

Mitschrieb: Algebraic Groups

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Vorwort

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1 Introduction

Let k be an algebraically closed field.

Definition 1. For $I \subseteq k[X] := k[X_1, \dots, X_n]$, we define its **vanishing set** by

$$V(I) := \{p \in k^n \mid \forall f \in I : f(p) = 0\}.$$

A set $S \subset k^n$ is called **algebraic**, if

$$S = V(I)$$

for some $I \subseteq k[X]$.

Example 1. The group $\mathrm{GL}_n(k)$ is not an algebraic subset of $k^{n \times n}$. But, we can identify it with an algebraic subset of $(k^{n \times n})^2$ by

$$\mathrm{GL}_n(k) \cong \{(x, y) \in k^{n \times n} \mid xy = 1_n\} = V(X \cdot Y - 1_n).$$

Definition 2. Let $\iota : \mathrm{GL}_n(k) \hookrightarrow k^{n \times n^2}$ be the injection

$$A \mapsto (A, A^{-1}).$$

A **linear algebraic group** over k is a subgroup $U \subseteq \mathrm{GL}_n(k)$ s.t. $\iota(k)$ is an algebraic subset of k^{2n^2} .

I.e., a linear algebraic group is a matrix-group which can be defined by polynomials over the entries of a matrix and its inverse.

Example 2. The following groups are linear algebraic groups:

1. The multiplicative group $\mathcal{G}_m(k) := k^\times = k \setminus \{0\} = \mathrm{GL}_1(k)$.
2. The general linear group $\mathrm{GL}_n(k)$.
3. The special linear group

$$\mathrm{SL}_n(k) := \{A \in \mathrm{GL}_n(k) \mid \det(A) = 1\}.$$

4. The orthogonal group

$$\mathrm{O}_n(k) := \{A \in \mathrm{GL}_n(k) \mid A^T \cdot A = 1\}.$$

5. The special orthogonal group

$$\mathrm{SO}_n(k) := \mathrm{O}_n(k) \cap \mathrm{SL}_n(k).$$

6. The upper triangle-matrix group

$$\left\{ \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ & \ddots & \vdots \\ & & a_{n,n} \end{pmatrix} \mid a_{i,j} \in k \right\} \cap \mathrm{GL}_n(k).$$

7. The normed upper triangle-matrix group

$$\left\{ \begin{pmatrix} 1 & \cdots & a_{1,n} \\ & \ddots & \vdots \\ & & 1 \end{pmatrix} \mid a_{i,j} \in k \right\} \cap \mathrm{GL}_n(k).$$

8. The group of n -th roots of unity

$$\mu_n(k) := \{x \in k \mid x^n = 1\}.$$

9. The additive group $(k, +)$ is not a subgroup of $\mathrm{GL}_n(k)$, but it can be identified with the linear algebraic group

$$\left\{ \begin{pmatrix} 1 & a \\ & 1 \end{pmatrix} \mid a \in k \right\} \subset \mathrm{GL}_2(k)$$

10. For $k = \mathbb{C}$, the unit sphere and the unitary groups are NOT linear algebraic groups.

2 Algebraic Groups and Hopf Algebras

Definition 3. A **morphism** $f : X \rightarrow Y$ of algebraic sets $X \subset k^m, Y \subset k^n$ is a map which is coordinatewise described by polynomials.

Definition 4. An **algebraic group** is an algebraic set $G \subset k^n$ together with a fixed element $e \in G$ and morphisms $m : G \times G \rightarrow G, i : G \rightarrow G$ s.t. (G, m, i, e) is a group.

A **morphism of algebraic groups** is a morphism of algebraic sets that is also a group homomorphism.

Definition 5. Let $V \subset k^n$ be any subset. Then, we define the vanishing ideal of V by

$$I(V) := \{f \in k[x] \mid f(V) = 0\}.$$

Definition 6. For a commutative ring R we define the **radical** of an ideal $I \subseteq R$ by

$$\sqrt{I} := \{r \in R \mid r^m \in I \text{ for some } m \in \mathbb{N}_0\}.$$

R is called **reduced**, if $\sqrt{0} = 0$.

Lemma 1 (Zariski's Lemma). *Let $L \supseteq k$ be fields. If L is finitely generated as a k -algebra, then the extension $L \supseteq k$ is finite, i.e., L is a finitely-generated k -vector space.*

Theorem 1 (Hilbert's Nullstellensatz). *For any ideal $I \subseteq k[x]$, we have*

$$I(V(I)) = \sqrt{I}.$$

Proof. It is easy to see that

$$I \subset \sqrt{I} \subset I(V(I)).$$

Now, let $f \in I(V(I))$ and assume – for the sake of contradiction – that $f \notin \sqrt{I}$. Since \sqrt{I} is the intersection of its upper prime ideals, there is a prime ideal $p \supset I$, s.t. $f \notin p$. Now, define the zero divisor-free ring

$$R := (k[x]/p)[f^{-1}].$$

And let $\phi : k[x] \rightarrow R$ be the corresponding ring homomorphism.

Let $m \subseteq R$ be a maximal ideal in R . Then, R/m is a field, which contains k and is finitely generated as k -algebra. According to Zariski's lemma, R/m is a finite (ergo algebraic) extension of k . Since k is algebraically closed, we have $R/m = k$. Let $\pi_m : R \rightarrow k$ be the corresponding ring homomorphism.

Now, for x_1, \dots, x_n , set

$$t_i := \pi_m(\phi(x_i)).$$

Then, $t = (t_1, \dots, t_n) \in k^n$. We now have

1. $t \in V(I)$: For each $g \in I$, we have $\phi(g) = 0$. On the other hand

$$g(t) = g(\pi_m \circ \phi(x)) = \pi_m \circ \phi(g) = 0.$$

2. $f(t) \neq 0$: $\phi(f)$ is invertible in R , therefore $\phi(f) \neq 0$ and $\phi(f) \notin m$. Ergo

$$f(t) = \pi_m \circ \phi(f) \neq 0.$$

Ergo, there is a point $t \in V(I)$ s.t. $f(t) \neq 0$. This yields a contradiction, since we assumed $f \in I(V(I))$. \square

Definition 7. For an algebraic set $X \subset k^n$, we define its **coordinate ring** by

$$k[X] := k[x_1, \dots, x_n]/I(X).$$

Lemma 2. For a morphism $f : X \rightarrow Y$ of algebraic sets define the following homomorphism of k -algebras.

$$\begin{aligned} f^* : k[Y] &\longrightarrow k[X] \\ p &\longmapsto p \circ f. \end{aligned}$$

We have a contravariant functor $_*$ from the categories of algebraic sets over k to the category of k -algebras:

$$\begin{aligned} X &\longmapsto k[X] \\ \text{Hom}(X, Y) &\longmapsto \text{Hom}_k(k[Y], k[X]) \\ f &\longmapsto f^*. \end{aligned}$$

Lemma 3. We have

$$k[X \times Y] \cong k[X] \otimes k[Y].$$

Proof.

$$k[X] \otimes k[Y] = k[x]/I(X) \otimes_k k[y]/I(Y) = k[x, y]/I(X) \otimes k[y] + k[x] \otimes I(Y).$$

But

$$V(I(X) \otimes k[y] + k[x] \otimes I(Y)) = V(I(X) \otimes k[y]) \cap V(k[x] \otimes I(Y)) = X \times Y.$$

\square

Theorem 2. *Every finitely generated reduced k -algebra A is isomorphic to some $k[X]$ for some algebraic X .*

Proof. Choose some $\pi : k[x_1, \dots, x_n] \twoheadrightarrow A$ and set $X := V(\ker \pi)$. Then $\ker \pi = I(X)$, since π 's kernel is radical since A is reduced. \square

Corollary 1. *The contravariant functor $_* : \mathcal{C}_{\text{algSets}} \rightarrow \mathcal{C}_{k\text{-alg.s}}$ gives an antiequivalence of categories.*

Lemma 4. *An algebraic set X is isomorphic to some algebraic subset of Y iff there is an epimorphism $k[Y] \twoheadrightarrow k[X]$.*

Lemma 5. *Let $G \subset k^n$ be an algebraic group. Then, we have maps*

$$\begin{aligned} m : G \times G &\longrightarrow G \\ i : G &\longrightarrow G \\ e : * &\longrightarrow G. \end{aligned}$$

They induce dual maps in the category of k -algebras:

$$\begin{aligned} \Delta &:= m^* : k[G] \longrightarrow k[G] \otimes_k k[G] \\ \iota &:= i^* : k[G] \longrightarrow k[G] \\ \varepsilon &:= e^* : k[G] \longrightarrow k \end{aligned}$$

Definition 8. A **Hopf-algebra** over k is a (reduced?!) k -algebra together with maps $\Delta, \varepsilon, \iota$ as above s.t. the following holds:

$$\begin{aligned} (\Delta \otimes \text{Id})\Delta &= (\text{Id} \otimes \Delta)\Delta \\ (\iota \otimes \text{Id})\Delta &= (\text{Id} \otimes \iota)\Delta = \varepsilon \\ (\varepsilon \otimes \text{Id})\Delta &= (\text{Id} \otimes \varepsilon)\Delta = \text{Id} \end{aligned}$$

A morphism of Hopf-algebras is a homomorphism of k -algebra $F : A \rightarrow B$ s.t.

$$\Delta \circ F = (F \otimes F) \circ \Delta.$$

Theorem 3. *The contravariant functor $_*$ gives an anti-equivalence of the categories of algebraic groups and the categories of finitely generated Hopf-algebras over k .*

Example 3. 1. Let $G = \mathcal{G}_a = (k, +)$. Then, $k[G] = k[x]$, since $I(x) = 0$. Then, we have

$$\begin{aligned} \Delta(x) &= x \otimes 1 + 1 \otimes x \\ \iota(x) &= -x \\ \varepsilon(x) &= 0. \end{aligned}$$

2. Let $G = \mathcal{G}_m = \{(a, a^{-1}) \mid a \neq 0\} \cong k^\times$. Then, $k[G] = k[x, y]/(xy - 1) = k[x, x^{-1}]$. Then, we have

$$\begin{aligned}\Delta(x) &= x \otimes x \\ \iota(x) &= x^{-1} \\ \varepsilon(x) &= 1.\end{aligned}$$

3. Let $G = \mathrm{GL}_n(k)$. Then, $k[G] = k[x, y]/(xy - 1_n) = k[x_{i,j}, \frac{1}{\det}]$. Then, we have

$$\begin{aligned}\Delta(x_{i,j}) &= \sum_k x_{i,k} \otimes x_{k,j} \\ \iota(x_{i,j}) &= (x^{-1})_{i,j} \\ \varepsilon(x_{i,j}) &= \delta_{i,j}.\end{aligned}$$

Recall: Last time we introduced the **Zariski-Topology** on X .

There, algebraic sets equal closed sets.

We called a set X **irreducible** iff each open subset lies dense in X .

Lemma 6. *For an algebraic set X , the following are equivalent:*

- (1) X is irreducible.
- (2) $k[X] = k[x_1, \dots, x_n]/I(X)$ is a domain.
- (3) $I(X)$ is a prime ideal.

The proof of (2) \iff (3) is a basic algebraic result.

Lemma 7. *An open base for the Zariski-Topology on an algebraic set X is given by sets:*

$$D(f) := \{p \in X \mid f(p) \neq 0\}$$

for each $f \in k[X]$. We call the $D(f)$ **basic open sets**.

Proof. Suppose $U \subseteq X$ is nonempty and open. Set

$$Z := X \setminus U$$

then Z is closed. Thus

$$Z = \{x \in X \mid f(x) = 0 \forall f \in I\}$$

for some ideal $I \subseteq k[X]$. Let $p \in U$, then there is an $f \in I$ s.t.

$$f(p) \neq 0.$$

Also, $D(f) \cap Z = \emptyset$, thus $p \in D(f) \subseteq U$. □

Lemma 1. It is clear that (2) is equivalent to (3).

(1) is equivalent to

$$\forall \text{ nonempty, open } U_1, U_2 \subset X : U_1 \cap U_2 \neq \emptyset$$

$$\stackrel{\text{Lemma 2}}{\iff} \forall \text{ nonempty, basic open } D(f_1), D(f_2) \subset X : D(f_1) \cap D(f_2) \neq \emptyset$$

Since $D(f_1) \cap D(f_2) = D(f_1 f_2)$, this is equivalent to the statement

$$f_1, f_2 \in k[X] : f_1, f_2 \neq 0 \implies f_1 f_2 \neq 0.$$

Which states that $k[X]$ is a domain. □

Lemma 8. *Let X be an algebraic set. We have bijections*

$$\{\text{closed subsets } Z \subseteq X\} \leftrightarrow \{\text{radical ideals } I \subset k[X]\}$$

and

$$\{\text{irreducible, closed subsets } Z \subseteq X\} \leftrightarrow \{\text{prime ideals } I \subset k[X]\}$$

and

$$\{\text{points of } X\} \leftrightarrow \{\text{maximum ideals } I \subset k[X]\}.$$

Lemma 9 (Primary Decompositions, Atiyah, Macdonald Ch. 4). *For an ideal I we call $P \supseteq I$ a **minimal prime** of I if P is a prime ideal and we have for each prime ideal Q :*

$$P \supseteq Q \supseteq I \implies P = Q.$$

*Any radical ideal I of $k[x_1, \dots, x_n]$ has only finitely many **minimal** primes P_1, \dots, P_r . In particular,*

$$I = \bigcap_{i=1}^n P_i$$

and for each i

$$P_i \not\supseteq \bigcap_{j:j \neq i} P_j.$$

Definition 9. An **(irreducible) component** Z of X is a maximal irreducible closed subset, i.e., an irreducible closed $Z \subseteq X$ s.t. there does not exist an irreducible closed $Y \subset X$ s.t. $Y \supsetneq Z$.

