

# Mitschrieb: Algebraic Groups

## SS 20

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### Vorwort

### Contents

1	Introduction	3
2	Algebraic Groups and Hopf Algebras	5
3	Actions	10
4	Connected Components	14
5	Jordan Decomposition	20
6	Non-Commutative Algebra	33
7	Unipotent Groups	38
8	Tori	42
9	Trigonalization	51
10	Commutative Groups	54
11	Connected Solvable Groups	57
12	Semisimple Elements of nilpotent Groups	62



# 1 Introduction

Let  $k$  be an algebraically closed field.

**Definition 1.** For  $I \subseteq k[X] := k[X_1, \dots, X_n]$ , we define its **vanishing set** by

$$V(I) := \{p \in k^n \mid \forall f \in I : f(p) = 0\}.$$

A set  $S \subset k^n$  is called **algebraic**, if

$$S = V(I)$$

for some  $I \subseteq k[X]$ .

**Example 1.** The group  $\mathrm{GL}_n(k)$  is not an algebraic subset of  $k^{n \times n}$ . But, we can identify it with an algebraic subset of  $(k^{n \times n})^2$  by

$$\mathrm{GL}_n(k) \cong \{(x, y) \in k^{n \times n} \mid xy = 1_n\} = V(X \cdot Y - 1_n).$$

**Definition 2.** Let  $\iota : \mathrm{GL}_n(k) \hookrightarrow k^{n \times n^2}$  be the injection

$$A \mapsto (A, A^{-1}).$$

A **linear algebraic group** over  $k$  is a subgroup  $U \subseteq \mathrm{GL}_n(k)$  s.t.  $\iota(k)$  is an algebraic subset of  $k^{2n^2}$ .

I.e., a linear algebraic group is a matrix-group which can be defined by polynomials over the entries of a matrix and its inverse.

**Example 2.** The following groups are linear algebraic groups:

1. The multiplicative group  $\mathcal{G}_m(k) := k^\times = k \setminus \{0\} = \mathrm{GL}_1(k)$ .
2. The general linear group  $\mathrm{GL}_n(k)$ .
3. The special linear group

$$\mathrm{SL}_n(k) := \{A \in \mathrm{GL}_n(k) \mid \det(A) = 1\}.$$

4. The orthogonal group

$$\mathcal{O}_n(k) := \{A \in \mathrm{GL}_n(k) \mid A^T \cdot A = 1\}.$$

5. The special orthogonal group

$$\mathrm{SO}_n(k) := \mathcal{O}_n(k) \cap \mathrm{SL}_n(k).$$

6. The upper triangle-matrix group

$$\left\{ \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ & \ddots & \vdots \\ & & a_{n,n} \end{pmatrix} \mid a_{i,j} \in k \right\} \cap \mathrm{GL}_n(k).$$

7. The normed upper triangle-matrix group

$$\left\{ \begin{pmatrix} 1 & \cdots & a_{1,n} \\ & \ddots & \vdots \\ & & 1 \end{pmatrix} \mid a_{i,j} \in k \right\} \cap \mathrm{GL}_n(k).$$

8. The group of  $n$ -th roots of unity

$$\mu_n(k) := \{x \in k \mid x^n = 1\}.$$

9. The additive group  $(k, +)$  is not a subgroup of  $\mathrm{GL}_n(k)$ , but it can be identified with the linear algebraic group

$$\left\{ \begin{pmatrix} 1 & a \\ & 1 \end{pmatrix} \mid a \in k \right\} \subset \mathrm{GL}_2(k)$$

10. For  $k = \mathbb{C}$ , the unit sphere and the unitary groups are NOT linear algebraic groups.

## 2 Algebraic Groups and Hopf Algebras

**Definition 3.** A **morphism**  $f : X \rightarrow Y$  of algebraic sets  $X \subset k^m, Y \subset k^n$  is a map which is coordinatewise described by polynomials.

**Definition 4.** An **algebraic group** is an algebraic set  $G \subset k^n$  together with a fixed element  $e \in G$  and morphisms  $m : G \times G \rightarrow G, i : G \rightarrow G$  s.t.  $(G, m, i, e)$  is a group.

A **morphism of algebraic groups** is a morphism of algebraic sets that is also a group homomorphism.

**Definition 5.** Let  $V \subset k^n$  be any subset. Then, we define the vanishing ideal of  $V$  by

$$I(V) := \{f \in k[x] \mid f(V) = 0\}.$$

**Definition 6.** For a commutative ring  $R$  we define the **radical** of an ideal  $I \subseteq R$  by

$$\sqrt{I} := \{r \in R \mid r^m \in I \text{ for some } m \in \mathbb{N}_0\}.$$

$R$  is called **reduced**, if  $\sqrt{0} = 0$ .

**Lemma 1** (Zariskis Lemma). *Let  $L \supseteq k$  be fields. If  $L$  is finitely generated as a  $k$ -algebra, then the extension  $L \supseteq k$  is finite, i.e.,  $L$  is a finitely-generated  $k$ -vector space.*

**Theorem 1** (Hilberts Nullstellensatz). *For any ideal  $I \subseteq k[x]$ , we have*

$$I(V(I)) = \sqrt{I}.$$

*Proof.* It is easy to see that

$$I \subset \sqrt{I} \subset I(V(I)).$$

Now, let  $f \in I(V(I))$  and assume – for the sake of contradiction – that  $f \notin \sqrt{I}$ . Since  $\sqrt{I}$  is the intersection of its upper prime ideals, there is a prime ideal  $p \supset I$ , s.t.  $f \notin p$ . Now, define the zero divisor-free ring

$$R := (k[x]/p)[f^{-1}].$$

And let  $\phi : k[x] \rightarrow R$  be the corresponding ring homomorphism.

Let  $m \subseteq R$  be a maximal ideal in  $R$ . Then,  $R/m$  is a field, which contains  $k$  and is finitely generated as  $k$ -algebra. According to Zariski's lemma,  $R/m$  is a finite (ergo algebraic) extension of  $k$ . Since  $k$  is algebraically closed, we have  $R/m = k$ . Let  $\pi_m : R \rightarrow k$  be the corresponding ring homomorphism.

Now, for  $x_1, \dots, x_n$ , set

$$t_i := \pi_m(\phi(x_i)).$$

Then,  $t = (t_1, \dots, t_n) \in k^n$ . We now have

1.  $t \in V(I)$ : For each  $g \in I$ , we have  $\phi(g) = 0$ . On the other hand

$$g(t) = g(\pi_m \circ \phi(x)) = \pi_m \circ \phi(g) = 0.$$

2.  $f(t) \neq 0$ :  $\phi(f)$  is invertible in  $R$ , therefore  $\phi(f) \neq 0$  and  $\phi(f) \notin m$ . Ergo

$$f(t) = \pi_m \circ \phi(f) \neq 0.$$

Ergo, there is a point  $t \in V(I)$  s.t.  $f(t) \neq 0$ . This yields a contradiction, since we assumed  $f \in I(V(I))$ .  $\square$

**Definition 7.** For an algebraic set  $X \subset k^n$ , we define its **coordinate ring** by

$$k[X] := k[x_1, \dots, x_n]/I(X).$$

**Lemma 2.** For a morphism  $f : X \rightarrow Y$  of algebraic sets define the following homomorphism of  $k$ -algebras.

$$\begin{aligned} f^* : k[Y] &\longrightarrow k[X] \\ p &\longmapsto p \circ f. \end{aligned}$$

We have a contravariant functor  $_*$  from the categories of algebraic sets over  $k$  to the category of  $k$ -algebras:

$$\begin{aligned} X &\longmapsto k[X] \\ \text{Hom}(X, Y) &\longmapsto \text{Hom}_k(k[Y], k[X]) \\ f &\longmapsto f^*. \end{aligned}$$

**Lemma 3.** We have

$$k[X \times Y] \cong k[X] \otimes k[Y].$$

*Proof.*

$$k[X] \otimes k[Y] = k[x]/I(X) \otimes_k k[y]/I(Y) = k[x, y]/I(X) \otimes k[y] + k[x] \otimes I(Y).$$

But

$$V(I(X) \otimes k[y] + k[x] \otimes I(Y)) = V(I(X) \otimes k[y]) \cap V(k[x] \otimes I(Y)) = X \times Y.$$

$\square$

**Theorem 2.** *Every finitely generated reduced  $k$ -algebra  $A$  is isomorphic to some  $k[X]$  for some algebraic  $X$ .*

*Proof.* Choose some  $\pi : k[x_1, \dots, x_n] \twoheadrightarrow A$  and set  $X := V(\ker \pi)$ . Then  $\ker \pi = I(X)$ , since  $\pi$ 's kernel is radical since  $A$  is reduced.  $\square$

**Corollary 1.** *The contravariant functor  $_* : \mathcal{C}_{\text{algSets}} \rightarrow \mathcal{C}_{k\text{-alg.s}}$  gives an antiequivalence of categories.*

**Lemma 4.** *An algebraic set  $X$  is isomorphic to some algebraic subset of  $Y$  iff there is an epimorphism  $k[Y] \twoheadrightarrow k[X]$ .*

**Lemma 5.** *Let  $G \subset k^n$  be an algebraic group. Then, we have maps*

$$\begin{aligned} m : G \times G &\longrightarrow G \\ i : G &\longrightarrow G \\ e : * &\longrightarrow G. \end{aligned}$$

*They induce dual maps in the category of  $k$ -algebras:*

$$\begin{aligned} \Delta &:= m^* : k[G] \longrightarrow k[G] \otimes_k k[G] \\ \iota &:= i^* : k[G] \longrightarrow k[G] \\ \varepsilon &:= e^* : k[G] \longrightarrow k \end{aligned}$$

**Definition 8.** A **Hopf-algebra** over  $k$  is a (reduced?!)  $k$ -algebra together with maps  $\Delta, \varepsilon, \iota$  as above s.t. the following holds:

$$\begin{aligned} (\Delta \otimes \text{Id})\Delta &= (\text{Id} \otimes \Delta)\Delta \\ s^* \circ (\iota \otimes \text{Id})\Delta &= s^* \circ (\text{Id} \otimes \iota)\Delta = \varepsilon \\ (\varepsilon \otimes \text{Id})\Delta &= (\text{Id} \otimes \varepsilon)\Delta = \text{Id} \end{aligned}$$

where  $s : G \rightarrow G \times G, g \mapsto (g, g)$  is the diagonal map.

A morphism of Hopf-algebras is a homomorphism of  $k$ -algebra  $F : A \rightarrow B$  s.t.

$$\Delta \circ F = (F \otimes F) \circ \Delta.$$

**Theorem 3.** *The contravariant functor  $_*$  gives an anti-equivalence of the categories of algebraic groups and the categories of finitely generated Hopf-algebras over  $k$ .*

**Example 3.** 1. Let  $G = \mathcal{G}_a = (k, +)$ . Then,  $k[G] = k[x]$ , since  $I(x) = 0$ . Then, we have

$$\begin{aligned}\Delta(x) &= x \otimes 1 + 1 \otimes x \\ \iota(x) &= -x \\ \varepsilon(x) &= 0.\end{aligned}$$

2. Let  $G = \mathcal{G}_m = \{(a, a^{-1}) \mid a \neq 0\} \cong k^\times$ . Then,  $k[G] = k[x, y]/(xy - 1) = k[x, x^{-1}]$ . Then, we have

$$\begin{aligned}\Delta(x) &= x \otimes x \\ \iota(x) &= x^{-1} \\ \varepsilon(x) &= 1.\end{aligned}$$

3. Let  $G = \mathrm{GL}_n(k)$ . Then,  $k[G] = k[x, y]/(xy - 1_n) = k[x_{i,j}, \frac{1}{\det}]$ . Then, we have

$$\begin{aligned}\Delta(x_{i,j}) &= \sum_k x_{i,k} \otimes x_{k,j} \\ \Delta\left(\frac{1}{\det(x)}\right) &= \frac{1}{\det(x)} \otimes \frac{1}{\det(x)} \iota(x_{i,j}) = (x^{-1})_{i,j} \\ \varepsilon(x_{i,j}) &= \delta_{i,j}.\end{aligned}$$

## 2.1 An Aside on the General Group

Let  $G = \mathrm{GL}_n(k) = \{(x, y) \mid xy = \mathrm{Id}_n\}$ . Since we have

$$x^{-1} = \frac{1}{\det(x)} \cdot \mathrm{adj}(x)$$

where the adjoint  $\mathrm{adj}(x)$  can be expressed by polynomials in the entries of  $x$ , we have isomorphisms

$$\begin{aligned}k[x, y]/(xy - 1_n) &\longrightarrow k[x, 1/\det(x)] = k[x, t]/(\det(x) \cdot t = 1) \\ (x, y) &\longmapsto (x, \det(y))\end{aligned}$$

and

$$\begin{aligned}k[x, 1/\det(x)] &\longrightarrow k[x, y]/(xy - 1_n) \\ (x, t) &\longmapsto (x, t \cdot \mathrm{adj}(x)).\end{aligned}$$



**Lemma 6.**

$$k[GL_n(k)] \cong k[x_{i,j}, \frac{1}{\det(x)}].$$

**Lemma 7.** *Let  $V$  be a finite-dimensional  $k$ -vector space. If we choose a basis for  $V$ , we get an isomorphism  $GL(V)$ . Hence,  $GL(V)$  is an algebraic group whose structure is up to a unique isomorphism independent of the choice of basis.*

### 3 Actions

*Remark 1.* Let  $G \curvearrowright M$  be a group action of algebraic sets, then the morphism

$$G \times M \longrightarrow M$$

yields an homomorphism

$$\Delta : k[M] \rightarrow k[G] \otimes k[M].$$

This turns  $k[M]$  to a **comodule** of the Hopf-Algebra  $k[G]$ .

**Definition 9.** Let  $V$  be vector space and  $G$  an algebraic group. A morphism  $r_V : G \rightarrow \mathrm{GL}(V)$  of groups is called **representation** of  $G$ , if there is a linear map

$$\Delta : V \rightarrow V \otimes_k k[G] (= \mathrm{Hom}_{alg}(G, V))$$

s.t. we have for each  $v \in V$  and  $g \in G$

$$r_V(g) \cdot v = \sum_i v_i \cdot f_i(g)$$

where  $\Delta v = \sum_i v_i \otimes f_i$ .

That is,  $V$  is a comodule for  $k[G]$ .

A map  $\phi : V \rightarrow W$  is called **equivariant** for two representations  $r_V, r_W$  of  $G$ , if

$$\phi(r_V(g)v) = r_W(g)\phi(v)$$

for all  $g, v$ .

**Example 4.** Let  $G = \mathrm{GL}_n(k)$ ,  $V = k^n$  and  $r_V$  be the canonical representation. For an orthonormal basis  $(b_i)_{i=1, \dots, n}$ , we for example can set

$$\Delta v = \sum_{i=1}^n b_i \otimes f_i$$

where

$$f_i(A) := b_i^T A v.$$

Then, we have

$$r_V(A) \cdot v = A \cdot v = \sum_{i=1}^n b_i \cdot b_i^T A v = \Delta(v)(A).$$

**Example 5.** Let  $M$  be a right  $G$ -set. Then,  $G$  also acts on  $k[M]$ , therefore we have a map

$$\rho : G \rightarrow \mathrm{GL}(k[M])$$

by, for  $v \in k[M]$ ,

$$(\rho(g)v)(m) := v(m.g).$$

Further, we have an algebra morphism

$$\Delta : k[M] \rightarrow k[M] \otimes k[G] = k[M \times G]$$

with

$$(\Delta v)(m, g) = v(m.g).$$

With  $\Delta v = \sum_i v_i \otimes f_i$

$$\rho(g)v(m) = v(mg) = \Delta v(m, g) = \sum_{i=1} f_i(g)v_i(m).$$

Ergo,  $\rho$  is a representation of  $G$ .

When  $M = G$  with action given by the right translation, then  $\rho : G \rightarrow \mathrm{GL}(k[G])$  is called the **right regular representation** of  $G$ .

**Lemma 8.** Let  $G$  be an algebraic group and  $V$  a finite-dimensional  $k$ -vector space. Then  $\rho : G \rightarrow \mathrm{GL}(V)$  is morphism of algebraic groups iff it is a representation.

**Definition 10.** Let  $G$  be an algebraic group and  $V$  a representation of  $G$ . A subspace  $W \subset V$  is called **invariant** or **subrepresentation**, if we have  $W.G = W$ .

**Lemma 9.** The following are equivalent:

1.  $W$  is invariant.
2.  $\Delta(W) \subseteq W \otimes k[G]$ .

**Lemma 10.** Any representation  $V$  is a filtered union of its finite-dim. subrepresentations:

1. Each  $v \in V$  is contained in some fin.-dim. subrep.
2. Any two finite-dim. subrep. are contained in some bigger fin.-dim. subrep.

**Theorem 4.** Every algebraic group  $G$  is isomorphic to a linear algebraic group.

*Proof.* Let  $\rho : G \rightarrow \mathbf{GL}(k[G])$  be the right regular representation.  $k[G]$  is a finitely-generated  $k$ -algebra. Then, there is a finite-dim. subrepresentation  $V \subseteq k[G]$  s.t.  $V$  generates  $k[G]$  as  $k$ -algebra. Then

$$\phi : G \longrightarrow \mathbf{GL}(V)$$

is morphism of algebraic groups.

