Mitschrieb: Algebraic Groups
SS 20

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Vorwort

Contents

1 Introduction

Let k be an algebraically closed field.

Definition 1. For $I \subseteq k[X] := k[X_1, \dots, X_n]$, we define its **vanishing set** by

$$V(I) := \{p \in k^n \mid \forall f \in I : \; f(p) = 0\}.$$

A set $S \subset k^n$ is called **algebraic**, if

$$S = V(I)$$

for some $I \subseteq k[X]$.

Example 1. The group $\mathsf{GL}_n(k)$ is not an algebraic subset of $k^{n\times n}$. But, we can identify it with an algebraic subset of $(k^{n\times n})^2$ by

$$\mathsf{GL}_n(k) \cong \left\{(x, y) \in k^{n\times n} \mid xy = 1_n\right\} = V(X \cdot Y - 1_n).$$

Definition 2. Let $\iota : \mathsf{GL}_n(k) \hookrightarrow k^{n\times n2}$ be the injection

$$A \mapsto (A, A^{-1}).$$

A **linear algebraic group** over k is a subgroup $U \subseteq \mathsf{GL}_n(k)$ s.t. $\iota(k)$ is an algebraic subset of k^{2n^2}.

I.e., a linear algebraic group is a matrix-group which can be defined by polynomials over the entries of a matrix and its inverse.

Example 2. The following groups are linear algebraic groups:

1. The multiplicative group $\mathcal{G}_m(k) := k^\times = k \setminus \{0\} = \mathsf{GL}_1(k)$.

2. The general linear group $\mathsf{GL}_n(k)$.

3. The special linear group

$$\mathsf{SL}_n(k) := \{A \in \mathsf{GL}_n(k) \mid \det(A) = 1\}.$$

4. The orthogonal group

$$\mathsf{O}_n(k) := \left\{A \in \mathsf{GL}_n(k) \mid A^T \cdot A = 1\right\}.$$

5. The special orthogonal group

$$\mathsf{SO}_n(k) := \mathsf{O}_n(k) \cap \mathsf{SL}_n(k).$$

6. The upper triangle-matrix group

$$\left\{ \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ & \ddots & \vdots \\ & & a_{n,n} \end{pmatrix} \mid a_{i,j} \in k \right\} \cap \mathsf{GL}_n(k).$$

7. The normed upper triangle-matrix group

$$\left\{ \begin{pmatrix} 1 & \cdots & a_{1,n} \\ & \ddots & \vdots \\ & & 1 \end{pmatrix} \mid a_{i,j} \in k \right\} \cap \mathsf{GL}_n(k).$$

8. The group of n-th roots of unity

$$\mu_n(k) := \{x \in k \mid x^n = 1\}.$$

9. The additive group $(k, +)$ is not a subgroup of $\mathsf{GL}_n(k)$, but it can be identified with the linear algebraic group

$$\left\{ \begin{pmatrix} 1 & a \\ & 1 \end{pmatrix} \mid a \in k \right\} \subset \mathsf{GL}_2(k)$$

10. For $k = \mathbb{C}$, the unit sphere and the unitary groups are NOT linear algebraic groups.

2 Algebraic Groups and Hopf Algebras

Definition 3. A **morphism** $f : X \to Y$ of algebraic sets $X \subset k^m, Y \subset k^n$ is a map which is coordinatewise described by polnomials.

Definition 4. An **algebraic group** is an algebraic set $G \subset k^n$ together with a fixed element $e \in G$ and morphisms $m : G \times G \to G, i : G \to G$ s.t. (G, m, i, e) is a group.

A **morphism of algebraic groups** is a morphism of algebraic sets that is also a group homomorphism.

Definition 5. Let $V \subset k^n$ be any subset. Then, we define the vanishing ideal of V by

$$I(V) := \{f \in k[x] \mid f(V) = 0\}.$$

Definition 6. For a commutative ring R we define the **radical** of an ideal $I \subseteq R$ by

$$\sqrt{I} := \{r \in R \mid r^m \in I \text{ for some } m \in \mathbb{N}_0\}.$$

R is called **reduced**, if $\sqrt{0} = 0$.

Lemma 1 (Zariskis Lemma)**.** *Let $L \supseteq k$ be fields. If L is finitely generated as a k-algebra, then the extension $L \supseteq k$ is finite, i.e., L is a finitely-generated k-vector space.*

Theorem 1 (Hilberts Nullstellensatz)**.** *For any ideal $I \subseteq k[x]$, we have*

$$I(V(I)) = \sqrt{I}.$$

Proof. It is easy to see that

$$I \subset \sqrt{I} \subset I(V(I)).$$

Now, let $f \in I(V(I))$ and assume – for the sake of contradiction – that $f \notin \sqrt{I}$. Since $\sqrt{I}$ is the intersection of its upper prime ideals, there is a prime ideal $p \supset I$, s.t. $f \notin p$. Now, define the zero divisor-free ring

$$R := (k[x]/p)[f^{-1}].$$

And let $\phi : k[x] \to R$ be the corresponding ring homomorphism.

Let $m \subseteq R$ be a maximal ideal in R. Then, R/m is a field, which contains k and is finitely generated as k-algebra. According to Zariski's lemma, R/m is a finite (ergo algebraic) extension of k. Since k is algebraically closed, we have $R/m = k$. Let $\pi_m : R \to k$ be the corresponding ring homomorphism.

Now, for $x_1, \ldots, x_n$, set

$$t_i := \pi_m(\phi(x_i)).$$

Then, $t = (t_1, \ldots, t_n) \in k^n$. We now have

1. $t \in V(I)$: For each $g \in I$, we have $\phi(g) = 0$. On the other hand

$$g(t) = g(\pi_m \circ \phi(x)) = \pi_m \circ \phi(g) = 0.$$

2. $f(t) \neq 0$: $\phi(f)$ is invertible in R, therefore $\phi(f) \neq 0$ and $\phi(f) \notin m$. Ergo

$$f(t) = \pi_m \circ \phi(f) \neq 0.$$

Ergo, there is a point $t \in V(I)$ s.t. $f(t) \neq 0$. This yields a contradiction, since we assumed $f \in I(V(I))$. $\square$

Definition 7. For an algebraic set $X \subset k^n$, we define its **coordinate ring** by

$$k[X] := k[x_1, \ldots, x_n]/I(X).$$

Lemma 2. *For a morphism $f : X \to Y$ of algebraic sets define the following homomorphism of k-algebras.*

$$\begin{aligned} f^* : k[Y] &\longrightarrow k[X] \\ p &\longmapsto p \circ f. \end{aligned}$$

We have a contravariant functor $\_^*$ from the categories of algebraic sets over k to the category of k-algebras:

$$\begin{aligned} X &\longmapsto k[X] \\ \mathsf{Hom}\,(X, Y) &\longmapsto \mathsf{Hom}_k\,(k[Y], k[X]) \\ f &\longmapsto f^*. \end{aligned}$$

Lemma 3. *We have*

$$k[X \times Y] \cong k[X] \otimes k[Y].$$

Proof.

$$k[X] \otimes k[Y] = k[x]/I(X) \otimes_k k[y]/I(Y) = k[x, y]/I(X) \otimes k[y] + k[x] \otimes I(Y).$$

But

$$V(I(X) \otimes k[y] + k[x] \otimes I(Y)) = V(I(X) \otimes k[y]) \cap V(k[x] \otimes I(Y)) = X \times Y.$$

$\square$

Theorem 2. *Every finitely generated reduced k-algebra A is isomorphic to some $k[X]$ for some algebraic X.*

Proof. Choose some $\pi : k[x_1, \dots, x_n] \twoheadrightarrow A$ and set $X := V(\ker \pi)$. Then $\ker \pi = I(X)$, since π's kernel is radical since A is reduced. $\square$

Corollary 1. *The contravariant functor $\_^* : \mathcal{C}_{\text{algSets}} \to \mathcal{C}_{k-\text{alg.s}}$ gives an antiequivalence of categories.*

Lemma 4. *An algebraic set X is isomorphic to some algebraic subset of Y iff there is an epimorphism $k[Y] \twoheadrightarrow k[X]$.*

Lemma 5. *Let $G \subset k^n$ be an algebraic group. Then, we have maps*

$$\begin{aligned} m : G \times G &\longrightarrow G \\ i : G &\longrightarrow G \\ e : * &\longrightarrow G. \end{aligned}$$

They induce dual maps in the category of k-algebras:

$$\begin{aligned} \Delta := m^* : k[G] &\longrightarrow k[G] \otimes_k k[G] \\ \iota := i^* : k[G] &\longrightarrow k[G] \\ \varepsilon := e^* : k[G] &\longrightarrow k \end{aligned}$$

Definition 8. A **Hopf-algebra** over k is a (reduced?!) k-algebra together with maps $\Delta, \varepsilon, \iota$ as above s.t. the following holds:

$$\begin{gathered} (\Delta \otimes \mathrm{Id})\Delta = (\mathrm{Id} \otimes \Delta)\Delta \\ s^* \circ (\iota \otimes \mathrm{Id})\Delta = s^* \circ (\mathrm{Id} \otimes \iota)\Delta = \varepsilon \\ (\varepsilon \otimes \mathrm{Id})\Delta = (\mathrm{Id} \otimes \varepsilon)\Delta = \mathrm{Id} \end{gathered}$$

where $s : G \to G \times G, g \mapsto (g, g)$ is the diagonal map.