Then, we have the bijection

$$\{\text{irreducible components of } X\} \leftrightarrow \{\text{minimal primes of } I(X)\}.$$

Lemma 10. *Any algebraic set X has finitely many components Z_1, \dots, Z_r . We have*

$$X = Z_1 \cup \dots \cup Z_r$$

and for each i

$$Z_i \not\subset \bigcup_{j:j \neq i} Z_j.$$

Example 4. 1. Let $X = V(x \cdot y) \subset k^2$. Then $X = Z_1 \cup Z_2$ where $Z_1 = V(x)$, $Z_2 = V(y)$.

X is connected, but not irreducible ($D(x)$ does not lie dense in X).

2. Let X be a **finite** algebraic set. It is easy to check that every subset of X is closed:

$$\{p\} = V(x_1 - p_1, \dots, x_n - p_n)$$

for each $p \in X$. Further

$$X = \{p_1\} \cup \dots \cup \{p_r\}.$$

Moreover: Any function $f : X \rightarrow k$ is regular (i.e. given by polynomials).

Lemma 11. *We call an element $e \in k[X]$ **idempotent** iff $e^2 = e$.*

Let X be an algebraic set. Then

$$\begin{aligned} X \text{ connected} &\iff \text{the only idempotents } e \in k[X] \text{ are } 0 \text{ and } 1 \\ &\iff k[X] \not\cong A \times B \text{ for any } k\text{-algebras } A, B. \end{aligned}$$

Lemma 12. *Morphisms of algebraic sets are continuous.*

Proof. Let $\phi : X \rightarrow Y$ be a morphism. It suffices to show that for all closed $Z \subset Y$ that $\phi^{-1}(Z) \subset X$ is closed.

But, if

$$Z = V_Y(S) := \{q \in Y \mid f(q) = 0 \forall f \in S\}$$

for some ideal $S \subset k[Y]$, then

$$\phi^{-1}(Z) = V_X(\phi^*(S)) = \{\phi^*(f) = f \circ \phi \mid f \in S\}.$$

□

Lemma 13. *Isomorphisms of algebraic sets are homeomorphisms. In particular, any isomorphism of algebraic sets $\phi : X \rightarrow X$ permutes the components Z_1, \dots, Z_r of X :*

$$\forall i \exists j : \phi(Z_i) = Z_j.$$

Theorem 4. *Let G be an algebraic group.*

- (i) *There is a unique component G^0 of G with $e \in G^0$.*
- (ii) *Every component Z of G is a coset gG^0 of G for some $g \in Z$.*
- (iii) *G^0 is a normal algebraic subgroup of G .*
- (iv) *G^0 is of finite index, i.e.*

$$[G : G^0] = \#(G/G^0) < \infty.$$

(v) *The irreducible components are also the connected components.*

Proof. Let $G = Z_1 \cup \dots \cup Z_r$ be the decomposition into components. We may assume that $e \in Z_1$.

Recall that $Z_1 \not\subset \bigcup_{j \geq 2} Z_j$. Then, there is an $x \in Z_1 \setminus \bigcup_{j \geq 2} Z_j$. Thus, for all algebraic set isomorphisms $\phi : G \rightarrow G$, we have by some previous lemma that $\phi(x)$ is likewise contained in some unique component of G . For example, we may take ϕ to be

$$\begin{aligned}\phi_g : G &\rightarrow G \\ y &\mapsto gy\end{aligned}$$

for any $g \in G$. Then, for all $g \in G$, the element $gx = \phi_g(x)$ is contained in only one component of G . Ergo, each $g \in G$ is contained in exactly one component.

(i) Take $g = e$.

(iii) G^0 is an algebraic subset, by construction. Denote by $m : G \times G \rightarrow G$ and $i : G \rightarrow G$ the continuous multiplication and inversion map on G . **Why is G^0 a subgroup?** We need to show

$$\begin{aligned}m(G^0 \times G^0) &\subseteq G^0. \\ i(G^0) &\subseteq G^0.\end{aligned}$$

We know that $i(G^0)$ is some component of G , since i is an isomorphism. But it contains the identity e , since $e^{-1} = e$. Therefore, $i(G^0) = G^0$.

If $g \in G$, then gG^0 is some component of G . Suppose $g \in G^0$. Then $gG^0 \cap G^0 \supseteq \{g\}$, therefore $gG^0 = G^0$. Ergo, G^0 is closed under multiplication.

Why is G^0 a normal? If $g \in G$, then gG^0g^{-1} is a component that contains e , therefore $G^0 = gG^0g^{-1}$.

(Alternative proof that $m(G^0 \times G^0) = G^0$: Consider

- any continuous image of an irreducible set is irreducible.
- the closure of any irreducible set is irreducible.

Ergo $\overline{m(G^0 \times G^0)}$ is a closed irreducible set containing e . Ergo, $\overline{m(G^0 \times G^0)} = G^0$.

(ii) Let $Z \subset G$ be a component. Let $g \in Z$. Then $g \in (gG^0 \cap Z)$, so $gG^0 = Z$.

- (iv) This follows from some previous lemma.
- (v) This is left as a topological exercise. It is true whenever the irreducible components do not intersect.

□

It now follows:

$$\{\text{finite algebraic groups}\} \longleftrightarrow \{\text{finite groups}\}$$

where the above arrow is an equivalence of categories.

Example 5. • Let $G = \{g_1, \dots, g_r\}$ be a finite algebraic group. Then,

$$G^0 = \{e\}.$$

- Without proofs:

$$G \in \{\mathrm{GL}_n(k), \mathrm{SO}_n(k), \mathrm{SL}_n(k)\} \implies G^0 = G.$$

Further,

$$G = \mathrm{O}_n(k) \implies G^0 = \mathrm{SO}_n(k)$$

(but only if $-1 = 1$ i.e. $\mathrm{char} k = 2$. Otherwise $[G : G^0] = 2$.)

3 Jordan Decomposition

As usual, $k = \bar{k}$ is an algebraically closed field.

Definition 10. Let V be a finite-dimensional vector space.

An element $x \in \mathrm{End}(V)$ is **semisimple**, if it is diagonalizable, i.e. it has a basis of eigenvectors, or equivalently, if the minimal polynomial of x is square-free.

Then, there is a decomposition $V = \bigoplus_{i=1}^r V_i$ and distinct elements $\lambda_1, \dots, \lambda_n \in k$ s.t.

$$x|_{V_i} = \lambda_i.$$

If $\dim(V_i) = n_i$, then

$$\text{char polynomial of } x = \prod_{i=1}^r (T_i - \lambda_i)_i^n \in k[T]$$

and

$$\text{minimal polynomial of } x = \prod_{i=1}^r (T_i - \lambda_i) \in k[T].$$

(Where the minimal polynomial of x is defined as the least degree monic m s.t. $m(x) = 0$ and Cayley-Hamilton $m|c$.)

Definition 11. $x \in \text{End}(V)$ is **nilpotent** if $x^n = 0$ for some n . (Equivalent to: characteristic polynomial of x is $T^{\dim(V)}$.)

x is **unipotent**, if $x - 1$ is nilpotent.

Lemma 14. *If x is semisimple and nilpotent, then $x = 0$.*

If x is semisimple and unipotent, then $x = 1$.

Lemma 15. *If x, y are commuting elements, that are semisimple resp. unipotent or nilpotent, then so is xy .*

Theorem 5 (Goal). *For all algebraic groups G and for all $g \in G$, there exist unique group elements $g_s, g_u \in G$ s.t.*

$$g = g_s g_u = g_u g_s$$

and for all finite-dimensional representations $\rho : G \rightarrow \text{GL}(V)$, $\rho(g_s)$ is semisimple and $\rho(g_u)$ is unipotent.

Example 6. If $g = \begin{pmatrix} \lambda & 1 & 0 \\ & \lambda & 1 \\ & & \lambda \end{pmatrix} \in G = \text{GL}_3(k)$, then $g_s = \begin{pmatrix} \lambda & 0 & 0 \\ & \lambda & 0 \\ & & \lambda \end{pmatrix}$, $g_u = \begin{pmatrix} 1 & \lambda^{-1} & 0 \\ & 1 & \lambda^{-1} \\ & & 1 \end{pmatrix}$.

Theorem 6 (Goal Theorem). *Let G an algebraic group. For all $g \in G$ there is exactly one pair $g_s, g_u \in G$ s.t.*

$$g = g_s g_u = g_u g_s$$

and for all finite-dimensional representations $r : G \rightarrow GL_n(V)$, the element $r(g_s)$ resp. $r(g_u)$ is semisimple resp. unipotent.

Last time, we saw:

Lemma 16. • *If g, h are commuting and semisimple resp. commuting and unipotent then so is gh .*

• *If g is semisimple and unipotent, then $g = 1$.*

Proposition 1. *Let V be a finite-dimensional vector space and $g \in GL(V)$. There exist unique elements $g_s, g_u \in GL(V)$ s.t.*

$$g = g_s g_u = g_u g_s$$

and g_s is semisimple and g_u is unipotent.

Moreover, $g_s, g_u \in k[g] = \{\sum_{i=1}^m a_i g^i \mid a_i \in k\} \subseteq \text{End}(V)$.

Proof. Existence (Sketch): Say

$$g = \begin{pmatrix} \lambda & 1 & 0 \\ & \lambda & 1 \\ & & \lambda \end{pmatrix}$$

then take

$$g_s = \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{pmatrix}, \quad g_u = \begin{pmatrix} 1 & \lambda^{-1} & 0 \\ & 1 & \lambda^{-1} \\ & & 1 \end{pmatrix}.$$

For $\lambda \in k$, define the **generalized λ -eigenspace** of g by

$$V_\lambda := \{v \in V \mid \exists n \in \mathbb{N}_0 : (g - \lambda)^n v = 0\}.$$

Then

$$V = \bigoplus_{\lambda \in k} V_\lambda.$$

Here V_λ = sum of domains of all Jordan blocks with λ s on the diagonal. (It follows from the Jordan Decomposition for matrices that such a decomposition exist.)

Let's define $g_s \in \text{GL}(V)$ by

$$g_s|_{V_\lambda} = \lambda \cdot \text{Id}.$$

Note that $gV_\lambda \subset V_\lambda$, hence g commutes with g_s , hence g, g_s commutes with $g_u := gg_s^{-1}$. Then, $g = g_s g_u = g_u g_s$.

Write $\det(T - g) = \prod_\lambda (T - \lambda)^{n(\lambda)}$, $n(\lambda) = \dim(V_\lambda)$. Since the polynomials $T - \lambda$ for $\lambda \in k$ are coprime, the chinese remainder theorem implies that there is a $Q \in k[T]$ s.t.

$$Q \equiv \lambda \pmod{(T - \lambda)^{n(\lambda)}}$$

for each $\lambda \in k$.

We claim that

$$Q(g) = g_s.$$

Indeed, since $gV_\lambda \subseteq V_\lambda$, we have

$$Q(g)V_\lambda \subseteq V_\lambda.$$

So, it suffices to show for all $v \in V_\lambda$

$$Q(g)v = g_s v = \lambda v.$$

Note that, by Cayley-Hamilton,

$$V_\lambda = \{v \in V \mid (g - \lambda)^{n(\lambda)} v = 0.\}$$

Write

$$Q = \lambda + R \cdot (T - \lambda)^{n(\lambda)}$$

for some $R \in k[T]$. Since $(g - \lambda)^{n(\lambda)} v = 0$, deduce that $Q(g)v = \lambda v$, as required.

If $Q' \equiv T(T - \lambda)^{n(\lambda)}$, then

$$Q'(g) = g_u.$$

If $Q'' \equiv \lambda^{-1}(T - \lambda)^{n(\lambda)}$, then $Q''(g) = g_s^{-1}$. check corresponding stuff for g_u .

Uniqueness: Suppose given some other decomposition

$$g = g'_s g'_u = g'_u g'_s$$

with g'_s semisimple and g'_u unipotent. Then g'_s commutes with g'_s and g'_u , hence with g , hence also with any element in $k[g]$. Ergo, g'_s commutes with g_s and g_u . Similarly, g'_u commutes with g_s and g_u .

Consider

$$h := g'_s g_s^{-1} = g'_s g'_u (g'_u)^{-1} g_s^{-1} = g(g'_u)^{-1} g_s^{-1} = g_u (g'_u)^{-1}.$$

Then $h = g'_s g_s^{-1}$ is a product of semisimple elements and $h = g_u (g'_u)^{-1}$ is a product of unipotent elements. By proceeding lemmas, h is semisimple and unipotent, ergo trivial. It follows $g'_s = g_s$ and $g'_u = g_u$. \square

Corollary 2. *Let $g \in GL(V)$, let $W \subset V$ be any g -invariant subspace, i.e. $gW \subseteq W$.*

Then, W is g_s -invariant and g_u -invariant.