Consider the dual map

$$\phi^* : k[\mathbf{GL}(V)] \rightarrow k[G].$$

We need to show that  $\phi^*$  is surjective. It is enough to show that  $V \subset \text{Im} \phi^*$ . Define

$$\begin{aligned} l : V \subset k[G] &\longrightarrow k \\ f &\longmapsto f(e). \end{aligned}$$

Let  $f \in V$  and set  $a(g) := l(g \cdot f)$  for  $g \in \mathbf{GL}(V)$ . Then  $a \in k[\mathbf{GL}(V)]$  is regular. Further,

$$\phi^*(a)(g) = a(\rho(g)) = l(\rho(g)f) = f(eg) = f(g).$$

Therefore,  $f = \phi^*(a) \in \text{Im}(\phi^*)$ . Since  $V$  generates  $k[G]$ , the surjectivity of  $\phi^*$  follows.  $\square$

**Theorem 5.** *Let  $H$  be an algebraic subgroup of an algebraic group  $G$ . There is a finite-dim. representation  $V$  of  $G$  and a line  $L \subset V$  s.t.  $H$  is the stabilizer in  $G$  of  $L$ , i.e.*

$$H = \{g \in G \mid L.g = g\}.$$

*Proof.* Let  $V$  be like in the previous proof. Consider

$$I \hookrightarrow k[G] \twoheadrightarrow k[H].$$

We can now set  $L' := V \cap I$ . We then have for  $g \in G$ .

$$I.g \subseteq I \iff g \in H.$$

Now, in general  $L'$  is not of dimension one. Set  $d = \dim(L')$  and consider the one-dimensional subspace  $L := \Lambda^d(L') \subseteq \Lambda^d(V)$ .  $G$  acts on  $\Lambda^d(V)$  in the natural way.

It is clear, that  $H$  stabilizes  $L$ . For the other direction, let  $g \notin H$  and let  $e_1, \dots, e_n$  be a basis of  $V$  s.t.  $L' = \langle e_1, \dots, e_d \rangle$ . Then,

$$L = \langle e_1 \wedge \dots \wedge e_d \rangle$$

and, since  $g$  does not stabilize  $L'$ , w.l.o.g. we can assume  $e_1.g = e_{d+1}$ . Then, we have  $g(e_1 \wedge \dots \wedge e_d) = g(e_1) \wedge \dots \wedge g(e_d) =: v$ . Now,  $v$  cannot be zero and it cannot lie in  $L$  because  $e_1.g = e_{d+1}$ . Therefore,  $g \notin H$  does not stabilize  $L$ .  $\square$

**Theorem 6.** *Let  $H$  be a normal algebraic subgroup of an algebraic group  $G$ . Then, there is a finite-dimensional  $\rho : G \rightarrow \mathbf{GL}(V)$  s.t.  $H = \ker(\rho)$ .*

*Proof.* Let  $V, L$  and  $\phi : G \rightarrow \mathbf{GL}(V)$  be like in the preceding theorem. Set

$$V_H := \{v \in V \mid H.v \subset \langle v \rangle\}.$$

Then,  $V_H$  is  $G$ -invariant, since

$$h.(g.v) = (hg).v = (gh').v = g.(h'v) = g.(\kappa \cdot v) = \kappa \cdot g.v$$

for all  $g \in G, h \in H, v \in V_H$  and fitting  $h' \in H, \kappa \in k^\times$ . W.l.o.g. we have  $V = V_H$ .  $V$  is not trivial, because  $L \subset V$ .

Let  $\chi$  range through all homomorphism  $H \rightarrow k^\times$ , then we have

$$V = \bigotimes_{\chi} V_{\chi}$$

where

$$V_{\chi} = \{v \in V \mid h.v = \chi(h) \cdot v\}.$$

Then each  $g \in G$  permutes those eigenspaces by

$$g.V_{\chi} = V_{\chi(g^{-1} \cdot g)}.$$

Now, let  $W := \bigoplus_{\chi} \text{End}(V_{\chi}) \subset \text{End}(V)$ . For  $g \in G$  and  $\lambda \in \text{End}(V)$ , define

$$\begin{aligned} \tilde{\gamma} : G &\longrightarrow \mathbf{GL}(\text{End}(V)) \\ g &\longmapsto \tilde{\gamma}(g) : [\lambda \mapsto \phi(g) \circ \lambda \circ \phi(g)^{-1}]. \end{aligned}$$

The action  $\tilde{\gamma}(g)$  stabilizes  $W$ , since each  $\phi(g)$  just permutes the  $V_{\chi}$  and  $\phi(g)^{-1}$  permutes them back. Therefore, we have a subrepresentation

$$\gamma : G \rightarrow \mathbf{GL}(W).$$

We now have to show

$$\ker(\gamma) = H.$$

Since elements of  $H$  don't permute  $V_{\chi}$ , we have  $\gamma(H) = \text{Id}$ .

One the other side, let  $g \in G$  with  $\gamma(g) = \text{Id}$ . Then, we can choose the projection  $\pi : V \twoheadrightarrow L$  in  $W$  and get

$$\phi(g) \circ \pi = \pi \circ \phi(g).$$

Therefore,  $g$  leaves each  $L$  invariant. But now, we have  $g \in H$ . □

## 4 Connected Components

**Lemma 11.** *Let  $I_1, I_2, I_\lambda \subset k[x]$  be ideals, then*

$$\begin{aligned} V(I_1 \cap I_2) &= V(I_1) \cup V(I_2) \\ V\left(\bigcup_{\lambda} I_{\lambda}\right) &= \bigcap_{\lambda} V(I_{\lambda}). \end{aligned}$$

**Definition 11.** A topological space  $X$  is called **connected**, if any of the following equivalent condition holds:

- There is no pair of non-empty closed subsets  $Z_1, Z_2 \subseteq X$ , s.t.  $X = Z_1 \dot{\cup} Z_2$ .
- There is no pair of non-empty open closed subsets  $U_1, U_2 \subseteq X$ , s.t.  $X = U_1 \dot{\cup} U_2$ .
- Each nonempty open subset of  $X$  is dense.

**Definition 12.** A topological space  $X$  is called **irreducibel**, if any of the following equivalent condition holds:

- There is no pair of proper closed subsets  $Z_1, Z_2 \subseteq X$ , s.t.  $X = Z_1 \cup Z_2$ .
- For each pair  $U_1, U_2 \subseteq X$  of non-empty open subsets we have  $U_1 \cap U_2 \neq \emptyset$ .
- Each nonempty open subset of  $X$  is dense.

**Example 6.**  $V(xy)$  is connected but not irreducible.

**Recall:** Last time we introduced the **Zariski-Topology** on  $X$ .

There, algebraic sets equal closed sets.

We called a set  $X$  **irreducible** iff each open subset lies dense in  $X$ .

**Lemma 12.** *For an algebraic set  $X$ , the following are equivalent:*

- (1)  $X$  is irreducible.
- (2)  $k[X] = k[x_1, \dots, x_n]/I(X)$  is an (integral) domain.
- (3)  $I(X)$  is a prime ideal.

The proof of (2)  $\iff$  (3) is a basic algebraic result.

**Lemma 13.** *An open base for the Zariski-Topology on an algebraic set  $X$  is given by sets:*

$$D(f) := \{p \in X \mid f(p) \neq 0\}$$

for each  $f \in k[X]$ . We call the  $D(f)$  **basic open sets**.

*Proof.* Suppose  $U \subseteq X$  is nonempty and open. Set

$$Z := X \setminus U$$

then  $Z$  is closed. Thus

$$Z = \{x \in X \mid f(x) = 0 \ \forall f \in I\} = V(I)$$

for some ideal  $I \subseteq k[X]$ . Let  $p \in U$ , then there is an  $f \in I$  s.t.

$$f(p) \neq 0.$$

Also,  $D(f) \cap Z = \emptyset$ , thus  $p \in D(f) \subseteq U$ . □

*Proof: Lemma 1.* It is clear that (2) is equivalent to (3).

(1) is equivalent to

$$\forall \text{ nonempty, open } U_1, U_2 \subset X : U_1 \cap U_2 \neq \emptyset$$

$$\stackrel{\text{Lemma 2}}{\iff} \forall \text{ nonempty, basic open } D(f_1), D(f_2) \subset X : D(f_1) \cap D(f_2) \neq \emptyset$$

Since  $D(f_1) \cap D(f_2) = D(f_1 f_2)$ , this is equivalent to the statement

$$\forall f_1, f_2 \in k[X] : f_1, f_2 \neq 0 \implies f_1 f_2 \neq 0.$$

Which states that  $k[X]$  is a domain. □

**Lemma 14.** *Let  $X$  be an algebraic set. We have bijections*

$$\{\text{closed subsets } Z \subseteq X\} \leftrightarrow \{\text{radical ideals } I \subset k[X]\}$$

*and*

$$\{\text{irreducible, closed subsets } Z \subseteq X\} \leftrightarrow \{\text{prime ideals } I \subset k[X]\}$$

*and*

$$\{\text{points of } X\} \leftrightarrow \{\text{maximum ideals } I \subset k[X]\}.$$

**Lemma 15** (Primary Decompositions, Atiyah, Macdonald Ch. 4). *For an ideal  $I$  we call  $P \supseteq I$  a **minimal prime** of  $I$  if  $P$  is a prime ideal and we have for each prime ideal  $Q$ :*

$$P \supseteq Q \supseteq I \implies P = Q.$$

*Any radical ideal  $I$  of  $k[x_1, \dots, x_n]$  has only finitely many **minimal** primes  $P_1, \dots, P_r$ . In particular,*

$$I = \bigcap_{i=1}^r P_i$$

*and for each  $i$*

$$P_i \not\supseteq \bigcap_{j:j \neq i} P_j.$$

**Definition 13.** An **(irreducible) component**  $Z$  of  $X$  is a maximal irreducible closed subset, i.e., an irreducible closed  $Z \subseteq X$  s.t. there does not exist an irreducible closed  $Y \subset X$  s.t.  $Y \supsetneq Z$ .

Then, we have the bijection

$$\{\text{irreducible components of } X\} \leftrightarrow \{\text{minimal primes of } I(X)\}.$$

**Lemma 16.** *Any algebraic set  $X$  has finitely many irreducible components  $Z_1, \dots, Z_r$ . We have*

$$X = Z_1 \cup \dots \cup Z_r$$

*and for each  $i$*

$$Z_i \not\subset \bigcup_{j:j \neq i} Z_j.$$

**Example 7.** 1. Let  $X = V(x \cdot y) \subset k^2$ . Then  $X = Z_1 \cup Z_2$  where  $Z_1 = V(x)$ ,  $Z_2 = V(y)$ .

$X$  is connected, but not irreducible ( $D(x)$  does not lie dense in  $X$ ).



2. Let  $X$  be a **finite** algebraic set. It is easy to check that every subset of  $X$  is closed:

$$\{p\} = V(x_1 - p_1, \dots, x_n - p_n)$$

for each  $p \in X$ . Further

$$X = \{p_1\} \cup \dots \cup \{p_r\}.$$

Moreover: Any function  $f : X \rightarrow k$  is regular (i.e. given by polynomials).

**Lemma 17.** *We call an element  $e \in k[X]$  **idempotent** iff  $e^2 = e$ .*

*Let  $X$  be an algebraic set. Then*

$$\begin{aligned} X \text{ connected} &\iff \text{the only idempotents } e \in k[X] \text{ are } 0 \text{ and } 1 \\ &\iff k[X] \not\cong A \times B \text{ for any } k\text{-algebras } A, B. \end{aligned}$$

**Lemma 18.** *Morphisms of algebraic sets are continuous.*

*Proof.* Let  $\phi : X \rightarrow Y$  be a morphism. It suffices to show that for all closed  $Z \subset Y$  that  $\phi^{-1}(Z) \subset X$  is closed.

But, if

$$Z = V_Y(S) := \{q \in Y \mid f(q) = 0 \forall f \in S\}$$

for some ideal  $S \subset k[Y]$ , then

$$\phi^{-1}(Z) = V_X(\phi^*(S)) = \{\phi^*(f) = f \circ \phi \mid f \in S\}.$$

□

**Lemma 19.** *Isomorphisms of algebraic sets are homeomorphisms. In particular, any isomorphism of algebraic sets  $\phi : X \rightarrow X$  permutes the irreducible components  $Z_1, \dots, Z_r$  of  $X$ :*

$$\forall i \exists j : \phi(Z_i) = Z_j.$$

**Theorem 7.** *Let  $G$  be an algebraic group.*

- (i) *There is a unique irreducible component  $G^0$  of  $G$  with  $e \in G^0$ .*
- (ii) *Every irreducible component  $Z$  of  $G$  is a coset  $gG^0$  of  $G$  for some  $g \in Z$ .*
- (iii)  *$G^0$  is a normal algebraic subgroup of  $G$ .*
- (iv)  *$G^0$  is of finite index, i.e.*

$$[G : G^0] = \#(G/G^0) < \infty.$$

(v) *The irreducible components are also the connected components.*

*Proof.* Let  $G = Z_1 \cup \dots \cup Z_r$  be the decomposition into components. We may assume that  $e \in Z_1$ .

Recall that  $Z_1 \not\subset \bigcup_{j \geq 2} Z_j$ . Then, there is an  $x \in Z_1 \setminus \bigcup_{j \geq 2} Z_j$ . Thus, for all algebraic set isomorphisms  $\phi : G \rightarrow G$ , we have by some previous lemma that  $\phi(x)$  is likewise contained in some unique component of  $G$ . For example, we may take  $\phi$  to be

$$\begin{aligned}\phi_g : G &\rightarrow G \\ y &\mapsto gy\end{aligned}$$

for any  $g \in G$ . Then, for all  $g \in G$ , the element  $gx = \phi_g(x)$  is contained in only one component of  $G$ . Ergo, each  $g \in G$  is contained in exactly one component.

(i) Take  $g = e$ .

(iii)  $G^0$  is an algebraic subset, by construction. Denote by  $m : G \times G \rightarrow G$  and  $i : G \rightarrow G$  the continuous multiplication and inversion map on  $G$ . **Why is  $G^0$  a subgroup?** We need to show

$$\begin{aligned}m(G^0 \times G^0) &\subseteq G^0. \\ i(G^0) &\subseteq G^0.\end{aligned}$$

We know that  $i(G^0)$  is some component of  $G$ , since  $i$  is an isomorphism. But it contains the identity  $e$ , since  $e^{-1} = e$ . Therefore,  $i(G^0) = G^0$ .

If  $g \in G$ , then  $gG^0$  is some component of  $G$ . Suppose  $g \in G^0$ . Then  $gG^0 \cap G^0 \supseteq \{g\}$ , therefore  $gG^0 = G^0$ . Ergo,  $G^0$  is closed under multiplication.

**Why is  $G^0$  a normal?** If  $g \in G$ , then  $gG^0g^{-1}$  is a component that contains  $e$ , therefore  $G^0 = gG^0g^{-1}$ .

(Alternative proof that  $m(G^0 \times G^0) = G^0$ : Consider

- any continuous image of an irreducible set is irreducible.
- the closure of any irreducible set is irreducible.

Ergo  $\overline{m(G^0 \times G^0)}$  is a closed irreducible set containing  $e$ . Ergo,  $\overline{m(G^0 \times G^0)} = G^0$ .

(ii) Let  $Z \subset G$  be a component. Let  $g \in Z$ . Then  $g \in (gG^0 \cap Z)$ , so  $gG^0 = Z$ .

- (iv) This follows from some previous lemma.
- (v) This is left as a topological exercise. It is true whenever the irreducible components do not intersect.

□

It now follows:

$$\{\text{finite algebraic groups}\} \longleftrightarrow \{\text{finite groups}\}$$

where the above arrow is an equivalence of categories.

**Example 8.** • Let  $G = \{g_1, \dots, g_r\}$  be a finite algebraic group. Then,

$$G^0 = \{e\}.$$

- Without proofs:

$$G \in \{\mathrm{GL}_n(k), \mathrm{SO}_n(k), \mathrm{SL}_n(k)\} \implies G^0 = G.$$

Further,

$$G = \mathrm{O}_n(k) \implies G^0 = \mathrm{SO}_n(k).$$

And if  $-1 = 1$  i.e.  $\mathrm{char} k = 2$ , then  $[G : G^0] = 1$ . Otherwise  $[G : G^0] = 2$ .

## 5 Jordan Decomposition

As usual,  $k = \bar{k}$  is an algebraically closed field.

**Definition 14.** Let  $V$  be a finite-dimensional vector space.

An element  $x \in \text{End}(V)$  is **semisimple**, if it is diagonalizable, i.e. it has a basis of eigenvectors, or equivalently, if the minimal polynomial of  $x$  is square-free.

Then, there is a decomposition  $V = \bigoplus_{i=1}^r V_i$  and distinct elements  $\lambda_1, \dots, \lambda_n \in k$  s.t.

$$x|_{V_i} = \lambda_i.$$

If  $\dim(V_i) = n_i$ , then

$$\text{char polynomial of } x = \prod_{i=1}^r (T_i - \lambda_i)^{n_i} \in k[T]$$

and

$$\text{minimal polynomial of } x = \prod_{i=1}^r (T_i - \lambda_i) \in k[T].$$

(Where the minimal polynomial of  $x$  is defined as the least degree monic polynomial  $m \in k[T]$  s.t.  $m(x) = 0$ .)

*Remark 2.* Let  $m(T) \in k[T]$  be the minimal polynomial of  $x \in k^{n \times n}$ .

The theorem of Cayley and Hamilton states that we have for each  $p \in k[T]$ :

$$p(x) = 0 \implies m|p.$$

**Definition 15.**  $x \in \text{End}(V)$  is **nilpotent** if  $x^n = 0$  for some  $n$ .

$x$  is **unipotent**, if  $x - 1$  is nilpotent.

**Lemma 20.**  $x$  is nilpotent iff the characteristic polynomial of  $x$  is  $T^{\dim(V)}$ . (Use Cayley-Hamilton for one of the directions).

**Lemma 21.** If  $x$  is semisimple and nilpotent, then  $x = 0$ .

If  $x$  is semisimple and unipotent, then  $x = 1$ .

**Lemma 22.** If  $x, y$  are commuting elements, that are semisimple resp. unipotent resp. nilpotent, then so is  $xy$ .

*Proof.* It is easy to see, that this is true for nilpotent  $x, y$ .