A morphism of Hopf-algebras is a homomorphism of k-algebra $F : A \to B$ s.t.

$$\Delta \circ F = (F \otimes F) \circ \Delta.$$

Theorem 3. *The contravariant functor $\_^*$ gives an anti-equivalence of the categories of algebraic groups and the categories of finitely generated Hopf-algebras over k.*

Example 3. 1. Let $G = \mathcal{G}_a = (k, +)$. Then, $k[G] = k[x]$, since $I(x) = 0$. Then, we have

$$\begin{aligned} \Delta(x) &= x \otimes 1 + 1 \otimes x \\ \iota(x) &= -x \\ \varepsilon(x) &= 0. \end{aligned}$$

2. Let $G = \mathcal{G}_m = \{(a, a^{-1}) \mid a \neq 0\} \cong k^\times$. Then, $k[G] = k[x, y]/(xy - 1) = k[x, x^{-1}]$. Then, we have

$$\begin{aligned} \Delta(x) &= x \otimes x \\ \iota(x) &= x^{-1} \\ \varepsilon(x) &= 1. \end{aligned}$$

3. Let $G = \mathsf{GL}_n(k)$. Then, $k[G] = k[x, y]/(xy - 1_n) = k[x_{i,j}, \frac{1}{\det}]$. Then, we have

$$\begin{aligned} \Delta(x_{i,j}) &= \sum_k x_{i,k} \otimes x_{k,j} \\ \Delta(\frac{1}{\det(x)}) &= \frac{1}{\det(x)} \otimes \frac{1}{\det(x)} \iota(x_{i,j}) \qquad\qquad = (x^{-1})_{i,j} \\ \varepsilon(x_{i,j}) &= \delta_{i,j}. \end{aligned}$$

2.1 An Aside on the General Group

Let $G = \mathsf{GL}_n(k) = \{(x, y) \mid xy = \mathrm{Id}_n\}$. Since we have

$$x^{-1} = \frac{1}{\det(x)} \cdot \mathrm{adj}(x)$$

where the adjoint $\mathrm{adj}(x)$ can be expressed by polnomials in the entries of x, we have isomorphisms

$$\begin{aligned} k[x, y]/(xy - 1_n) &\longrightarrow k[x, 1/\det(x)] = k[x, t]/(\det(x) \cdot t = 1) \\ (x, y) &\longmapsto (x, \det(y)) \end{aligned}$$

and

$$\begin{aligned} k[x, 1/\det(x)] &\longrightarrow k[x, y]/(xy - 1_n) \\ (x, t) &\longmapsto (x, t \cdot \mathrm{adj}(x)). \end{aligned}$$

Lemma 6.

$$k[\mathsf{GL}_n(k)] \cong k[x_{i,j}, \frac{1}{\det(x)}].$$

Lemma 7. *Let V be a finite-dimensional k-vector space. If we choose a basis for V, we get an isomorphism $\mathsf{GL}(V)$. Hence, $\mathsf{GL}(V)$ is an algebraic group whose structure is up to a unique isomorphism independent of the choice of basis.*

3 Actions

Remark 1. Let $G \curvearrowright M$ be a group action of algebraic sets, then the morphism

$$G \times M \longrightarrow M$$

yields an homomorphism

$$\Delta : k[M] \to k[G] \otimes k[M].$$

This turns $k[M]$ to a **comodule** of the Hopf-Algebra $k[G]$.

Definition 9. Let V be vector space and G an algebraic group. A morphism $r_V : G \to \mathsf{GL}(V)$ of groups is called **representation** of G, if there is a linear map

$$\Delta : V \to V \otimes_k k[G](= \mathsf{Hom}_{alg}(G, V))$$

s.t. we have for each $v \in V$ and $g \in G$

$$r_V(g) \cdot v = \sum_i v_i \cdot f_i(g)$$

where $\Delta v = \sum_i v_i \otimes f_i$.

That is, V is a comodule for $k[G]$.

A map $\phi : V \to W$ is called **equivariant** for two representations r_V, r_W of G, if

$$\phi(r_V(g)v) = r_W(g)\phi(v)$$

for all g, v.

Example 4. Let $G = \mathsf{GL}_n(k)$, $V = k^n$ and r_V be the canonical representation. For an orthonormal basis $(b_i)_{i=1,\ldots,n}$, we for example can set

$$\Delta v = \sum_{i=1}^{n} b_i \otimes f_i$$

where

$$f_i(A) := b_i^T A v.$$

Then, we have

$$r_V(A) \cdot v = A \cdot v = \sum_{i=1}^{n} b_i \cdot b_i^T A v = \Delta(v)(A).$$

Example 5. Let M be a right G-set. Then, G also acts on $k[M]$, therefore we have a map

$$\rho : G \to \mathsf{GL}(k[M])$$

by, for $v \in k[M]$,

$$(\rho(g)v)(m) := v(m.g).$$

Further, we have an algebra morphism

$$\Delta : k[M] \to k[M] \otimes k[G] = k[M \times G]$$

with

$$(\Delta v)(m, g) = v(m.g).$$

With $\Delta v = \sum_i v_i \otimes f_i$

$$\rho(g)v(m) = v(mg) = \Delta v(m, g) = \sum_{i=1} f_i(g)v_i(m).$$

Ergo, g is a representation of G.

When $M = G$ with action given by the right translation, then $\rho : G \to \mathsf{GL}(k[G])$ is called the **right regular representation** of G.

Lemma 8. *Let G be an algebraic group and V a finite-dimensional k-vector space. Then $\rho : G \to \mathsf{GL}(V)$ is morphism of algebraic groups iff it is a representation.*

Definition 10. Let G be an algebraic group and V a representation of G. A subspace $W \subset V$ is called **invariant** or **subrepresentation**, if we have $W.G = W$.

Lemma 9. *The following are equivalent:*

1. *W is invariant.*
2. *$\Delta(W) \subseteq W \otimes k[G]$.*

Lemma 10. *Any representation V is a filtered union of its finite-dim. subrepresentations:*

1. *Each $v \in V$ is contained in some fin.-dim. subrep.*
2. *Any two finite-dim. subrep. are contained in some bigger fin.-dim. subrep.*

Theorem 4. *Every algebraic group G is isomorphic to a linear algebraic group.*

Proof. Let $\rho : G \to \mathsf{GL}(k[G])$ be the right regular representation. $k[G]$ is a finitely-generated k-algebra. Then, there is a finite-dim. subrepresentation $V \subseteq k[G]$ s.t. V generates $k[G]$ as k-algebra. Then

$$\phi : G \longrightarrow \mathsf{GL}(V)$$

is morphism of algebraic groups.

Consider the dual map

$$\phi^* : k[\mathsf{GL}(V)] \to k[G].$$

We need to show that ϕ^* is surjective. It is enough to show that $V \subset \mathsf{Img}\phi^*$. Define

$$\begin{aligned} l : V \subset k[G] &\longrightarrow k \\ f &\longmapsto f(e). \end{aligned}$$

Let $f \in V$ and set $a(g) := l(g \cdot f)$ for $g \in \mathsf{GL}(V)$. Then $a \in k[\mathsf{GL}(V)]$ is regular. Further,

$$\phi^*(a)(g) = a(\rho(g)) = l(\rho(g)f) = f(eg) = f(g).$$

Therefore, $f = \phi^*(a) \in \mathsf{Img}(\phi^*)$. Since V generates $k[G]$, the surjectivity of ϕ^* follows. □

Theorem 5. *Let H be an algebraic subgroup of an algebraic group G. There is a finite-dim. representation V of G and a line $L \subset V$ s.t. H is the stabilizer in G of L, i.e.*

$$H = \{g \in G \mid L.g = g\}.$$

Proof. Let V be like in the previous proof. Consider

$$I \hookrightarrow k[G] \twoheadrightarrow k[H].$$

We can now set $L' := V \cap I$. We then have for $g \in G$.

$$I.g \subseteq I \iff g \in H.$$

Now, in general L' is not of dimension one. Set $d = \dim(L')$ and consider the one-dimensional subspace $L := \Lambda^d(L') \subseteq \Lambda^d(V)$. G acts on $\Lambda^d(V)$ in the natural way.

It is clear, that H stabilizes L. For the other direction, let $g \notin H$ and let $e_1, \ldots, e_n$ be a basis of V s.t. $L' = \langle e_1, \ldots, e_d \rangle$. Then,

$$L = \langle e_1 \wedge \ldots \wedge e_d \rangle$$

and, since g does not stabilize L', w.l.o.g. we can assume $e_1.g = e_{d+1}$. Then, we have $g(e_1 \wedge \ldots \wedge e_d) = g(e_1) \wedge \ldots \wedge g(e_d) =: v$. Now, v cannot be zero and it cannot lie L because $e_1.g = e_{d+1}$. Therefore, $g \notin H$ does not stabilize L. □

Theorem 6. *Let H be a normal algebraic subgroup of an algebraic group G. Then, there is a finite-dimensional $\rho : G \to \mathsf{GL}(V)$ s.t. $H = \ker(\rho)$.*

Proof. Let V, L and $\phi : G \to \mathsf{GL}(V)$ be like in the preceding theorem. Set

$$V_H := \{v \in V \mid H.v \subset \langle v \rangle\}.$$

Then, V_H is G-invariant, since

$$h.(g.v) = (hg).v = (gh').v = g.(h'v) = g.(\kappa \cdot v) = \kappa \cdot g.v$$

for all $g \in G, h \in H, v \in V_H$ and fitting $h' \in H, \kappa \in k^\times$. W.l.o.g. we have $V = V_H$. V is not trivial, because $L \subset V$.

Let χ range through all homorphism $H \to k^\times$, then we have

$$V = \otimes_\chi V_\chi$$

where

$$V_\chi = \{v \in V \mid h.v = \chi(h) \cdot v\}.$$

Then each $g \in G$ permutes those eigenspaces by

$$g.V_\chi = V_{\chi(g^{-1}\_g)}.$$

Now, let $W := \bigoplus_\chi \mathsf{End}(V_\chi) \subset \mathsf{End}(V)$. For $g \in G$ and $\lambda \in \mathsf{End}(V)$, define

$$\begin{aligned} \widetilde{\gamma} : G &\longrightarrow \mathsf{GL}(\mathsf{End}(V)) \\ g &\longmapsto \widetilde{\gamma}(g) : [\lambda \mapsto \phi(g) \circ \lambda \circ \phi(g)^{-1}]. \end{aligned}$$

The action $\widetilde{\gamma}(g)$ stabilizes W, since each $\phi(g)$ just permutes the V_χ and $\phi(g)^{-1}$ permutes them back. Therefore, we have a subrepresentation

$$\gamma : G \to \mathsf{GL}(W).$$

We now have to show

$$\ker(\gamma) = H.$$

Since elements of H don't permute V_χ, we have $\gamma(H) = \mathrm{Id}$.