Proof. This is clear, since g_s and g_u are algebraically generated by g over g . \square

Lemma 17. *Let $\phi : V \rightarrow W$ be a linear map between finite-dimensional vector spaces.*

Let $\alpha \in GL(W)$ and $\beta \in GL(W)$ s.t.

$$\begin{array}{ccc} V & \xrightarrow{\alpha} & V \\ \downarrow \phi & & \downarrow \phi \\ W & \xrightarrow{\beta} & W, \end{array}$$

i.e. $\phi \circ \alpha = \beta \circ \phi$.

Then,

$$\begin{aligned} \phi \circ \alpha_s &= \beta_s \circ \phi, \\ \phi \circ \alpha_u &= \beta_u \circ \phi. \end{aligned}$$

Proof. Write $V = \bigoplus_{\lambda \in k} V_\lambda$, $W = \bigoplus_{\lambda \in k} W_\lambda$ where V_λ are the generalized α -eigenspaces and W_λ are the generalized β -eigenspaces.

We claim that

$$\phi(V_\lambda) \subset W_\lambda.$$

Indeed, let $v \in V_\lambda$, then

$$(\beta - \lambda)^n \phi(v) = \phi((\alpha - \lambda)^n v) = 0.$$

Since $(\alpha - \lambda)^n v = 0$, the claim follows.

Since, $\alpha_s|_{V_\lambda} = \lambda \text{Id}$ and $\beta_s|_{W_\lambda} = \lambda \text{Id}$, deduce that

$$\phi \circ \alpha_s = \beta_s \circ \phi.$$

Indeed, both sides are given on V_λ by $\lambda \cdot \phi$. Thus

$$\begin{aligned}\phi \circ \alpha_u &= \phi \circ \alpha \alpha_s^{-1} \\ &= \beta \beta_s^{-1} \circ \phi \\ &= \beta_u \circ \phi.\end{aligned}$$

□

Lemma 18. *Let $\alpha \in GL(V)$, $\beta \in GL(W)$. Then the **tensor** $\alpha \otimes \beta \in GL(V \otimes W)$ is defined by*

$$(\alpha \otimes \beta)(u \otimes v) = \alpha(u) \otimes \beta(v).$$

Then, we have

$$\begin{aligned}(\alpha \otimes \beta)_s &\stackrel{(1)}{=} \alpha_s \otimes \beta_s \\ (\alpha \otimes \beta)_u &\stackrel{(2)}{=} \alpha_u \otimes \beta_u.\end{aligned}$$

Proof. It suffices to prove (1), since

$$\begin{aligned}(\alpha \otimes \beta)_u &= (\alpha \otimes \beta) \circ (\alpha \otimes \beta)_s^{-1} \\ &\stackrel{(1)}{=} (\alpha \otimes \beta) \circ (\alpha_s \otimes \beta_s)^{-1} \\ &= \alpha \alpha_s^{-1} \otimes \beta \beta_s^{-1} \\ &= \alpha_u^{-1} \otimes \beta_u^{-1}\end{aligned}$$

For (1), consider

$$\begin{aligned}V &= \bigoplus_{\lambda \in k} V_\lambda, \\ W &= \bigoplus_{\lambda \in k} W_\lambda.\end{aligned}$$

It follows

$$V \otimes W = \bigoplus_{\lambda, \mu \in k} V_\lambda \otimes W_\mu.$$

Now,

$$\alpha_s \otimes \beta_s|_{V_\lambda \otimes W_\mu} = \lambda \mu \cdot \text{Id}.$$

Ergo, $\alpha_s \otimes \beta_s$ is semisimple. By Proposition, we reduce to checking that $\alpha_u \otimes \beta_u$ is unipotent. Indeed,

$$\alpha_u \otimes \beta_u - 1 = (\alpha_u - 1) \otimes (\beta_u - 1) + 1 \otimes (\beta_u - 1) + (\alpha_u - 1) \otimes 1$$

is nilpotent (You can also check that $\alpha_u \otimes \beta_u = (\alpha_u \otimes 1) \circ (1 \otimes \beta_u)$ is unipotent.) □

Example 7. Let $1 \in GL(V)$. Then $1_s = 1$ and $1_u = 1$.

Summary : Let G be an algebraic group. Let $r_V : G \rightarrow \mathrm{GL}(V)$ be a finite-dimensional representation. Also, fix $g \in G$.

Let $\lambda_V := r_V(g)_s$ (or $r_V(g)_u$).

We get a family of operators $\lambda_V \in \mathrm{End}(V)$ with the following properties:

- (i) if $V = k$ and $r_V(g') = 1$ for all $g' \in G$, then $\lambda_V = 1$.
- (ii) for any two representations in V and W , we have

$$\lambda_{V \otimes W} = \lambda_V \otimes \lambda_W.$$

- (iii) for all G -equivariant $\phi : V \rightarrow W$ we have

$$\phi \circ \lambda_V = \lambda_W \circ \phi.$$

Theorem 7. *Let G be an algebraic group. Let $\lambda_V \in \mathrm{End}(V)$ (i.e. $V = (r_V, V)$ is a finite-dim. representation of G) be a family of operations satisfying (i), (ii), (iii).*

Then, there is exactly one $g \in G$ s.t. $\lambda_V = r_V(g)$ for all V .

Note, that this theorem implies our goal theorem.

Applying the theorem to $\lambda_V = r_V(g_s)$ implies

$$\exists g_s \in G : r_V(g_s) = r_V(g)_s$$

and

$$\exists g_u \in G : r_V(g_u) = r_V(g)_u.$$

Proof of Goal Theorem. There exist unique $g_s, g_u \in G$ s.t.

$$g \stackrel{(*)}{=} g_u g_s = g_s g_u,$$

Then, $r_V(g) = r_V(g_s)r_V(g_u) = r_V(g_u)r_V(g_s)$.

Since $r_V(g_u)$ is unipotent and $r_V(g_s)$ is semisimple, it follows $r_V(g_u) = r_V(g)_u$ and $r_V(g_s) = r_V(g)_s$.

To deduce $(*)$, take any $r_V : G \hookrightarrow \mathrm{GL}(V)$. We know for each V

$$r_V(g) = r_V(g_s)r_V(g_u) = r_V(g_u)r_V(g_s).$$

□

Proof of Theorem. We first extend the assignment

$$V \mapsto \lambda_V$$

to all representations of G .

Say $V = \bigcup_j W_j$ where each W_j is a finite-dimensional G -invariant subspace. Try to define $\lambda_V \in \text{End}(V)$ by

$$\lambda_V|_{W_j} := \lambda_{W_j}.$$

For this to be well-defined, we need to show for each i, j

$$\lambda_{W_i}|_{W_i \cap W_j} \stackrel{(*)}{=} \lambda_{W_j}|_{W_i \cap W_j}.$$

Proof of (*): Apply assumption (iii) to the G -equivariant linear maps

$$\begin{aligned} W_i \cap W_j &\xrightarrow{\phi} W_i, \\ W_i \cap W_j &\xrightarrow{\phi'} W_j. \end{aligned}$$

Then,

$$\begin{aligned} \lambda_{W_i}|_{W_i \cap W_j} &= \lambda_{W_i} \circ \phi \\ &\stackrel{(iii)}{=} \phi \circ \lambda_{W_i \cap W_j} \\ &= \phi' \circ \lambda_{W_i \cap W_j} \end{aligned}$$

and

$$\lambda_{W_j}|_{W_i \cap W_j} = \lambda_{W_j} \circ \phi' = \phi' \circ \lambda_{W_i \cap W_j}.$$

Recall here that any finite-dimensional G -invariant $W \subset V$ is a representation. \square

⁰Not necessarily finite-dimensional, but may be written as a filtered union of finite-dimensional G -invariant subspaces of W .

Let G be an algebraic group.

Easy Exercise : If V_1, V_2 are representations r_1, r_2 of G , then $V_1 \otimes V_2$ is also a representation with

$$r = r_1 \otimes r_2 : G \rightarrow \mathrm{GL}(V_1 \otimes V_2)$$

given by

$$r(g)(v_1 \otimes v_2) = (r_1(g)v_1) \otimes (r_2(g)v_2).$$

Proof. Given $\Delta_j : V_j \rightarrow V_j \otimes k[G]$, define

$$\Delta : V_1 \otimes V_2 \longrightarrow V_1 \otimes V_2 \otimes k[G]$$

by: if

$$\Delta_1 u = \sum u_i \otimes f_i, \quad \Delta_2 v = \sum v_j \otimes h_j,$$

then

$$\Delta(u \otimes v) \sum \sum u_i \otimes v_j \otimes f_i h_j.$$

Set $A := k[G]$, then

$$r_A := \text{right regular representation with } r_A(g)f(x) = f(xg).$$

The map

$$\begin{aligned} A \otimes A &\xrightarrow{m} A \\ f_1 \otimes f_2 &\longmapsto f_1 f_2 \end{aligned}$$

defines a morphism of representations

$$(A, r_A) \otimes (A, r_A) \rightarrow (A, r_A).$$

Indeed,

$$\begin{aligned} m((r_A \otimes r_A)(g)(f_1 \otimes f_2))(x) &= f_1(xg)f_2(xg), \\ &= f_1 f_2(xg) = r_A(g)(m(f_1 \otimes f_2))(x), \end{aligned}$$

since $f_1(_g) \otimes f_2(_g) = (r_A \otimes r_A)(g)(f_1 \otimes f_2)$.

Ergo $m \circ (r_A \otimes r_A)(g) = r_A(g) \circ m$. □

Recall: We stated the following theorem

Theorem 8. Let $\lambda_V \in \text{End}(V)$ be given s.t. for all finite-dim. rep.s V of G s.t.:

(i) $\lambda_k = 1$

(ii) $\lambda_{V \otimes W} = \lambda_V \otimes \lambda_W$

(iii) for all morphisms of rep.s $\phi : V \rightarrow W$ we have

$$\phi \circ \lambda_V = \lambda_W \circ \phi.$$

Then, there is exactly one $g \in G$ s.t. $\lambda_V = r_V(g)$ for all V .

Proof. Last time, we saw that any such family $V \mapsto \lambda_V$ extends to **all** rep.s V of G .

Let's note also that, if (V_0, r_0) is any representation of G with trivial action, i.e. $r(g) = 1$ for all g , then $\lambda_{V_0} = 1$. Indeed, let $v \in V_0$. We must check that $\lambda_{V_0} v = v$. Since the action is trivial, any subspace of V_0 is G -invariant.

Consider the map

$$\begin{aligned} \phi : k &\longrightarrow V_0 \\ \alpha &\longmapsto \alpha v \end{aligned}$$

where $v = \phi(1)$. Then, ϕ is a morphism of rep.s because the action is trivial.

Thus,

$$\lambda_V v = (\lambda_V \circ \phi)(1) \stackrel{(iii)}{=} (\phi \circ \lambda_k)(1) \stackrel{(i)}{=} \phi(1) = v.$$

Consider $\lambda_A \in \text{End}(A)$. Then,

$$\lambda_{A \otimes A} = \lambda_A \otimes \lambda_A.$$

It is an easy exercise to see that $m : (A, r_A) \otimes (A, r_A) \rightarrow (A, r_A)$ is a morphism of rep.s.

By (iii) it follows, $m \circ (\lambda_A \otimes \lambda_A) = \lambda_A \circ m$, i.e.

$$\lambda_A(f_1 f_2) = \lambda_A(f_1) \lambda_A(f_2)$$

for all $f_1, f_2 \in A$. Thus, λ_A is an algebra morphism (check, using the morphism $k \hookrightarrow A$, that $\lambda_A(1) = 1$).

Thus, $\lambda_A = \phi^*$ for some unique morphism ϕ of algebraic sets $\phi : G \rightarrow G$.

We claim that ϕ commutes left multiplication i.e.

$$\phi(hx) = h\phi(x)$$

for all $h, x \in G$. Indeed, let's consider the map

$$\begin{aligned} A &\longrightarrow A \\ f &\longmapsto f(h \cdot _). \end{aligned}$$

This induces a morphism

$$(A, r_A) \xrightarrow{\psi} (A, r_A).$$

By (ii), $\psi \circ \lambda_A = \lambda_A \circ \psi$.

Since $\lambda_A = \phi^*$, this implies the claim.

Now, set $g := \phi(e)$. Then for all $h \in G$,

$$\phi(h) = \phi(h e) = h g.$$

Thus, $\lambda_A = \phi^* = r_A(g)$.

(It remains to show that

$$\lambda_V = r_V(g)$$

for each finite-dim. rep. V .)

Let $V = (V, r)$ be any rep. This induces a map

$$\Delta : V \longrightarrow V \otimes A.$$

If $\Delta v = \sum v_i \otimes f_i$, then

$$h v = \sum f_i(h_i) \otimes v_i.$$

Let

$$\begin{aligned} \varepsilon : V \otimes A &\longrightarrow V \\ v \otimes f &\longmapsto f(1)v. \end{aligned}$$

It follows $\varepsilon \circ \Delta : V \rightarrow V$ is the identity map.