Now, let  $x, y$  be unipotent and commuting. Then, we have

$$xy - 1 = (x + 1)(y - 1) + (x - y).$$

Since  $x, y$  commute,  $(x+1)(y-1)$  must be nilpotent.  $(x-y)$  must be nilpotent because the sum of commuting nilpotent elements must be nilpotent. Because everything commutes, also  $xy - 1$  as the sum of two commuting, nilpotent elements must be nilpotent.

Now, let  $A, B \in k^{n \times n}$  be two diagonalizable and commuting matrices. Let  $\lambda_1, \dots, \lambda_r$  be different eigenvalues of  $A$  and let  $E_i$  be the corresponding eigenspaces. We then have

$$A \cdot (BE_i) = BAE_i = \lambda_i \cdot BE_i.$$

Ergo, each  $E_i$  is invariant under  $B$ . Since  $B|_{E_i}$  stays diagonalizable, we can simply choose a basis of eigenvectors  $b_1, \dots, b_n \in \bigcup_i E_i$  of  $B$ . Since each  $b_i$  lies in a  $E_j$ , those vectors are also eigenvectors for  $A$ . Therefore,  $b_1, \dots, b_n$  is basis of eigenvectors for both matrices.  $\square$

**Theorem 8** (Goal). *For all algebraic groups  $G$  and for all  $g \in G$ , there exist unique group elements  $g_s, g_u \in G$  s.t.*

$$g = g_s g_u = g_u g_s$$

*and for all finite-dimensional representations  $\rho : G \rightarrow GL(V)$ ,  $\rho(g_s)$  is semisimple and  $\rho(g_u)$  is unipotent.*

**Example 9.** If  $g = \begin{pmatrix} \lambda & 1 & 0 \\ & \lambda & 1 \\ & & \lambda \end{pmatrix} \in G = GL_3(k)$ , then  $g_s = \begin{pmatrix} \lambda & 0 & 0 \\ & \lambda & 0 \\ & & \lambda \end{pmatrix}$ ,  $g_u = \begin{pmatrix} 1 & \lambda^{-1} & 0 \\ & 1 & \lambda^{-1} \\ & & 1 \end{pmatrix}$ .

**Theorem 9** (Goal Theorem). *Let  $G$  an algebraic group. For all  $g \in G$  there is exactly one pair  $g_s, g_u \in G$  s.t.*

$$g = g_s g_u = g_u g_s$$

*and for all finite-dimensional representations  $r : G \rightarrow GL_n(V)$ , the element  $r(g_s)$  resp.  $r(g_u)$  is semisimple resp. unipotent.*

Last time, we saw:

**Lemma 23.** • *If  $g, h$  are commuting and semisimple resp. commuting and unipotent then so is  $gh$ .*

• *If  $g$  is semisimple and unipotent, then  $g = 1$ .*

**Proposition 1.** *Let  $V$  be a finite-dimensional vector space and  $g \in GL(V)$ . There exist unique elements  $g_s, g_u \in GL(V)$  s.t.*

$$g = g_s g_u = g_u g_s$$

*and  $g_s$  is semisimple and  $g_u$  is unipotent.*

*Moreover,  $g_s, g_u \in k[g] = \{\sum_{i=1}^m a_i g^i \mid a_i \in k\} \subseteq \text{End}(V)$ .*

*Proof. Existence* (Sketch): Say

$$g = \begin{pmatrix} \lambda & 1 & 0 \\ & \lambda & 1 \\ & & \lambda \end{pmatrix}$$

then take

$$g_s = \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{pmatrix}, \quad g_u = \begin{pmatrix} 1 & \lambda^{-1} & 0 \\ & 1 & \lambda^{-1} \\ & & 1 \end{pmatrix}.$$

For  $\lambda \in k$ , define the **generalized  $\lambda$ -eigenspace** of  $g$  by

$$V_\lambda := \{v \in V \mid \exists n \in \mathbb{N}_0 : (g - \lambda)^n v = 0\}.$$

Then

$$V = \bigoplus_{\lambda \in k} V_\lambda.$$

Here  $V_\lambda$  = sum of domains of all Jordan blocks with  $\lambda$ s on the diagonal. (It follows from the Jordan Decomposition for matrices that such a decomposition exist.)

Let's define  $g_s \in \text{GL}(V)$  by

$$g_s|_{V_\lambda} = \lambda \cdot \text{Id}.$$

Note that  $gV_\lambda \subset V_\lambda$ , hence  $g$  commutes with  $g_s$ , hence  $g, g_s$  commutes with  $g_u := gg_s^{-1}$ . Then,  $g = g_s g_u = g_u g_s$ .

Write  $\det(T - g) = \prod_\lambda (T - \lambda)^{n(\lambda)}$ ,  $n(\lambda) = \dim(V_\lambda)$ . Since the polynomials  $T - \lambda$  for  $\lambda \in k$  are coprime, the chinese remainder theorem implies that there is a  $Q \in k[T]$  s.t.

$$Q \equiv \lambda \pmod{(T - \lambda)^{n(\lambda)}}$$

for each  $\lambda \in k$ .

We claim that

$$Q(g) = g_s.$$

Indeed, since  $gV_\lambda \subseteq V_\lambda$ , we have

$$Q(g)V_\lambda \subseteq V_\lambda.$$

So, it suffices to show for all  $v \in V_\lambda$

$$Q(g)v = g_s v = \lambda v.$$

Note that, by Cayley-Hamilton,

$$V_\lambda = \{v \in V \mid (g - \lambda)^{n(\lambda)} v = 0.\}$$

Write

$$Q = \lambda + R \cdot (T - \lambda)^{n(\lambda)}$$

for some  $R \in k[T]$ . Since  $(g - \lambda)^{n(\lambda)} v = 0$ , deduce that  $Q(g)v = \lambda v$ , as required.

If  $P \equiv \lambda^{-1} \pmod{(T - \lambda)^{n(\lambda)}}$ , then  $P(g) = g_s^{-1}$ .

Therefore,

$$g_u = g \cdot P(g)$$

for  $T \cdot P(T) \in k[T]$ .

**Uniqueness:** Suppose given some other decomposition

$$g = g'_s g'_u = g'_u g'_s$$

with  $g'_s$  semisimple and  $g'_u$  unipotent. Then  $g'_s$  commutes with  $g'_s$  and  $g'_u$ , hence with  $g$ , hence also with any element in  $k[g]$ . Ergo,  $g'_s$  commutes with  $g_s$  and  $g_u$ . Similarly,  $g'_u$  commutes with  $g_s$  and  $g_u$ .

Consider

$$h := g'_s g_s^{-1} = g'_s g'_u (g'_u)^{-1} g_s^{-1} = g(g'_u)^{-1} g_s^{-1} = g_u (g'_u)^{-1}.$$

Then  $h = g'_s g_s^{-1}$  is a product of semisimple elements and  $h = g_u (g'_u)^{-1}$  is a product of unipotent elements. By proceeding lemmas,  $h$  is semisimple and unipotent, ergo trivial. It follows  $g'_s = g_s$  and  $g'_u = g_u$ .  $\square$

**Corollary 2.** *Let  $g \in GL(V)$ , let  $W \subset V$  be any  $g$ -invariant subspace, i.e.  $gW \subseteq W$ .*

*Then,  $W$  is  $g_s$ -invariant and  $g_u$ -invariant.*

*Proof.* This is clear, since  $g_s$  and  $g_u$  are algebraically generated by  $g$  over  $g$ .  $\square$

**Lemma 24.** *Let  $\phi : V \rightarrow W$  be a linear map between finite-dimensional vector spaces.*

*Let  $\alpha \in GL(W)$  and  $\beta \in GL(W)$  s.t.*

$$\begin{array}{ccc} V & \xrightarrow{\alpha} & V \\ \downarrow \phi & & \downarrow \phi \\ W & \xrightarrow{\beta} & W, \end{array}$$

*i.e.  $\phi \circ \alpha = \beta \circ \phi$ .*

*Then,*

$$\begin{aligned} \phi \circ \alpha_s &= \beta_s \circ \phi, \\ \phi \circ \alpha_u &= \beta_u \circ \phi. \end{aligned}$$

*Proof.* Write  $V = \bigoplus_{\lambda \in k} V_\lambda$ ,  $W = \bigoplus_{\lambda \in k} W_\lambda$  where  $V_\lambda$  are the generalized  $\alpha$ -eigenspaces and  $W_\lambda$  are the generalized  $\beta$ -eigenspaces.

We claim that

$$\phi(V_\lambda) \subset W_\lambda.$$

Indeed, let  $v \in V_\lambda$ , then

$$(\beta - \lambda)^n \phi(v) = \phi((\alpha - \lambda)^n v) = 0.$$

Since  $(\alpha - \lambda)^n v = 0$ , the claim follows.

Since,  $\alpha_s|_{V_\lambda} = \lambda \text{Id}$  and  $\beta_s|_{W_\lambda} = \lambda \text{Id}$ , deduce that

$$\phi \circ \alpha_s = \beta_s \circ \phi.$$



Indeed, both sides are given on  $V_\lambda$  by  $\lambda \cdot \phi$ . Thus

$$\begin{aligned}\phi \circ \alpha_u &= \phi \circ \alpha \alpha_s^{-1} \\ &= \beta \beta_s^{-1} \circ \phi \\ &= \beta_u \circ \phi.\end{aligned}$$

□

**Lemma 25.** *Let  $\alpha \in GL(V)$ ,  $\beta \in GL(W)$ . Then the **tensor**  $\alpha \otimes \beta \in GL(V \otimes W)$  is defined by*

$$(\alpha \otimes \beta)(u \otimes v) = \alpha(u) \otimes \beta(v).$$

Then, we have

$$\begin{aligned}(\alpha \otimes \beta)_s &\stackrel{(1)}{=} \alpha_s \otimes \beta_s \\ (\alpha \otimes \beta)_u &\stackrel{(2)}{=} \alpha_u \otimes \beta_u.\end{aligned}$$

*Proof.* It suffices to prove (1), since

$$\begin{aligned}(\alpha \otimes \beta)_u &= (\alpha \otimes \beta) \circ (\alpha \otimes \beta)_s^{-1} \\ &\stackrel{(1)}{=} (\alpha \otimes \beta) \circ (\alpha_s \otimes \beta_s)^{-1} \\ &= \alpha \alpha_s^{-1} \otimes \beta \beta_s^{-1} \\ &= \alpha_u^{-1} \otimes \beta_u^{-1}\end{aligned}$$

For (1), consider

$$\begin{aligned}V &= \bigoplus_{\lambda \in k} V_\lambda, \\ W &= \bigoplus_{\lambda \in k} W_\lambda.\end{aligned}$$

It follows

$$V \otimes W = \bigoplus_{\lambda, \mu \in k} V_\lambda \otimes W_\mu.$$

Now,

$$\alpha_s \otimes \beta_s|_{V_\lambda \otimes W_\mu} = \lambda \mu \cdot \text{Id}.$$

Ergo,  $\alpha_s \otimes \beta_s$  is semisimple. By Proposition, we reduce to checking that  $\alpha_u \otimes \beta_u$  is unipotent. Indeed,

$$\alpha_u \otimes \beta_u - 1 = (\alpha_u - 1) \otimes (\beta_u - 1) + 1 \otimes (\beta_u - 1) + (\alpha_u - 1) \otimes 1$$

is nilpotent (You can also check that  $\alpha_u \otimes \beta_u = (\alpha_u \otimes 1) \circ (1 \otimes \beta_u)$  is unipotent.) □

**Example 10.** Let  $1 \in GL(V)$ . Then  $1_s = 1$  and  $1_u = 1$ .

**Summary** : Let  $G$  be an algebraic group. Let  $r_V : G \rightarrow \mathrm{GL}(V)$  be a finite-dimensional representation. Also, fix  $g \in G$ .

Let  $\lambda_V := r_V(g)_s$  (or  $r_V(g)_u$ ).

We get a family of operators  $\lambda_V \in \mathrm{End}(V)$  with the following properties:

- (i) if  $V = k$  and  $r_V(g') = 1$  for all  $g' \in G$ , then  $\lambda_V = 1$ .
- (ii) for any two representations in  $V$  and  $W$ , we have

$$\lambda_{V \otimes W} = \lambda_V \otimes \lambda_W.$$

- (iii) for all  $G$ -equivariant  $\phi : V \rightarrow W$  we have

$$\phi \circ \lambda_V = \lambda_W \circ \phi.$$

**Theorem 10.** *Let  $G$  be an algebraic group. Let  $\lambda_V \in \mathrm{End}(V)$  (i.e.  $V = (r_V, V)$  is a finite-dim. representation of  $G$ ) be a family of operations satisfying (i), (ii), (iii).*

*Then, there is exactly one  $g \in G$  s.t.  $\lambda_V = r_V(g)$  for all  $V$ .*

Note, that this theorem implies our goal theorem.

Applying the theorem to  $\lambda_V = r_V(g)_s$  implies

$$\exists_1 g_s \in G : r_V(g_s) = r_V(g)_s$$

and

$$\exists_1 g_u \in G : r_V(g_u) = r_V(g)_u.$$

*Proof of Goal Theorem.* There exist unique  $g_s, g_u \in G$  s.t.

$$g \stackrel{(*)}{=} g_u g_s = g_s g_u,$$

Then,  $r_V(g) = r_V(g_s)r_V(g_u) = r_V(g_u)r_V(g_s)$ .

Since  $r_V(g_u)$  is unipotent and  $r_V(g_s)$  is semisimple, it follows  $r_V(g_u) = r_V(g)_u$  and  $r_V(g_s) = r_V(g)_s$ .

To deduce  $(*)$ , take any  $r_V : G \hookrightarrow \mathrm{GL}(V)$ . We know for each  $V$

$$r_V(g) = r_V(g_s)r_V(g_u) = r_V(g_u)r_V(g_s).$$

□

*Proof of Theorem 10.* We first extend the assignment

$$V \mapsto \lambda_V$$

to all representations of  $G$ .

Say  $V = \bigcup_j W_j$  where each  $W_j$  is a finite-dimensional  $G$ -invariant subspace. Try to define  $\lambda_V \in \text{End}(V)$  by

$$\lambda_V|_{W_j} := \lambda_{W_j}.$$

For this to be well-defined, we need to show for each  $i, j$

$$\lambda_{W_i}|_{W_i \cap W_j} \stackrel{(*)}{=} \lambda_{W_j}|_{W_i \cap W_j}.$$

**Proof of (\*):** Apply assumption (iii) to the  $G$ -equivariant linear maps

$$W_i \cap W_j \xrightarrow{\phi} W_i,$$

$$W_i \cap W_j \xrightarrow{\phi'} W_j.$$

Then,

$$\begin{aligned} \lambda_{W_i}|_{W_i \cap W_j} &= \lambda_{W_i} \circ \phi \\ &\stackrel{(iii)}{=} \phi \circ \lambda_{W_i \cap W_j} \\ &= \phi' \circ \lambda_{W_i \cap W_j} \end{aligned}$$

and

$$\lambda_{W_j}|_{W_i \cap W_j} = \lambda_{W_j} \circ \phi' = \phi' \circ \lambda_{W_i \cap W_j}.$$

Recall here that any finite-dimensional  $G$ -invariant  $W \subset V$  is a representation.  $\square$

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<sup>0</sup>Not necessarily finite-dimensional, but may be written as a filtered union of finite-dimensional  $G$ -invariant subspaces of  $W$ .

Let  $G$  be an algebraic group.

**Easy Exercise** : If  $V_1, V_2$  are representations  $r_1, r_2$  of  $G$ , then  $V_1 \otimes V_2$  is also a representation with

$$r = r_1 \otimes r_2 : G \rightarrow \mathbf{GL}(V_1 \otimes V_2)$$

given by

$$r(g)(v_1 \otimes v_2) = (r_1(g)v_1) \otimes (r_2(g)v_2).$$

*Proof.* Given  $\Delta_j : V_j \rightarrow V_j \otimes k[G]$ , define

$$\Delta : V_1 \otimes V_2 \longrightarrow V_1 \otimes V_2 \otimes k[G]$$

by: if

$$\Delta_1 u = \sum_i u_i \otimes f_i, \quad \Delta_2 v = \sum_j v_j \otimes h_j,$$

then

$$\Delta(u \otimes v) = \sum_i \sum_j u_i \otimes v_j \otimes f_i h_j.$$

Set  $A := k[G]$ , then

$$r_A := \text{right regular representation with } r_A(g)f(x) = f(xg).$$

The map

$$\begin{aligned} A \otimes A &\xrightarrow{m} A \\ f_1 \otimes f_2 &\longmapsto f_1 f_2 \end{aligned}$$

defines a morphism of representations

$$(A, r_A) \otimes (A, r_A) \rightarrow (A, r_A).$$

Indeed,

$$\begin{aligned} m((r_A \otimes r_A)(g)(f_1 \otimes f_2))(x) &= f_1(xg)f_2(xg), \\ &= f_1 f_2(xg) = r_A(g)(m(f_1 \otimes f_2))(x), \end{aligned}$$

since  $f_1(\_g) \otimes f_2(\_g) = (r_A \otimes r_A)(g)(f_1 \otimes f_2)$ .

Ergo  $m \circ (r_A \otimes r_A)(g) = r_A(g) \circ m$ . □

Recall: We stated to prove the following theorem

**Theorem 11.** *Let  $\lambda_V \in \text{End}(V)$  be given s.t. for all finite-dim. rep.s  $V$  of  $G$  s.t.:*

$$(i) \lambda_k = 1$$

$$(ii) \lambda_{V \otimes W} = \lambda_V \otimes \lambda_W$$

(iii) *for all morphisms of rep.s  $\phi : V \rightarrow W$  we have*

$$\phi \circ \lambda_V = \lambda_W \circ \phi.$$

*Then, there is exactly one  $g \in G$  s.t.  $\lambda_V = r_V(g)$  for all  $V$ .*

*Proof.* Last time, we saw that any such family  $V \mapsto \lambda_V$  extends to **all** rep.s  $V$  of  $G$ .

