

$$A_n := \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & * \end{pmatrix} \right\} \subseteq \mathsf{GL}_n(k),$$

$$U_n := \left\{ \begin{pmatrix} 1 & \dots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & 1 \end{pmatrix} \right\} \subseteq \mathsf{GL}_n(k) \text{ normal algebraic subgroup of } B_n,$$

$$U := G \cap U_n \text{ normal unipotent algebraic subgroup of } G.$$

Let V be an irreducible representation of G, then V is not zero. Consider the subspace of V fixed by U

$$V^U := \{ v \in V \mid r(u)v = v \forall u \in U \}.$$

Then, we get a representation

$$r|_U : U \longrightarrow \mathsf{GL}(V).$$

Then, $r(U)$ is a unipotent algebraic group of $\mathsf{GL}(V)$. Ergo,

$$r(U) \subseteq \left\{ \begin{pmatrix} 1 & \dots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Ergo, $V^U \neq 0$. Since U is normal in G, the subspace V^U of V is G-invariant: if $v \in V^U, g \in G$, then for all $u \in U$ we have

$$r(u)r(g)v = r(g)r(g^{-1}ug)v = r(g)v$$

since $v \in V^U$. Ergo $r(g)v \in V^U$.

Since V is irreducible, $V = V^U$, i.e. U acts trivially on V. Ergo, r descends to a representation of the group G/U.

But $G/U \hookrightarrow B_n/U_n \cong A_n$. Therefore, G/U and $r(G)$ are commutative. Moreover, for all $g \in G$, $r(g) \in \mathsf{GL}(V)$ is semisimple:

if $g = g_s g_u$, then $g_u \in U$, because U_n is the group of unipotent elements of B_n.

Hence, $r(g) = r(g_s)r(g_u) = r_(g_s)$ is semisimple.

It follows that $r(G)$ is commutative and consists of semisimple elements. By some HW: $r(G)$ is trigonalizable. It is easy to show now that V is one-dimensional. (Since V is irreducible and ke_1 is G-invariant.)

□

Definition 21. G is **trigonalizable**, if it satisfies one of the above equivalent conditions.

Later, we will see, that if G is connected, then being trigonalizable implies being solvable.

0.3.2 Commutative Groups

Let G be an algebraic group. Denote by G_s resp. G_u the subsets of semisimple resp. unipotent elements of G.

Then, G_u is always algebraical i.e. closed: if $G \hookrightarrow \mathsf{GL}_n(k)$, then $G_u = \{g \mid (g-1)^n = 0\}$. G_u does not need to be closed under multiplication (for example, take $G = \mathsf{SL}_2(k)$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$).

G_s needs not to be algebraic: for example, take $G = \mathsf{SL}_2(k)$ and if G_s were algebraic, then

$$\left\{\lambda \in k^\times \mid \begin{pmatrix} \lambda & 1 \\ 0 & \lambda^{-1} \end{pmatrix} \in G_s \right\} = \{\lambda \mid \lambda \neq \lambda^{-1}\}$$

but the last set is not algebraic. Also, G_s does not need to be a subgroup.

We have the a surjective map of sets Lecture from 30.03.2020

$$\begin{aligned} G_s \times G_u &\longrightarrow G \\ (g_1, g_2) &\longmapsto g_1 g_2. \end{aligned}$$

Example 10 (Non-Example)**.** Take generic $g \in G_s, h \in G_u$ for $G = \mathsf{SL}_2(k)$. Then, g, h do not commute and we have

$$((gh)_s, (gh_u)) \neq (g, h)$$

because Jordan components commute.

Theorem 9. *Let G be a commutative algebraic group. Then:*

(i) G_s, G_u are closed subgroups and the multiplicative map $G_s \times G_u \to G$ is an isomorphism of algebraic groups.

(ii) G is trigonalizable. Moreover, for each finite dimensional representation $r : G \to \mathsf{GL}(V)$ there is a basis s.t.

$$r(G_s) \subseteq \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & * \end{pmatrix} \right\} \qquad r(G_u) \subseteq \left\{ \begin{pmatrix} 1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

(iii) G_s is diagonalizable.