Let (V_0, r_0) be the representation of G with $V_0 := V$ and r_0 the trivial action.

Then, $\Delta : V \rightarrow V_0 \otimes A$ is a morphism of representations.

(Indeed, if $\Delta v = \sum v_i \otimes f_i$, then

$$\Delta(r(h)v) \stackrel{?}{=} (r_0(h) \otimes r_A(h))\Delta v$$

since

$$\begin{aligned}
\Delta v &= \sum v_i \otimes f_i \\
\iff xv &= \sum f_i(x_i)v_i \quad \forall x \in G \\
\iff xhv &= \sum f_i(xh)v_i \quad \forall x, h \in G.
\end{aligned}$$

Since $r(h)v = hv$, it follows

$$\Delta(hv) = \sum v_i \otimes f_i(\cdot h) \implies (?).$$

We want to show

$$\lambda_V = r_V(g).$$

We have

$$\begin{aligned}
\Delta \circ \lambda_V &\stackrel{(iii)}{=} \lambda_{V_0 \otimes A} \circ \Delta \\
&\stackrel{(ii)}{=} \lambda_{V_0} \otimes \lambda_A \\
&= 1 \otimes \lambda_A = 1 \otimes r_A(g).
\end{aligned}$$

This implies

$$\Delta \circ \lambda_V = (1 \otimes r_A(g)) \circ \Delta$$

but also

$$\Delta \circ r_V(g) = (1 \otimes r_A(g)) \circ \Delta.$$

Because of the injectivity of Δ it now follows

$$\lambda_V = r_V(g).$$

□

Corollary 3. *Let $\phi : G \rightarrow H$ be any morphism of algebraic groups. Then, for all $g \in G$*

$$\begin{aligned}
\phi(g)_s &= \phi(g_s) \\
\phi(g)_u &= \phi(g_u).
\end{aligned}$$

Proof. Let V be any **faithful** representation of H , i.e. $r_V : H \rightarrow \text{GL}(V)$ is injective, (for a finite-dim. V).

Then, $r_V \circ \phi$ is a rep. of G . To prove (i), it suffices to show

$$r_V(\phi(g)_s) = r_V(\phi(g_s))$$

since H operates faithfully on V .

We know that

$$r_V(\phi(g)_s) = r_V(\phi(g))_s$$

(characterizing property of h_s for $h \in H$). On the other hand,

$$r_V(\phi(g_s)) = (r_V \circ \phi)(g_s) = r_V(\phi(g))_s.$$

Therefore, claim (i) follows. (ii) works analogously. \square

Definition 12. Let $g \in G$ where G is an algebraic group. We call g **semisimple**, if $g = g_s$.

We call g **unipotent**, if $g = g_u$.

Lemma 19. For $g \in G$, the following are equivalent:

- (i) g is semisimple.
- (ii) $r_V(g)$ is semisimple for all finite-dim. rep. V .
- (iii) $r_V(g)$ is semisimple for at least one faithful f.d. rep. V of G .

We get an analogous lemma for unipotent group elements.

Proof. We have

$$\begin{aligned}
(i) &\iff g = g_s \\
&\stackrel{\text{Def. of } g_s \text{ by goal thm.}}{\iff} r_V(g) = r_V(g)_s \forall \text{ f.d. } V \\
&\iff r_V(g) \text{ is semisimple} \\
&\iff (ii) \implies (iii).
\end{aligned}$$

On the other hand,

$$(iii) \implies \exists \text{ faithful f.d. } V \text{ s.t. } r_V(g) = r_V(g)_s = r_V(g_s) \implies g = g_s.$$

\square

4 Non-Commutative Algebra

Definition 13. A **ring** R (for now) is unital, associative but not necessarily commutative.

Example 8. The ring of matrices over some field or ring.

Definition 14. A **left ideal** $I \subset R$ is a subset that is an abelian subgroup of $(R, +)$ s.t. $ra \in I$ for all $r \in R, a \in I$.

A **right ideal** $I \subset R$ is a subset that is an abelian subgroup with

$$IR \subset I.$$

A two-sided ideal I is a subset that is a left and a right ideal of R .

It is easy to check that for any homomorphism of rings $\phi : R \rightarrow S$, $\text{Kern}\phi$ is a two-sided ideal. Also, if $J \subset R$ is any two-sided ideal, then there exists a unique ring structure on R/J s.t. the projection $R \rightarrow R/J$ is a ring homomorphism.

Definition 15. A **left module** M for R is an abelian group equipped with a ring homomorphism

$$R \xrightarrow{\alpha} \text{End}(M)$$

where $\text{End}(M)$ acts on the left of M . We write

$$rm := \alpha(r)m.$$

We have

$$(r_1 r_2)(m) = r_1(r_2(m)).$$

If R acts on M by the right, we write

$$R \curvearrowright M.$$

Example 9. $M_n(k) \curvearrowright k^n$ where k^n is the space of column vectors.

If k^n denotes the space of row vectors, we have $k^n \curvearrowleft M_n(k)$.

Definition 16. A **(left) submodule** $N \subset M$ is an algebraic subgroup s.t.

$$RN \subset N.$$

It follows that N is itself is a left module.

Definition 17. A (left) module M of R is **simple** (or irreducible) if it has exactly the two submodules: $0 = \{0\}$ and M .

Definition 18. A ring R is a **division ring** if it satisfies any of the following equivalent requirements:

- (i) $R^\times = R \setminus \{0\}$ where $R^\times = \{r \in R \mid \exists a, b \in R : ar = rb = 1.\}$
- (ii) R has no nontrivial left or right ideals.

Definition 19. If $R \curvearrowright M$, then we can define

$$\text{End}_R(M) := \{\phi \in \text{End}(M) \mid \phi(rm) = r\phi(m) \forall r \in R, m \in M\}.$$

Note, that $\text{End}_R(M)$ is a ring.

Lemma 20 (Schur's Lemma). *If M is simple, then $\text{End}_R(M)$ is a division ring.*

Lemma 21. *Let k be a field. Then, $M_n(k)$ has no nontrivial twosided ideals.*

Theorem 9 (Jacobson Density Theorem (Double Commutant Theorem)). *Suppose M is a simple left module which is finitely generated as a right D -module for $D = \text{End}_R(M)$.*

Assume that R acts faithfully on M , i.e. $R \rightarrow \text{End}_R(M)$ is injective.

Then, the map $R \rightarrow \text{End}_D(M)$ is an isomorphism.

⁰If $ar = rb = 1$, then $a = arb = b$.

Recap:

- Basics: definitions, Hopf-algebras, ...
- Jordan decomposition
- Primer on non-commutative algebra
 - Jacobson density theorem
- Unipotent groups
- Tori

4.1 Jacobson Density Theorem

We had last week

$$\text{End}_D(M) := \{\phi \in \text{End}(M) \mid \phi \circ d = d \circ \phi \forall d \in D\}.$$

Let k be an algebraically closed field, V a non-trivial finite-dimensional k -vector space and let G be a subgroup of $\text{GL}(V)$ that acts **irreducibly** on V , i.e., V is **G -irreducible**, i.e., the only G -invariant subspaces of V are 0 and V .

Set

$$D := \{d \in \text{End}_k(V) \mid dg = gd \forall g \in G\} = \text{span}(G) = \left\{ \sum_{i=1}^n c_i g_i \mid c_i \in k, g_i \in G, n \in \mathbb{N}_0 \right\}.$$

Then,

$$D = \text{End}_R(V)$$

where R is the k -subalgebra of $\text{End}(V)$ that is generated by G .

Lemma 22 (Schur's Lemma). *We understand $k \xrightarrow{\text{End}} (V)$ as the inclusion of operations which operate by scalar multiplication*

$$k \xrightarrow{\cong} \{\phi : V \rightarrow V \mid \phi : v \mapsto t \cdot v \text{ for some } t \in k\}.$$

Then, we have

$$D \cong k.$$

Proof. Let $d \in D$. Since $V \neq 0$, there is an eigenspace $V_\lambda \neq 0$ for d . Observe that V_λ has to be G -invariant:

if $g \in G$ and $v \in V_\lambda$, then $gv \in V_\lambda$, since

$$dgv = gdv = g(\lambda v) = \lambda gv.$$

Since V_λ is a non-trivial G -invariant subspace and V is irreducible under G , we have

$$V_\lambda = V.$$

Ergo $d = \lambda$ in the sense of $k \hookrightarrow \text{End}(V)$. □

Consequence of the Jacobson Density Theorem: $R = \text{End}_k(V)$, i.e., G generates all linear operations on V , if V is G -irreducible.

We will prove this after a lemma.

Lemma 23. *Let $n \in \mathbb{N}$. Set*

$$V^n := V \oplus V \oplus \dots \oplus V = V_1 \oplus \dots \oplus V_n$$

where each $V_i = V$.

Let $v = (v_1, \dots, v_n) \in V^n$ and set

$$Rv := \{(rv_1, \dots, rv_n) \mid r \in R\} = \text{span}\{(gv_1, \dots, gv_n) \mid g \in G\}.$$

Then, $Rv \neq V^n$ iff the v_j are linearly dependent over k .

Consequence: Take $n := \dim(V)$. Let $\{e_1, \dots, e_n\}$ be a basis of V and set

$$e := (e_1, \dots, e_n) \in V^n.$$

Since the $(e_i)_i$ are linearly independent, the lemma states that $Re = V^n$.

Now, let $x \in \text{End}_k(V)$. Choose $r \in R$ s.t.

$$re = (xe_1, \dots, xe_n).$$

Then $re_i = xe_i$ for all i , thus $x = r$. Hence, $R = \text{End}_k(V)$.

Proof. Choose $J \in \{1, \dots, n\}$ as large as possible with

$$Rv + V_1 + V_2 + \dots + V_{J-1} =: U \neq V^n$$

. Such an J does exist, since we know that $Rv \neq V^n$.

Then, $V_J \not\subseteq U$, otherwise we may increase J . Also, U is invariant by the diagonal action of G on V^n . Thus, $V_J \cap U \subseteq V_J$ is a proper G -invariant subspace of the G -irreducible $V_J \cong V$. Therefore, $V_J \cap U = 0$.

On the other hand, by maximality of J , we have

$$U \oplus V_J = V^n.$$

Ergo, the map (composition)

$$V \cong V_J \hookrightarrow V^n \twoheadrightarrow V^n/U$$

is a G -equivariant isomorphism, since U is G -invariant.

Let $z : V^n/U \xrightarrow{\cong} V$ be the inverse isomorphism. Let l be the G -equivariant map given by

$$\begin{array}{ccc} V^n & \xrightarrow{l} & V \\ \downarrow & \nearrow z & \\ V^n/U & & \end{array}$$

and let l_j be the G -equivariant maps by restricting l on V_j . Then $l_j \in D \cong k$.

Say $l_j = t_j \in k$. Then,

$$l(w) = t_1 w_1 + \dots t_n w_n.$$

Since z is an isomorphism, l is nonzero and $(t_1, \dots, t_n) \neq (0, \dots, 0)$.

Since $l|_U = 0$, we can deduce for all $u \in U$

$$t_1 u_1 + \dots + t_n u_n = 0.$$

But $v \in Rv \subseteq U$, so we may conclude – as required – that the $(v_i)_i$ are linearly dependent ($l(v) = 0$). \square

4.2 Unipotent Groups

Let G be a subgroup of $\mathrm{GL}(V)$ where V is a finite-dimensional vector space and k an algebraically closed field.

Definition 20. We say that G is **unipotent** if one of the following equivalent conditions hold:

- each $g \in G$ is unipotent (i.e. $(g - 1)^n = 0$ for some $n \in \mathbb{N}$).
- all eigenvalues of g are 1.
- g is conjugate to $\begin{pmatrix} 1 & \star & \star \\ 0 & \ddots & \star \\ 0 & 0 & 1 \end{pmatrix}$.

Theorem 10. Any unipotent subgroup of $\mathrm{GL}_n(k)$ is conjugate to a subgroup of

$$U_n := \left\{ \begin{pmatrix} 1 & \star & \star \\ 0 & \ddots & \star \\ 0 & 0 & 1 \end{pmatrix} \right\} = \left\{ U \in M_n(k) \mid U_{i,j} = \begin{cases} 0, & \text{if } i > j \\ 1, & \text{if } i = j \\ \text{arbitrary,} & \text{otherwise.} \end{cases} \right\}.$$

Definition 21. For two subgroups G, H of some common supergroup, define their **commutator** by

$$[G, H] := \langle ghg^{-1}h^{-1} \mid g \in G, h \in H \rangle.$$

A group G is called **nilpotent**, if one of its commutators is trivial, i.e. if we set

$$G_0 := G \text{ and } G_{i+1} := [G_i, G],$$

then G is called nilpotent iff there is an $j \in \mathbb{N}$ with $G_j = 1$.

Corollary 4. Any unipotent subgroup of $\mathrm{GL}(V)$ is nilpotent.

Definition 22. A group G is called **solvable**, if $G^{(n)} = 1$ for some n where

$$\begin{aligned} G^{(0)} &:= G, \\ G^{(i+1)} &:= [G^{(i)}, G^{(i)}]. \end{aligned}$$

Notation 1. In the following, we will write $G' := [G, G]$.