Let's note also that, if  $(V_0, r_0)$  is any representation of  $G$  with trivial action, i.e.  $r(g) = 1$  for all  $g$ , then  $\lambda_{V_0} = 1$ . Indeed, let  $v \in V_0$ . We must check that  $\lambda_{V_0} v = v$ . Since the action is trivial, any subspace of  $V_0$  is  $G$ -invariant.

Consider the map

$$\begin{aligned} \phi : k &\longrightarrow V_0 \\ \alpha &\longmapsto \alpha v \end{aligned}$$

where  $v = \phi(1)$ . Then,  $\phi$  is a morphism of rep.s because the action is trivial.

Thus,

$$\lambda_V v = (\lambda_V \circ \phi)(1) \stackrel{(iii)}{=} (\phi \circ \lambda_k)(1) \stackrel{(i)}{=} \phi(1) = v.$$

Consider  $\lambda_A \in \text{End}(A)$ . Then,

$$\lambda_{A \otimes A} = \lambda_A \otimes \lambda_A.$$

It is an easy exercise to see that  $m : (A, r_A) \otimes (A, r_A) \rightarrow (A, r_A)$  is a morphism of rep.s.

By (iii) it follows,  $m \circ (\lambda_A \otimes \lambda_A) = \lambda_A \circ m$ , i.e.

$$\lambda_A(f_1 f_2) = \lambda_A(f_1) \lambda_A(f_2)$$

for all  $f_1, f_2 \in A$ . Thus,  $\lambda_A$  is an algebra morphism (check, using the morphism  $k \hookrightarrow A$ , that  $\lambda_A(1) = 1$ ).

Thus,  $\lambda_A = \phi^*$  for some unique morphism  $\phi$  of algebraic sets  $\phi : G \rightarrow G$ .

We claim that  $\phi$  commutes left multiplication i.e.

$$\phi(hx) = h\phi(x)$$

for all  $h, x \in G$ . Indeed, let's consider the map

$$\begin{aligned} A &\longrightarrow A \\ f &\longmapsto f(h \cdot \_). \end{aligned}$$

This induces a morphism

$$(A, r_A) \xrightarrow{\psi} (A, r_A).$$

By (ii),  $\psi \circ \lambda_A = \lambda_A \circ \psi$ .

Since  $\lambda_A = \phi^*$ , this implies the claim.

Now, set  $g := \phi(e)$ . Then for all  $h \in G$ ,

$$\phi(h) = \phi(h e) = h g.$$

Thus,  $\lambda_A = \phi^* = r_A(g)$ .

(It remains to show that

$$\lambda_V = r_V(g)$$

for each finite-dim. rep.  $V$ .)

Let  $V = (V, r)$  be any rep. This induces a map

$$\Delta : V \longrightarrow V \otimes A.$$

If  $\Delta v = \sum v_i \otimes f_i$ , then

$$h v = \sum f_i(h_i) \otimes v_i.$$

Let

$$\begin{aligned} \varepsilon : V \otimes A &\longrightarrow V \\ v \otimes f &\longmapsto f(1)v. \end{aligned}$$

It follows  $\varepsilon \circ \Delta : V \rightarrow V$  is the identity map.

Let  $(V_0, r_0)$  be the representation of  $G$  with  $V_0 := V$  and  $r_0$  the trivial action. Then,  $\Delta : V \rightarrow V_0 \otimes A$  is a morphism of representations.

(Indeed, if  $\Delta v = \sum v_i \otimes f_i$ , then

$$\Delta(r(h)v) \stackrel{?}{=} (r_0(h) \otimes r_A(h))\Delta v$$

since

$$\begin{aligned}
\Delta v &= \sum v_i \otimes f_i \\
\iff xv &= \sum f_i(x_i)v_i \quad \forall x \in G \\
\iff xhv &= \sum f_i(xh)v_i \quad \forall x, h \in G.
\end{aligned}$$

Since  $r(h)v = hv$ , it follows

$$\Delta(hv) = \sum v_i \otimes f_i(\cdot h) \implies (?).$$

We want to show

$$\lambda_V = r_V(g).$$

We have

$$\begin{aligned}
\Delta \circ \lambda_V &\stackrel{(iii)}{=} \lambda_{V_0 \otimes A} \circ \Delta \\
&\stackrel{(ii)}{=} \lambda_{V_0} \otimes \lambda_A \\
&= 1 \otimes \lambda_A = 1 \otimes r_A(g).
\end{aligned}$$

This implies

$$\Delta \circ \lambda_V = (1 \otimes r_A(g)) \circ \Delta$$

but also

$$\Delta \circ r_V(g) = (1 \otimes r_A(g)) \circ \Delta.$$

Because of the injectivity of  $\Delta$  it now follows

$$\lambda_V = r_V(g).$$

□

**Corollary 3.** *Let  $\phi : G \rightarrow H$  be any morphism of algebraic groups. Then, for all  $g \in G$*

$$\begin{aligned}
\phi(g)_s &= \phi(g_s) \\
\phi(g)_u &= \phi(g_u).
\end{aligned}$$

*Proof.* Let  $V$  be any **faithful** representation of  $H$ , i.e.  $r_V : H \rightarrow \text{GL}(V)$  is injective, (for a finite-dim.  $V$ ).

Then,  $r_V \circ \phi$  is a rep. of  $G$ . To prove (i), it suffices to show

$$r_V(\phi(g)_s) = r_V(\phi(g_s))$$

since  $H$  operates faithfully on  $V$ .

We know that

$$r_V(\phi(g)_s) = r_V(\phi(g))_s$$

(characterizing property of  $h_s$  for  $h \in H$ ). On the other hand,

$$r_V(\phi(g_s)) = (r_V \circ \phi)(g_s) = r_V(\phi(g))_s.$$

Therefore, claim (i) follows. (ii) works analogously.  $\square$

**Definition 16.** Let  $g \in G$  where  $G$  is an algebraic group. We call  $g$  **semisimple**, if  $g = g_s$ .

We call  $g$  **unipotent**, if  $g = g_u$ .

**Lemma 26.** For  $g \in G$ , the following are equivalent:

- (i)  $g$  is semisimple.
- (ii)  $r_V(g)$  is semisimple for all finite-dim. rep.  $V$ .
- (iii)  $r_V(g)$  is semisimple for at least one faithful f.d. rep.  $V$  of  $G$ .

We get an analogous lemma for unipotent group elements.

*Proof.* We have

$$\begin{aligned}
(i) &\iff g = g_s \\
&\stackrel{\text{Def. of } g_s \text{ by goal thm.}}{\iff} r_V(g) = r_V(g)_s \forall \text{ f.d. } V \\
&\iff r_V(g) \text{ is semisimple} \\
&\iff (ii) \implies (iii).
\end{aligned}$$

On the other hand,

$$(iii) \implies \exists \text{ faithful f.d. } V \text{ s.t. } r_V(g) = r_V(g)_s = r_V(g_s) \implies g = g_s.$$

$\square$



## 6 Non-Commutative Algebra

**Definition 17.** A ring  $R$  (for now) is unital, associative but not necessarily commutative.

**Example 11.** The ring of matrices over some field or ring.

**Definition 18.** A **left ideal**  $I \subset R$  is a subset that is an abelian subgroup of  $(R, +)$  s.t.  $ra \in I$  for all  $r \in R, a \in I$ .

A **right ideal**  $I \subset R$  is a subset that is an abelian subgroup with

$$IR \subset I.$$

A two-sided ideal  $I$  is a subset that is a left and a right ideal of  $R$ .

It is easy to check that for any homomorphism of rings  $\phi : R \rightarrow S$ ,  $\text{Kern}\phi$  is a two-sided ideal. Also, if  $J \subset R$  is any two-sided ideal, then there exists a unique ring structure on  $R/J$  s.t. the projection  $R \rightarrow R/J$  is a ring homomorphism.

**Definition 19.** A **left module**  $M$  for  $R$  is an abelian group equipped with a ring homomorphism

$$R \xrightarrow{\alpha} \text{End}(M)$$

where  $\text{End}(M)$  acts on the left of  $M$ . We write

$$rm := \alpha(r)m.$$

We have

$$(r_1 r_2)(m) = r_1(r_2(m)).$$

If  $R$  acts on  $M$  by the left, we write

$$R \curvearrowright M.$$

**Example 12.**  $M_n(k) \curvearrowright k^n$  where  $k^n$  is the space of column vectors.

If  $k^n$  denotes the space of row vectors, we have  $k^n \curvearrowleft M_n(k)$ .

**Definition 20.** A **(left) submodule**  $N \subset M$  is an algebraic subgroup s.t.

$$RN \subset N.$$

It follows that  $N$  is itself is a left module.

**Definition 21.** A (left) module  $M$  of  $R$  is **simple** (or irreducible) if it has exactly the two submodules:  $0 = \{0\}$  and  $M$ .

**Definition 22.** A ring  $R$  is a **division ring** (aka **skew field**) if it satisfies any of the following equivalent requirements:

- (i)  $R^\times = R \setminus \{0\}$  where<sup>1</sup>  $R^\times = \{r \in R \mid \exists a, b \in R : ar = rb = 1.\}$
- (ii)  $R$  has no nontrivial left or right ideals.

**Definition 23.** If  $R \curvearrowright M$ , then we can define

$$\text{End}_R(M) := \{\phi \in \text{End}(M) \mid \phi(rm) = r\phi(m) \forall r \in R, m \in M\}.$$

Note, that  $\text{End}_R(M)$  is a ring.

**Lemma 27** (Schur's Lemma). *If  $M$  is simple, then  $\text{End}_R(M)$  is a division ring.*

**Lemma 28.** *Let  $k$  be a field. Then,  $M_n(k)$  has no nontrivial twosided ideals.*

**Theorem 12** (Jacobson Density Theorem (Double Commutant Theorem)). *Suppose  $M$  is a simple left module which is finitely generated as a right  $D$ -module for  $D = \text{End}_R(M)$ .*

*Assume that  $R$  acts faithfully on  $M$ , i.e.  $R \rightarrow \text{End}_R(M)$  is injective.*

*Then, the map  $R \rightarrow \text{End}_D(M)$  is an isomorphism.*

---

<sup>1</sup>If  $ar = rb = 1$ , then  $a = arb = b$ .

**Recap:**

- Basics: definitions, Hopf-algebras, ...
- Jordan decomposition
- Primer on non-commutative algebra
  - Jacobson density theorem
- Unipotent groups
- Tori

We had last week

$$\text{End}_D(M) := \{\phi \in \text{End}(M) \mid \phi \circ d = d \circ \phi \ \forall d \in D\}.$$

Let  $k$  be an algebraically closed field,  $V$  a non-trivial finite-dimensional  $k$ -vector space and let  $G$  be a subgroup of  $\text{GL}(V)$  that acts **irreducibly** on  $V$ , i.e.,  $V$  is  **$G$ -irreducible**, i.e., the only  $G$ -invariant subspaces of  $V$  are 0 and  $V$ .

Set

$$D := \{d \in \text{End}_k(V) \mid dg = gd \ \forall g \in G\} = \text{span}(G) = \left\{ \sum_{i=1}^n c_i g_i \mid c_i \in k, g_i \in G, n \in \mathbb{N}_0 \right\}.$$

Then,

$$D = \text{End}_R(V)$$

where  $R$  is the  $k$ -subalgebra of  $\text{End}(V)$  that is generated by  $G$ .

**Lemma 29** (Schur's Lemma). *We understand  $k \hookrightarrow \text{End}(V)$  as the inclusion of operations which operate by scalar multiplication*

$$k \xrightarrow{\cong} \{\phi : V \rightarrow V \mid \phi : v \mapsto t \cdot v \text{ for some } t \in k\}.$$

Let  $V$  be  $G$ -irreducible. Then, we have

$$D \cong k.$$

*Proof.* Let  $d \in D$ . Since  $V \neq 0$ , there is an eigenspace  $V_\lambda \neq 0$  for  $d$ . Observe that  $V_\lambda$  has to be  $G$ -invariant:  
if  $g \in G$  and  $v \in V_\lambda$ , then  $gv \in V_\lambda$ , since

$$dgv = gdv = g(\lambda v) = \lambda gv.$$

Since  $V_\lambda$  is a non-trivial  $G$ -invariant subspace and  $V$  is irreducible under  $G$ , we have

$$V_\lambda = V.$$

Ergo  $d = \lambda$  in the sense of  $k \hookrightarrow \text{End}(V)$ . □

**Consequence of the Jacobson Density Theorem:**  $R = \text{End}_k(V)$ , i.e.,  $G$  generates all linear operations on  $V$ , if  $V$  is  $G$ -irreducible.

We will prove this after a lemma.

**Lemma 30.** *Let  $V$  be  $G$ -irreducible.*

*Let  $n \in \mathbb{N}$ . Set*

$$V^n := V \oplus V \oplus \dots \oplus V = V_1 \oplus \dots \oplus V_n$$

*where each  $V_i = V$ .*

*Let  $v = (v_1, \dots, v_n) \in V^n$  and set*

$$Rv := \{(rv_1, \dots, rv_n) \mid r \in R\} = \text{span}\{(gv_1, \dots, gv_n) \mid g \in G\}.$$

*Then,  $Rv \neq V^n$  iff the  $v_j$  are linearly dependent over  $k$ .*

**Consequence:** Take  $n := \dim(V)$ . Let  $\{e_1, \dots, e_n\}$  be a basis of  $V$  and set

$$e := (e_1, \dots, e_n) \in V^n.$$

Since the  $(e_i)_i$  are linearly independent, the lemma states that  $Re = V^n$ .

Now, let  $x \in \text{End}_k(V)$ . Choose  $r \in R$  s.t.

$$re = (xe_1, \dots, xe_n).$$

Then  $re_i = xe_i$  for all  $i$ , thus  $x = r$ . Hence,  $R = \text{End}_k(V)$ .

*Proof.* For  $v = (v_1, \dots, v_n) \in V^n$  choose  $J \in \{1, \dots, n\}$  as large as possible with

$$Rv + V_1 + V_2 + \dots + V_{J-1} =: U \neq V^n.$$

Such an  $J$  does exist, since we know that  $Rv \neq V^n$ .

Then,  $V_J \not\subseteq U$ , otherwise we may increase  $J$ . Also,  $U$  is invariant by the diagonal action of  $G$  on  $V^n$ . Thus,  $V_J \cap U \subseteq V_J$  is a proper  $G$ -invariant subspace of the  $G$ -irreducible  $V_J \cong V$ . Therefore,  $V_J \cap U = 0$ .

On the other hand, by maximality of  $J$ , we have

$$U \oplus V_J = V^n.$$

Ergo, the map (composition)

$$V \cong V_J \hookrightarrow V^n \twoheadrightarrow V^n/U$$

is a  $G$ -equivariant isomorphism, since  $U$  is  $G$ -invariant.

Let  $z : V^n/U \xrightarrow{\cong} V$  be the inverse isomorphism. Let  $l$  be the  $G$ -equivariant map given by

$$\begin{array}{ccc} V^n & \xrightarrow{l} & V \\ \downarrow & \nearrow z & \\ V^n/U & & \end{array}$$

and let  $l_j$  be the  $G$ -equivariant maps by restricting  $l$  on  $V_j$ . Then  $l_j \in D \cong k$ .

Say  $l_j = t_j \in k$ . Then,

$$l(w) = t_1 w_1 + \dots + t_n w_n.$$

Since  $z$  is an isomorphism,  $l$  is nonzero and  $(t_1, \dots, t_n) \neq (0, \dots, 0)$ .

Since  $l|_U = 0$ , we can deduce for all  $u \in U$

$$t_1 u_1 + \dots + t_n u_n = 0.$$

But  $v \in Rv \subseteq U$ , so we may conclude – as required – that the  $(v_i)_i$  are linearly dependent ( $l(v) = 0$ ).  $\square$

## 7 Unipotent Groups

Let  $G$  be a subgroup of  $\mathrm{GL}(V)$  where  $V$  is a finite-dimensional vector space and  $k$  an algebraically closed field.

**Definition 24.** We say that  $G$  is **unipotent** if one of the following equivalent conditions hold for each  $g \in G$ :

- $g$  is unipotent (i.e.  $(g - 1)^n = 0$  for some  $n \in \mathbb{N}$ ).
- all eigenvalues of  $g$  are 1.
- $g$  is conjugate to  $\begin{pmatrix} 1 & \star & \star \\ 0 & \ddots & \star \\ 0 & 0 & 1 \end{pmatrix}$ .

**Theorem 13.** Any unipotent subgroup of  $\mathrm{GL}_n(k)$  is conjugate to a subgroup of

$$U_n := \left\{ \begin{pmatrix} 1 & \star & \star \\ 0 & \ddots & \star \\ 0 & 0 & 1 \end{pmatrix} \right\} = \left\{ U \in M_n(k) \mid U_{i,j} = \begin{cases} 0, & \text{if } i > j \\ 1, & \text{if } i = j \\ \text{arbitrary,} & \text{otherwise.} \end{cases} \right\}.$$

**Definition 25.** For two subgroups  $G, H$  of some common supergroup, define their **commutator** by

$$[G, H] := \langle ghg^{-1}h^{-1} \mid g \in G, h \in H \rangle.$$

A group  $G$  is called **nilpotent**, if one of its commutators is trivial, i.e. if we set

$$G_0 := G \text{ and } G_{i+1} := [G_i, G],$$

then  $G$  is called nilpotent iff there is an  $j \in \mathbb{N}$  with  $G_j = 1$ .

**Corollary 4.** Any unipotent subgroup of  $\mathrm{GL}(V)$  is nilpotent.

**Definition 26.** A group  $G$  is called **solvable**, if  $G^{(n)} = 1$  for some  $n$  where

$$\begin{aligned} G^{(0)} &:= G, \\ G^{(i+1)} &:= [G^{(i)}, G^{(i)}]. \end{aligned}$$

Note that nilpotent groups are solvable, since  $G^{(i)} \subset G_i$ .