Proof. (ii) Let V be any irreducible representation of G. We have seen that commuting semisimple operators may be simultanously diagonalizable, then

$$V = \bigoplus_{\chi : G_s \to \mathcal{G}_m} V_\chi$$

where

$$V_\chi = \{ v \in V \mid r(h)v = \chi(h)v \ \forall h \in G_s \}.$$

Since G is commutative, each subspace V_χ is G-invariant $(r(h)r(g)v = r(g)r(h)v = r(g)\chi(h)v = \chi(h)r(g)v)$.

Since V is irreducible, we must have $V = V_\chi$ for some χ.

Recall that $G \cong G_s \times G_u$ as abstract groups. We have seen that $r(G_s) \subseteq k^\times$. We proved a while ago that any unipotent group, such as G_u, is trigonalizable. Ergo, V is trigonalizable. Since V is irreducible, we have $\dim V = 1$.

If we apply the same argument without assuming that V is irreducible, then we see that V is the coproduct of V_χ's as above and that each V_χ admits a basis s.t.

$$r(G_s)|_{V_\chi} \subseteq \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & * \end{pmatrix} \right\} \qquad r(G_u)|_{V_\chi} \subseteq \left\{ \begin{pmatrix} 1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

This yields the same conclusion for V.

(i) We have to show that G_s and G_u are closed and $j : G_s \times G_u \to G$ is an isomorphism of groups. Take any faithful representation

$$G \xrightarrow{\cong, r} r(G) \subseteq \mathsf{GL}(V)$$

and apply (ii). Then we have

$$r(G) \subseteq \left\{ \begin{pmatrix} * & * & * \\ 0 & \ddots & * \\ 0 & 0 & * \end{pmatrix} \right\} =: B$$

$$B_u = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & 1 \end{pmatrix} \right\},$$

$$r(G_s) \subseteq \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & * \end{pmatrix} \right\} =: A.$$

In fact, $r(G_s) = r(G) \cap A$, because if $g \in G$ with $r(g) \in A$, then $r(g)$ is semisimple, so $g \in G_s$.

Therefore, G_s is closed in G. Ergo, G_s and G_u are closed subgroups.

Then, the map j is a morphism of algebraic groups.

We need to show that j^{-1} is a morphism of algebraic groups. For this, it suffices to verify that the projection $G \to G_s$ is a morphism. But this map is given under r by the morphism:

$$t := \begin{pmatrix} a_1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & a_n \end{pmatrix} \longmapsto \begin{pmatrix} a_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_n \end{pmatrix} =: t_s.$$

This suffices because if $g = g_s g_u$, then $g_u = g_s^{-1} g$, so if the map $g \mapsto g_s$ is a morphism, so is $g \mapsto g g_s^{-1} = g_u$, hence so is $g \mapsto (g_s, g_u)$.

(iii) We have seen that G_s is a closed subgroup. Hence G_s is a commutative algebraic group where elements are semisimple. Ergo, G_s is diagonalizable.

□

Theorem 10 (Lie-Kolchin)**.** *Let G be a connected solvable algebraic group. Then G is trigonalizable.*

(By comparison, recall that we have seen that far that, if G is commutative or unipotent, then G is trigonalizable.) We can reformulate this theorem as: Any connected solvable subgroup of $\mathsf{GL}(V)$ stabilizes some complete flag $\mathcal{F} = (V_0 \subsetneq \ldots \subsetneq V_n)$.

Generalization (Borel's Fixed Point Theorem): Any connected algebraic group G acting on a projective variety X has a fixed point in X.

We get a relation between complete flags and projective varieties.

Proof. Induct on the number n s.t. $G^{(n)} = 1$.

For $n = 0$, there is nothing to show.

If $n = 1$, $(G, G) = 1$, then G is commutative, ergo trigonalizable.