Definition 23. Let $n := \dim(V)$. A **complete flag** is a maximal strictly increasing chain of subspaces

$$0 = V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_n = V.$$

Any complete flag is of the form

$$V_j := \text{span}\{e_1, \dots, e_j\}$$

for some basis e_1, \dots, e_n of V .

Let B be the basis of some flag $0 = V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_n = V$. For $x \in \text{End}(V)$, we have that x is upper-triangular with respect to B iff x leaves each member V_i of the flag invariant, i.e. $xV_i \subseteq V_i$.

Proposition 2 (Key Proposition). *Let G be a unipotent subgroup of $\text{GL}(V)$. Then there is a complete flag $V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_n$ consisting of G -invariant subspaces, i.e., each V_i is G -invariant.*

Proof. Recall, that G is a unipotent subgroup of $\text{GL}_n(V)$. We will give an induction on $n = \dim V$.

If $n = 0$, there is nothing to show.

Let $n \geq 1$. We may assume that V is G -irreducible. Because, if not, there is a G -invariant subspace $0 \neq W \subset V$ s.t. W and V/W have dimension $< n$. Then there exist complete G -invariant flags in W and V/W and the claim – that there is a complete G -invariant flag in V – follows by the induction hypothesis.

By Jacobson Density Theorem, we have

$$R := \text{span}(G) = \text{End}(V) := \text{End}_k(V).$$

Since G is unipotent, we have for each $g \in G$

$$\text{trace}(g) = n.$$

Ergo, for $g, h \in G$

$$\text{trace}(gh) = \text{trace}(h)$$

and

$$\text{trace}((g - 1)h) = \text{trace}(gh) - \text{trace}(h) = 0.$$

Since $\text{span}(G) = \text{End}(V)$, it now in particular follows for all $g \in G, \phi \in \text{End}(V)$

$$\text{trace}((g - 1)\phi) = 0.$$

Since the above holds for all $\phi \in \text{End}(V)$, it must hold

$$g - 1 = 0$$

for all $g \in G$ (take for example the elementary matrices $\phi = E_{i,j}$). Ergo, G is trivial. Then, any complete flag is trivially G -invariant. \square

Remark 1. This gives the group analogue of Engel's Theorem.

Proof Goal Theorem. Let B be a basis of V s.t. G leaves each subspace in the corresponding flag invariant. Then, G is upper-triangle with respect to this basis.

On the other hand, each $g \in G$ is unipotent, hence its diagonal (i.e. eigenvalues) are all 1. Thus, with respect to B

$$G \subseteq \left\{ \begin{pmatrix} 1 & \star & \star \\ 0 & \ddots & \star \\ 0 & 0 & 1 \end{pmatrix} \right\} = U_n.$$

□

Remark 2. Tori are of the form $(k^\times)^n$. In the case $k = \mathbb{C}$, $(\mathbb{C}^\times)^n$ are the complexification of $U(1)^n$. This equals tori in top. sense.

$$\begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix} \subseteq \mathrm{GL}_2(\mathbb{C})$$

is a non-algebraic unipotent group.

Exercise. (to be discussed next time)

it would have sufficed to prove the Goal theorem in the special case that G is algebraic.

Corollary of Proof: If $G \subset \mathrm{GL}(V)$ (with $V \neq 0$) is unipotent and acts irreducibly (?), then $G = 1$, $\dim V = 1$.

Answer to last Exercise: Recall that the main point was to show that any unipotent subgroup $G \subseteq \mathrm{GL}(V)$ leaves invariant some complete flag $\mathcal{F} = (V_0 \subset V_1 \dots)$. But by some homework (problem 1), the group

$$\mathrm{GL}(V)_{\mathcal{F}} := \{g \in \mathrm{GL}(V) \mid g\mathcal{F} = \mathcal{F}\}$$

is algebraic.

Proof: If \mathcal{F} is the standard flag with $V_i = \mathrm{span}(e_1, \dots, e_i)$ for the standard basis $\{e_1, \dots, e_n\}$, then

$$\mathrm{GL}(V)_{\mathcal{F}} = \{A \in \mathrm{GL}(V) \mid A \text{ is upper-triangle}\}.$$

The condition that A is upper triangle can be realized by polynomials. □

Thus,

$$\begin{aligned} G \text{ fixes } \mathcal{F} \\ \iff G \subseteq \mathrm{GL}(V)_{\mathcal{F}} \\ \mathrm{GL}(V)_{\mathcal{F}} \text{ is algebraic} \iff \overline{G} \subseteq \mathrm{GL}(V)_{\mathcal{F}} \\ \iff \overline{G} \text{ fixes } \mathcal{F}. \end{aligned}$$

Now, the Zariski-Closure \overline{G} of any group G is an algebraic group (shown in some homework).

Further, if G is unipotent, then \overline{G} is unipotent.

5 Tori

Definition 24. A **torus** is an algebraic group that is isomorphic to \mathcal{G}_m^n for some $n \in \mathbb{N}_0$ where $\mathcal{G}_m = k^\times = \mathrm{GL}_1(k)$ is the unit group of k .

We think of $\mathcal{G}_m^n \subseteq \mathrm{GL}_n(k)$ as the subgroup of diagonal matrices.

Lemma 24. *Let G be a commutative algebraic group. Then the following are equivalent:*

- (i) *each $g \in G$ is semisimple.*
- (ii) *for each finite-dimensional representation V of G and for each $g \in G$, the operator $r_V(g)$ is diagonalizable.*

(iii) for all finite-dimensional representations V of G , there is a basis of common eigenvectors for $r_V(G)$, i.e. a basis s.t.

$$r_V(G) \subseteq \mathcal{G}_m^n.$$

(iv) G is isomorphic to an algebraic subgroup of a torus.

(i) ~~Proof~~ \Rightarrow (ii): This follows from the Jordan decomposition and definition of semisimple.

(ii) \Rightarrow (iii) : This is homework. Note that any commutative subset S of $\mathrm{GL}(V)$ consisting of semisimple operators may be diagonalized simultaneously.

(iii) \Rightarrow (iv) : Take any faithful representation V of G and diagonalize it simultaneously. Then, $G \cong r_V(G) \subseteq \mathcal{G}_m^n$.

(iv) \Rightarrow (i) : Any diagonal matrix is semisimple.

□

Definition 25. A commutative algebraic group G is called **diagonalizable**, if it satisfies one of the above equivalent conditions.

Definition 26. A character χ of any algebraic group F is an element $\chi \in \mathrm{Hom}_{\mathrm{alg.grp.}}(F, k^\times)$, i.e., a homomorphism $\chi : F \rightarrow k^\times$ of algebraic groups.

Notation 2. For an algebraic group G , set $\Xi(G) := \mathrm{Hom}_{\mathrm{alg.grp.}}(G, k^\times)$.

Also denote now by $\mathcal{O}(X) := k[T]/I(X)$ the coordinate ring of an algebraic set X (rather than $k[X]$).

Lemma 25. *There is a bijection*

$$\Xi(G) = \{\text{characters } \chi \text{ of } G\} \longleftrightarrow \{x \in \mathcal{O}(G)^\times \mid \Delta(x) = x \otimes x\}.$$

Proof. Note, that any $x \in \mathcal{O}(G)^\times$ can be thought of as a map $x : G \rightarrow k^\times \subset k$.

We have

$$\begin{aligned} \mathrm{Hom}_{\mathrm{alg.grp.}}(G, \mathcal{G}_m) &= \{\phi \in \mathrm{Hom}_{\mathrm{alg.sets}}(G, \mathcal{G}_m) \mid \phi(gh) = \phi(g)\phi(h) \ \forall g, h\} \\ &= \{\phi \in \mathrm{Hom}_{k\text{-alg.}}(\mathcal{O}(\mathcal{G}_m), \mathcal{O}(G)) \mid (\phi \otimes \phi) \circ \Delta = \Delta \circ \phi\}. \end{aligned}$$

Recall: $\mathcal{O}(\mathcal{G}_m) \cong k[t, \frac{1}{t}]$ with $\Delta(t) = t \otimes t$.

Thus for any k -algebra A , $\text{Hom}_{k\text{-alg.}}(\mathcal{O}(\mathcal{G}_m), A) \cong^{A^\times}$ via

$$[t \mapsto a, (t^{-1} \mapsto a^{-1})] \longleftrightarrow a.$$

Thus,

$$\text{Hom}_{\text{alg.grp.}}(G, \mathcal{G}_m) \cong \{a \in \mathcal{O}(G)^\times \mid a \otimes a = \Delta(a)\}.$$

Therefore, it suffices to test the condition $(\phi \otimes \phi) \circ \Delta = \Delta \circ \phi$ on the generators t, t^{-1} of $\mathcal{O}(\mathcal{G}_m)$. Now, the above isomorphism is given by

$$\phi \mapsto a = \phi(t)$$

which is equivalent or regarding $\chi : G \rightarrow \mathcal{G}_m$ as a map $\chi : G \rightarrow k$. \square

Example 10. Let $G = \mathcal{G}_m$, then $\mathcal{O}(G) = k[t, \frac{1}{t}]$. Which $x = \sum_{m \in \mathbb{Z}} c_m t^m \in \mathcal{O}(G)$, almost all $c_m = 0$, but not all of them, have the property

$$\Delta(x) = x \otimes x.$$

We have

$$\begin{aligned} x \otimes x &= \sum_{m, n \in \mathbb{Z}} c_m c_n t^m \otimes t^n, \\ \Delta(x) &= \sum_{m \in \mathbb{Z}} c_m t^m \otimes t^m. \end{aligned}$$

Those sums equal, if

$$\begin{aligned} c_m c_n &= 0 \text{ for all } m \neq n, \\ c_m^2 &= c_m \text{ for all } m. \end{aligned}$$

By those conditions, it follows

$$x = t^m.$$

Therefore

$$\Xi(G) = \{\chi_m \mid m \in \mathbb{Z}\} \cong \mathbb{Z}$$

with

$$\chi_m(y) = y^m.$$

Example 11. Let $T \cong \mathcal{G}_m^n$ be a torus. Then,

$$\Xi(T) = \{\chi_m \mid m \in \mathbb{Z}^n\} \cong \mathbb{Z}^n$$

where $\chi_m(y) = y^m = y_1^{m_1} \cdots y_n^{m_n}$.

Note: For each algebraic group G , $\Xi(G)$ is naturally an abelian group:

$$(\chi_1 \cdot \chi_2)(g) := \chi_1(g) \cdot \chi_2(g).$$

Given a morphism of algebraic groups $f : G \rightarrow H$, we get a morphism of abelian groups

$$\begin{aligned} f^* : \Xi(H) &\longrightarrow \Xi(G) \\ \chi &\longmapsto \chi \circ f =: f^*(\chi). \end{aligned}$$

This induces a contravariant functor from the category of algebraic groups to the category of abelian groups.

Lemma 26. *Let G be a diagonalizable algebraic group. Then, $\Xi(G)$ is a k -basis for $\mathcal{O}(G)$.*

Example 12. Let $G = \mathcal{G}_m^n$ be a torus. Then, we have the embedding

$$\begin{aligned} \Xi(G) &\hookrightarrow \mathcal{O}(G) \\ \chi_m &\longmapsto t^m. \end{aligned}$$

The lemma is obvious in this case: each element of $\mathcal{O}(G) = k[t_1, \dots, t_m, t_1^{-1}, \dots, t_m^{-1}]$ can be written uniquely as a linear combination of monomials.

Proof. (i) $\Xi(G)$ spans $\mathcal{O}(G)$:

Choose an embedding $G \subset \mathcal{G}_m^n$ of algebraic groups. Then, by restriction, we get

$$\mathcal{O}(\mathcal{G}_m^n) \twoheadrightarrow \mathcal{O}(G).$$

Since the $\chi_m, m \in \mathbb{Z}^n$, span $\mathcal{O}(\mathcal{G}_m^n)$, their images $\chi_m|_G \in \Xi(G)$ span $\mathcal{O}(G)$.

(ii) $\Xi(G)$ is linearly independent:

Suppose otherwise and let ϕ_1, \dots, ϕ_m be a linearly dependent subset of $\Xi(G)$ with $m \geq 1$ chosen minimally, with $c_1, \dots, c_m \in k^\times$ s.t.

$$\sum_{i=1}^m c_i \phi_i = 0.$$

We distinguish the following cases:

$m = 1$: In this case, we have $\phi_1 = 0$, but $\phi_1(1) = 1$, a contradiction.