**Notation 1.** In the following, we will write  $G' := [G, G]$ .

**Definition 27.** Let  $n := \dim(V)$ . A **complete flag** is a maximal strictly increasing chain of subspaces

$$0 = V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_n = V.$$

Any complete flag is of the form

$$V_j := \text{span}\{e_1, \dots, e_j\}$$

for some basis  $e_1, \dots, e_n$  of  $V$ .

Let  $B$  be the basis of some flag  $0 = V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_n = V$ . For  $x \in \text{End}(V)$ , we have that  $x$  is upper-triangular with respect to  $B$  iff  $x$  leaves each member  $V_i$  of the flag invariant, i.e.  $xV_i \subseteq V_i$ .

**Proposition 2** (Key Proposition). *Let  $G$  be a unipotent subgroup of  $GL(V)$ . Then there is a complete flag  $V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_n$  consisting of  $G$ -invariant subspaces, i.e., each  $V_i$  is  $G$ -invariant.*

*Proof.* Recall, that  $G$  is a unipotent subgroup of  $GL_n(V)$ . We will give an induction on  $n = \dim V$ .

If  $n = 0$ , there is nothing to show.

Let  $n \geq 1$ . We may assume that  $V$  is  $G$ -irreducible. Because, if not, there is a  $G$ -invariant subspace  $0 \neq W \subset V$  s.t.  $W$  and  $V/W$  have dimension  $< n$ . Then there exist complete  $G$ -invariant flags in  $W$  and  $V/W$  and the claim – that there is a complete  $G$ -invariant flag in  $V$  – follows by the induction hypothesis.

By Jacobson Density Theorem, we have

$$R := \text{span}(G) = \text{End}(V) := \text{End}_k(V).$$

Since  $G$  is unipotent, we have for each  $g \in G$

$$\text{trace}(g) = n.$$

Ergo, for  $g, h \in G$

$$\text{trace}(gh) = \text{trace}(h)$$

and

$$\text{trace}((g - 1)h) = \text{trace}(gh) - \text{trace}(h) = 0.$$

Since  $\text{span}(G) = \text{End}(V)$ , it now in particular follows for all  $g \in G, \phi \in \text{End}(V)$

$$\text{trace}((g - 1)\phi) = 0.$$

Since the above holds for all  $\phi \in \mathbf{End}(V)$ , it must hold

$$g - 1 = 0$$

for all  $g \in G$  (take for example the elementary matrices  $\phi = E_{i,j}$ ). Ergo,  $G$  is trivial. Then, any complete flag is trivially  $G$ -invariant.  $\square$

*Remark 3.* This gives the group analogue of Engel's Theorem.

*Proof Goal Theorem.* Let  $B$  be a basis of  $V$  s.t.  $G$  leaves each subspace in the corresponding flag invariant. Then,  $G$  is upper-triangle with respect to this basis.

On the other hand, each  $g \in G$  is unipotent, hence its diagonal (i.e. eigenvalues) are all 1. Thus, with respect to  $B$

$$G \subseteq \left\{ \begin{pmatrix} 1 & \star & \star \\ 0 & \ddots & \star \\ 0 & 0 & 1 \end{pmatrix} \right\} = U_n.$$

$\square$

*Remark 4.* Tori are of the form  $(k^\times)^n$ . In the case  $k = \mathbb{C}$ ,  $(\mathbb{C}^\times)^n$  are the complexification of  $U(1)^n$ . This equals tori in top. sense.

$$\begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix} \subseteq \mathbf{GL}_2(\mathbb{C})$$

is a non-algebraic unipotent group.

**Exercise.** (to be discussed next time)

it would have sufficed to prove the Goal theorem in the special case that  $G$  is algebraic.

**Corollary of Proof:** If  $G \subset \mathbf{GL}(V)$  (with  $V \neq 0$ ) is unipotent and  $V$  is  $G$ -irreducible, then  $G = 1$ ,  $\dim V = 1$ .



**Answer to last Exercise:** Recall that the main point was to show that any unipotent subgroup  $G \subseteq \mathrm{GL}(V)$  leaves invariant some complete flag  $\mathcal{F} = (V_0 \subset V_1 \dots)$ . But by some homework (problem 1), the group

$$\mathrm{GL}(V)_{\mathcal{F}} := \{g \in \mathrm{GL}(V) \mid g\mathcal{F} = \mathcal{F}\}$$

is algebraic.

**Proof:** If  $\mathcal{F}$  is the standard flag with  $V_i = \mathrm{span}(e_1, \dots, e_i)$  for the standard basis  $\{e_1, \dots, e_n\}$ , then

$$\mathrm{GL}(V)_{\mathcal{F}} = \{A \in \mathrm{GL}(V) \mid A \text{ is upper-triangle}\}.$$

The condition that  $A$  is upper triangle can be realized by polynomials. □

Thus,

$$\begin{aligned} & G \text{ fixes } \mathcal{F} \\ & \iff G \subseteq \mathrm{GL}(V)_{\mathcal{F}} \\ & \stackrel{\mathrm{GL}(V)_{\mathcal{F}} \text{ is algebraic}}{\iff} \overline{G} \subseteq \mathrm{GL}(V)_{\mathcal{F}} \\ & \iff \overline{G} \text{ fixes } \mathcal{F}. \end{aligned}$$

Now, the Zariski-Closure  $\overline{G}$  of any group  $G$  is an algebraic group (shown in some homework).

Further, if  $G$  is unipotent, then  $\overline{G}$  is unipotent.

## 8 Tori

**Definition 28.** A **torus** is an algebraic group that is isomorphic to  $\mathcal{G}_m^n$  for some  $n \in \mathbb{N}_0$  where  $\mathcal{G}_m = k^\times = \mathrm{GL}_1(k)$  is the unit group of  $k$ .

We think of  $\mathcal{G}_m^n \subseteq \mathrm{GL}_n(k)$  as the subgroup of diagonal matrices.

**Lemma 31.** *Let  $G$  be a commutative algebraic group. Then the following are equivalent:*

- (i) *each  $g \in G$  is semisimple.*
- (ii) *for each finite-dimensional representation  $V$  of  $G$  and for each  $g \in G$ , the operator  $r_V(g)$  is diagonalizable.*
- (iii) *for all finite-dimensional representations  $V$  of  $G$ , there is a basis of common eigenvectors for  $r_V(G)$ , i.e. a basis s.t.*

$$r_V(G) \subseteq \mathcal{G}_m^n.$$

- (iv)  *$G$  is isomorphic to an algebraic subgroup of a torus.*

*Proof.* We show:

- (i)  $\iff$  (ii): This follows from the Jordan decomposition and definition of semisimple.
- (ii)  $\implies$  (iii) : This is homework. Note that any commutative subset  $S$  of  $\mathrm{GL}(V)$  consisting of semisimple operators may be diagonalized simultaneously.
- (iii)  $\implies$  (iv) : Take any faithful representation  $V$  of  $G$  and diagonalize it simultaneously. Then,  $G \cong r_V(G) \subseteq \mathcal{G}_m^n$ .
- (iv)  $\implies$  (i) : Any diagonal matrix is semisimple.

□

**Definition 29.** A commutative algebraic group  $G$  is called **diagonalizable**, if it satisfies one of the above equivalent conditions.

**Definition 30.** A **character**  $\chi$  of an algebraic group  $G$  is an element  $\chi \in \mathrm{Hom}_{\mathrm{alg.grp.}}(G, k^\times)$ , i.e., a homomorphism  $\chi : G \rightarrow k^\times$  of algebraic groups.

**Notation 2.** For an algebraic group  $G$ , set  $\mathfrak{X}(G) := \mathrm{Hom}_{\mathrm{alg.grp.}}(G, k^\times)$ .

Also denote now by  $\mathcal{O}(X) := k[T]/I(X)$  the coordinate ring of an algebraic set  $X$  (rather than  $k[X]$ ).

**Lemma 32.** *There is a bijection*

$$\mathfrak{X}(G) = \{\text{characters } \chi \text{ of } G\} \longleftrightarrow \{x \in \mathcal{O}(G)^\times \mid \Delta(x) = x \otimes x\}.$$

*Proof.* Note, that any  $x \in \mathcal{O}(G)^\times$  can be thought of as a map  $x : G \rightarrow k^\times \subset k$ .

We have

$$\begin{aligned} \text{Hom}_{\text{alg.grp.}}(G, \mathcal{G}_m) &= \{\phi \in \text{Hom}_{\text{alg.sets}}(G, \mathcal{G}_m) \mid \phi(gh) = \phi(g)\phi(h) \ \forall g, h\} \\ &= \{\phi \in \text{Hom}_{k\text{-alg.}}(\mathcal{O}(\mathcal{G}_m), \mathcal{O}(G)) \mid (\phi \otimes \phi) \circ \Delta = \Delta \circ \phi\}. \end{aligned}$$

**Recall:**  $\mathcal{O}(\mathcal{G}_m) \cong k[t, \frac{1}{t}]$  with  $\Delta(t) = t \otimes t$ .

Thus for any  $k$ -algebra  $A$ ,  $\text{Hom}_{k\text{-alg.}}(\mathcal{O}(\mathcal{G}_m), A) \cong^{A^\times}$  via

$$[t \mapsto a, (t^{-1} \mapsto a^{-1})] \longleftrightarrow a.$$

Thus,

$$\text{Hom}_{\text{alg.grp.}}(G, \mathcal{G}_m) \cong \{a \in \mathcal{O}(G)^\times \mid a \otimes a = \Delta(a)\}.$$

Therefore, it suffices to test the condition  $(\phi \otimes \phi) \circ \Delta = \Delta \circ \phi$  on the generators  $t, t^{-1}$  of  $\mathcal{O}(\mathcal{G}_m)$ . Now, the above isomorphism is given by

$$\phi \mapsto a = \phi(t)$$

which is equivalent or regarding  $\chi : G \rightarrow \mathcal{G}_m$  as a map  $\chi : G \rightarrow k$ . □

**Example 13.** Let  $G = \mathcal{G}_m$ , then  $\mathcal{O}(G) = k[t, \frac{1}{t}]$ .

Which  $x = \sum_{m \in \mathbb{Z}} c_m t^m \in \mathcal{O}(G)$  – with almost all  $c_m = 0$ , but not all of them – have the property

$$\Delta(x) = x \otimes x?$$

We have

$$\begin{aligned} x \otimes x &= \sum_{m, n \in \mathbb{Z}} c_m c_n t^m \otimes t^n, \\ \Delta(x) &= \sum_{m \in \mathbb{Z}} c_m t^m \otimes t^m. \end{aligned}$$

Those sums equal, if

$$\begin{aligned} c_m c_n &= 0 \text{ for all } m \neq n, \\ c_m^2 &= c_m \text{ for all } m. \end{aligned}$$

By those conditions, it follows

$$x = t^m.$$

Therefore

$$\mathfrak{X}(G) = \{\chi_m \mid m \in \mathbb{Z}\} \cong \mathbb{Z}$$

with

$$\chi_m(y) = y^m.$$

**Example 14.** Let  $T \cong \mathcal{G}_m^n$  be a torus. Then,

$$\mathfrak{X}(T) = \{\chi_m \mid m \in \mathbb{Z}^n\} \cong \mathbb{Z}^n$$

where  $\chi_m(y) = y^m = y_1^{m_1} \cdots y_n^{m_n}$ .

**Note:** For each algebraic group  $G$ ,  $\mathfrak{X}(G)$  is naturally an abelian group:

$$(\chi_1 \cdot \chi_2)(g) := \chi_1(g) \cdot \chi_2(g).$$

Given a morphism of algebraic groups  $f : G \rightarrow H$ , we get a morphism of abelian groups

$$\begin{aligned} f^* : \mathfrak{X}(H) &\longrightarrow \mathfrak{X}(G) \\ \chi &\longmapsto \chi \circ f =: f^*(\chi). \end{aligned}$$

This induces a contravariant functor from the category of algebraic groups to the category of abelian groups.

**Lemma 33.** *Let  $G$  be a diagonalizable algebraic group. Then,  $\mathfrak{X}(G)$  is a  $k$ -vector space basis for  $\mathcal{O}(G)$ .*

**Example 15.** Let  $G = \mathcal{G}_m^n$  be a torus. Then, we have the embedding

$$\begin{aligned} \mathfrak{X}(G) &\hookrightarrow \mathcal{O}(G) \\ \chi_{(m_1, \dots, m_n)} &\longmapsto t^{(m_1, \dots, m_n)}. \end{aligned}$$

The lemma is obvious in this case: each element of  $\mathcal{O}(G) = k[t_1, \dots, t_n, t_1^{-1}, \dots, t_n^{-1}]$  can be written uniquely as a linear combination of monomials.

*Proof.* (i)  $\mathfrak{X}(G)$  spans  $\mathcal{O}(G)$ :

Choose an embedding  $G \subset \mathcal{G}_m^n$  of algebraic groups. Then, by restriction, we get

$$\mathcal{O}(\mathcal{G}_m^n) \twoheadrightarrow \mathcal{O}(G).$$

Since the  $\chi_m, m \in \mathbb{Z}^n$ , span  $\mathcal{O}(\mathcal{G}_m^n)$ , their images  $\chi_m|_G \in \mathfrak{X}(G)$  span  $\mathcal{O}(G)$ .

(ii)  $\mathfrak{X}(G)$  is linearly independent:

Suppose otherwise and let  $\phi_1, \dots, \phi_m$  be a linearly dependent subset of  $\mathfrak{X}(G)$  with  $m \geq 1$  chosen minimally, with  $c_1, \dots, c_m \in k^\times$  s.t.

$$\sum_{i=1}^m c_i \phi_i = 0.$$

We distinguish the following cases:

$m = 1$ : In this case, we have  $\phi_1 = 0$ , but  $\phi_1(1) = 1$ , a contradiction.

$m > 1$ : We can assume  $\phi_1 \neq \phi_2$ , so there is an  $h \in G$  s.t.  $\phi_1(h) \neq \phi_2(h)$ . Then,

$$\phi_1(h) \sum_{i=1}^m c_i \phi_i = 0,$$

but also for all  $h, g \in G$

$$\sum_{i=1}^m c_i \phi_i(hg) = \sum_{i=1}^m c_i \phi_i(h) \phi_i(g) = 0.$$

This implies

$$\sum_{i=1}^m c_i \phi_i(h) \phi = 0.$$

Ergo

$$\sum_{i=1}^m c_j (\phi_i(h) - \phi_1(h)) \phi_i = \sum_{i=2}^m c_j (\phi_i(h) - \phi_1(h)) \phi_i = 0.$$

Now,  $\phi_i(h) - \phi_1(h)$  is zero if  $i = 1$  and non-zero, if  $i = 2$ . Therefore, this yields a shorter linear dependency for the elements

$$\phi_2, \dots, \phi_m,$$

which contradicts our requirement.

□

**Definition 31.** Let  $M$  be an abelian group. The **group algebra** on  $M$  is the  $k$ -algebra  $k[M]$  (not a coordinate ring!) defined as follows:

$$\begin{aligned} k[M] &:= \text{the } k\text{-vectorspace with basis } M \\ &:= \left\{ \sum_{m \in M} c_m \cdot m \mid c_m \in k, \text{ almost all } c_m = 0 \right\}, \end{aligned}$$

where the multiplication on  $k[M]$  extends that on  $M$ :

$$\left(\sum_{m \in M} c_m m\right) \left(\sum_{n \in M} d_n n\right) = \sum_{m, n \in M} c_m d_n mn.$$

**Corollary 5.** *For a diagonalizable  $G$ , we have*

$$\mathcal{O}(G) \cong k[\mathfrak{X}(G)].$$

**Fact:** For an abelian group  $M$ , there is exactly one Hopf algebra structure on  $k[M]$  given by  $\Delta(m) = m \otimes m$  for all  $m \in M$ .

With this definition, the above isomorphism is one of Hopf algebras.

**Lemma 34.** *If  $G, H$  are diagonalizable algebraic groups, then*

$$\mathrm{Hom}_{\mathrm{alg. grp.s}}(G, H) \xrightarrow{f \mapsto f^*} \mathrm{Hom}_{\mathrm{grp.s}}(\mathfrak{X}(H), \mathfrak{X}(G))$$

*is a bijection.*

*Proof.*

$$\begin{aligned} \mathrm{Hom}(G, H) &\cong \mathrm{Hom}_{\mathrm{Hopf-alg.}}(\mathcal{O}(H), \mathcal{O}(G)) \\ &\cong \{\phi \in \mathrm{Hom}_{k\text{-alg.}}(\mathcal{O}(H), \mathcal{O}(G)) \mid (\phi \otimes \phi) \circ \Delta = \Delta \circ \phi\}. \end{aligned}$$

Since  $\mathrm{Hom}_{k\text{-alg.}}(\mathcal{O}(H), \mathcal{O}(G)) \cong \mathrm{Hom}(k[\mathfrak{X}(H)], k[\mathfrak{X}(G)])$ , this reduces to the following lemma:

**Lemma 35.** *Let  $M_1, M_2$  be two abelian groups. Then*

$$\begin{aligned} \mathrm{Hom}(M_1, M_2) &\xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Hopf-alg.}}(k[M_1], k[M_2]) \\ \phi &\longmapsto \left[ \sum c_m m \mapsto \sum c_m \phi(m) \right]. \end{aligned}$$

*Proof.* We have to show that

$$M = \{x \in K[M]^\times \mid \Delta(x) = x \otimes x\}.$$

Then, by this, it follows for each  $\phi \in \mathrm{Hom}_{\mathrm{Hopf-alg.}}(k[M_1], k[M_2])$ ,

$$\phi(M_1) \subseteq M_2.$$

Ergo,  $\phi|_{M_1} \in \mathbf{Hom}(M_1, M_2)$ . Therefore, the surjectivity of the claimed bijection is shown. The injectivity is clear, since  $M$  generates  $k[M]$  as a  $k$ -algebra.