One the other side, let $g \in G$ with $\gamma(g) = \mathrm{Id}$. Then, we can choose the projection $\pi : V \twoheadrightarrow L$ in W and get

$$\phi(g) \circ \pi = \pi \circ \phi(g).$$

Therefore, g leaves each L invariant. But now, we have $g \in H$. $\square$

4 Connected Components

Lemma 11. *Let $I_1, I_2, I_\lambda \subset k[x]$ be ideals, then*

$$V(I_1 \cap I_2) = V(I_1) \cup V(I_2)$$
$$V(\bigcup_\lambda I_\lambda) = \bigcap_\lambda V(I_\lambda).$$

Definition 11. A topological space X is called **connected**, if any of the following equivalent condition holds:

- There is no pair of non-empty closed subsets $Z_1, Z_2 \subseteq X$, s.t. $X = Z_1 \dot{\cup} Z_2$.
- There is no pair of non-empty open closed subsets $U_1, U_2 \subseteq X$, s.t. $X = U_1 \dot{\cup} U_2$.
- Each nonempty open subset of X is dense.

Definition 12. A topological space X is called **irreducibel**, if any of the following equivalent condition holds:

- There is no pair of proper closed subsets $Z_1, Z_2 \subseteq X$, s.t. $X = Z_1 \cup Z_2$.
- For each pair $U_1, U_2 \subseteq X$ of non-empty open subsets we have $U_1 \cap U_2 \neq \emptyset$.
- Each nonempty open subset of X is dense.

Example 6. $V(xy)$ is connected but not irreducible.

Lecture from 03.03.2020

Recall: Last time we introduced the **Zariski-Topology** on X.
There, algebraic sets equal closed sets.
We called a set X **irreducible** iff each open subset lies dense in X.

Lemma 12. *For an algebraic set X, the following are equivalent:*

(1) X is irreducible.

(2) $k[X] = k[x_1, \dots, x_n]/I(X)$ is an (integral) domain.

(3) $I(X)$ is a prime ideal.

The proof of (2) $\iff$ (3) is a basic algebraic result.

Lemma 13. *An open base for the Zariski-Topology on an algebraic set X is given by sets:*

$$D(f) := \{p \in X \mid f(p) \neq 0\}$$

for each $f \in k[X]$. We call the $D(f)$ ***basic open sets****.*

Proof. Suppose $U \subseteq X$ is nonempty and open. Set

$$Z := X \setminus U$$

then Z is closed. Thus

$$Z = \{x \in X \mid f(x) = 0 \ \forall f \in I\} = V(I)$$

for some ideal $I \subseteq k[X]$. Let $p \in U$, then there is an $f \in Z$ s.t.

$$f(p) \neq 0.$$

Also, $D(f) \cap Z = 0$, thus $p \in D(f) \subseteq U$. □

Proof: Lemma 1. It is clear that (2) is equivalent to (3).
(1) is equivalent to

$$\forall \text{ nonempty, open } U_1, U_2 \subset X : U_1 \cap U_2 \neq \emptyset$$

$$\overset{\text{Lemma 2}}{\iff} \forall \text{ nonempty, basic open } D(f_1), D(f_2) \subset X : D(f_1) \cap D(f_2) \neq \emptyset$$

Since $D(f_1) \cap D_(f_2) = D(f_1 f_2)$, this is equivalent to the statement

$$\forall f_1, f_2 \in k[X] : \ f_1, f_2 \neq 0 \implies f_1 f_2 \neq 0.$$

Which states that $k[X]$ is a domain. □

Lemma 14. *Let X be an algebraic set. We have bijections*

$$\{\textit{closed subsets } Z \subseteq X\} \leftrightarrow \{\textit{ radical ideals } I \subset k[X]\}$$

and

$$\{\textit{irreducible, closed subsets } Z \subseteq X\} \leftrightarrow \{\textit{ prime ideals } I \subset k[X]\}$$

and

$$\{\textit{points of } X\} \leftrightarrow \{\textit{ maximum ideals } I \subset k[X]\}.$$

Lemma 15 (Primary Decompositions, Atiyah, Macdonald Ch. 4)**.** *For an ideal I we call $P \supseteq I$ a **minimal prime** of I if P is a prime ideal and we have for each prime ideal Q:*

$$P \supseteq Q \supseteq I \implies P = Q.$$

*Any radical ideal I of $k[x_1, \ldots, x_n]$ has only finitely many **minimal** primes $P_1, \ldots, P_r$. Inparticular,*

$$I = \bigcap_{i=1}^{r} P_i$$

and for each i

$$P_j \not\supseteq \bigcap_{j:j\neq i} P_j.$$

Definition 13. An **(irreducible) component** Z of X is a maximal irreducible closed subset, i.e., an irreducible closed $Z \subseteq X$ s.t. there does not exist an irreducible closed $Y \subset X$ s.t. $Y \supsetneq Z$.

Then, we have the bijection

$$\{\text{irreducible components of } X\} \leftrightarrow \{\text{ mimimal primes of } I(X)\}.$$

Lemma 16. *Any algebraic set X has finitely many irreducible components $Z_1, \ldots, Z_r$. We have*

$$X = Z_1 \cup \ldots \cup Z_r$$

and for each i

$$Z_i \not\subset \bigcup_{j:j\neq i} Z_j.$$

Example 7. 1. Let $X = V(x \cdot y) \subset k^2$. Then $X = Z_1 \cup Z_2$ where $Z_1 = V(x), Z_2 = V(y)$.

X is connected, but not irreducible ($D(x)$ does not lie dense in X).

2. Let X be a **finite** algebraic set. It is easy to check that every subset of X is closed:

$$\{p\} = V(x_1 - p_1, \ldots, x_n - p_n)$$

for each $p \in X$. Further

$$X = \{p_1\} \cup \ldots \cup \{p_r\}.$$

Moreover: Any function $f : X \to k$ is regular (i.e. given by polynomials).

Lemma 17. *We call an element $e \in k[X]$ **idempotent** iff $e^2 = e$.*
Let X be an algebraic set. Then

$$\begin{aligned} X \textit{ connected} &\iff \textit{the only idempotents } e \in k[X] \textit{ are 0 and 1} \\ &\iff k[X] \ncong A \times B \textit{ for any } k\textit{-algebras } A, B. \end{aligned}$$

Lemma 18. *Morphisms of algebraic sets are continuous.*

Proof. Let $\phi : X \to Y$ be a morphism. It suffices to show that for all closed $Z \subset Y$ that $\phi^{-1}(Z) \subset X$ is closed.

But, if

$$Z = V_Y(S) := \{q \in Y \mid f(q) = 0 \forall f \in S\}$$

for some ideal $S \subset k[Y]$, then

$$\phi^{-1}(Z) = V_X(\phi^*(S)) = \{\phi^*(f) = f \circ \phi \mid f \in S\}.$$

□

Lemma 19. *Isomorphisms of algebraic sets are homeorphisms. In particular, any isomorphism of algebraic sets $\phi : X \to X$ permutes the irreducible components $Z_1, \ldots, Z_r$ of X:*

$$\forall i \exists j : \phi(Z_i) = Z_j.$$

Theorem 7. *Let G be an algebraic group.*

(i) There is a unique irreducible component G^0 of G with $e \in G^0$.

(ii) Every irreducible component Z of G is a coset gG^0 of G for some $g \in Z$.

(iii) G^0 is a normal algebraic subgroup of G.

(iv) G^0 is of finite index, i.e.

$$[G : G^0] = \#\left(G/G^0\right) < \infty.$$

(v) The irreducible components are also the connected components.

Proof. Let $G = Z_1 \cup \ldots \cup Z_r$ be the decomposition into components. We may assume that $e \in Z_1$.

Recall that $Z_1 \not\subset \bigcup_{j\geq 2} Z_j$. Then, there is an $x \in Z_1 \setminus \bigcup_{j\geq 2} Z_j$. Thus, for all algebraic set isomorphisms $\phi : G \to G$, we have by some previous lemma that $\phi(x)$ is likewise contained in some unique component of G. For example, we may take ϕ to be

$$\begin{aligned} \phi_g : G &\to G \\ y &\longmapsto gy \end{aligned}$$

for any $g \in G$. Then, for all $g \in G$, the element $gx = \phi_g(x)$ is contained in only one component of G. Ergo, each $g \in G$ is contained in exactly one component.

(i) Take $g = e$.

(iii) G^0 is an algebraic subset, by construction. Denote by $m : G \times G \to G$ and $i : G \to G$ the continuous multiplication and inversion map on G. **Why is G^0 a subgroup?** We need to show

$$\begin{aligned} m(G^0 \times G^0) &\subseteq G^0. \\ i(G^0) &\subseteq G^0. \end{aligned}$$

We know that $i(G^0)$ is some component of G, since i is an isomorphism. But it contains the identity e, since $e^{-1} = e$. Therefore, $i(G^0) = G^0$.

If $g \in G$, then gG^0 is some component of G. Suppose $g \in G^0$. Then $gG^0 \cap G^0 \supseteq \{g\}$, therefore $gG^0 = G^0$. Ergo, G^0 is closed under multiplication.

Why is G^0 a normal? If $g \in G$, then gG^0g^{-1} is a component that contains e, therefore $G^0 = gG^0g^{-1}$.

(Alternative proof that $m(G^0 \times G^0) = G^0$: Consider

- any continuous image of an irreducible set is irreducible.
- the closure of any irreducible set is irreducible.

Ergo $\overline{m(G^0 \times G^0)}$ is a closed irreducible set containing e. Ergo, $\overline{m(G^0 \times G^0)} = G^0$).

(ii) Let $Z \subset G$ be a component. Let $g \in Z$. Then $g \in (gG^0 \cap Z)$, so $gG^0 = Z$.

(iv) This follows from some previous lemma.