Let $n \geq 2$. Then, we have $G' := (G, G) \neq 1$. We will show the following lemma:

Lemma 26. *If G is connected, then the abstract group G' with the induced topology is connected ($\iff$ the Zariski Closure of G' is connected).*

Proof. We have the following facts:

- An increasing union of connected spaces is connected.
- A continuous image of a connected space is connected.

We have

$$\begin{aligned} G' &= \left\langle (g,h) := ghg^{-1}h^{-1} \mid g, h \in G \right\rangle \\ &= \bigcup_{j \geq 0} \bigcup_{g_1, h_1, \ldots, g_j, h_j \in G} \{(g_1, h_1) \cdots (g_j, h_j)\}. \end{aligned}$$

Since

$$\bigcup_{g_1, h_1, \ldots, g_j, h_j \in G} \{(g_1, h_1) \cdots (g_j, h_j)\} = \mathsf{Img}\phi_j$$

for some continuous map $\phi_j : G^{2j} \to G$, the claim follows. □

Ergo, G' is connected.

Note: It is equivalent to show that $(*)$ any subgroup of $\mathsf{GL}(V)$ s.t. G is connected and solvable is trigonalizable in $\mathsf{GL}(V)$.

Indeed, the theorem implies $(*)$: the Zariski closure of G is a connected algebraic group that is solvable (which extends by continuity). If $Zcl(G)$ is trigonalizable, then also G is trigonalizable.

$(*)$ implies the theorem, since if G is given as in the theorem, apply $(*)$ to $r(G) \subseteq \mathsf{GL}(V)$.

If $G^{(n)} = 1$, then $(G')^{(n-1)} = G^{(n)} = 1$. By induction, we may assume that G' satisfies the following:

For all finite dimensional representations $r : G \to \mathsf{GL}(V)$, $r(G')$ is trigonalizable.

Our aim is to show that any irreducible representation V of G has dimension 1.

The induction hypothesis implies that $r(G')$ is trigonalizable. In particular, there exist an eigenspace $V_\chi \subseteq V$ for G' for some character $\chi : G' \to k^\times$. Since G' is normal in G we know that G acts from the left on

$$\{\text{eigenspaces } V_\chi \text{ in } V \text{ for } G'\}.$$

Ergo, $\bigoplus_{\chi:G'\to k^\times} V_\chi$ is G-invariant. Ergo, $V = \bigoplus_{\chi:G'\to k^\times} V_\chi = \bigoplus_{\chi\in\Xi'} V_\chi$ for some finite subset $\Xi' = \{\chi \mid V_\chi \neq 0\}$ of $\mathsf{Hom}\,(G', \mathcal{G}_m)$, since V is finite dimensional.

Claim: Let $h \in G'$. Then, the map

$$\begin{aligned} G &\longrightarrow \mathsf{GL}(V) \\ g &\longmapsto r(ghg^{-1}) \end{aligned}$$

has a finite map.

Proof. Denote by $\chi \mapsto \chi^g$ the action of $g \in G$ in $\mathsf{Hom}\,(G', \mathcal{G}_m)$ given by $\chi^g(h) := \chi(ghg^{-1})$. This is an action, since G' is normal.

This descends to an action $G \curvearrowright \Xi$, because r is a homomorphism. Since $r(h)$ is determined by $\{\chi(h) \mid \chi \in \Xi\}$, hence similarly $r(ghg^{-1}) \in r(G')$ by $\{\chi(ghg^{-1}) \mid \chi \in \Xi\}$.

Hence,

$$\#\left\{r(ghg^{-1}) \mid g \in G\right\} \leq \#\text{representations of the finite set } \Xi < \infty.$$

$\square$

Lemma 27. *Let G be an algebraic group. Then, G is connected iff for each finite algebraic set X, and for each morphism $f : G \to X$ of algebraic sets, we have that f is constant.*

Claim with the Lemma implies that the map $g \mapsto t(ghg^{-1})$ is constant. This implies that $r(ghg^{-1}) = r(h)$ for all $g \in G, h \in G'$. Ergo, G stabilizes each eigenspace V_χ for G'. Ergo, $V = V_{\chi_0}$, since V is irreducible.