To show

$$M = \{x \in K[M]^\times \mid \Delta(x) = x \otimes x\},$$

let

$$\begin{aligned} x &= \sum c_m m \in K[M]^\times \\ \Delta(x) &= \sum c_m m \otimes m \\ x \otimes x &= \sum c_m c_n m \otimes n. \end{aligned}$$

If  $\Delta(x) = x \otimes x$ , then it follows

$$x = m$$

for some  $m \in M$ .

□

□

**Recall:** We have seen that for diagonalizable algebraic groups  $G, H$

$$\mathrm{Hom}(G, H) \cong \mathrm{Hom}(\mathfrak{X}(H), \mathfrak{X}(G)).$$

If  $G$  is diagonalizable, then

$$\mathcal{O}(G) \cong k[\mathfrak{X}(G)].$$

**Theorem 14.** *The functor*

$$\begin{aligned} G &\longrightarrow \mathfrak{X}(G) \\ f &\longmapsto f^* \end{aligned}$$

*defines an equivalence of categories:*

$$\{\text{diagonalizable alg. groups}\} \cong \{\text{finite-dim. abelian groups with no char}(k)\text{-torsion}\}.$$

This amounts to the bijection above between Hom-spaces and the following lemma.

**Lemma 36.** (i) *Let  $G$  be a diagonalizable alg. group. Then,  $\mathfrak{X}(G)$  is a finitely generated abelian group with no char( $k$ )-torsion.*

(ii) *Let  $\Gamma$  be a finitely generated abelian group with no char( $k$ )-torsion. Then, there is a diagonalizable algebraic group  $G$  s.t.  $\mathfrak{X}(G) \cong \Gamma$ .*

*Proof.* We will use the following facts:

- Let  $n \in \mathbb{N}$ . Then,  $t^n - 1$  is square-free in  $k[t]$  iff the ideal  $(t^n - 1)$  is radical in  $k[t]$  iff  $t^n - 1$  has not repetitive root iff either  $\mathrm{char}(k) = 0$  or  $\mathrm{char}(k) = p > 0$  and  $p \nmid n$ .

(Proof: Galois Theory, separable/inseparable extensions.)

- Let  $M := \mathbb{Z}/n\mathbb{Z}$ . Then, the  $k$ -group-algebra generated by  $M$

$$k[M] \cong k[t]/(t^n - 1)$$

is reduced iff either  $\mathrm{char}(k) = 0$  or  $\mathrm{char}(k) = p > 0, p \nmid n$ .



- If  $M_1, M_2$  are abelian groups, then we have the following isomorphism of Hopf algebras

$$\begin{aligned} k[M_1] \otimes_k k[M_2] &\xrightarrow{\cong} k[M_1 \oplus M_2] \\ m_1 \otimes m_2 &\longmapsto m_1 m_2 \end{aligned}$$

where  $M_1 \oplus M_2 \cong M_1 \times M_2$ .

- (i) Embed  $G \hookrightarrow T := \mathcal{G}_m^n$  for some  $n$ . Then, we have a surjection  $\mathbb{Z}^n \cong \mathfrak{X}(T) \twoheadrightarrow \Xi(G)$ . Ergo,  $\mathfrak{X}(G)$  is finitely generated.

Suppose  $\text{char}(k) = p > 0$ . Let  $\chi \in \mathfrak{X}(G)$  with  $\chi^p = 1$ . Then, for all  $g \in G$ ,  $\chi^p(g) = \chi(g^p) = 1$ . The unit group  $k^\times$  has not  $p$ -torsion, therefore  $G \hookrightarrow T = (k^\times)^n$  has also no  $p$ -torsion. Therefore, the Frobenius  $g \mapsto g^p$  is an isomorphism on  $G$ . Therefore,  $\chi = 1$  is a trivial character. Ergo  $\mathfrak{X}(G)$  has no  $p$ -torsion.

- (ii) Let  $\Gamma$  be a finitely generated abelian group with no  $\text{char}(k)$ -torsion. Then,

$$\Gamma \cong \mathbb{Z}^r \oplus \mathbb{Z}/n_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/n_l\mathbb{Z}$$

where  $\text{char}(k) \nmid n_1, \dots, n_l$ . We may reduce to the cases:

- (a)  $\Gamma = \mathbb{Z}$ : take  $G = \mathcal{G}_m$ , then  $\Xi(G) \cong \mathbb{Z} \cong \Gamma$ .  
(b)  $\Gamma = \mathbb{Z}/n\mathbb{Z}$  with  $\text{char}(k) =: p \nmid n$ :  
take  $G := \mu_n := \{y \in k^\times \mid y^n = 1\}$ . Then, since  $p \nmid n$ ,  $(t^n - 1)$  is radical. So,

$$\mathcal{O}(\mu_n) \stackrel{\text{Nullstellensatz}}{=} k[t]/(t^n - 1) \stackrel{\text{as Hopf algebras}}{\cong} k[\Gamma]$$

where  $t$  gets mapped to the generator of  $\Gamma$ .

□

**Corollary 6.** *We have the bijection*

$$\{\text{tori}\} \cong \{\text{finitely generated free abelian groups}(\cong \mathbb{Z}^n)\}.$$

*Remark 5.*

$$\{\text{algebraic group schemes}/k\} \stackrel{\text{not necessarily natural}}{\cong} \{\text{f.g. Hopf algebras}\}.$$

by

$$G \mapsto \mathcal{O}(G)$$

and

$$\{\text{diagonalizable algebraic group schemes}/k\} \cong \{\text{f.g. abelian groups}\}.$$

by

$$G \mapsto \Xi(G).$$

Where  $\mu_p$  in the left hand term gets mapped to  $\mathcal{O}(\mu_p) = k[t]/(t^p - 1)$  with  $p = \text{char } k$ .

## 9 Trigonalization

We say a representation  $r : G \rightarrow \mathrm{GL}(V)$  of a group  $G$  on a finite-dimensional  $k$ -vectorspace  $V$  is **trigonalizable** if it admits a basis with respect to which  $r(V)$  is upper-triangular:

$$r(G) \subseteq \left\{ \begin{pmatrix} * & \dots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & * \end{pmatrix} \right\}$$

**Definition 32.** We call a subgroup  $G \subseteq \mathrm{GL}(V)$  **trigonalizable**, if the identity representation is.

**Lemma 37.** *Let  $G$  be an algebraic group. The following are equivalent:*

- (i) *Every finite-dimensional representation  $r : G \rightarrow \mathrm{GL}(V)$  is trigonalizable.*
- (ii) *Every irreducible representation of  $G$  is 1-dimensional.*
- (iii)  *$G$  is isomorphic to an algebraic subgroup of*

$$B_n := \left\{ \begin{pmatrix} * & \dots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & * \end{pmatrix} \right\} \subseteq \mathrm{GL}_n(k).$$

- (iv) *There is a normal unipotent algebraic subgroup  $U$  of  $G$  s.t.  $G/U$  is diagonalizable.*

*Proof.* We prove as follows:

- (i)  $\implies$  (ii): Let  $V$  be an irreducible representation. Then,  $V \neq 0$ . Choose a basis  $e_1, \dots, e_n$  of  $V$  s.t.

$$r(G) \subseteq B_n.$$

Then,  $r(G)e_1 \subseteq ke_1$ , so  $V_0 := ke_1$  is  $G$ -invariant. Ergo  $V = V_0$  is 1-dimensional.

- (ii)  $\implies$  (i): Let  $V$  be a f.d. representation. We show by induction on  $\dim(V)$  that  $r : G \rightarrow \mathrm{GL}(V)$  is trigonalizable:

In the cases  $\dim(V) = 0, 1$ , there is nothing to show.

In the case  $\dim(V) \geq 2$ , assume that  $V$  is not irreducible. Then, there is a  $G$ -invariant  $V_0$  with  $0 \neq V_0 \neq V$ .

By the induction hypothesis,  $V_0$  and  $V/V_0$  are trigonalizable. Ergo,  $V$  is trigonalizable.

(**Recall:** we used this criterion above in the proof that unipotent groups are trigonalizable by showing that every ??? of each  $G$  is trivial.)

(i)  $\implies$  (iii): Choose a faithful representation  $V$  of  $G$ . Then,  $G \cong r(G)$ . Since  $r$  is trigonalizable, there is a basis of  $V$  s.t.

$$r(G) \subseteq B_n \subseteq \mathrm{GL}_n(k).$$

(iii)  $\implies$  (ii): Suppose  $G \subseteq B_n \subseteq \mathrm{GL}_n(k)$ . Set

$$A_n := \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & * \end{pmatrix} \right\} \subseteq \mathrm{GL}_n(k),$$

$$U_n := \left\{ \begin{pmatrix} 1 & \dots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & 1 \end{pmatrix} \right\} \subseteq \mathrm{GL}_n(k) \text{ normal algebraic subgroup of } B_n,$$

$$U := G \cap U_n \text{ normal unipotent algebraic subgroup of } G.$$

Let  $V$  be an irreducible representation of  $G$ , then  $V$  is not zero. Consider the subspace of  $V$  fixed by  $U$

$$V^U := \{v \in V \mid r(u)v = v \forall u \in U\}.$$

Then, we get a representation

$$r|_U : U \longrightarrow \mathrm{GL}(V).$$

Then,  $r(U)$  is a unipotent algebraic group of  $\mathrm{GL}(V)$ . Ergo,

$$r(U) \subseteq \left\{ \begin{pmatrix} 1 & \dots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Ergo,  $V^U \neq 0$ . Since  $U$  is normal in  $G$ , the subspace  $V^U$  of  $V$  is  $G$ -invariant: if  $v \in V^U, g \in G$ , then for all  $u \in U$  we have

$$r(u)r(g)v = r(g)r(g^{-1}ug)v = r(g)v$$

since  $v \in V^U$ . Ergo  $r(g)v \in V^U$ .

Since  $V$  is irreducible,  $V = V^U$ , i.e.  $U$  acts trivially on  $V$ . Ergo,  $r$  descends to a representation of the group  $G/U$ .

But  $G/U \hookrightarrow B_n/U_n \cong A_n$ . Therefore,  $G/U$  and  $r(G)$  are commutative. Moreover, for all  $g \in G$ ,  $r(g) \in \mathbf{GL}(V)$  is semisimple:

if  $g = g_s g_u$ , then  $g_u \in U$ , because  $U_n$  is the group of unipotent elements of  $B_n$ .

Hence,  $r(g) = r(g_s)r(g_u) = r(g_s)$  is semisimple.

It follows that  $r(G)$  is commutative and consists of semisimple elements. By some HW:  $r(G)$  is trigonalizable. It is easy to show now that  $V$  is one-dimensional. (Since  $V$  is irreducible and  $ke_1$  is  $G$ -invariant.)

□

**Definition 33.**  $G$  is **trigonalizable**, if it satisfies one of the above equivalent conditions.

Later, we will see, that if  $G$  is connected, then being trigonalizable implies being solvable.

## 10 Commutative Groups

Let  $G$  be an algebraic group. Denote by  $G_s$  resp.  $G_u$  the subsets of semisimple resp. unipotent elements of  $G$ .

Then,  $G_u$  is always algebraic i.e. closed: if  $G \hookrightarrow \mathrm{GL}_n(k)$ , then  $G_u = \{g \mid (g - 1)^n = 0\}$ .  $G_u$  does not need to be closed under multiplication (for example, take  $G = \mathrm{SL}_2(k)$ ,  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  and  $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ ).

$G_s$  needs not to be algebraic: for example, take  $G = \mathrm{SL}_2(k)$  and if  $G_s$  were algebraic, then

$$\left\{ \lambda \in k^\times \mid \begin{pmatrix} \lambda & 1 \\ 0 & \lambda^{-1} \end{pmatrix} \in G_s \right\} = \{ \lambda \mid \lambda \neq \lambda^{-1} \}$$

but the last set is not algebraic. Also,  $G_s$  does not need to be a group.

We have the a surjective map of sets

$$\begin{aligned} G_s \times G_u &\longrightarrow G \\ (g_1, g_2) &\longmapsto g_1 g_2. \end{aligned}$$

**Example 16** (Non-Example). Take generic  $g \in G_s, h \in G_u$  for  $G = \mathrm{SL}_2(k)$ . Then,  $g, h$  do not commute and we have

$$((gh)_s, (gh_u)) \neq (g, h)$$

because Jordan components don't commute.

**Theorem 15.** *Let  $G$  be a commutative algebraic group. Then:*

- (i)  $G_s, G_u$  are closed subgroups and the multiplicative map  $G_s \times G_u \rightarrow G$  is an isomorphism of algebraic groups.
- (ii)  $G$  is trigonalizable. Moreover, for each finite dimensional representation  $r : G \rightarrow \mathrm{GL}(V)$  there is a basis s.t.

$$r(G_s) \subseteq \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & * \end{pmatrix} \right\} \quad r(G_u) \subseteq \left\{ \begin{pmatrix} 1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

- (iii)  $G_s$  is diagonalizable.

*Proof.* (ii) Let  $V$  be any irreducible representation of  $G$ . We have seen that commuting semisimple operators may be simultaneously diagonalizable, then

$$V = \bigoplus_{\chi: G_s \rightarrow \mathcal{G}_m} V_\chi$$

where

$$V_\chi = \{v \in V \mid r(h)v = \chi(h)v \ \forall h \in G_s\}.$$

Since  $G$  is commutative, each subspace  $V_\chi$  is  $G$ -invariant ( $r(h)r(g)v = r(g)r(h)v = r(g)\chi(h)v = \chi(h)r(g)v$ ).

Since  $V$  is irreducible, we must have  $V = V_\chi$  for some  $\chi$ .

Recall that  $G \cong G_s \times G_u$  as abstract groups. We have seen that  $r(G_s) \subseteq \mathcal{G}_m^n$ . We proved a while ago that any unipotent group, such as  $G_u$ , is trigonalizable. Ergo,  $V$  is trigonalizable. Since  $V$  is irreducible, we have  $\dim V = 1$ .

If we apply the same argument without assuming that  $V$  is irreducible, then we see that  $V$  is the coproduct of  $V_\chi$ 's as above and that each  $V_\chi$  admits a basis s.t.

$$r(G_s)|_{V_\chi} \subseteq \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & * \end{pmatrix} \right\} \quad r(G_u)|_{V_\chi} \subseteq \left\{ \begin{pmatrix} 1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

This yields the same conclusion for  $V$ .

- (i) We have to show that  $G_s$  and  $G_u$  are closed and  $j : G_s \times G_u \rightarrow G$  is an isomorphism of groups. Take any faithful representation

$$G \xrightarrow{\cong, r} r(G) \subseteq \mathbf{GL}(V)$$

and apply (ii). Then we have

$$\begin{aligned} r(G) &\subseteq \left\{ \begin{pmatrix} * & * & * \\ 0 & \ddots & * \\ 0 & 0 & * \end{pmatrix} \right\} =: B \\ B_u &= \left\{ \begin{pmatrix} 1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & 1 \end{pmatrix} \right\}, \\ r(G_s) &\subseteq \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & * \end{pmatrix} \right\} =: A. \end{aligned}$$

In fact,  $r(G_s) = r(G) \cap A$ , because if  $g \in G$  with  $r(g) \in A$ , then  $r(g)$  is semisimple, so  $g \in G_s$ .

Therefore,  $G_s$  is closed in  $G$ . Ergo,  $G_s$  and  $G_u$  are closed subgroups.

Then, the map  $j$  is a morphism of algebraic groups.

We need to show that  $j^{-1}$  is a morphism of algebraic groups. For this, it suffices to verify that the projection  $G \rightarrow G_s$  is a morphism. But this map is given under  $r$  by the morphism:

$$t := \begin{pmatrix} a_1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & a_n \end{pmatrix} \mapsto \begin{pmatrix} a_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_n \end{pmatrix} =: t_s.$$



This suffices because if  $g = g_s g_u$ , then  $g_u = g_s^{-1} g$ , so if the map  $g \mapsto g_s$  is a morphism, so is  $g \mapsto g g_s^{-1} = g_u$ , hence so is  $g \mapsto (g_s, g_u)$ .

- (iii) We have seen that  $G_s$  is a closed subgroup. Hence  $G_s$  is a commutative algebraic group where elements are semisimple. Ergo,  $G_s$  is diagonalizable.  $\square$

## 11 Connected Solvable Groups

**Theorem 16** (Lie-Kolchin). *Let  $G$  be a connected solvable algebraic group. Then  $G$  is trigonalizable.*

(By comparison, recall that we have seen so far that, if  $G$  is commutative or unipotent, then  $G$  is trigonalizable.) We can reformulate this theorem as: Any connected solvable subgroup of  $\mathrm{GL}(V)$  stabilizes some complete flag  $\mathcal{F} = (V_0 \subsetneq \dots \subsetneq V_n)$ .

**Generalization (Borel's Fixed Point Theorem):** Any connected algebraic group  $G$  acting on a projective variety  $X$  has a fixed point in  $X$ .

We get a relation between complete flags and projective varieties.

*Proof.* Induct on the number  $n$  s.t.  $G^{(n)} = 1$ .

For  $n = 0$ , there is nothing to show.

If  $n = 1$ ,  $(G, G) = 1$ , then  $G$  is commutative, ergo trigonalizable.

Let  $n \geq 2$ . Then, we have  $G' := (G, G) \neq 1$ . We will show the following lemma:  $\square$

**Lemma 38.** *Let  $G \subseteq \mathrm{GL}(V)$  be a subgroup.*

*If  $G$  is connected, then the group  $G'$  with the induced topology is connected ( $\iff$  the Zariski Closure of  $G'$  is connected).*

*Proof.* We have the following facts:

- An increasing union of connected spaces is connected.
- A continuous image of a connected space is connected.