(v) This is left as a topological exercise. It is true whenever the irreducible components do not intersect.

$\square$

It now follows:

$$\{\text{finite algebraic groups}\} \longleftrightarrow \{\text{finite groups}\}$$

where the above arrow is an equivalence of categories.

Example 8.

- Let $G = \{g_1, \ldots, g_r\}$ be a finite algebraic group. Then,

$$G^0 = \{e\}.$$

- Without proofs:

$$G \in \{\mathsf{GL}_n(k), \mathsf{SO}_n(k), \mathsf{SL}_n(k)\} \implies G^0 = G.$$

Further,

$$G = O_n(k) \implies G^0 = \mathsf{SO}_n(k).$$

And if $-1 = 1$ i.e. $\mathsf{char} k = 2$, then $[G : G^0] = 1$. Otherwise $[G : G^0] = 2$.

5 Jordan Decomposition

As usual, $k = \overline{k}$ is an algebraically closed field.

Definition 14. Let V be a finite-dimensional vector space.

An element $x \in \mathsf{End}(V)$ is **semisimple**, if it is diagonalizable, i.e. it has a basis of eigenvectors, or equivalently, if the minimal polynomial of x is square-free.

Then, there is a decomposition $V = \bigoplus_{i=1}^r V_i$ and distinct elements $\lambda_1, \ldots, \lambda_n \in k$ s.t.

$$x|_{V_i} = \lambda_i.$$

If $\dim(V_i) = n_i$, then

$$\text{char polynomial of } x = \prod_{i=1}^{r} (T_i - \lambda_i)^{n_i} \in k[T]$$

and

$$\text{minimal polynomial of } x = \prod_{i=1}^{r} (T_i - \lambda_i) \in k[T].$$

(Where the minimal polynomial of x is defined as the least degree monic polynomial $m \in k[T]$ s.t. $m(x) = 0$.)

Remark 2. Let $m(T) \in k[T]$ be the minimal polynomial of $x \in k^{n \times n}$.

The theorem of Cayley and Hamilton states that we have for each $p \in k[T]$:

$$p(x) = 0 \implies m | p.$$

Definition 15. $x \in End(V)$ is **nilpotent** if $x^n = 0$ for some n.

x is **unipotent**, if $x - 1$ is nilpotent.

Lemma 20. *x is nilpotent iff the characteristic polynomial of x is $T^{\dim(V)}$. (Use Cayley-Hamilton for one of the directions).*

Lemma 21. *If x is semisimple and nilpotent, then $x = 0$.*

If x is semisimple and unipotent, then $x = 1$.

Lemma 22. *If x, y are commuting elements, that are semisimple resp. unipotent resp. nilpotent, then so is xy.*

Proof. It is easy to see, that this is true for nilpotent x, y.

Now, let x, y be unipotent and commuting. Then, we have

$$xy - 1 = (x+1)(y-1) + (x-y).$$

Since x, y commute, $(x+1)(y-1)$ must be nilpotent. $(x-y)$ must be nilpotent because the sum of commuting nilpotent elements must be nilpotent. Because everything commutes, also $xy-1$ as the sum of two commuting, nilpotent elements must be nilpotent.

Now, let $A, B \in k^{n\times n}$ be two diagonalizable and commuting matrices. Let $\lambda_1, \ldots, \lambda_r$ be different eigenvectors of A and let E_i be the corresponding eigenspaces. We then have

$$A \cdot (BE_i) = BAE_i = \lambda_i \cdot BE_i.$$

Ergo, each E_i is invariant under B. Since $B_{|E_i}$ stays diagonalizable, we can simply choose a basis of eigenvectors $b_1, \ldots, b_n \in \bigcup_i E_i$ of B. Since each b_i lies in a E_j, those vectors are also eigenvectors for A. Therefore, $b_1, \ldots, b_n$ is basis of eigenvectors for both matrices. $\square$

Theorem 8 (Goal)**.** *For all algebraic groups G and for all $g \in G$, there exist unique group elements $g_s, g_u \in G$ s.t.*

$$g = g_s g_u = g_u g_s$$

and for all finite-dimensional representations $\rho : G \to \mathsf{GL}(V)$, $\rho(g_s)$ is semisimple and $\rho(g_u)$ is unipotent.

Example 9. If $g = \begin{pmatrix} \lambda & 1 & 0 \\ & \lambda & 1 \\ & & \lambda \end{pmatrix} \in G = \mathsf{GL}_3(k)$, then $g_s = \begin{pmatrix} \lambda & 0 & 0 \\ & \lambda & 0 \\ & & \lambda \end{pmatrix}$, $g_u = \begin{pmatrix} 1 & \lambda^{-1} & 0 \\ & 1 & \lambda^{-1} \\ & & 1 \end{pmatrix}$.

Theorem 9 (Goal Theorem)**.** *Let G an algebraic group. For all $g \in G$ there is exactly one pair $g_s, g_u \in G$ s.t.* Lecture from 09.03.2020

$$g = g_s g_u = g_u g_s$$

and for all finite-dimensional representations $r : G \to \mathsf{GL}_n(V)$, the element $r(g_s)$ resp. $r(g_u)$ is semisimple resp. unipotent.

Last time, we saw:

Lemma 23.

- *If g, h are commuting and semisimple resp. commuting and unipotent then so is gh.*
- *If g is semisimple and unipotent, then $g = 1$.*

Proposition 1. *Let V be a finite-dimensional vector space and $g \in \mathsf{GL}(V)$. There exist unique elements $g_s, g_u \in \mathsf{GL}(V)$ s.t.*

$$g = g_s g_u = g_u g_s$$

and g_s is semisimple and g_u is unipotent.

Moreover, $g_s, g_u \in k[g] = \{\sum_{i=1}^{m} a_i g^i \mid a_i \in k\} \subseteq \mathsf{End}(V)$.

Proof. **Existence** (Sketch): Say

$$g = \begin{pmatrix} \lambda & 1 & 0 \\ & \lambda & 1 \\ & & \lambda \end{pmatrix}$$

then take

$$g_s = \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{pmatrix}, \quad g_u = \begin{pmatrix} 1 & \lambda^{-1} & 0 \\ & 1 & \lambda^{-1} \\ & & 1 \end{pmatrix}.$$

For $\lambda \in k$, define the **generalized λ-eigenspace** of g by

$$V_\lambda := \{v \in V \mid \exists n \in \mathbb{N}_0 : \ (g - \lambda)^n v = 0\}.$$

Then

$$V = \bigoplus_{\lambda \in k} V_\lambda.$$

Here V_λ = sum of domains of all Jordan blocks with λs on the diagonal. (It follows from the Jordan Decomposition for matrices that such a decomposition exist.)

Let's define $g_s \in \mathsf{GL}(V)$ by

$$g_s|_{V_\lambda} = \lambda \cdot \mathrm{Id}.$$

Note that $gV_\lambda \subset V_\lambda$, hence g commutes with g_s, hence g, g_s commutes with $g_u := gg_s^{-1}$. Then, $g = g_s g_u = g_u g_s$.

Write $\det(T-g) = \prod_\lambda (T-\lambda)^{n(\lambda)}$, $n(\lambda) = \dim(V_\lambda)$. Since the polynomials $T-\lambda$ for $\lambda \in k$ are coprime, the chinese remainder theorem implies that there is a $Q \in k[T]$ s.t.

$$Q \equiv \lambda \mod (T-\lambda)^{n(\lambda)}$$

for each $\lambda \in k$.

We claim that

$$Q(g) = g_s.$$

Indeed, since $gV_\lambda \subseteq V_\lambda$, we have

$$Q(g)V_\lambda \subseteq V_\lambda.$$

So, it suffices to show for all $v \in V_\lambda$

$$Q(g)v = g_s v = \lambda v.$$

Note that, by Cayley-Hamilton,

$$V_\lambda = \left\{ v \in V \mid (g-\lambda)^{n(\lambda)} v = 0. \right\}$$

Write

$$Q = \lambda + R \cdot (T-\lambda)^{n(\lambda)}$$

for some $R \in k[T]$. Since $(g-\lambda)^{n(\lambda)} v = 0$, deduce that $Q(g)v = \lambda v$, as required.

If $P \equiv \lambda^{-1} \mod (T-\lambda)^{n(\lambda)}$, then $P(g) = g_s^{-1}$.

Therefore,

$$g_u = g \cdot P(g)$$

for $T \cdot P(T) \in k[T]$.

Uniqueness: Suppose given some other decomposition

$$g = g'_s g'_u = g'_u g'_s$$

with g'_s semisimple and g'_u unipotent. Then g'_s commutes with g'_s and g'_u, hence with g, hence also with any element in $k[g]$. Ergo, g'_s commutes with g_s and g_u. Similarly, g'_u commutes with g_s and g_u.

Consider

$$h := g'_s g_s^{-1} = g'_s g'_u (g'_u)^{-1} g_s^{-1} = g(g'_u)^{-1} g_s^{-1} = g_u (g'_u)^{-1}.$$

Then $h = g'_s g_s^{-1}$ is a product of semisimple elements and $h = g_u(g'_u)^{-1}$ is a product of unipotent elements. By proceeding lemmas, h is semisimple and unipotent, ergo trivial. It follows $g'_s = g_s$ and $g'_u = g_u$. $\square$

Corollary 2. *Let $g \in \mathsf{GL}(V)$, let $W \subset V$ be any g-**invariant subspace**, i.e. $gW \subseteq W$.*

Then, W is g_s-invariant and g_u-invariant.

Proof. This is clear, since g_s and g_u are algebraically generated by g over g. $\square$

Lemma 24. *Let $\phi : V \to W$ be a linear map between finite-dimensional vector spaces.*

Let $\alpha \in \mathsf{GL}(W)$ and $\beta \in \mathsf{GL}(W)$ s.t.

$$\begin{array}{ccc} V & \xrightarrow{\alpha} & V \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ W & \xrightarrow{\beta} & W, \end{array}$$

i.e. $\phi \circ \alpha = \beta \circ \phi$.