$\square$

Lemma 28. *Let G be any group with a finite dimensional representation $r : G \to \mathsf{GL}(V)$. Then, the subspaces V_χ for $\chi \in \mathsf{Hom}(G, k^\times)$ are independent, i.e., the map*

$$\oplus V_\chi \longrightarrow V$$

is injective.

Proof. The spaces V_χ are G-invariant. Suppose, there exist distinct $\chi_1, \ldots, \chi_n$ of non-zero $v_j \in V_{\chi_j}$ s.t. $\sum_j v_j = 0$.

We may assume that n, the number of v_j, is minimal. W.l.o.g., $n \geq 2$.

Choose $g \in G$ s.t. $\chi_1(g) \neq \chi_2(g)$. Use that $0 = g \sum_j v_j = \sum_j g v_j$ and take the linear combination as in the proof of linear indpendence of characters to contradict the minimality of n. $\square$

Lecture from 01.04.2020

Since $G' = \langle ghg^{-1}h^{-1} \mid g, h \in G\rangle$, so $\det(r(G')) = 1$.
On the other hand, for each $g \in G'$, we have

$$r(g) = \begin{pmatrix} \chi_0(g) & & \\ & \ddots & \\ & & \chi_0(g) \end{pmatrix}$$

since $V = V_{\chi_0}$. This implies

$$1 = \det(r(g)) = \chi_0(g)^d.$$

Ergo, χ_0 defines a morphism

$$\chi_0 : G' \longrightarrow \mu_d \subseteq \mathcal{G}_m.$$

But G' is connected and μ_d is finite. Since χ_0 is a morphism, χ_0 must be constant, ergo the trivial character.

As a consequence, we get $r(G') = 1$ on $V = V_{\chi_0}$.

Lemma 29. *Let G be an algebraic group, $r : G \to \mathsf{GL}(V)$ a representation. $v \in V$ shall be a simultaneous non-zero eigenvector for $r(G)$.*

Then, for each $g \in G$, there is a value $\chi(g) \in k^\times$ s.t.

$$r(g)v =: \chi(g)v.$$

Then, the mapping $\chi : G \to \mathcal{G}_m$ is a morphism of algebraic groups.

Therefore, r descends to a representation of the commutative group

$$\overline{r} : G/G' \longrightarrow \mathsf{GL}(V).$$

Ergo, $r(G/G') = r(G)$ is commutative and therefore trigonalizable (because of irreducibility). □

Example 11 (Non-Example)**.**
- Take $G = D_4 \hookrightarrow \mathsf{GL}_2(\mathbb{C})$ which is solvable and has an irreducible and faithful representation over $\mathbb{C}^2$.
- Consider the solvable group

$$G = \left\langle \begin{pmatrix} \pm 1 & \\ & \pm 1 \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \right\rangle$$

which is a finite subgroup of $\mathsf{GL}_2(\mathbb{C})$, s.t. $\mathbb{C}^2$ define an irreducible representation of G.

Lemma 30 (Form of Schur's Lemma)**.** *If S is any commutative subset of $\mathsf{GL}(V)$ for a finite-dimensional $0 \neq V$ over an algebraically closed field k. Let V be S-irreducible.*

Then, $\dim V = 1$.

Proof. There is notthing to show if S is empty.

Let $s \in S$ and denote by $V_\lambda \subseteq V$ the λ-eigenspace for s. Then, since S is commutative, V_λ is S-invariant. Therefore, $V = V_\lambda$ for one $\lambda \in k^\times$.