We have

$$\begin{aligned} G' &= \langle (g, h) := ghg^{-1}h^{-1} \mid g, h \in G \rangle \\ &= \bigcup_{j \geq 0} \bigcup_{g_1, h_1, \dots, g_j, h_j \in G} \{(g_1, h_1) \cdots (g_j, h_j)\}. \end{aligned}$$

Since

$$\bigcup_{g_1, h_1, \dots, g_j, h_j \in G} \{(g_1, h_1) \cdots (g_j, h_j)\} = \text{img} \phi_j$$

for some continuous map  $\phi_j : G^{2j} \rightarrow G$ , the claim follows. Ergo,  $G'$  is connected.  $\square$

*Remark 6.* It is equivalent to show that (\*) any subgroup  $G$  of  $\text{GL}(V)$  – s.t.  $G$  is connected and solvable – is trigonalizable in  $\text{GL}(V)$ .

Indeed, the theorem implies (\*): the Zariski closure of  $G$  is a connected algebraic group that is solvable (which extends by continuity). If  $Zcl(G)$  is trigonalizable, then also  $G$  is trigonalizable.

On the other hand: (\*) implies the theorem, since if  $G$  is given as in the theorem, apply (\*) to  $r(G) \subseteq \text{GL}(V)$ .

*Proof of Theorem.* If  $G^{(n)} = 1$ , then  $(G')^{(n-1)} = G^{(n)} = 1$ . By induction, we may assume that  $G'$  satisfies the following:

For all finite dimensional representations  $r : G \rightarrow \text{GL}(V)$ ,  $r(G')$  is trigonalizable.

Our aim is to show that any irreducible representation  $V$  of  $G$  has dimension 1.

The induction hypothesis implies that  $r(G')$  is trigonalizable. In particular, there exists an eigenspace  $V_\chi \subseteq V$  for  $G'$  for some character  $\chi : G' \rightarrow k^\times$ . Since  $G'$  is normal in  $G$  we know that  $G$  acts from the left on

$$\{\text{eigenspaces } V_\chi \text{ in } V \text{ for } G'\}.$$

Ergo,  $\bigoplus_{\chi: G' \rightarrow k^\times} V_\chi$  is  $G$ -invariant. Since  $V$  is  $G$ -irreducible, we have

$$V = \bigoplus_{\chi: G' \rightarrow k^\times} V_\chi = \bigoplus_{\chi \in \mathfrak{X}'} V_\chi$$

for some finite subset  $\mathfrak{X}' = \{\chi \mid V_\chi \neq 0\}$  of  $\text{Hom}(G', \mathcal{G}_m)$ , since  $V$  is finite dimensional.

**Claim:** Let  $h \in G'$ . Then, the map

$$\begin{aligned} G &\longrightarrow \text{GL}(V) \\ g &\longmapsto r(ghg^{-1}) \end{aligned}$$

has a finite image.

*Proof.* Denote by  $\chi \mapsto \chi^g$  the action of  $g \in G$  in  $\text{Hom}(G', \mathcal{G}_m)$  given by  $\chi^g(h) := \chi(ghg^{-1})$ . This is an action, since  $G'$  is normal.

Note, that  $\mathfrak{X}' \subseteq \text{Hom}(G', \mathcal{G}_m)$  is a finite subset.

Also note, that the action  $\chi \mapsto \chi^g$  descends to an action  $G \curvearrowright \mathfrak{X}'$ .

Now, let  $\mathfrak{X}' = \{\chi_1, \dots, \chi_r\}$ . The matrix  $r(h)$  is totally determined by the values  $\chi_1(h), \dots, \chi_r(h)$ . Then, the element  $r(ghg^{-1})$  is totally determined by the values  $\chi_1^g(h), \dots, \chi_r^g(h)$ . It follows

$$\#\{r(ghg^{-1}) \mid g \in G\} \leq r!.$$

□

The following lemma is easy to show:

**Lemma 39.** *Let  $G$  be an algebraic set. Then,  $G$  is connected iff for each finite algebraic set  $X$ , and for each morphism  $f : G \rightarrow X$  of algebraic sets, we have that  $f$  is constant.*

Claim with the Lemma implies that the map  $g \mapsto t(ghg^{-1})$  is constant. This implies that  $r(ghg^{-1}) = r(h)$  for all  $g \in G, h \in G'$ . Ergo,  $G$  stabilizes each eigenspace  $V_\chi$  for  $G'$ . Ergo,  $V = V_{\chi_0}$ , since  $V$  is irreducible. □

**Lemma 40.** *Let  $G$  be any group with a finite dimensional representation  $r : G \rightarrow \text{GL}(V)$ . Then, the subspaces  $V_\chi$  for  $\chi \in \text{Hom}(G, k^\times)$  are linearly independent, i.e., the map*

$$\oplus V_\chi \longrightarrow V$$

*is injective.*

*Proof.* The spaces  $V_\chi$  are  $G$ -invariant. Suppose, there exist distinct  $\chi_1, \dots, \chi_n$  of non-zero  $v_j \in V_{\chi_j}$  s.t.  $\sum_j v_j = 0$ .

We may assume that  $n$ , the number of  $v_j$ , is minimal. W.l.o.g.,  $n \geq 2$ .

Choose  $g \in G$  s.t.  $\chi_1(g) \neq \chi_2(g)$ . Use that  $0 = g \sum_j v_j = \sum_j g v_j$  and take the linear combination as in the proof of linear independence of characters to contradict the minimality of  $n$ .

( $g - \chi_1(g)$  is not zero, but reduces  $\sum_j v_j$  by one summand.) □

*Finishing Proof of Theorem.* Since  $G' = \langle ghg^{-1}h^{-1} \mid g, h \in G \rangle$ , so  $\det(r(G')) = 1$ .

On the other hand, for each  $g \in G'$ , we have

$$r(g) = \begin{pmatrix} \chi_0(g) & & \\ & \ddots & \\ & & \chi_0(g) \end{pmatrix}$$

since  $V = V_{\chi_0}$ . This implies

$$1 = \det(r(g)) = \chi_0(g)^d.$$

Ergo,  $\chi_0$  defines a morphism

$$\chi_0 : G' \longrightarrow \mu_d \subseteq \mathcal{G}_m.$$

But  $G'$  is connected and  $\mu_d$  is finite. Since  $\chi_0$  is a morphism,  $\chi_0$  must be constant, ergo the trivial character.

As a consequence, we get  $r(G') = 1$  on  $V = V_{\chi_0}$ .

**Lemma 41.** *Let  $G$  be an algebraic group,  $r : G \rightarrow \mathbf{GL}(V)$  a representation.  $v \in V$  shall be a simultaneous non-zero eigenvector for  $r(G)$ .*

*Then, for each  $g \in G$ , there is a value  $\chi(g) \in k^\times$  s.t.*

$$r(g)v =: \chi(g)v.$$

*Then, the mapping  $\chi : G \rightarrow \mathcal{G}_m$  is a morphism of algebraic groups.*

Therefore,  $r$  descends to a representation of the commutative group

$$\bar{r} : G/G' \longrightarrow \mathbf{GL}(V).$$

Ergo,  $r(G/G') = r(G)$  is commutative and therefore trigonalizable (because of irreducibility).

□

**Example 17** (Non-Example). • Take  $G = D_4 \hookrightarrow \mathbf{GL}_2(\mathbb{C})$  which is solvable and has an irreducible and faithful representation over  $\mathbb{C}^2$ .

- Consider the solvable group

$$G = \left\langle \begin{pmatrix} \pm 1 & \\ & \pm 1 \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \right\rangle$$

which is a finite subgroup of  $\mathrm{GL}_2(\mathbb{C})$ , s.t.  $\mathbb{C}^2$  defines an irreducible representation of  $G$ .

**Lemma 42** (Form of Schur's Lemma). *Let  $S$  be any commutative subset of  $\mathrm{GL}(V)$  for a finite-dimensional  $0 \neq V$  over an algebraically closed field  $k$ . Let  $V$  be  $S$ -irreducible. Then,  $\dim V = 1$ .*

*Proof.* There is nothing to show if  $S$  is empty.

Let  $s \in S$  and denote by  $V_\lambda \subseteq V$  the  $\lambda$ -eigenspace for  $s$ . Then, since  $S$  is commutative,  $V_\lambda$  is  $S$ -invariant. Therefore,  $V = V_\lambda$  for one  $\lambda \in k^\times$ .

Thus, every  $s \in S$  acts by scaling, therefore every subspace of  $V$  is  $S$ -invariant. Since  $V$  is invariant, we get  $\dim V = 1$ .  $\square$

**Corollary 7.** *Let  $G$  be a connected algebraic group. Then,  $G$  is solvable iff  $G$  is trigonalizable.*

**Proposition 3.** *If  $G$  is trigonalizable, then  $G_u$  is a normal algebraic subgroup.*

*Proof.* We have

$$G \hookrightarrow B := \left\{ \begin{pmatrix} * & \dots & * \\ & \ddots & \vdots \\ & & * \end{pmatrix} \right\} \subseteq \mathrm{GL}_n(k).$$

$B$  has the normal subgroup  $U := \left\{ \begin{pmatrix} 1 & \dots & * \\ & \ddots & \vdots \\ & & 1 \end{pmatrix} \right\}$  and we have  $G_u = G \cap U$ . Now,

$U$  is the kernel of the multiplicative morphism

$$\begin{pmatrix} a_1 & \dots & * \\ & \ddots & \vdots \\ & & a_n \end{pmatrix} \mapsto \begin{pmatrix} a_1 & \\ & a_n \end{pmatrix}.$$

$\square$

**Corollary 8.** *If  $G$  is connected and solvable, then  $G_u$  is a normal algebraic subgroup.*

## 12 Semisimple Elements of nilpotent Groups

**Theorem 17.** *Let  $G$  be a connected nilpotent algebraic group. Then, we have*

$$G_s \subseteq Z(G)$$

where  $Z(G)$  denotes the **center** of  $G$ , i.e.

$$Z(G) = \{g \in G \mid \forall h \in G : gh = hg\}.$$

**Theorem 18** (Lie-algebraic Analogue). *Let  $V$  be a finite-dimensional vectorspace. Let  $\mathfrak{g}$  be the Lie-Subalgebra of  $\text{End}(V)$ , i.e.  $\mathfrak{g}$  is a subspace s.t. we have for each  $x, y \in \mathfrak{g}$*

$$[x, y] := xy - yx \in \mathfrak{g}.$$

*Assume that  $\mathfrak{g}$  is nilpotent, i.e. there is an  $n \in \mathbb{N}_0$  s.t.*

$$[x_1, [x_2, [\dots, [x_{n-1}, x_n]]]] = 0$$

*for all  $x_1, \dots, x_n \in \mathfrak{g}$ .*

*Then, any semisimple (semisimple in  $\text{End}(V)$  that is)  $x \in \mathfrak{g}$  is **central** in  $\mathfrak{g}$ , i.e.  $[x, y] = 0$  for each  $y \in \mathfrak{g}$ .*

*Remark 7.* The Lie-algebraic Analogue implies the general theorem if – for example –  $k = \mathbb{C}$ .

*Proof.* Let  $g \in G_s$ . We want to show  $Z_G(g) = G$ .

**Fact from the theory of Lie-Algebras:** For the Lie-Algebra  $\text{Lie}Z_G(g)$  we have

$$\text{Lie}Z_G(g) = \ker(\text{Ad}(g))$$

where  $\text{Ad}$  is the map

$$\begin{aligned} \text{Ad} : G &\longrightarrow \text{GL}(\mathfrak{g}) \\ x &\longmapsto gxg^{-1}. \end{aligned}$$

Since  $G$  is connected, it suffices to verify

$$\ker(\text{Ad}(g)) = \mathfrak{g}$$

i.e.  $\text{Ad}(g) = 1$ .

Since  $g$  is semisimple, we have for suitable basis

$$g = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$$

with  $a_j \in \mathbb{C}^\times$ . This is  $\exp(x)$  for a suitable diagonal matrix  $x = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} \in \mathrm{GL}_n(\mathbb{C})$ .

**Fact:** We may assume that  $x \in \mathfrak{g} := \mathrm{Lie}(G)$ .

Since  $G$  is nilpotent, it can be shown that  $\mathfrak{g}$  is nilpotent.

By the theorem,  $x$  is central in  $\mathfrak{g}$ . By the properties of  $\exp$  we have

$$\mathrm{Ad}(g) = \exp(\mathrm{ad}(g)) = 1$$

ergo  $\mathrm{ad}(x) = 0$  where  $\mathrm{ad} : \mathfrak{g} \rightarrow \mathfrak{g}$  is defined by

$$\mathrm{ad}(x) \cdot y := [x, y].$$

□

*Proof.* If  $\mathfrak{g}$  is nilpotent, then  $\mathrm{ad}(x) \in \mathrm{End}(\mathfrak{g})$  is nilpotent.

Since  $x$  is semisimple,  $\mathrm{ad}(x)$  is semisimple, because  $\mathrm{ad}(x)$  is the restriction to  $\mathfrak{g}$  of the map

$$\begin{aligned} \mathrm{End}(V) &\longrightarrow \mathrm{End}(V) \\ y &\longmapsto [x, y] \end{aligned}$$

and, if  $e_1, \dots, e_n$  are a basis of eigenvectors for  $x$ , then  $E_{i,j}$  is a basis of eigenvectors for  $\ell$ .

So,  $\mathrm{ad}(x)$  is nilpotent and semisimple, therefore  $\mathrm{ad}(x) = 0$ . □

*Proof Theorem.* Let  $G$  be a connected nilpotent algebraic group,  $G \xrightarrow{\mathrm{GL}} (V)$ .

Let  $g \in G_s$ , we want to show that  $g \in Z(G)$ .

Assume otherwise, then we have a  $h \in G$  s.t.  $(g, h) = ghg^{-1}h^{-1} \neq 1$ .

Since  $G$  is connected and nilpotent (ergo solvable), we know by Lie-Kolchin that  $G$  stabilizes some complete flag  $V_0 \subset \dots \subset V_n$ .

We have  $g|_{V_i}, h|_{V_i} \in \text{GL}(V_i)$ . They commute, if  $i = 0$ , but not if  $i = n$ .

So, there is an  $i$  s.t.  $g|_{V_i}, h|_{V_i}$  commute but  $g|_{V_{i+1}}, h|_{V_{i+1}}$  don't commute. W.l.o.g.  $V = V_{i+1}, g = g|_{V_{i+1}}, h = h|_{V_{i+1}}$ . Set  $a := g|_{V_i}, b := h|_{V_i} \in \text{GL}(V_i)$ .  $a$  will be semisimple, since  $g$  is.

Since  $g$  is semisimple, there is an eigenvector  $v \in V_{i+1}$  for  $g$  s.t.

$$V_{i+1} = V_i \oplus \langle v \rangle.$$

We have an isomorphism of vector spaces

$$\text{End}(V_{i+1}) \cong \text{End}(V_i) \oplus \text{Hom}(\langle v \rangle, V_i) \oplus \text{Hom}(V_i, \langle v \rangle) \oplus \text{End}(\langle v \rangle)$$

with

$$\text{End}(\langle v \rangle) \cong k \text{ and } \text{Hom}(\langle v \rangle, V_i) \cong V_i.$$

So, we can write  $g|_{V_{i+1}}, h|_{V_{i+1}}$  write as

$$g = \begin{pmatrix} a & \\ & * \in k \end{pmatrix} \text{ and } h = \begin{pmatrix} b & c \in V_i \\ & * \end{pmatrix}.$$

We may replace  $g, h$  with scalar multiples to reduce to the case that  $* = 1$ . Then, So, we can write  $g|_{V_{i+1}}, h|_{V_{i+1}}$  write as

$$g = \begin{pmatrix} a & \\ & 1 \end{pmatrix} \text{ and } h = \begin{pmatrix} b & c \\ & 1 \end{pmatrix}.$$

Then,

$$h \neq ghg^{-1} = \begin{pmatrix} b & ac \\ & 1 \end{pmatrix}.$$

Ergo,  $c \neq ac$ , i.e.  $c \notin \ker(a - 1)$ . Define

$$h_1 := h^{-1}ghg^{-1} = \begin{pmatrix} 1 & b^{-1}(a-1)c \\ & 1 \end{pmatrix}.$$

We claim that  $h_1$  does not commute with  $g$ . This claim implies the theorem, since we can iterate the claim to obtain elements  $h_i$  by  $h_{i+1} := h_i^{-1}gh_i g^{-1}$ . Then,  $h_i$  does not commute with  $g$ . But  $G$  is nilpotent, therefore  $h_i = 1$  for some large enough  $i$ .

We can prove the claim as follows: By some calculation as for  $h$  and  $g$ , we see, that  $h_1$  and  $g$  don't commute iff  $b^{-1}(a-1)c \notin \ker(a-1)$ . This is equivalent to

$$\begin{aligned} &\iff (a-1)b^{-1}(a-1)c \neq 0 \\ &\iff b^{-1}(a-1)^2c \neq 0 \\ &\iff (a-1)^2c \neq 0 \\ &\iff c \notin \ker((a-1)^2). \end{aligned}$$



But  $a$  being semisimple implies  $a - 1$  being semisimple, therefore

$$\ker((a - 1)^2) = \ker(a - 1).$$

So  $h_1, g$  don't commute iff  $c \in \ker(a - 1)$  iff  $h, g$  don't commute. □

## 13 Algebraic Geometry

### 13.1 Projective Algebraic Sets

Let  $V$  be a finite-dimensional vector space. Then  $\mathcal{G}_m = k^\times$  acts on  $V$  by scalar multiplication.  $\{0\}$  is a  $\mathcal{G}_m$ -invariant subspace of  $V$ . We are interested on the orbits of  $\mathcal{G}_m$  on  $V \setminus \{0\}$ .

Define the **projective space** over  $V$  by

$$\mathbb{P}V := \mathcal{G}_m \backslash (V - 0) = (V - 0) / \sim \cong \{\text{lines in } V\}$$

where for  $a, b \in V - 0$  we set

$$a \sim b : \iff \exists \lambda \in k^\times : \lambda a = b.$$

If  $V = k^{n+1}$ , we denote the  $n$ -dimensional projective space by  $\mathbb{P}^n := \mathbb{P}V$ .