$m > 1$: We can assume $\phi_1 \neq \phi_2$, so there is an $h \in G$ s.t. $\phi_1(h) \neq \phi_2(h)$. Then,

$$\phi_1(h) \sum_{i=1}^m c_i \phi_i = 0,$$

but also for all $h, g \in G$

$$\sum_{i=1}^m c_i \phi_i(hg) = \sum_{i=1}^m c_i \phi_i(h) \phi_i(g) = 0.$$

This implies

$$\sum_{i=1}^m c_i \phi_i(h) \phi = 0.$$

Ergo

$$\sum_{i=1}^m c_j (\phi_i(h) - \phi_1(h)) \phi_i = \sum_{i=2}^m c_j (\phi_i(h) - \phi_1(h)) \phi_i = 0.$$

Now, $\phi_i(h) - \phi_1(h)$ is zero if $i = 1$ and non-zero, if $i = 2$. Therefore, this yields a shorter linear dependency for the elements

$$\phi_2, \dots, \phi_m,$$

which contradicts our requirement. □

Definition 27. Let M be an abelian group. The **group algebra** on M is the k -algebra $k[M]$ (not a coordinate ring!) defined as follows:

$$\begin{aligned} k[M] &:= \text{the } k\text{-vectorspace with basis } M \\ &:= \left\{ \sum_{m \in M} c_m \cdot m \mid c_m \in k, \text{ almost all } c_m = 0 \right\}, \end{aligned}$$

where the multiplication on $k[M]$ extends that on M :

$$\left(\sum_{m \in M} c_m m \right) \left(\sum_{n \in M} d_n n \right) = \sum_{m, n \in M} c_m d_n mn.$$

Corollary 5. For a diagonalizable G , we have

$$\mathcal{O}(G) \cong k[\Xi(G)].$$

Fact: For an abelian group M , there is exactly one Hopf algebra structure on $k[M]$ given by $\Delta(m) = m \otimes m$ for all $m \in M$.

With this definition, the above isomorphism is one of Hopf algebras.

Lemma 27. *If G, H are diagonalizable algebraic groups, then*

$$\text{Hom}_{\text{alg.grp.s}}(G, H) \xrightarrow{f \mapsto f^*} \text{Hom}_{\text{grp.s}}(\Xi(H), \Xi(G))$$

is a bijection.

Proof.

$$\begin{aligned} \text{Hom}(G, H) &\cong \text{Hom}_{\text{Hopf-alg.}}(\text{O}(H), \text{O}(G)) \\ &\cong \{\phi \in \text{Hom}_{k\text{-alg.}}(\text{O}(H), \text{O}(G)) \mid (\phi \otimes \phi) \circ \Delta = \Delta \circ \phi\}. \end{aligned}$$

Since $\text{Hom}_{k\text{-alg.}}(\text{O}(H), \text{O}(G)) \cong \text{Hom}(k[\Xi(H)], k[\Xi(G)])$, this reduces to the following lemma:

Lemma 28. *Let M_1, M_2 be two abelian groups. Then*

$$\begin{aligned} \text{Hom}(M_1, M_2) &\xrightarrow{\cong} \text{Hom}_{\text{Hopf-alg.}}(k[M_1], k[M_2]) \\ \phi &\longmapsto \left[\sum c_m m \mapsto \sum c_m \phi(m) \right]. \end{aligned}$$

Proof. We have to show that

$$M = \{x \in K[M]^\times \mid \Delta(x) = x \otimes x\}.$$

Then, by this, it follows for each $\phi \in \text{Hom}_{\text{Hopf-alg.}}(k[M_1], k[M_2])$,

$$\phi(M_1) \subseteq M_2.$$

Ergo, $\phi|_{M_1} \in \text{Hom}(M_1, M_2)$. Therefore, the surjectivity of the claimed bijection is shown. The injectivity is clear, since M generates $k[M]$ as a k -algebra.

To show

$$M = \{x \in K[M]^\times \mid \Delta(x) = x \otimes x\},$$

let

$$\begin{aligned} x &= \sum c_m m \in K[M]^\times \\ \Delta(x) &= \sum c_m m \otimes m \\ x \otimes x &= \sum c_m c_n m \otimes n. \end{aligned}$$

If $\Delta(x) = x \otimes x$, then it follows

$$x = m$$

for some $m \in M$.

□

□

Recall: We have seen that for diagonalizable algebraic groups G, H

$$\mathrm{Hom}(G, H) \cong \mathrm{Hom}(\Xi(H), \Xi(G)).$$

If G is diagonalizable, then

$$\mathrm{O}(G) \cong k[\Xi(G)].$$

Theorem 11. *The functor*

$$\begin{aligned} G &\longrightarrow \Xi(G) \\ f &\longmapsto f^* \end{aligned}$$

defines an equivalence of categories:

$$\{\text{diagonalizable alg. groups}\} \cong \{\text{finite-dim. abelian groups with no char}(k)\text{-torsion}\}.$$

This amounts to the bijection above between Hom-spaces and the following lemma.

Lemma 29. (i) *Let G be a diagonalizable alg. group. Then, $\Xi(G)$ is a finitely generated abelian group with no char(k)-torsion.*

(ii) *Let Γ be a finitely generated abelian group with no char(k)-torsion. Then, there is a diagonalizable algebraic group G s.t. $\Xi(G) \cong \Gamma$.*

Proof. We will use the following facts:

- Let $n \in \mathbb{N}$. Then, $t^n - 1$ is square-free in $k[t]$ iff the ideal $(t^n - 1)$ is radical in $k[t]$ iff $t^n - 1$ has not repetitive root iff either $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) = p > 0$ and $p \nmid n$.

(Proof: Galois Theory, separable/inseparable extensions.)

- Let $M := \mathbb{Z}/n\mathbb{Z}$. Then, the k -group-algebra generated by M

$$k[M] \cong k[t]/(t^n - 1)$$

is reduced iff either $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) = p > 0, p \nmid n$.

- If M_1, M_2 are abelian groups, then we have the following isomorphism of Hopf algebras

$$\begin{aligned} k[M_1] \otimes_k k[M_2] &\xrightarrow{\cong} k[M_1 \oplus M_2] \\ m_1 \otimes m_2 &\longmapsto m_1 m_2 \end{aligned}$$

where $M_1 \oplus M_2 \cong M_1 \times M_2$.

- (i) Embed $G \hookrightarrow T := \mathcal{G}_m^n$ for some n . Then, we have a surjection $\mathbb{Z}^n \cong \Xi(T) \twoheadrightarrow \Xi(G)$. Ergo, $\Xi(G)$ is finitely generated.

Suppose $\text{char}(k) = p > 0$. Let $\chi \in \Xi(G)$ with $\chi^p = 1$. Then, for all $g \in G$, $\chi^p(g) = \chi(g^p) = 1$. The unit group k^\times has not p -torsion, therefore $G \hookrightarrow T = (k^\times)^n$ has also no p -torsion. Therefore, the Frobenius $g \mapsto g^p$ is an isomorphism on G . Therefore, $\chi = 1$ is a trivial character. Ergo $\Xi(G)$ has no p -torsion.

- (ii) Let Γ be a finitely generated abelian group with no $\text{char}(k)$ -torsion. Then,

$$\Gamma \cong \mathbb{Z}^r \oplus \mathbb{Z}/n_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/n_l\mathbb{Z}$$

where $\text{char}(k) \nmid n_1, \dots, n_l$. We may reduce to the cases:

- (a) $\Gamma = \mathbb{Z}$: take $G = \mathcal{G}_m$, then $\Xi(G) \cong \mathbb{Z} \cong \Gamma$.
(b) $\Gamma = \mathbb{Z}/n\mathbb{Z}$ with $\text{char}(k) =: p \nmid n$:
take $G := \mu_n := \{y \in k^\times \mid y^n = 1\}$. Then, since $p \nmid n$, $(t^n - 1)$ is radical. So,

$$\mathcal{O}(\mu_n) \stackrel{\text{Nullstellensatz}}{=} k[t]/(t^n - 1) \stackrel{\text{as Hopf algebras}}{\cong} k[\Gamma]$$

where t gets mapped to the generator of Γ .

□

Corollary 6. *We have the bijection*

$$\{\text{tori}\} \cong \{\text{finitely generated free abelian groups}(\cong \mathbb{Z}^n)\}.$$

Remark 3.

$$\{\text{algebraic group schemes}/k\} \stackrel{\text{not necessarily natural}}{\cong} \{\text{f.g. Hopf algebras}\}.$$

by

$$G \mapsto \mathcal{O}(G)$$

and

$$\{\text{diagonalizable algebraic group schemes}/k\} \cong \{\text{f.g. abelian groups}\}.$$

by

$$G \mapsto \Xi(G).$$

Where μ_p in the left hand term gets mapped to $O(\mu_p) = k[t]/(t^p - 1)$ with $p = \text{char } k$.

5.1 Trigonalization

We say a representation $r : G \rightarrow \text{GL}(V)$ of a group G on a finite-dimensional k -vector space V is **trigonalizable** if it admits a basis with respect to which $r(V)$ is upper-triangular:

$$r(G) \subseteq \left\{ \begin{pmatrix} * & \dots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & * \end{pmatrix} \right\}$$

Definition 28. We call a subgroup $G \subseteq \text{GL}(V)$ **trigonalizable**, if the identity representation is.

Lemma 30. *Let G be an algebraic group. The following are equivalent:*

- (i) *Every finite-dimensional representation $r : G \rightarrow \text{GL}(V)$ is trigonalizable.*
- (ii) *Every irreducible representation of G is 1-dimensional.*
- (iii) *G is isomorphic to an algebraic subgroup of*

$$B_n := \left\{ \begin{pmatrix} * & \dots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & * \end{pmatrix} \right\} \subseteq \text{GL}_n(k).$$

- (iv) *There is a normal unipotent algebraic subgroup U of G s.t. G/U is diagonalizable.*

Proof. We prove as follows:

- (i) \implies (ii): Let V be an irreducible representation. Then, $V \neq 0$. Choose a basis e_1, \dots, e_n of V s.t.

$$r(G) \subseteq B_n.$$

Then, $r(G)e_1 \subseteq ke_1$, so $V_0 := ke_1$ is G -invariant. Ergo $V = V_0$ is 1-dimensional.

(ii) \implies (i): Let V be a f.d. representation. We show by induction on $\dim(V)$ that $r : G \rightarrow \mathrm{GL}(V)$ is trigonalizable:

In the cases $\dim(V) = 0, 1$, there is nothing to show.

In the case $\dim(V) \geq 2$, assume that V is not irreducible. Then, there is a G -invariant V_0 with $0 \neq V_0 \neq V$.

By the induction hypothesis, V_0 and V/V_0 are trigonalizable. Ergo, V is trigonalizable.

(**Recall:** we used this criterion above in the proof that unipotent groups are trigonalizable by showing that every ??? of each G is trivial.)

(i) \implies (iii): Choose a faithful representation V of G . Then, $G \cong r(G)$. Since r is trigonalizable, there is a basis of V s.t.

$$r(G) \subseteq B_n \subseteq \mathrm{GL}_n(k).$$

(iii) \implies (ii): Suppose $G \subseteq B_n \subseteq \mathrm{GL}_n(k)$. Set

$$A_n := \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & * \end{pmatrix} \right\} \subseteq \mathrm{GL}_n(k),$$

$$U_n := \left\{ \begin{pmatrix} 1 & \dots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & 1 \end{pmatrix} \right\} \subseteq \mathrm{GL}_n(k) \text{ normal algebraic subgroup of } B_n,$$

$$U := G \cap U_n \text{ normal unipotent algebraic subgroup of } G.$$

Let V be an irreducible representation of G , then V is not zero. Consider the subspace of V fixed by U

$$V^U := \{v \in V \mid r(u)v = v \forall u \in U\}.$$

Then, we get a representation

$$r|_U : U \longrightarrow \mathrm{GL}(V).$$

Then, $r(U)$ is a unipotent algebraic group of $\mathrm{GL}(V)$. Ergo,

$$r(U) \subseteq \left\{ \begin{pmatrix} 1 & \dots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Ergo, $V^U \neq 0$. Since U is normal in G , the subspace V^U of V is G -invariant: if $v \in V^U, g \in G$, then for all $u \in U$ we have

$$r(u)r(g)v = r(g)r(g^{-1}ug)v = r(g)v$$

since $v \in V^U$. Ergo $r(g)v \in V^U$.

Since V is irreducible, $V = V^U$, i.e. U acts trivially on V . Ergo, r descends to a representation of the group G/U .

But $G/U \hookrightarrow B_n/U_n \cong A_n$. Therefore, G/U and $r(G)$ are commutative. Moreover, for all $g \in G$, $r(g) \in \text{GL}(V)$ is semisimple:

if $g = g_s g_u$, then $g_u \in U$, because U_n is the group of unipotent elements of B_n .

Hence, $r(g) = r(g_s)r(g_u) = r(g_s)$ is semisimple.

It follows that $r(G)$ is commutative and consists of semisimple elements. By some HW: $r(G)$ is trigonalizable. It is easy to show now that V is one-dimensional. (Since V is irreducible and ke_1 is G -invariant.)

□

Definition 29. G is **trigonalizable**, if it satisfies one of the above equivalent conditions.

Later, we will see, that if G is connected, then being trigonalizable implies being solvable.

5.2 Commutative Groups

Let G be an algebraic group. Denote by G_s resp. G_u the subsets of semisimple resp. unipotent elements of G .

Then, G_u is always algebraical i.e. closed: if $G \hookrightarrow \text{GL}_n(k)$, then $G_u = \{g \mid (g - 1)^n = 0\}$. G_u does not need to be closed under multiplication (for example, take $G = \text{SL}_2(k)$,

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

G_s needs not to be algebraic: for example, take $G = \text{SL}_2(k)$ and if G_s were algebraic, then

$$\left\{ \lambda \in k^\times \mid \begin{pmatrix} \lambda & 1 \\ 0 & \lambda^{-1} \end{pmatrix} \in G_s \right\} = \{ \lambda \mid \lambda \neq \lambda^{-1} \}$$

but the last set is not algebraic. Also, G_s does not need to be a subgroup.