Then,

$$\phi \circ \alpha_s = \beta_s \circ \phi,$$
$$\phi \circ \alpha_u = \beta_u \circ \phi.$$

Proof. Write $V = \bigoplus_{\lambda \in k} V_\lambda$, $W = \bigoplus_{\lambda \in k} W_\lambda$ where V_λ are the generalized α-eigenspaces and W_λ are the generalized β-eigenspaces.

We claim that

$$\phi(V_\lambda) \subset W_\lambda.$$

Indeed, let $v \in V_\lambda$, then

$$(\beta - \lambda)^n \phi(v) = \phi((\alpha - \lambda)^n v) = 0.$$

Since $(\alpha - \lambda)^n v = 0$, the claim follows.

Since, $\alpha_s|_{V_\lambda} = \lambda \mathrm{Id}$ and $\beta_s|_{W_\lambda} = \lambda \mathrm{Id}$, deduce that

$$\phi \circ \alpha_s = \beta_s \circ \phi.$$

Indeed, both sides are given on V_λ by $\lambda \cdot \phi$. Thus

$$\begin{aligned} \phi \circ \alpha_u &= \phi \circ \alpha \alpha_s^{-1} \\ &= \beta \beta_s^{-1} \circ \phi \\ &= \beta_u \circ \phi. \end{aligned}$$

□

Lemma 25. *Let $\alpha \in \mathit{GL}(V)$, $\beta \in \mathit{GL}(W)$. Then the* ***tensor*** *$\alpha \otimes \beta \in \mathit{GL}(V \otimes W)$ is defined by*

$$(\alpha \otimes \beta)(u \otimes v) = \alpha(u) \otimes \beta(v).$$

Then, we have

$$(\alpha \otimes \beta)_s \overset{(1)}{=} \alpha_s \otimes \beta_s$$

$$(\alpha \otimes \beta)_u \overset{(2)}{=} \alpha_u \otimes \beta_u.$$

Proof. It suffices to prove (1), since

$$\begin{aligned} (\alpha \otimes \beta)_u &= (\alpha \otimes \beta) \circ (\alpha \otimes \beta)_s^{-1} \\ &\overset{(1)}{=} (\alpha \otimes \beta) \circ (\alpha_s \otimes \beta_s)^{-1} \\ &= \alpha \alpha_s^{-1} \otimes \beta \beta_s^{-1} \\ &= \alpha_u^{-1} \otimes \beta_u^{-1} \end{aligned}$$

For (1), consider

$$V = \bigoplus_{\lambda \in k} V_\lambda,$$

$$W = \bigoplus_{\lambda \in k} W_\lambda.$$

It follows

$$V \otimes W = \bigoplus_{\lambda, \mu \in k} V_\lambda \otimes W_\mu.$$

Now,

$$\alpha_s \otimes \beta_s|_{V_\lambda \otimes W_\mu} = \lambda\mu \cdot \mathrm{Id}.$$

Ergo, $\alpha_s \otimes \beta_s$ is semisimple. By Proposition, we reduce to checking that $\alpha_u \otimes \beta_u$ is unipotent. Indeed,

$$\alpha_u \otimes \beta_u - 1 = (\alpha_u - 1) \otimes (\beta_u - 1) + 1 \otimes (\beta_u - 1) + (\alpha_u - 1) \otimes 1$$

is nilpotent (You can also check that $\alpha_u \otimes \beta_u = (\alpha_u \otimes 1) \circ (1 \otimes \beta_u)$ is unipotent.) □

Example 10. Let $1 \in \mathsf{GL}(V)$. Then $1_s = 1$ and $1_u = 1$.

Summary : Let G be an algebraic group. Let $r_V : G \to \mathsf{GL}(V)$ be a finite-dimensional representation. Also, fix $g \in G$.

Let $\lambda_V := r_V(g)_s$ (or $r_V(g)_u$).

We get a family of operators $\lambda_V \in \mathsf{End}(V)$ with the following properties:

(i) if $V = k$ and $r_V(g') = 1$ for all $g' \in G$, then $\lambda_V = 1$.

(ii) for any two representations in V and W, we have

$$\lambda_{V \otimes W} = \lambda_V \otimes \lambda_W.$$

(iii) for all G-equivariant $\phi : V \to W$ we have

$$\phi \circ \lambda_V = \lambda_W \circ \phi.$$

Theorem 10. *Let G be an algebraic group. Let $\lambda_V \in \mathsf{End}(V)$ (i.e. $V = (r_V, V)$ is a finite-dim. representation of G) be a family of operations satisfying (i), (ii), (iii).*

Then, there is exactly one $g \in G$ s.t. $\lambda_V = r_V(g)$ for all V.

Note, that this theorem implies our goal theorem.

Applying the theorem to $\lambda_V = r_V(g)_s$ implies

$$\exists_1 g_s \in G : r_V(g_s) = r_V(g)_s$$

and

$$\exists_1 g_u \in G : r_V(g_u) = r_V(g)_u.$$

Proof of Goal Theorem. There exist unique $g_s, g_u \in G$ s.t.

$$g \overset{(*)}{=} g_u g_s = g_s g_u,$$

Then, $r_V(g) = r_V(g_s) r_V(g_u) = r_V(g_u) r_V(g_s)$.

Since $r_V(g_u)$ is unipotent and $r_V(g_s)$ is semisimple, it follows $r_V(g_u) = r_V(g)_u$ and $r_V(g_s) = r_V(g)_s$.

To deduce $(*)$, take any $r_V : G \hookrightarrow \mathsf{GL}(V)$. We know for each V

$$r_V(g) = r_V(g_s) r_V(g_u) = r_V(g_u) r_V(g_s).$$

□

Proof of Theorem 10. We first extend the assignment

$$V \mapsto \lambda_V$$

to all representations of G.

Say $V = \bigcup_j W_j$ where each W_j is a finite-dimensional G-invariant subspace. Try to define $\lambda_V \in \mathsf{End}(V)$ by

$$\lambda_V|_{W_j} := \lambda_{W_j}.$$

For this to be well-defined, we need to show for each i, j

$$\lambda_{W_i}|_{W_i \cap W_j} \overset{(*)}{=} \lambda_{W_j}|_{W_i \cap W_j}.$$

Proof of $(*)$**:** Apply assumption (iii) to the G-equivariant linear maps

$$W_i \cap W_j \overset{\phi}{\hookrightarrow} W_i,$$
$$W_i \cap W_j \overset{\phi'}{\hookrightarrow} W_j.$$

Then,

$$\begin{aligned} \lambda_{W_i}|_{W_i \cap W_j} &= \lambda_{W_i} \circ \phi \\ &\overset{(iii)}{=} \phi \circ \lambda_{W_i \cap W_j} \\ &= \phi' \circ \lambda_{W_i \cap W_j} \end{aligned}$$

and

$$\lambda_{W_j}|_{W_i \cap W_j} = \lambda_{W_i} \circ \phi' = \phi' \circ \lambda_{W_i \cap W_j}.$$

Recall here that any finite-dimensional G-invariant $W \subset V$ is a representation. $\square$

[0]Not necessarily finite-dimensional, but may be written as a filtered union of finite-dimensional G-invariant subspaces of W.

Let G be an algebraic group. Lecture from 11.03.2020

Easy Exercise : If V_1, V_2 are representations r_1, r_2 of G, then $V_1 \otimes V_2$ is also a representation with

$$r = r_1 \otimes r_2 : G \to \mathsf{GL}(V_1 \otimes V_2)$$

given by

$$r(g)(v_1 \otimes v_2) = (r_1(g)v_1) \otimes (r_2(g)v_2).$$

Proof. Given $\Delta_j : V_j \to V_j \otimes k[G]$, define

$$\Delta : V_1 \otimes V_2 \longrightarrow V_1 \otimes V_2 \otimes k[G]$$

by: if

$$\Delta_1 u = \sum_i u_i \otimes f_i, \qquad \Delta_2 v = \sum_j v_j \otimes h_j,$$

then

$$\Delta(u \otimes v) = \sum_i \sum_j u_i \otimes v_j \otimes f_i h_j.$$

Set $A := k[G]$, then

$$r_A := \text{right regular representation with } r_A(g)f(x) = f(xg).$$

The map

$$\begin{aligned} A \otimes A &\xrightarrow{m} A \\ f_1 \otimes f_2 &\longmapsto f_1 f_2 \end{aligned}$$

defines a morphism of representations

$$(A, r_A) \otimes (A, r_A) \to (A, r_A).$$

Indeed,

$$\begin{aligned} m((r_A \otimes r_A)(g)(f_1 \otimes f_2))(x) &= f_1(xg) f_2(xg), \\ &= f_1 f_2(xg) = r_A(g)(m(f_1 \otimes f_2))(x), \end{aligned}$$

since $f_1(\_g) \otimes f_2(\_g) = (r_A \otimes r_A)(g)(f_1 \otimes f_2)$.

Ergo $m \circ (r_A \otimes r_A)(g) = r_A(g) \circ m$. $\square$

Recall: We stated to prove the following theorem

Theorem 11. *Let $\lambda_V \in \mathit{End}(V)$ be given s.t. for all finite-dim. rep.s V of G s.t.:*

(i) $\lambda_k = 1$

(ii) $\lambda_{V\otimes W} = \lambda_V \otimes \lambda_W$

(iii) for all morphisms of rep.s $\phi : V \to W$ we have

$$\phi \circ \lambda_V = \lambda_W \circ \phi.$$

Then, there is exactly one $g \in G$ s.t. $\lambda_V = r_V(g)$ for all V.

Proof. Last time, we saw that any such family $V \mapsto \lambda_V$ extends to **all** rep.s V of G.

Let's note also that, if (V_0, r_0) is any representation of G with trivial action, i.e. $r(g) = 1$ for all g, then $\lambda_{V_0} = 1$. Indeed, let $v \in V_0$. We must check that $\lambda_{V_0} v = v$. Since the action is trivial, any subsapce of V_0 is G-invariant.