Thus, every $s \in S$ acts by scaling, therefore every subspace of V is S-invariant. Since V is invariant, we get $\dim V = 1$. □

Corollary 6. *Let G be a connected algebraic group. Then, G is solvable iff G is trigonalizable.*

Proposition 3. *If G is trigonalizable, then G_u is a normal algebraic subgroup.*

Proof. We have

$$G \hookrightarrow B := \left\{ \begin{pmatrix} * & \dots & * \\ & \ddots & \vdots \\ & & * \end{pmatrix} \right\} \subseteq \mathsf{GL}_n(k).$$

B has the normal subgroup $U := \left\{ \begin{pmatrix} 1 & \dots & * \\ & \ddots & \vdots \\ & & 1 \end{pmatrix} \right\}$ and we have $G_u = G \cap U$. Now, U is the kernel of the multiplicative morphism

$$\begin{pmatrix} a_1 & \dots & * \\ & \ddots & \vdots \\ & & a_n \end{pmatrix} \longmapsto \begin{pmatrix} a_1 & & \\ & & \\ & & a_n \end{pmatrix}.$$

□

Corollary 7. *If G is connected and solvable, then G_u is a normal algebraic subgroup.*

0.3.3 Semisimple Elements of nilpotent Groups

Theorem 11. *Let G be a connected nilpotent algebraic group. Then, we have*

$$G_s \subseteq Z(G)$$

where $Z(G)$ denotes the center of G.

Theorem 12 (Lie-algebraic Analogue)**.** *Let V be a finite-dimensional vectorspace. Let $\mathfrak{g}$ be the Lie-Subalgebra of $\mathsf{End}(V)$, i.e. $\mathfrak{g}$ is a subspace s.t. we have for each $x, y \in \mathfrak{g}$*

$$[x, y] := xy - yx \in \mathfrak{g}.$$

Assume that $\mathfrak{g}$ is nilpotent, i.e. there is an $n \in \mathbb{N}_0$ s.t.

$$[x_1, [x_2, [\ldots, [x_{n-1}, x_n]]]] = 0$$

for all $x_1, \ldots, x_n \in \mathfrak{g}$.

*Then, any semisimple (semisimple in $\mathsf{End}(V)$ that is) $x \in \mathfrak{g}$ is **central** in $\mathfrak{g}$, i.e. $[x, y] = 0$ for each $y \in \mathfrak{g}$.*

Remark 4. The Lie-algebraic Analogue implies the general theorem if – for example – $k = \mathbb{C}$.

Proof. Let $g \in G_s$. We want to show $Z_G(g) = G$.

Fact from the theory of Lie-Algebras: For the Lie-Algebra $\mathrm{Lie}Z_G(g)$ we have

$$\mathrm{Lie}Z_G(g) = \ker(\mathsf{Ad}(g))$$

where Ad is the map

$$\begin{aligned} \mathsf{Ad} : G &\longrightarrow \mathsf{GL}(\mathfrak{g}) \\ x &\longmapsto gxg^{-1}. \end{aligned}$$

Since G is connected, it suffices to verify

$$\ker(\mathsf{Ad}(g)) = \mathfrak{g}$$

i.e. $\mathsf{Ad}(g) = 1$.

Since g is semisimple, we have for suitable basis

$$g = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$$

with $a_j \in \mathbb{C}^\times$. This is $\exp(x)$ for a suitable diagonal matrix $x \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} \in \mathsf{GL}_n(\mathbb{C})$.

Fact: We may assume that $x \in \mathfrak{g} := \mathrm{Lie}(G)$.

Since G is nilpotent, it can be shown that $\mathfrak{g}$ is nilpotent.

By the theorem, x is central in $\mathfrak{g}$. By the properties of exp we have

$$\mathsf{Ad}(g) = \exp(\mathrm{ad}(g)) = 1$$

ergo $\mathrm{ad}(x) = 0$ where $\mathrm{ad} : \mathfrak{g} \to \mathfrak{g}$ is defined by

$$\mathrm{ad}(x) \cdot y := [x, y].$$

□

Proof. If $\mathfrak{g}$ is nilpotent, then $\mathrm{ad}(x) \in \mathsf{End}(\mathfrak{g})$ is nilpotent.

Since x is semisimple, $\mathrm{ad}(x)$ is semisimple, because $\mathrm{ad}(x)$ is the restriction to $\mathfrak{g}$ of the map

$$\begin{aligned} \mathsf{End}(V) &\longrightarrow \mathsf{End}(V) \\ y &\longmapsto [x, y] \end{aligned}$$

and, if $e_1, \ldots, e_n$ are a basis of eigenvectors for x, then $E_{i,j}$ is a basis of eigenvectors for ℓ.