We have the a surjective map of sets

$$\begin{aligned} G_s \times G_u &\longrightarrow G \\ (g_1, g_2) &\longmapsto g_1 g_2. \end{aligned}$$

Example 13 (Non-Example). Take generic $g \in G_s, h \in G_u$ for $G = \mathrm{SL}_2(k)$. Then, g, h do not commute and we have

$$((gh)_s, (gh_u)) \neq (g, h)$$

because Jordan components commute.

Theorem 12. *Let G be a commutative algebraic group. Then:*

- (i) G_s, G_u are closed subgroups and the multiplicative map $G_s \times G_u \rightarrow G$ is an isomorphism of algebraic groups.
- (ii) G is trigonalizable. Moreover, for each finite dimensional representation $r : G \rightarrow \mathrm{GL}(V)$ there is a basis s.t.

$$r(G_s) \subseteq \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & * \end{pmatrix} \right\} \quad r(G_u) \subseteq \left\{ \begin{pmatrix} 1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

- (iii) G_s is diagonalizable.

Proof. (ii) Let V be any irreducible representation of G . We have seen that commuting semisimple operators may be simultaneously diagonalizable, then

$$V = \bigoplus_{\chi: G_s \rightarrow \mathcal{G}_m} V_\chi$$

where

$$V_\chi = \{v \in V \mid r(h)v = \chi(h)v \ \forall h \in G_s\}.$$

Since G is commutative, each subspace V_χ is G -invariant ($r(h)r(g)v = r(g)r(h)v = r(g)\chi(h)v = \chi(h)r(g)v$).

Since V is irreducible, we must have $V = V_\chi$ for some χ .

Recall that $G \cong G_s \times G_u$ as abstract groups. We have seen that $r(G_s) \subseteq k^\times$. We proved a while ago that any unipotent group, such as G_u , is trigonalizable. Ergo, V is trigonalizable. Since V is irreducible, we have $\dim V = 1$.

If we apply the same argument without assuming that V is irreducible, then we see that V is the coproduct of V_χ 's as above and that each V_χ admits a basis s.t.

$$r(G_s)|_{V_\chi} \subseteq \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & * \end{pmatrix} \right\} \quad r(G_u)|_{V_\chi} \subseteq \left\{ \begin{pmatrix} 1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

This yields the same conclusion for V .

- (i) We have to show that G_s and G_u are closed and $j : G_s \times G_u \rightarrow G$ is an isomorphism of groups. Take any faithful representation

$$G \xrightarrow{\cong, r} r(G) \subseteq \mathrm{GL}(V)$$

and apply (ii). Then we have

$$\begin{aligned} r(G) &\subseteq \left\{ \begin{pmatrix} * & * & * \\ 0 & \ddots & * \\ 0 & 0 & * \end{pmatrix} \right\} =: B \\ B_u &= \left\{ \begin{pmatrix} 1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & 1 \end{pmatrix} \right\}, \\ r(G_s) &\subseteq \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & * \end{pmatrix} \right\} =: A. \end{aligned}$$

In fact, $r(G_s) = r(G) \cap A$, because if $g \in G$ with $r(g) \in A$, then $r(g)$ is semisimple, so $g \in G_s$.

Therefore, G_s is closed in G . Ergo, G_s and G_u are closed subgroups.

Then, the map j is a morphism of algebraic groups.

We need to show that j^{-1} is a morphism of algebraic groups. For this, it suffices to verify that the projection $G \rightarrow G_s$ is a morphism. But this map is given under r by the morphism:

$$t := \begin{pmatrix} a_1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & a_n \end{pmatrix} \mapsto \begin{pmatrix} a_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_n \end{pmatrix} =: t_s.$$

This suffices because if $g = g_s g_u$, then $g_u = g_s^{-1} g$, so if the map $g \mapsto g_s$ is a morphism, so is $g \mapsto g g_s^{-1} = g_u$, hence so is $g \mapsto (g_s, g_u)$.

- (iii) We have seen that G_s is a closed subgroup. Hence G_s is a commutative algebraic group where elements are semisimple. Ergo, G_s is diagonalizable. □

Theorem 13 (Lie-Kolchin). *Let G be a connected solvable algebraic group. Then G is trigonalizable.*

(By comparison, recall that we have seen that far that, if G is commutative or unipotent, then G is trigonalizable.) We can reformulate this theorem as: Any connected solvable subgroup of $\mathrm{GL}(V)$ stabilizes some complete flag $\mathcal{F} = (V_0 \subsetneq \dots \subsetneq V_n)$.

Generalization (Borel's Fixed Point Theorem): Any connected algebraic group G acting on a projective variety X has a fixed point in X .

We get a relation between complete flags and projective varieties.

Proof. Induct on the number n s.t. $G^{(n)} = 1$.

For $n = 0$, there is nothing to show.

If $n = 1$, $(G, G) = 1$, then G is commutative, ergo trigonalizable.

Let $n \geq 2$. Then, we have $G' := (G, G) \neq 1$. We will show the following lemma:

Lemma 31. *If G is connected, then the abstract group G' with the induced topology is connected (\iff the Zariski Closure of G' is connected).*

Proof. We have the following facts:

- An increasing union of connected spaces is connected.
- A continuous image of a connected space is connected.

We have

$$\begin{aligned} G' &= \langle (g, h) := ghg^{-1}h^{-1} \mid g, h \in G \rangle \\ &= \bigcup_{j \geq 0} \bigcup_{g_1, h_1, \dots, g_j, h_j \in G} \{(g_1, h_1) \cdots (g_j, h_j)\}. \end{aligned}$$

Since

$$\bigcup_{g_1, h_1, \dots, g_j, h_j \in G} \{(g_1, h_1) \cdots (g_j, h_j)\} = \mathrm{Im} \phi_j$$

for some continuous map $\phi_j : G^{2j} \rightarrow G$, the claim follows. □

Ergo, G' is connected.

Note: It is equivalent to show that (*) any subgroup of $\mathrm{GL}(V)$ s.t. G is connected and solvable is trigonalizable in $\mathrm{GL}(V)$.

Indeed, the theorem implies (*): the Zariski closure of G is a connected algebraic group that is solvable (which extends by continuity). If $Zcl(G)$ is trigonalizable, then also G is trigonalizable.

(*) implies the theorem, since if G is given as in the theorem, apply (*) to $r(G) \subseteq \mathrm{GL}(V)$.

If $G^{(n)} = 1$, then $(G')^{(n-1)} = G^{(n)} = 1$. By induction, we may assume that G' satisfies the following:

For all finite dimensional representations $r : G \rightarrow \mathrm{GL}(V)$, $r(G')$ is trigonalizable.

Our aim is to show that any irreducible representation V of G has dimension 1.

The induction hypothesis implies that $r(G')$ is trigonalizable. In particular, there exist an eigenspace $V_\chi \subseteq V$ for G' for some character $\chi : G' \rightarrow k^\times$. Since G' is normal in G we know that G acts from the left on

$$\{\text{eigenspaces } V_\chi \text{ in } V \text{ for } G'\}.$$

Ergo, $\bigoplus_{\chi: G' \rightarrow k^\times} V_\chi$ is G -invariant. Ergo, $V = \bigoplus_{\chi: G' \rightarrow k^\times} V_\chi = \bigoplus_{\chi \in \Xi'} V_\chi$ for some finite subset $\Xi' = \{\chi \mid V_\chi \neq 0\}$ of $\mathrm{Hom}(G', \mathcal{G}_m)$, since V is finite dimensional.

Claim: Let $h \in G'$. Then, the map

$$\begin{aligned} G &\longrightarrow \mathrm{GL}(V) \\ g &\longmapsto r(ghg^{-1}) \end{aligned}$$

has a finite map.

Proof. Denote by $\chi \mapsto \chi^g$ the action of $g \in G$ in $\mathrm{Hom}(G', \mathcal{G}_m)$ given by $\chi^g(h) := \chi(ghg^{-1})$. This is an action, since G' is normal.

This descends to an action $G \curvearrowright \Xi$, because r is a homomorphism. Since $r(h)$ is determined by $\{\chi(h) \mid \chi \in \Xi\}$, hence similarly $r(ghg^{-1}) \in r(G')$ by $\{\chi(ghg^{-1}) \mid \chi \in \Xi\}$.

Hence,

$$\#\{r(ghg^{-1}) \mid g \in G\} \leq \#\text{representations of the finite set } \Xi < \infty.$$

□

Lemma 32. *Let G be an algebraic group. Then, G is connected iff for each finite algebraic set X , and for each morphism $f : G \rightarrow X$ of algebraic sets, we have that f is constant.*

Claim with the Lemma implies that the map $g \mapsto t(ghg^{-1})$ is constant. This implies that $r(ghg^{-1}) = r(h)$ for all $g \in G, h \in G'$. Ergo, G stabilizes each eigenspace V_χ for G' . Ergo, $V = V_{\chi_0}$, since V is irreducible. \square

Lemma 33. *Let G be any group with a finite dimensional representation $r : G \rightarrow GL(V)$. Then, the subspaces V_χ for $\chi \in \text{Hom}(G, k^\times)$ are independent, i.e., the map*

$$\oplus V_\chi \longrightarrow V$$

is injective.

Proof. The spaces V_χ are G -invariant. Suppose, there exist distinct χ_1, \dots, χ_n of non-zero $v_j \in V_{\chi_j}$ s.t. $\sum_j v_j = 0$.

We may assume that n , the number of v_j , is minimal. W.l.o.g., $n \geq 2$.

Choose $g \in G$ s.t. $\chi_1(g) \neq \chi_2(g)$. Use that $0 = g \sum_j v_j = \sum_j g v_j$ and take the linear combination as in the proof of linear independence of characters to contradict the minimality of n . \square

Since $G' = \langle ghg^{-1}h^{-1} \mid g, h \in G \rangle$, so $\det(r(G')) = 1$.
On the other hand, for each $g \in G'$, we have

$$r(g) = \begin{pmatrix} \chi_0(g) & & \\ & \ddots & \\ & & \chi_0(g) \end{pmatrix}$$

since $V = V_{\chi_0}$. This implies

$$1 = \det(r(g)) = \chi_0(g)^d.$$

Ergo, χ_0 defines a morphism

$$\chi_0 : G' \longrightarrow \mu_d \subseteq \mathcal{G}_m.$$

But G' is connected and μ_d is finite. Since χ_0 is a morphism, χ_0 must be constant, ergo the trivial character.

As a consequence, we get $r(G') = 1$ on $V = V_{\chi_0}$.

Lemma 34. *Let G be an algebraic group, $r : G \rightarrow \mathrm{GL}(V)$ a representation. $v \in V$ shall be a simultaneous non-zero eigenvector for $r(G)$.*

Then, for each $g \in G$, there is a value $\chi(g) \in k^\times$ s.t.

$$r(g)v =: \chi(g)v.$$

Then, the mapping $\chi : G \rightarrow \mathcal{G}_m$ is a morphism of algebraic groups.

Therefore, r descends to a representation of the commutative group

$$\bar{r} : G/G' \longrightarrow \mathrm{GL}(V).$$

Ergo, $r(G/G') = r(G)$ is commutative and therefore trigonalizable (because of irreducibility). \square

Example 14 (Non-Example). • Take $G = D_4 \hookrightarrow \mathrm{GL}_2(\mathbb{C})$ which is solvable and has an irreducible and faithful representation over \mathbb{C}^2 .

- Consider the solvable group

$$G = \left\langle \begin{pmatrix} \pm 1 & \\ & \pm 1 \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \right\rangle$$

which is a finite subgroup of $\mathrm{GL}_2(\mathbb{C})$, s.t. \mathbb{C}^2 define an irreducible representation of G .

Lemma 35 (Form of Schur's Lemma). *If S is any commutative subset of $GL(V)$ for a finite-dimensional $0 \neq V$ over an algebraically closed field k . Let V be S -irreducible. Then, $\dim V = 1$.*

Proof. There is nothing to show if S is empty.

Let $s \in S$ and denote by $V_\lambda \subseteq V$ the λ -eigenspace for s . Then, since S is commutative, V_λ is S -invariant. Therefore, $V = V_\lambda$ for one $\lambda \in k^\times$.

Thus, every $s \in S$ acts by scaling, therefore every subspace of V is S -invariant. Since V is invariant, we get $\dim V = 1$. \square

Corollary 7. *Let G be a connected algebraic group. Then, G is solvable iff G is trigonalizable.*

Proposition 3. *If G is trigonalizable, then G_u is a normal algebraic subgroup.*

Proof. We have

$$G \hookrightarrow B := \left\{ \begin{pmatrix} * & \dots & * \\ & \ddots & \vdots \\ & & * \end{pmatrix} \right\} \subseteq GL_n(k).$$

B has the normal subgroup $U := \left\{ \begin{pmatrix} 1 & \dots & * \\ & \ddots & \vdots \\ & & 1 \end{pmatrix} \right\}$ and we have $G_u = G \cap U$. Now,

U is the kernel of the multiplicative morphism

$$\begin{pmatrix} a_1 & \dots & * \\ & \ddots & \vdots \\ & & a_n \end{pmatrix} \mapsto \begin{pmatrix} a_1 & \\ & a_n \end{pmatrix}.$$

\square

Corollary 8. *If G is connected and solvable, then G_u is a normal algebraic subgroup.*

5.3 Semisimple Elements of nilpotent Groups

Theorem 14. *Let G be a connected nilpotent algebraic group. Then, we have*

$$G_s \subseteq Z(G)$$

where $Z(G)$ denotes the center of G .