Consider the map

$$\begin{aligned} \phi : k &\longrightarrow V_0 \\ \alpha &\longmapsto \alpha v \end{aligned}$$

where $v = \phi(1)$. Then, ϕ is a morphism of rep.s because the action is trivial.

Thus,

$$\lambda_V v = (\lambda_V \circ \phi)(1) \overset{(iii)}{=} (\phi \circ \lambda_k)(1) \overset{(i)}{=} \phi(1) = v.$$

Consider $\lambda_A \in \mathsf{End}(A)$. Then,

$$\lambda_{A\otimes A} = \lambda_A \otimes \lambda_A.$$

It is an easy exercise to see that $m : (A, r_A) \otimes (A, r_A) \to (A, r_A)$ is a morphism of rep.s.

By (iii) it follows, $m \circ (\lambda_A \otimes \lambda_A) = \lambda_A \circ m$, i.e.

$$\lambda_A(f_1 f_2) = \lambda_A(f_1)\lambda_A(f_2)$$

for all $f_1, f_2 \in A$. Thus, λ_A is an algebra morhism (check, using the morphism $k \hookrightarrow A$, that $\lambda_A(1) = 1$).

Thus, $\lambda_A = \phi^*$ for some unique morphism ϕ of algebraic sets $\phi : G \to G$.

We claim that ϕ commutes left multiplication i.e.

$$\phi(hx) = h\phi(x)$$

for all $h, x \in G$. Indeed, let's consider the map

$$\begin{aligned} A &\longrightarrow A \\ f &\longmapsto f(h \cdot \_). \end{aligned}$$

This induces a morphism

$$(A, r_A) \xrightarrow{\psi} (A, r_A).$$

By (ii), $\psi \circ \lambda_A = \lambda_A \circ \psi$.

Since $\lambda_A = \phi^*$, this implies the claim.

Now, set $g := \phi(e)$. Then for all $h \in G$,

$$\phi(h) = \phi(he) = hg.$$

Thus, $\lambda_A = \phi^* = r_A(g)$.

(It remains to show that

$$\lambda_V = r_V(g)$$

for each finite-dim. rep. V.)

Let $V = (V, r)$ be any rep. This induces a map

$$\Delta : V \longrightarrow V \otimes A.$$

If $\Delta v = \sum v_i \otimes f_i$, then

$$hv = \sum f_i(h_i) \otimes v_i.$$

Let

$$\begin{aligned} \varepsilon : V \otimes A &\longrightarrow V \\ v \otimes f &\longmapsto f(1)v. \end{aligned}$$

It follows $\varepsilon \circ \Delta : V \to V$ is the identity map.

Let (V_0, r_0) be the representation of G with $V_0 := V$ and r_0 the trivial action. Then, $\Delta : V \to V_0 \otimes A$ is a morphism of representations.

(Indeed, if $\Delta v = \sum v_i \otimes f_i$, then

$$\Delta(r(h)v) \stackrel{?}{=} (r_0(h) \otimes r_A(h))\Delta v$$

since

$$\begin{aligned}
\Delta v &= \sum v_i \otimes f_i \\
\iff xv &= \sum f_i(x_i) v_i \ \forall x \in G \\
\iff xhv &= \sum f_i(xh) v_i \ \forall x, h \in G.
\end{aligned}$$

Since $r(h)v = hv$, it follows

$$\Delta(hv) = \sum v_i \otimes f_i(\cdot h) \implies (?).)$$

We want to show

$$\lambda_V = r_V(g).$$

We have

$$\begin{aligned}
\Delta \circ \lambda_V &\overset{(iii)}{=} \lambda_{V_0 \otimes A} \circ \Delta \\
&\overset{(ii)}{=} \lambda_{V_0} \otimes \lambda_A \\
&= 1 \otimes \lambda_A = 1 \otimes r_A(g).
\end{aligned}$$

This implies

$$\Delta \circ \lambda_V = (1 \otimes r_A(g)) \circ \Delta$$

but also

$$\Delta \circ r_V(g) = (1 \otimes r_A(g)) \circ \Delta.$$

Because of the injectivity of Δ it now follows

$$\lambda_V = r_V(g).$$

$\square$

Corollary 3. *Let $\phi : G \to H$ be any morphism of algebraic groups. Then, for all $g \in G$*

$$\begin{aligned}
\phi(g)_s &= \phi(g_s) \\
\phi(g)_u &= \phi(g_u).
\end{aligned}$$

Proof. Let V be any **faithful** representation of H, i.e. $r_V : H \to \mathsf{GL}(V)$ is injective, (for a finite-dim. V).

Then, $r_V \circ \phi$ is a rep. of G. To prove (i), it suffices to show

$$r_V(\phi(g)_s) = r_V(\phi(g_s))$$

since H operates faithfully on V.

We know that

$$r_V(\phi(g)_s) = r_V(\phi(g))_s$$

(characterizing property of h_s for $h \in H$). On the other hand,

$$r_V(\phi(g_s)) = (r_V \circ \phi)(g_s) = r_V(\phi(g))_s.$$

Therefore, claim (i) follows. (ii) works analogously. □

Definition 16. Let $g \in G$ where G is an algebraic group. We call g **semisimple**, if $g = g_s$.

We call g **unipotent**, if $g = g_u$.

Lemma 26. *For $g \in G$, the following are equivalent:*

(i) g is semisimple.

(ii) $r_V(g)$ is semisimple for all finite-dim. rep. V.

(iii) $r_V(g)$ is semisimple for at least one faithful f.d. rep. V of G.

We get an analogous lemma for unipotent group elements.

Proof. We have

$$\begin{aligned} (i) \iff & g = g_s \\ \overset{\text{Def. of } g_s \text{ by goal thm.}}{\iff} & r_V(g) = r_V(g)_s \forall \text{ f.d. } V \\ \iff & r_V(g) \text{ is semisimple} \\ \iff & (ii) \implies (iii). \end{aligned}$$

On the other hand,

$$(iii) \implies \exists \text{ faithful f.d. } V \text{ s.t. } r_V(g) = r_V(g)_s = r_V(g_s) \implies g = g_s.$$

□

6 Non-Commutative Algebra

Definition 17. A **ring** R (for now) is unital, associative but not necessarily commutative.

Example 11. The ring of matrices over some field or ring.

Definition 18. A **left ideal** $I \subset R$ is a subset that is an abelian subgroup of $(R, +)$ s.t. $ra \in I$ for all $r \in R, a \in I$.

A **right ideal** $I \subset R$ is a subset that is an abelian subgroup with

$$IR \subset I.$$

A two-sided ideal I is a subset that is a left and a right ideal of R.

It is easy to check that for any homomorphism of rings $\phi : R \to S$, $\mathsf{Kern}\phi$ is a two-sided ideal. Also, if $J \subset R$ is any two-sided ideal, then there exists a unique ring structure on R/J s.t. the projection $R \to R/J$ is a ring homomorphism.

Definition 19. A **left module** M for R is an abelian group equipped with a ring homomorphism

$$R \xrightarrow{\alpha} \mathsf{End}(M)$$

where $\mathsf{End}(M)$ acts on the left of M. We write

$$rm := \alpha(r)m.$$

We have

$$(r_1r_2)(m) = r_1(r_2(m)).$$

If R **acts** on M by the left, we write

$$R \curvearrowright M.$$

Example 12. $M_n(k) \curvearrowright k^n$ where k^n is the space of column vectors.

If k^n denotes the space of row vectors, we have $k^n \curvearrowleft M_n(k)$.

Definition 20. A **(left) submodule** $N \subset M$ is an algebraic subgroup s.t.

$$RN \subset N.$$

It follows that N is itself is a left module.

Definition 21. A (left) module M of R is **simple** (or irreducible) if it has exactly the two submodules: $0 = \{0\}$ and M.

Definition 22. A ring R is a **division ring** (aka **skew field**) if it satisfies any of the following equivalent requirements:

(i) $R^\times = R \setminus \{0\}$ where[1] $R^\times = \{r \in R \mid \exists a, b \in R : ar = rb = 1.\}$

(ii) R has no nontrivials left or right ideals.

Definition 23. If $R \curvearrowright M$, then we can define

$$\mathsf{End}_R(M) := \{\phi \in \mathsf{End}(M) \mid \phi(rm) = r\phi(m)\ \forall r \in R, m \in M\}.$$

Note, that $\mathsf{End}_R(M)$ is a ring.

Lemma 27 (Schur's Lemma)**.** *If M is simple, then $\mathsf{End}_R(M)$ is a division ring.*

Lemma 28. *Let k be a field. Then, $M_n(k)$ has no nontrivial twosided ideals.*

Theorem 12 (Jacobson Density Theorem (Double Commutant Theorem))**.** *Suppose M is a simple left module which is finitely generated as a right D-module for $D = \mathsf{End}_R(M)$.*

Assume that R acts faithfully on M, i.e. $R \to \mathsf{End}_R(M)$ is injective.

Then, the map $R \to \mathsf{End}_D(M)$ is an isomorphism.

[1] If $ar = rb = 1$, then $a = arb = b$.

Recap:

Lecture from 16.03.2020 (Corona-Madness started here...)

– Basics: definitions, Hopf-algebras, ...

– Jordan decomposition

– Primer on non-commutative algebra

- Jacobson density theorem

– Unipotent groups

– Tori

We had last week

$$\mathsf{End}_D(M) := \{\phi \in \mathsf{End}(M) \mid \phi \circ d = d \circ \phi \ \forall d \in D\}.$$

Let k be an algebraically cloesd field, V a non-trivial finite-dimensional k-vector space and let G be a subgroup of $\mathsf{GL}(V)$ that acts **irreducibly** on V, i.e., V is G**-irreducible**, i.e., the only G-invariant subspaces of V are 0 and V.

Set

$$D := \{d \in \mathsf{End}_k(V) \mid dg = gd \ \forall g \in G\} = \operatorname{span}(G) = \left\{\sum_{i=1}^{n} c_i g_i \mid c_i \in k, g_i \in G, n \in \mathbb{N}_0\right\}.$$

Then,

$$D = \mathsf{End}_R(V)$$

where R is the k-subalgebra of $\mathsf{End}(V)$ that is generated by G.