Given  $a = (a_0, a_1, \dots, a_n) \in k^{n+1} - 0$ , we denote the  $\sim$ -class of  $a$  by

$$[a] = [a_0, \dots, a_n] \in \mathbb{P}^n.$$

Define  $S$  to be the graded algebra of polynomials in  $k$

$$S := k[x_0, \dots, x_n] = \bigoplus_{d \geq 0} S_d$$

where each  $S_d$  is the space of homogenous polynomials of degree  $d$ , i.e.

$$S_d = \bigoplus_{i_1, \dots, i_d \in \{0, \dots, n\}} k \cdot x_{i_1} \cdots x_{i_d}.$$

We identify  $k$  with the space of constant polynomials  $S_0 \subseteq S$ .

We have

$$S_d = \{f \in S \mid f(\lambda X) = \lambda^d f(X) \ \forall \lambda \in k^\times\}.$$

Given  $f \in S_d$ , the set

$$\{a \in k^{n+1} \mid f(a) = 0\}$$

is  $\mathcal{G}_m$ -invariant. In other words, given  $a \in \mathbb{P}^n$  and  $f \in S^d$ , it is well-defined to state  $f(a) = 0$  and  $f(a) \neq 0$ .

**Definition 34.** A **projective algebraic subset**  $X \subseteq \mathbb{P}^n$  is a set of the form

$$X = V(\Sigma) := V_{\mathbb{P}^n}(\Sigma)$$

where  $\Sigma$  is a collection of homogenous elements of  $S$ , where

$$V_{\mathbb{P}^n}(\Sigma) := \{a \in \mathbb{P}^n \mid f(a) = 0 \ \forall f \in \Sigma\}.$$

**Facts:**

- Hilbert basis theorem states

$$V(\Sigma) = V(f_1, \dots, f_m)$$

for some finite collection  $f_1, \dots, f_m \in \Sigma$ .

- It is useful to extend the meaning of " $f(a) = 0$ " for  $a \in \mathbb{P}^n$  to *general* elements  $f \in S$  by requiring that  $f(a') = 0$  for each  $a' \in [a]$ .

If we write  $f = \sum_{d \geq 0} f_d$ ,  $f_d \in S_d$ , then we have

$$f(a) = 0 \iff f_d(a) = 0 \ \forall d \geq 0.$$

Therefore, we can extend the definition of  $V(\Sigma)$  to any  $\Sigma \subseteq S$ .

- We have  $V(\Sigma) = V((\Sigma))$  where  $(\Sigma)$  is the ideal generated by some finite subset of  $\Sigma$ .
- We call an ideal  $I \subseteq S$  **homogenous** if it is the direct sum of its  $d$ -homogeneous components, i.e.

$$I = \sum_{d \geq 0} I_d$$

where  $I_d = \{f \in I \mid f \text{ is homogenous of degree } d\}$ .

$I$  is homogeneous iff it is generated by homogeneous elements.

- We have the following *Nullstellensatz*:

For any  $X \subseteq \mathbb{P}^n$ , set  $I(X)$  to be the ideal generated by all homogeneous polynomials of  $S$  vanishing on  $X$ .

Let  $I \subseteq S$  be a *homogeneous* ideal which is *not equal* to  $(x_0, \dots, x_n)$ . Then, we have

$$I(V_{\mathbb{P}^n}(I)) = \sqrt{I}.$$

**Example 18** (Anti-example). The second property is necessary:

Set  $I = (x_0, \dots, x_n)$ . Then  $V_{k^{n+1}}(I) = 0$ . Therefore,  $V_{\mathbb{P}^n}(I) = \emptyset$ . However,

$$I(V_{\mathbb{P}^n}(I)) = S.$$

- The above point induces a bijection between algebraic subsets of  $\mathbb{P}^n$  and radical ideals  $I \subset S$  which are not  $(x_0, \dots, x_n)$ .

For  $i = 0, \dots, n$ , set  $D(x_i) := \{a \in \mathbb{P}^n \mid a_i \neq 0\}$ .  $D(x_i)$  is an open set homeomorphic to  $k^n$  by mapping

$$\phi_i : a \mapsto \left( \frac{a_0}{a_i}, \dots, \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i}, \dots, \frac{a_n}{a_i} \right).$$

The  $D(x_i)$  cover  $\mathbb{P}^n = \bigcup_i D(x_i)$ .

Given a projective algebraic subset  $X \subset \mathbb{P}^n$ , define  $X^{(i)} \subset k^n$  by

$$X^{(i)} := \phi_i(X \cap D(x_i)).$$

If  $X = V_{\mathbb{P}^n}(I)$ , then

$$X^{(i)} = V_{k^n}(I^{(i)})$$

where

$$I^{(i)} := \{f^{(i)} \mid f \in I\}$$

where  $f^{(i)}(t_1, \dots, t_n) := f(t_1, \dots, t_{i-1}, 1, t_i, \dots, t_n)$ . Thus,  $X^{(i)}$  is an algebraic subset of  $k^n$ .

**Definition 35.** The **Zariski topology** on  $\mathbb{P}^n$  is defined by setting the set of closed sets to be the set of projective algebraic sets.

**Facts:**

- $D(x_i)$  is open in  $\mathbb{P}^n$ , since  $D(x_i) = \mathbb{P}^n - V(x_i)$ .
- The bijections  $D(x_i) \cong k^n$  are homeomorphisms.

**Definition 36.** A **quasi-projective** algebraic set  $Y$  is an open subset of a projective algebraic set  $X \subseteq \mathbb{P}^n$ .

**Example 19.** Any algebraic set in  $k^n$  is quasi-projective.

**Definition 37.** A **quasi-projective variety** is defined as an irreducible quasi-projective algebraic set.

**Lemma 43** (Products). *Define the **Segre-embedding** by*

$$\begin{aligned} S^{n,m} : \mathbb{P}^n \times \mathbb{P}^m &\hookrightarrow \mathbb{P}^{nm+n+m} \\ (a, b) &\mapsto [(a_i b_j)_{i,j=0,\dots,n}]. \end{aligned}$$

*We have:*

1.  $S^{n,m}$  is injective.
2.  $S^{n,m}$  has a closed image.
3.  $k^n \times k^m \cong D(z_{00}) \cap S^{n,m}(\mathbb{P}^n \times \mathbb{P}^m) = S^{n,m}(D(x_0) \times D(y_0))$ .

**Definition 38.** For quasi-projective algebraic sets  $X \subset \mathbb{P}^n, Y \subset \mathbb{P}^m$ , we define their product by

$$X \times Y := S^{n,m}(X, Y) \subseteq \mathbb{P}^{nm+n+m}.$$

Then,  $X \times Y$  is a quasi-projective algebraic subset of  $\mathbb{P}^{nm+n+m}$ .

## 13.2 Flag Varieties

**Definition 39.** We define the **Grassmannian manifold** by

$$G(n, d) := \{W \subset k^n \mid W \text{ is a } d\text{-dimensional subvectorspace}\}.$$

Then, we have the **Plücker-embedding** by

$$\begin{aligned} P_d : G(n, d) &\longrightarrow \mathbb{P} \left( \bigwedge^d k^n \right) = \mathbb{P}^{\binom{n}{d}} \\ W &\longmapsto [w_1 \wedge \dots \wedge w_d] \end{aligned}$$

where  $w_1, \dots, w_d$  is a basis of  $W$ .

**Lemma 44.**  $P_d$  is injective and has a closed image.

Therefore, we can see  $G(n, d)$  as a projective algebraic set.

**Definition 40.** Let  $V$  be a finite-dimensional vector space of dimension  $n$ . Set

$$\mathrm{Gr}_d(V) := \{d\text{-dim. subspaces of } V\} \cong G(n, d).$$

Define further the **flag manifold** to be

$$\mathrm{Flag}(V) := \{\text{complete flags } \mathcal{F} = (0 = V_0 \subseteq V_1 \subseteq \dots \subseteq V_n = V)\}.$$

Then, we have a map

$$\begin{aligned} P_v : \mathrm{Flag}(V) &\longrightarrow \mathrm{Gr}_0(V) \times \dots \times \mathrm{Gr}_n(V) \\ \mathcal{F} &\longmapsto (V_0, \dots, V_n). \end{aligned}$$

**Lemma 45.**  $P_v$  has a closed image and is injective.

Thus, we can see  $\mathrm{Flag}(V)$  as a projective algebraic set.

**Lemma 46.**  $\mathrm{Gr}_d(V)$  and  $\mathrm{Flag}(V)$  are both irreducible, hence projective alg. varieties.

$\mathrm{Flag}(V)$  is called the **variety of complete flags**.

### 13.3 Local Rings and Function Fields

**Definition 41.** An **affine variety** is an irreducible algebraic subset of  $k^n$ .

**Definition 42.** If  $X$  is an affine variety, then the coordinate ring  $\mathcal{O}(X)$  is a domain. Define the **function field** of  $X$  by

$$k(X) := \text{Frac}(\mathcal{O}(X)) := \left\{ \frac{a}{b} \mid a, b \in \mathcal{O}(X), b \neq 0 \right\}.$$

**Definition 43.** Let  $p \in X$ . We define the **local ring** of  $\mathcal{O}(X)$  at  $p$  by

$$\mathcal{O}_{X,p} := \left\{ \frac{a}{b} \mid a \in \mathcal{O}(X), 0 \neq b \in \mathcal{O}(X), b(p) \neq 0 \right\} \subset k(X).$$

We have an **evaluation** map

$$\begin{aligned} \text{eval}_p : \mathcal{O}_{X,p} &\longrightarrow k \\ \frac{a}{b} &\longmapsto \frac{a(p)}{b(p)}. \end{aligned}$$

**Lemma 47.** Let  $X$  be an affine variety. Then

$$\mathcal{O}(X) = \bigcap_{p \in X} \mathcal{O}_{X,p}.$$

**Definition 44.** Let  $X \subset \mathbb{P}^n$  be a projective variety. Denote by  $I_{\mathbb{P}}(X)$  its homogenous vanishing ideal.

Define its **function field** by

$$k(X) := R/M,$$

where

$$\begin{aligned} R &:= \left\{ \frac{f}{g} \mid f, g \in k[x_0, \dots, x_n] \text{ homogen.}, \deg f = \deg g, g \notin I_{\mathbb{P}}(X) \right\}, \\ M &:= \left\{ \frac{f}{g} \in R \mid f \in I_{\mathbb{P}}(X) \right\}. \end{aligned}$$

**Lemma 48.**  $M$  is a maximal ideal in  $R$  and  $R/M$  is a field.

**Lemma 49.** If  $X$  is a projective variety, then  $X^{(i)} \subset k^n$  is an affine variety.

If  $X^{(i)} \neq \emptyset$ , then

$$k(X) \cong k(X^{(i)}).$$

**Definition 45.** Let  $X$  be a projective variety. For  $p \in X$ , we define its **local ring** at  $p$  by

$$\mathcal{O}_{X,p} := \left\{ \frac{f}{g} \in k(X) \mid g(p) \neq 0 \right\} \subset k(X).$$

**Lemma 50.** *For a projective variety  $X$ , we have:*

1. *For  $p \in X^{(i)}$ :  $\mathcal{O}_{X,p} \cong \mathcal{O}_{X^{(i)},p}$ .*
2. *For  $p \in X^{(i)} \cap X^{(j)}$ :  $\mathcal{O}_{X^{(j)},p} \cong \mathcal{O}_{X^{(i)},p}$ .*

**Definition 46.** If  $X \subset \mathbb{P}^n$  is quasi-projective variety, there is a minimal projective variety  $\overline{X} \subset \mathbb{P}^n$  which contains  $X$  as an open subset.

Then, we can set

$$\begin{aligned} k(X) &:= k(\overline{X}) \\ \mathcal{O}_{X,p} &:= \mathcal{O}_{\overline{X},p}. \end{aligned}$$



### 13.4 Regular Functions and Morphisms

**Definition 47.** Let  $X$  be quasi-projective variety. Let  $U \subseteq X$  be open. Then, we define the **ring of regular functions** on  $U$  by

$$\mathcal{O}(U) := \bigcap_{P \in U} \mathcal{O}_{X,P} \subseteq k(X).$$

**Definition 48.** Let  $X, Y$  be two quasi-projective varieties. A map  $f : X \rightarrow Y$  is called a **morphism**, if  $f$  is continuous and we have

$$f^* \mathcal{O}(U) := \{h \circ f \mid h \in \mathcal{O}(U)\} \subseteq \mathcal{O}(f^{-1}(U)).$$

*Remark 8.* Let  $X, Y$  be affine varieties and  $f : X \rightarrow Y$  be a map. Then we have

$$f^* \mathcal{O}(U) \subseteq \mathcal{O}(f^{-1}(U))$$

iff  $f$  is given by polynomials.

**Lemma 51.** *Let  $X$  be a quasi-projective variety and let  $p \in X$ .*

*Then there is an open neighborhood  $U$  of  $p$  in  $X$  s.t.  $U$  is isomorphic (as quasi-projective varieties) to an affine variety  $U' \subset k^n$ .*

*Proof.* Let  $Y$  be a projective variety s.t.  $X$  lies open in  $Y$ . By replacing  $X \hookrightarrow Y$  with  $X^{(i)} \hookrightarrow Y^{(i)}$ , we may assume that  $X$  is an open subset of an affine variety  $Y$  in  $k^n$ .

Since the sets  $D(f)$  give an open basis of  $k^n$ , there is a  $f \in \mathcal{O}(Y)$  s.t.

$$p \in D_Y(f) := \{y \in Y \mid f(y) \neq 0\} \subset X.$$

Now,  $D_Y(f)$  is affine, because the map

$$\begin{aligned} D_Y(f) &\longrightarrow \{(q, r) \in k^{n+1} \mid q \in Y, f(q)r = 1\} \\ q &\longmapsto (q, \frac{1}{f(q)}) \end{aligned}$$

is an isomorphism of quasi-projective varieties. □

## 13.5 Dimensions

**Definition 49.** Let  $X$  be a quasi-projective variety. We define its **dimension** as the transcendency degree of its function field, i.e.

$$\dim(X) := \text{tr.-deg}_k(k(X)).$$

*Remark 9.* If  $X$  is affine, then

$$\begin{aligned} \dim(X) &= \text{tr.-deg}_k(k(X)) \\ &= \dim_{\text{Krull}}(\mathcal{O}(X)) \\ &= \sup \{n \in \mathbb{N}_0 \mid P_0 \subsetneq P_1 \subsetneq \dots \subsetneq P_n, P_i \text{ prime in } \mathcal{O}(X)\} \\ &= \sup \{n \in \mathbb{N}_0 \mid Z_n \subsetneq \dots \subsetneq Z_0, Z_i \text{ closed, irreducible in } X\} \end{aligned}$$

*Remark 10.* If  $U \subset X$  is open, then  $k(U) = k(X)$  and  $\dim(U) = \dim(X)$ .

**Lemma 52.** Let  $\phi : X \rightarrow Y$  be a surjective morphism of quasi-projective varieties. Then,

$$\dim X \geq \dim Y.$$

*Proof.*  $\phi$  induces an inclusion

$$\begin{aligned} \phi^* : k(Y) &\hookrightarrow k(X) \\ [(U_i, \alpha_i)_i] &\longmapsto [(\phi^{-1}(U_i), \alpha_i \circ \phi)]. \end{aligned}$$

This map is indeed injective, since  $\phi$  is surjective. Therefore, the claim follows.  $\square$

**Lemma 53.** Let  $X$  be a quasi-projective variety and  $Y$  a proper, closed subvariety. Then,

$$\dim(Y) < \dim(X).$$

*Proof.* By going from  $X$  to its closure  $\tilde{X}$  and from there to  $\tilde{X}^{(i)}$ , we can assume that  $X$  is affine.

Then,  $I_X(Y)$  is a non-trivial prime ideal in  $\mathcal{O}(X)$ . Therefore, we have

$$\dim_{\text{Krull}}(\mathcal{O}(Y)) = \dim_{\text{Krull}}(\mathcal{O}(X)/I_X(Y)) \leq \dim_{\text{Krull}}(\mathcal{O}(X)),$$

since  $A$  is a finitely generated  $k$ -algebra and a domain.  $\square$

**Lemma 54.** *Let  $X$  be an affine variety and  $f \in \mathcal{O}(X)$  be non-zero.*

*Then, the set*

$$V_X(f) := \{p \in X \mid f(p) = 0\}$$

*is a proper, closed subset of  $X$  and we can decompose it into irreducible components*

$$V_X(f) = Z_1 \cup \dots \cup Z_l.$$

*For each of those  $Z_i$ , we have*

$$\dim(Z_i) = \dim(X) - 1.$$

*Proof.* The  $Z_i$  correspond to minimal prime ideals  $P_i$  in  $\mathcal{O}(X)$  which contain  $(f)$ . Since, they are minimal, we have

$$\text{height}(P_i) = 1.$$

□

**Lemma 55.** *Let  $X$  be an quasi-projective algebraic set. Then – as in the affine case – we may write*

$$X = Z_1 \cup \dots \cup Z_l$$

*where each  $Z_i \subseteq X$  is an **irreducible component**, i.e. a maximal closed irreducible subset.*

*We then define*

$$\dim(X) := \max_i \dim(Z_i).$$

**Lemma 56.** *Let  $\phi : X \xrightarrow{Y}$  be a morphism of quasi-projective varieties. Further, let  $\phi$  be **dominant**, i.e.,  $\text{Im}\phi$  is dense.*

*Then, for all  $p \in \text{Im}(\phi)$ , we have the following for the fiber of  $\phi$  along  $p$ :*

$$\dim(\phi^{-1}(p)) \geq \dim(X) - \dim(Y).$$