Theorem 15 (Lie-algebraic Analogue). *Let V be a finite-dimensional vectorspace. Let \mathfrak{g} be the Lie-Subalgebra of $\text{End}(V)$, i.e. \mathfrak{g} is a subspace s.t. we have for each $x, y \in \mathfrak{g}$*

$$[x, y] := xy - yx \in \mathfrak{g}.$$

Assume that \mathfrak{g} is nilpotent, i.e. there is an $n \in \mathbb{N}_0$ s.t.

$$[x_1, [x_2, [\dots, [x_{n-1}, x_n]]]] = 0$$

for all $x_1, \dots, x_n \in \mathfrak{g}$.

*Then, any semisimple (semisimple in $\text{End}(V)$ that is) $x \in \mathfrak{g}$ is **central** in \mathfrak{g} , i.e. $[x, y] = 0$ for each $y \in \mathfrak{g}$.*

Remark 4. The Lie-algebraic Analogue implies the general theorem if – for example – $k = \mathbb{C}$.

Proof. Let $g \in G_s$. We want to show $Z_G(g) = G$.

Fact from the theory of Lie-Algebras: For the Lie-Algebra $\text{Lie}Z_G(g)$ we have

$$\text{Lie}Z_G(g) = \ker(\text{Ad}(g))$$

where Ad is the map

$$\begin{aligned} \text{Ad} : G &\longrightarrow \text{GL}(\mathfrak{g}) \\ x &\longmapsto gxg^{-1}. \end{aligned}$$

Since G is connected, it suffices to verify

$$\ker(\text{Ad}(g)) = \mathfrak{g}$$

i.e. $\text{Ad}(g) = 1$.

Since g is semisimple, we have for suitable basis

$$g = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$$

with $a_j \in \mathbb{C}^\times$. This is $\exp(x)$ for a suitable diagonal matrix $x = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} \in \text{GL}_n(\mathbb{C})$.

Fact: We may assume that $x \in \mathfrak{g} := \text{Lie}(G)$.

Since G is nilpotent, it can be shown that \mathfrak{g} is nilpotent.

By the theorem, x is central in \mathfrak{g} . By the properties of \exp we have

$$\text{Ad}(g) = \exp(\text{ad}(g)) = 1$$

ergo $\text{ad}(x) = 0$ where $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{g}$ is defined by

$$\text{ad}(x) \cdot y := [x, y].$$

□

Proof. If \mathfrak{g} is nilpotent, then $\text{ad}(x) \in \text{End}(\mathfrak{g})$ is nilpotent.

Since x is semisimple, $\text{ad}(x)$ is semisimple, because $\text{ad}(x)$ is the restriction to \mathfrak{g} of the map

$$\begin{aligned} \text{End}(V) &\longrightarrow \text{End}(V) \\ y &\longmapsto [x, y] \end{aligned}$$

and, if e_1, \dots, e_n are a basis of eigenvectors for x , then $E_{i,j}$ is a basis of eigenvectors for ℓ .

So, $\text{ad}(x)$ is nilpotent and semisimple, therefore $\text{ad}(x) = 0$. □

Proof Theorem. Let G be a connected nilpotent algebraic group, $G \xrightarrow{\text{GL}} (V)$.

Let $g \in G_s$, we want to show that $g \in Z(G)$.

Assume otherwise, then we have a $h \in G$ s.t. $(g, h) = ghg^{-1}h^{-1} \neq 1$.

Since G is connected and nilpotent (ergo solvable), we know by Lie-Kolchin that G stabilizes some complete flag $V_0 \subset \dots \subset V_n$.

We have $g|_{V_i}, h|_{V_i} \in \text{GL}(V_i)$. They commute, if $i = 0$, but not if $i = n$.

So, there is an i s.t. $g|_{V_i}, h|_{V_i}$ commute but $g|_{V_{i+1}}, h|_{V_{i+1}}$ don't commute. W.l.o.g. $V = V_{i+1}, g = g|_{V_{i+1}}, h = h|_{V_{i+1}}$. Set $a := g|_{V_i}, b := h|_{V_i} \in \text{GL}(V_i)$. a will be semisimple, since g is.

Since g is semisimple, there is an eigenvector $v \in V_{i+1}$ for g s.t.

$$V_{i+1} = V_i \oplus \langle v \rangle.$$

We have an isomorphism of vector spaces

$$\text{End}(V_{i+1}) \cong \text{End}(V_i) \oplus \text{Hom}(\langle v \rangle, V_i) \oplus \text{Hom}(V_i, \langle v \rangle) \oplus \text{End}(\langle v \rangle)$$

with

$$\text{End}(\langle v \rangle) \cong k \text{ and } \text{Hom}(\langle v \rangle, V_i) \cong V_i.$$

So, we can write $g|_{V_{i+1}}, h|_{V_{i+1}}$ write as

$$g = \begin{pmatrix} a & \\ & * \in k \end{pmatrix} \text{ and } h = \begin{pmatrix} b & c \in V_i \\ & * \end{pmatrix}.$$

We may replace g, h with scalar multiples to reduce to the case that $* = 1$. Then, So, we can write $g|_{V_{i+1}}, h|_{V_{i+1}}$ write as

$$g = \begin{pmatrix} a & \\ & 1 \end{pmatrix} \text{ and } h = \begin{pmatrix} b & c \\ & 1 \end{pmatrix}.$$

Then,

$$h \neq ghg^{-1} = \begin{pmatrix} b & ac \\ & 1 \end{pmatrix}.$$

Ergo, $c \neq ac$, i.e. $c \notin \ker(a - 1)$. Let $h_1 := h^{-1}ghg^{-1}$. Check

$$h_1 = \begin{pmatrix} 1 & b^{-1}(a - 1)c \\ & 1 \end{pmatrix}.$$

We claim that h_1 does not commute with g . This claim implies the theorem, since we can iterate the claim to obtain elements h_i by $h_{i+1} := h_i^{-1}gh_i g^{-1}$. Then, h_i does not commute with g . But G is nilpotent, therefore $h_i = 1$ for some large enough i .

We can prove the claim as follows: By some calculation as for h and g , we see, that h_1 and g don't commute iff $b^{-1}(a - 1)c \notin \ker(a - 1)$. This is equivalent to

$$\begin{aligned} &\iff (a - 1)b^{-1}(a - 1)c \neq 0 \\ &\iff b^{-1}(a - 1)^2c \neq 0 \\ &\iff (a - 1)^2c \neq 0 \\ &\iff c \in \ker((a - 1)^2). \end{aligned}$$

But a being semisimple implies $a - 1$ being semisimple, therefore $\ker((a - 1)^2) = \ker(a - 1)$. So h_1, g don't commute iff $c \in \ker(a - 1)$ iff h, g don't commute. \square

6 Projective Space

Let V be a finite-dimensional vector space. Then $\mathcal{G}_m = k^\times$ acts on V by scalar multiplication. $\{0\}$ is a \mathcal{G}_m -invariant subspace of V . We are interested on the orbits of \mathcal{G}_m on $V \setminus \{0\}$.

Define the **projective space** over V by

$$\mathbb{P}V := \mathcal{G}_m \backslash (V - 0) = (V - 0) / \sim \cong \{\text{lines in } V\}$$

where for $a, b \in V - 0$ we set

$$a \sim b : \Longleftrightarrow \exists \lambda \in k^\times : \lambda a = b.$$

If $V = k^{n+1}$, we denote the n -dimensional projective space by $\mathbb{P}^n := \mathbb{P}V$.

Given $a = (a_0, a_1, \dots, a_n) \in k^{n+1} - 0$, we denote the \sim -class of a by

$$[a] = [a_0, \dots, a_n] \in \mathbb{P}^n.$$

Define S to be the graded algebra of polynomials in k

$$S := k[x_0, \dots, x_n] = \bigoplus_{d \geq 0} S_d$$

where each S_d is the space of homogenous polynomials of degree d , i.e.

$$S_d = \bigoplus_{i_1, \dots, i_d \in \{0, \dots, n\}} k \cdot x_{i_1} \cdots x_{i_d}.$$

We identify k with the space of constant polynomials $S_0 \subseteq S$.

We have

$$S_d = \{f \in S \mid f(\lambda X) = \lambda^d f(X) \ \forall \lambda \in k^\times\}.$$

Given $f \in S_d$, the set

$$\{a \in k^{n+1} \mid f(a) = 0\}$$

is \mathcal{G}_m -invariant. In other words, given $a \in \mathbb{P}^n$ and $f \in S^d$, it is well-defined to state $f(a) = 0$ and $f(a) \neq 0$.

Definition 30. A **projective** algebraic subset $X \subseteq \mathbb{P}^n$ is a set of the form

$$X = V(\Sigma) := V_{\mathbb{P}^n}(\Sigma)$$

where Σ is a collection of homogenous elements of S , where

$$V_{\mathbb{P}^n}(\Sigma) := \{a \in \mathbb{P}^n \mid f(a) = 0 \ \forall f \in \Sigma\}.$$

Facts:

- Hilbert basis theorem states

$$V(\Sigma) = V(f_1, \dots, f_m)$$

for some finite collection $f_1, \dots, f_m \in \Sigma$.

- It is useful to extend the meaning of " $f(a) = 0$ " for $a \in \mathbb{P}^n$ to *general* elements $f \in S$ by requiring that $f(a') = 0$ for each $a' \in [a]$.

If we write $f = \sum_{d \geq 0} f_d$, $f_d \in S_d$, then we have

$$f(a) = 0 \iff f_d(a) = 0 \ \forall d \geq 0.$$

Therefore, we can extend the definition of $V(\Sigma)$ to any $\Sigma \subseteq S$.

- We have $V(\Sigma) = V((\Sigma))$ where (Σ) is the ideal generated by some finite subset of Σ .
- We call an ideal $I \subseteq S$ **homogenous** if it is the direct sum of its d -homogeneous components, i.e.

$$I = \sum_{d \geq 0} I_d$$

where $I_d = \{f \in I \mid f \text{ is homogenous of degree } d\}$.

I is homogeneous iff it is generated by homogeneous elements.

- We have the following *Nullstellensatz*:

For any $X \subseteq \mathbb{P}^n$, set $I(X)$ to be the ideal generated by all homogeneous polynomials of S vanishing on X . Then, we have

$$I(V_{\mathbb{P}^n}(I)) = I$$

for each homogeneous ideal $I \subseteq S$ for which we have:

1. I is radical.
2. I is not (x_0, \dots, x_n) .

Example 15 (Anti-example). The second property is necessary:

Set $I = (x_0, \dots, x_n)$. Then $V_{k^{n+1}}(I) = 0$. Therefore, $V_{\mathbb{P}^n}(I) = \emptyset$. However,

$$I(V_{\mathbb{P}^n}(I)) = S.$$

- The above point induces a bijection between algebraic subsets of \mathbb{P}^n and radical ideals $I \subset S$ which are not (x_0, \dots, x_n) .

For $i = 0, \dots, n$, set $D(x_i) := \{a \in \mathbb{P}^n \mid a_i \neq 0\}$. $D(x_i)$ is an open set homeomorphic to k^n by mapping

$$\phi_i : a \mapsto \left(\frac{a_0}{a_i}, \dots, \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i}, \dots, \frac{a_n}{a_i} \right).$$

The $D(x_i)$ cover $\mathbb{P}^n = \bigcup_i D(x_i)$.

Given a projective algebraic subset $X \subset \mathbb{P}^n$, define $X^{(i)} \subset k^n$ by

$$X^{(i)} := \phi_i(X \cap D(x_i)).$$

If $X = V_{\mathbb{P}^n}(I)$, then

$$X^{(i)} = V_{k^n}(I^{(i)})$$

where

$$I^{(i)} := \{f^{(i)} \mid f \in I\}$$

where $f^{(i)}(t_1, \dots, t_n) := f(t_1, \dots, t_{i-1}, 1, t_i, \dots, t_n)$. Thus, $X^{(i)}$ is an algebraic subset of k^n .

Definition 31. The **Zariski topology** on \mathbb{P}^n is defined by setting the set of closed sets to be the set of projective algebraic sets.

Facts:

- $D(x_i)$ is open in \mathbb{P}^n , since $D(x_i) = \mathbb{P}^n - V(x_i)$.
- The bijections $D(x_i) \cong k^n$ are homeomorphisms.

Definition 32. A **quasi-projective** algebraic set Y is an open subset of a projective algebraic set $X \subseteq \mathbb{P}^n$.

Example 16. Any algebraic set in k^n is quasi-projective.

Definition 33. A quasi-projective variety is an irreducible quasi-projective algebraic set.

Lecture
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