Lemma 29 (Schur's Lemma)**.** *We understand $k \hookrightarrow \mathsf{End}(V)$ as the inclusion of operations which operate by scalar multiplication*

$$k \xrightarrow{\cong} \{\phi : V \to V \mid \phi : v \mapsto t \cdot v \ \ \textit{for some } t \in k\}.$$

Let V be G-irreducible. Then, we have

$$D \cong k.$$

Proof. Let $d \in D$. Since $V \neq 0$, there is an eigenspace $V_\lambda \neq 0$ for d. Observe that V_λ has to be G-invariant:
if $g \in G$ and $v \in V_\lambda$, then $gv \in V_\lambda$, since

$$dgv = gdv = g(\lambda v) = \lambda gv.$$

Since V_λ is a non-trivial G-invariant subspace and V is irreducible under G, we have

$$V_\lambda = V.$$

Ergo $d = \lambda$ in the sense of $k \hookrightarrow \mathsf{End}(V)$. □

Consequence of the Jacobson Density Theorem: $R = \mathsf{End}_k(V)$, i.e., G generates all linear operations on V, if V is G-irreducible.

We will prove this after a lemma.

Lemma 30. *Let V be G-irreducible.*

Let $n \in \mathbb{N}$. Set

$$V^n := V \oplus V \oplus \ldots \oplus V = V_1 \oplus \ldots \oplus V_n$$

where each $V_i = V$.

Let $v = (v_1, \ldots, v_n) \in V^n$ and set

$$Rv := \{(rv_1, \ldots, rv_n) \mid r \in R\} = \text{span}\{(gv_1, \ldots, gv_n) \mid g \in G\}.$$

Then, $Rv \neq V^n$ iff the v_j are linearly dependent over k.

Consequence: Take $n := \dim(V)$. Let $\{e_1, \ldots, e_n\}$ be a basis of V and set

$$e := (e_1, \ldots, e_n) \in V^n.$$

Since the $(e_i)_i$ are linearly independent, the lemma states that $Re = V^n$.

Now, let $x \in \mathsf{End}_k(V)$. Choose $r \in R$ s.t.

$$re = (xe_1, \ldots, xe_n).$$

Then $re_i = xe_i$ for all i, thus $x = r$. Hence, $R = \mathsf{End}_k(V)$.

Proof. For $v = (v_1, \ldots, v_n) \in V^n$ choose $J \in \{1, \ldots, n\}$ as large as possible with

$$Rv + V_1 + V_2 + \ldots + V_{J-1} =: U \neq V^n.$$

Such an J does exist, since we know that $Rv \neq V^n$.

Then, $V_J \not\subseteq U$, otherwise we may increase J. Also, U is invariant by the diagonal action of G on V^n. Thus, $V_J \cap U \subsetneq V_J$ is a proper G-invariant subspace of the G-irreducible $V_J \cong V$. Therefore, $V_J \cap U = 0$.

On the other hand, by maximality of J, we have

$$U \oplus V_J = V^n.$$

Ergo, the map (composition)

$$V \cong V_J \hookrightarrow V^n \twoheadrightarrow V^n/U$$

is a G-equivariant isomorphism, since U is G-invariant.

Let $z : V^n/U \overset{\cong}{\to} V$ be the inverse isomorphism. Let l be the G-equivariant map given by

$$\begin{tikzcd} V^n \arrow[r, "l"] \arrow[d] & V \\ V^n/U \arrow[ur, "z"] & \end{tikzcd}$$

and let l_j be the G-equivariant maps by restricting l on V_j. Then $l_j \in D \cong k$.

Say $l_j = t_j \in k$. Then,

$$l(w) = t_1 w_1 + \ldots t_n w_n.$$

Since z is an isomorphism, l is nonzero and $(t_1, \ldots, t_n) \neq (0, \ldots, 0)$.

Since $l|_U = 0$, we can deduce for all $u \in U$

$$t_1 u_1 + \ldots + t_n u_n = 0.$$

But $v \in Rv \subseteq U$, so we may conclude – as required – that the $(v_i)_i$ are linearly dependent ($l(v) = 0$). $\square$

7 Unipotent Groups

Let G be a subgroup of $\mathsf{GL}(V)$ where V is a finite-dimensional vector space and k an algebrically closed field.

Definition 24. We say that G is **unipotent** if one of the following equivalent conditions hold for each $g \in G$:

- g is unipotent (i.e. $(g-1)^n = 0$ for some $n \in \mathbb{N}$).
- all eigenvalues of g are 1.
- g is conjugate to $\begin{pmatrix} 1 & \star & \star \\ 0 & \ddots & \star \\ 0 & 0 & 1 \end{pmatrix}$.

Theorem 13. *Any unipotent subgroup of $\mathsf{GL}_n(k)$ is conjugate to a subgroup of*

$$U_n := \left\{ \begin{pmatrix} 1 & \star & \star \\ 0 & \ddots & \star \\ 0 & 0 & 1 \end{pmatrix} \right\} = \left\{ U \in M_n(k) \mid U_{i,j} = \begin{cases} 0, & \text{if } i > j \\ 1, & \text{if } i = j \\ \text{arbitrary}, & \text{otherwise.} \end{cases} \right\}.$$

Definition 25. For two subgroups G, H of some common supergroup, define their **commutator** by

$$[G,H] := \left\langle ghg^{-1}h^{-1} \mid g \in G, h \in H \right\rangle.$$

A group G is called **nilpotent**, if one of its commutators is trivial, i.e. if we set

$$G_0 := G \text{ and } G_{i+1} := [G_i, G],$$

then G is called nilpotent iff there is an $j \in \mathbb{N}$ with $G_j = 1$.

Corollary 4. *Any unipotent subgroup of $\mathsf{GL}(V)$ is nilpotent.*

Definition 26. A group G is called **solvable**, if $G^{(n)} = 1$ for some n where

$$G^{(0)} := G,$$
$$G^{(i+1)} := [G^{(i)}, G^{(i)}].$$

Note that nilpotent groups are solvable, since $G^{(i)} \subset G_i$.

Notation 1. In the following, we will write $G' := [G,G]$.

Definition 27. Let $n := \dim(V)$. A **complete flag** is a maximal strictly increasing chain of subspaces

$$0 = V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_n = V.$$

Any complete flag is of the form

$$V_j := \operatorname{span}\{e_1, \ldots, e_j\}$$

for some basis $e_1, \ldots, e_n$ of V.

Let B be the basis of some flag $0 = V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_n = V$. For $x \in \mathsf{End}(V)$, we have that x is upper-triangle with respect to B iff x leaves each member V_i of the flag invariant, i.e. $xV_i \subseteq V_i$.

Proposition 2 (Key Proposition)**.** *Let G be a unipotent subgroup of $\mathsf{GL}(V)$. Then there is a complete flag $V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_n$ consisting of G-invariant subspaces, i.e., each V_i is G-invariant.*

Proof. Recall, that G is a unipotent subgroup of $\mathsf{GL}_n(V)$. We will give an induction on $n = \dim V$.

If $n = 0$, there is nothing to show.

Let $n \geq 1$. We may assume that V is G-irreducible. Because, if not, there is a G-invariant subspace $0 \neq W \subset V$ s.t. W and V/W have dimension $< n$. Then there exist complete G-invariant flags in W and V/W and the claim – that there is a complete G-invariant flag in V – follows by the induction hypothesis.

By Jacobson Density Theorem, we have

$$R := \operatorname{span}(G) = \mathsf{End}(V) := \mathsf{End}_k(V).$$

Since G is unipotent, we have for each $g \in G$

$$\operatorname{trace}(g) = n.$$

Ergo, for $g, h \in G$

$$\operatorname{trace}(gh) = \operatorname{trace}(h)$$

and

$$\operatorname{trace}((g-1)h) = \operatorname{trace}(gh) - \operatorname{trace}(h) = 0.$$

Since $\operatorname{span}(G) = \mathsf{End}(V)$, it now in particularly follows for all $g \in G, \phi \in \mathsf{End}(V)$

$$\operatorname{trace}((g-1)\phi) = 0.$$

Since the above holds for all $\phi \in \mathsf{End}(V)$, it must hold

$$g - 1 = 0$$

for all $g \in G$ (take for example the elementary matrices $\phi = E_{i,j}$). Ergo, G is trivial. Then, any complete flag is trivially G-invariant. $\square$

Remark 3. This gives the group analogue of Engel's Theorem.

Proof Goal Theorem. Let B be a basis of V s.t. G leaves each subspace in the corresponding flag invariant. Then, G is upper-triangle with respect to this basis.

On the other hand, each $g \in G$ is unipotent, hence its diagonal (i.e. eigenvalues) are all 1. Thus, with respect to B

$$G \subseteq \left\{ \begin{pmatrix} 1 & \star & \star \\ 0 & \ddots & \star \\ 0 & 0 & 1 \end{pmatrix} \right\} = U_n.$$

$\square$

Remark 4. Tori are of the form $(k^\times)^n$. In the case $k = \mathbb{C}$, $(\mathbb{C}^\times)^n$ are the complexification of $U(1)^n$. This equals tori in top. sense.

$$\begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix} \subseteq \mathsf{GL}_2(\mathbb{C})$$

is a non-algebraic unipotent group.

Exercise. (to be discussed next time)

it would have sufficed to prove the Goal theorem in the special case that G is algebraic.

Corollary of Proof: If $G \subset \mathsf{GL}(V)$ (with $V \neq 0$) is unipotent and V is G-irreducible, then $G = 1$, $\dim V = 1$.

Lecture from 18.03.2020

Answer to last Exercise: Recall that the main point was to show that any unipotent subgroup $G \subseteq \mathsf{GL}(V)$ leaves invariant some complete flag $\mathcal{F} = (V_0 \subset V_1 \ldots)$. But by some homework (problem 1), the group

$$\mathsf{GL}(V)_{\mathcal{F}} := \{g \in \mathsf{GL}(V) \mid g\mathcal{F} = \mathcal{F}\}$$

is algebraic.

