Example $X = S^1 \times [0,1]/\sim$, $\sim$ generated by

$$(z,0) \sim (e^{2\pi/n}z, 0); (z,1) \sim (e^{2\pi/m}z, 1)$$

identifying points that are an angle $2\pi/n$, $2\pi/m$ apart
It follows:

$$top \cong S^1; bottom \cong S^1$$

Pushout: (1)

The inclusions

$$top \hookrightarrow X_T, bottom \hookrightarrow X_B$$

are deformation retracts, in particular homotopy equivalences.

In the first case,

$$r : X_T \longrightarrow Top, [(z,t)] \longmapsto [(z,1)]$$

$$h : X_T \times [0,1] \longrightarrow X_T, ([(z,s)],t) \longmapsto [(z, s \cdot t)]$$

provides the data of a deformation retract. Similiar for bottom.

van Kampen yields a pushout of groups, when applying π_1 to (1). (2)

How do the induced morphisms look like?
In the case of top:

$$r_* : \pi_1(X_T) \longrightarrow \pi_1(Top), \gamma \longmapsto \gamma^n$$

m One gains

$$\pi_1(S^1 \times \{\frac{1}{2}\}) \longrightarrow \pi_1(X_T) \xrightarrow{\sim} \pi_1(Top) \xrightarrow{\sim} \pi_1(S^1) \cong \mathbb{Z}$$

Be γ a generator of $\pi_1(Top)$, r applied to the generator of $\pi_1(S^1 \times \frac{1}{2})$ wraps around m times the top circle.

Because of (2) we obtain a group presentation

$$\pi_1(X) \cong \langle a, b \mid a^m = b^n \rangle$$

We have an epimorphism

$$\pi_1(X) \longrightarrow> \mathbb{Z}/m \star \mathbb{Z}/n$$

$$a \longmapsto 1_{\mathbb{Z}/m}$$

$$b \longmapsto 1_{\mathbb{Z}/n}$$

Kapitel 1

Homology - the axiomatic approach 2

1.1 2.1 The Eilenberg-Steenrod axioms

Let R be a commutative Ring.

A sequence of moprhisms of R-modules

$$M_{i+1} \xrightarrow{f_{i+1}} M_i \xrightarrow{f_i} M_{i-1} \xrightarrow{f_{i-1}}$$

is called **exact** if

$$Kern f_i = im f_{i+1} \forall i$$

Definition

A **homology theory** (H_*, ∂_*) with values in R-modules consists of a family $(H_n)_{n\in\mathbb{Z}}$ of functors

$$H_n : \mathbf{Top}^2 \longrightarrow R - \mathbf{Mod}$$

from the category of pairs of spaces to the category of R-modules and a family of natural transformations $(\partial_n)_{n\in\mathbb{Z}}$

$$\partial_n : H_n \longrightarrow H_{n-1} \circ J$$

where J is the functor

$$J : \mathbf{Top}^2 \longrightarrow \mathbf{Top}^2$$
$$(X, A) \longmapsto (X, \emptyset)$$

such that the following axioms are true

- **Homotopy invariance**
  If $f, g : (X, A) \to (Y, B)$ are maps of pairs of spaces and $h_t : f \simeq g$ is a homotopy with $h_t(A) \subset B \forall t \in [0, 1]$ then
  $$H_n(f) = H_n(g)$$

- **Long exact sequence**
  For every pair (X, A) the following sequence of R-modules is exact:

$$\ldots \longrightarrow H_{n+1}(X,A) \overset{\partial_{n+1}(X,A)}{\longrightarrow} H_n(A,\emptyset) \overset{H_n(\iota)}{\longrightarrow} H_n(X,\emptyset) \overset{H_n(j)}{\longrightarrow} H_n(X,A) \longrightarrow H_{n-1}(A,\emptyset) \longrightarrow \ldots$$

  is exact where

$$\iota : (A,\emptyset) \longrightarrow (X,\emptyset)$$
$$j : (X,\emptyset) \longrightarrow (X,A)$$

  This maps $\partial_i(X,A)$ are called **boundary homomorphisms**.

- **Excision axiom**
  Let X be a space and $A, B \subset X$ be a subspace such that $\overline{A} \subset B^o$. Then the R-homomorphisms

$$H_n(X \setminus A, B \setminus A) \longrightarrow H_n(X,B)$$

  induced by

$$(X \setminus A, B \setminus A) \hookrightarrow (X,A)$$

  is an isomorphism for all n.

If (H_*, ∂_*) in addition satisfies the following, we say (H_*, ∂_*) satisfies the **dimension axiom**:

$$H_n(\{*\},\emptyset) \cong R, ifn = 0; 0ifn \neq 0$$

Notation: In the sequel we write $H_n(X)$ instead of $H_n(X,\emptyset)$.

Remark

In a nutshell, (H_*, ∂_*) is the following:

- $(X,A) \rightsquigarrow R - \text{Modules} H_n(X,A), n \in \mathbb{Z}$
- $(X,A) \overset{f}{\to} (Y,B) \rightsquigarrow H_n(X,A) \overset{H_n(f)}{\to} H_n(Y,B)$
- boundary homom. (3)

Remark

Long exact sequence for (X, X)

$$H_{n+1}(X,X) \overset{\partial_{n+1}}{\longrightarrow} H_n(X) \overset{\sim}{\longrightarrow} H_n(X) \overset{H_nj}{\longrightarrow} H_n(X,X) \overset{\partial_n}{\longrightarrow} H_{n-1}(X) \overset{\sim}{\longrightarrow} H_{n-1}(X)$$

$$Kernid = im(\partial_n) = 0 \Longrightarrow \partial_n = 0$$
$$imH_n(j) = Kern\partial_n = Hn(X,X)$$
$$KerH_n(j) = imid = H_n(X)$$

Therefore $H_n(X,X) = 0$

1.2 2.2 First conclusions from the axioms

Fünferlemma

Consider the following commuting diagram of R-modules (4) such that both rows are exact and f_1 is surjective, f_5 is injective and f_2 and f_4 are isomorphisms. Then f_3 is an isomorphism.

Proof by Diagrammjagd f_3 is injective: Let $x \in M_3$, if $f_3(x) = 0$