So, $\mathrm{ad}(x)$ is nilpotent and semisimple, therefore $\mathrm{ad}(x) = 0$. □

Proof Theorem. Let G be a connected nilpotent algebraic group, $G \overset{\mathsf{GL}}{\hookrightarrow} (V)$.

Let $g \in G_s$, we want to show that $g \in Z(G)$.

Assume otherwise, then we have a $h \in G$ s.t. $(g, h) = ghg^{-1}h^{-1} \neq 1$.

Since G is connected and nilpotent (ergo solvable), we know by Lie-Kolchin that G stabilizes some complete flag $V_0 \subset \ldots \subset V_n$.

We have $g|_{V_i}, h|_{V_i} \in \mathsf{GL}(V_i)$. They commute, if $i = 0$, but not if $i = n$.

So, there is an i s.t. $g|_{V_i}, h|_{V_i}$ commute but $g|_{V_{i+1}}, h|_{V_{i+1}}$ don't commute. W.l.o.g. $V = V_{i+1}, g = g|_{V_{i+1}}, h = h|_{V_{i+1}}$. Set $a := g|_{V_i}, b := h|_{V_i} \in \mathsf{GL}(V_i)$. a will be semisimple, since g is.

Since g is semisimple, there is an eigenvector $v \in V_{i+1}$ for g s.t.

$$V_{i+1} = V_i \oplus \langle v \rangle .$$

We have an isomorphism of vector spaces

$$\mathsf{End}(V_{i+1}) \cong \mathsf{End}(V_i) \oplus \mathsf{Hom}\left(\langle v \rangle, V_i\right) \oplus \mathsf{Hom}\left(V_i, \langle v \rangle\right) \oplus \mathsf{End}(\langle v \rangle)$$

with

$$\mathsf{End}(\langle v \rangle) \cong k \text{ and } \mathsf{Hom}\left(\langle v \rangle, V_i\right) \cong V_i.$$

So, we can write $g|_{V_{i+1}}, h|_{V_{i+1}}$ write as

$$g = \begin{pmatrix} a & \\ & * \in k \end{pmatrix} \text{ and } h = \begin{pmatrix} b & c \in V_i \\ & * \end{pmatrix}.$$

We may replace g, h with scalar multiples to reduce to the case that $* = 1$. Then, So, we can write $g|_{V_{i+1}}, h|_{V_{i+1}}$ write as

$$g = \begin{pmatrix} a & \\ & 1 \end{pmatrix} \text{ and } h = \begin{pmatrix} b & c \\ & 1 \end{pmatrix}.$$

Then,

$$h \neq ghg^{-1} = \begin{pmatrix} b & ac \\ & 1 \end{pmatrix}.$$

Ergo, $c \neq ac$, i.e. $c \notin \ker(a-1)$. Let $h_1 := h^{-1}ghg^{-1}$. Check

$$h_1 = \begin{pmatrix} 1 & b^{-1}(a-1)c \\ & 1 \end{pmatrix}.$$

We claim that h_1 does not commute with g. This claim implies the theorem, since we can iterate the claim to obtain elements h_i by $h_{i+1} := h_i^{-1}gh_ig^{-1}$. Then, h_i does not commute with g. But G is nilpotent, therefore $h_i = 1$ for some large enough i.

We can prove the claim as follows: By some calculation as for h and g, we see, that h_1 and g don't commute iff $b^{-1}(a-1)c \notin \ker(a-1)$. This is equivalent to

$$\begin{aligned} &\iff (a-1)b^{-1}(a-1)c \neq 0 \\ &\iff b^{-1}(a-1)^2c \neq 0 \\ &\iff (a-1)^2c \neq 0 \\ &\iff c \in \ker((a-1)^2). \end{aligned}$$

But a being semisimple implies $a-1$ being semisimple, therefore $\ker((a-1)^2) = \ker(a-1)$. So h_1, g don't commute iff $c \in \ker(a-1)$ iff h, g don't commute. $\square$