Proof: If $\mathcal{F}$ is the standard flag with $V_i = \mathrm{span}(e_1, \ldots, e_i)$ for the standard basis $\{e_1, \ldots, e_n\}$, then

$$\mathsf{GL}(V)_{\mathcal{F}} = \{A \in \mathsf{GL}(V) \mid A \text{ is upper-triangle}\}.$$

The condition that A is upper triangle can be realized by polynomials. □

Thus,

$$\begin{aligned} & G \text{ fixes } \mathcal{F} \\ \iff & G \subseteq \mathsf{GL}(V)_{\mathcal{F}} \\ \overset{\mathsf{GL}(V)_{\mathcal{F}} \text{ is algebraic}}{\iff} & \overline{G} \subseteq \mathsf{GL}(V)_{\mathcal{F}} \\ \iff & \overline{G} \text{ fixes } \mathcal{F}. \end{aligned}$$

Now, the Zariski-Closure $\overline{G}$ of any group G is an algebraic group (shown in some homework).

Further, if G is unipotent, then $\overline{G}$ is unipotent.

8 Tori

Definition 28. A **torus** is an algebraic group that is isomorphic to $\mathcal{G}_m^n$ for some $n \in \mathbb{N}_0$ where $\mathcal{G}_m = k^\times = \mathsf{GL}_1(k)$ is the unit group of k.

We think of $\mathcal{G}_m^n \subseteq \mathsf{GL}_n(k)$ as the subgroup of diagonal matrices.

Lemma 31. *Let G be a commutative algebraic group. Then the following are equivalent:*

(i) each $g \in G$ is semisimple.

(ii) for each finite-dimensional representation V of G and for each $g \in G$, the operator $r_V(g)$ is diagonalizable.

(iii) for all finite-dimensional representations V of G, there is a basis of common eigenvectors for $r_V(G)$, i.e. a basis s.t.

$$r_V(G) \subseteq \mathcal{G}_m^n.$$

(iv) G is isomorphic to an algebraic subgroup of a torus.

Proof. We show:

(i) $\iff$ (ii): This follows from the Jordan decomposition and definition of semisimple.

(ii) $\implies$ (iii) : This is homework. Note that any commutative subset S of $\mathsf{GL}(V)$ consisting of semisimple operators may be diagonalized simultaneously.

(iii) $\implies$ (iv) : Take any faithful representation V of G and diagonalize it simultaneously. Then, $G \cong r_V(G) \subseteq \mathcal{G}_m^n$.

(iv) $\implies$ (i) : Any diagonal matrix is semisimple.

□

Definition 29. A commutative algebraic group G is called **diagonalizable**, if it satisfies one of the above equivalent conditions.

Definition 30. A **character** χ of an algebraic group G is an element $\chi \in \mathsf{Hom}_{\text{alg.grp.}}(G, k^\times)$, i.e., a homomorphism $\chi : G \to k^\times$ of algebraic groups.

Notation 2. For an algebraic group G, set $\mathfrak{X}(G) := \mathsf{Hom}_{\text{alg.grp.}}(G, k^\times)$.

Also denote now by $\mathsf{O}(X) := k[T]/I(X)$ the coordinate ring of an algebraic set X (rather than $k[X]$).

Lemma 32. *There is a bijection*

$$\mathfrak{X}(G) = \{\textit{characters } \chi \textit{ of } G\} \longleftrightarrow \{x \in \mathsf{O}(G)^\times \mid \Delta(x) = x \otimes x\}.$$

Proof. Note, that any $x \in O(G)^\times$ can be thought of as a map $x : G \to k^\times \subset k$.

We have

$$\begin{aligned}\mathsf{Hom}_{\text{alg.grp.}}(G, \mathcal{G}_m) &= \{\phi \in \mathsf{Hom}_{\text{alg.sets}}(G, \mathcal{G}_m) \mid \phi(gh) = \phi(g)\phi(h)\ \forall g, h\} \\ &= \{\phi \in \mathsf{Hom}_{k-\text{alg.}}(\mathsf{O}(\mathcal{G}_m), \mathsf{O}(G)) \mid (\phi \otimes \phi) \circ \Delta = \Delta \circ \phi\}.\end{aligned}$$

Recall: $\mathsf{O}(\mathcal{G}_m) \cong k[t, \frac{1}{t}]$ with $\Delta(t) = t \otimes t$.

Thus for any k-algebra A, $\mathsf{Hom}_{k-\text{alg.}}(\mathsf{O}(\mathcal{G}_m), A) \overset{A^\times}{\cong}$ via

$$[t \mapsto a, (t^{-1} \mapsto a^{-1})] \longleftrightarrow a.$$

Thus,

$$\mathsf{Hom}_{\text{alg.grp.}}(G, \mathcal{G}_m) \cong \left\{a \in \mathsf{O}(G)^\times \mid a \otimes a = \Delta(a)\right\}.$$

Therefore, it suffices to test the condition $(\phi \otimes \phi) \circ \Delta = \Delta \circ \phi$ on the generators t, t^{-1} of $\mathsf{O}(\mathcal{G}_m)$. Now, the above isomorphism is given by

$$\phi \mapsto a = \phi(t)$$

which is equivalent or regarding $\chi : G \to \mathcal{G}_m$ as a map $\chi : G \to k$. □

Example 13. Let $G = \mathcal{G}_m$, then $\mathsf{O}(G) = k[t, \frac{1}{t}]$.

Which $x = \sum_{m \in \mathbb{Z}} c_m t^m \in \mathsf{O}(G)$ – with almost all $c_m = 0$, but not all of them – have the property

$$\Delta(x) = x \otimes x?$$

We have

$$\begin{aligned} x \otimes x &= \sum_{m,n \in \mathbb{Z}} c_m c_n t^m \otimes t^n, \\ \Delta(x) &= \sum_{m \in \mathbb{Z}} c_m t^m \otimes t^m. \end{aligned}$$

Those sums equal, if

$$\begin{aligned} &c_m c_n = o \text{ for all } m \neq n, \\ &c_m^2 = c_m \text{ for all m.} \end{aligned}$$

By those conditions, it follows

$$x = t^m.$$

Therefore

$$\mathfrak{X}(G) = \{\chi_m \mid m \in \mathbb{Z}\} \cong \mathbb{Z}$$

with

$$\chi_m(y) = y^m.$$

Example 14. Let $T \cong \mathcal{G}_m^n$ be a torus. Then,

$$\mathfrak{X}(T) = \{\chi_m \mid m \in \mathbb{Z}^n\} \cong \mathbb{Z}^n$$

where $\chi_m(y) = y^m = y_1^{m_1} \cdots y_n^{m_n}$.

Note: For each algebraic group G, $\mathfrak{X}(G)$ is naturally an abelian group:

$$(\chi_1 \cdot \chi_2)(g) := \chi_1(g) \cdot \chi_2(g).$$

Given a morphism of algebraic groups $f : G \to H$, we get a morphism of abelian groups

$$\begin{aligned} f^* : \mathfrak{X}(H) &\longrightarrow \mathfrak{X}(G) \\ \chi &\longmapsto \chi \circ f =: f^*(\chi). \end{aligned}$$

This induces a contravariant functor from the category of algebraic groups to the category of abelian groups.

Lemma 33. *Let G be a diagonalizable algebraic group. Then, $\mathfrak{X}(G)$ is a k-vector space basis for $\mathsf{O}(G)$.*

Example 15. Let $G = \mathcal{G}_m^n$ be a torus. Then, we have the embedding

$$\begin{aligned} \mathfrak{X}(G) &\hookrightarrow \mathsf{O}(G) \\ \chi_{(m_1,\ldots,m_n)} &\longmapsto t^{(m_1,\ldots,m_n)}. \end{aligned}$$

The lemma is obvious in this case: each elment of $\mathsf{O}(G) = k[t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1}]$ can be written uniquely as a linear combination of monomials.

Proof. (i) $\mathfrak{X}(G)$ spans $\mathsf{O}(G)$:

Choose an embedding $G \subset \mathcal{G}_m^n$ of algebraic groups. Then, by restriction, we get

$$\mathsf{O}(\mathcal{G}_m^n) \twoheadrightarrow \mathsf{O}(G).$$

Since the $\chi_m, m \in \mathbb{Z}^n$, span $\mathsf{O}(\mathcal{G}_m^n)$, their images $\chi_m|_G \in \mathfrak{X}(G)$ span $\mathsf{O}(G)$.

(ii) $\mathfrak{X}(G)$ is linearly independent:
Suppose otherwise and let $\phi_1, \dots, \phi_m$ be a linearly dependent subset of $\mathfrak{X}(G)$ with $m \geq 1$ chosen minimally, with $c_1, \dots, c_m \in k^\times$ s.t.

$$\sum_{i=1}^{m} c_i \phi_i = 0.$$

We distinguish the following cases:

$m = 1$: In this case, we have $\phi_1 = 0$, but $\phi_1(1) = 1$, a contradiction.

$m > 1$: We can assume $\phi_1 \neq \phi_2$, so there is an $h \in G$ s.t. $\phi_1(h) \neq \phi_2(h)$. Then,

$$\phi_1(h) \sum_{i=1}^{m} c_i \phi_i = 0,$$

but also for all $h, g \in G$

$$\sum_{i=1}^{m} c_i \phi_i(hg) = \sum_{i=1}^{m} c_i \phi_i(h) \phi_i(g) = 0.$$

This implies

$$\sum_{i=1}^{m} c_i \phi_i(h) \phi = 0.$$

Ergo

$$\sum_{i=1}^{m} c_j (\phi_i(h) - \phi_1(h)) \phi_i = \sum_{i=2}^{m} c_j (\phi_i(h) - \phi_1(h)) \phi_i = 0.$$

Now, $\phi_i(h) - \phi_1(h)$ is zero if $i = 1$ and non-zero, if $i = 2$. Therefore, this yields a shorter linear dependency for the elements

$$\phi_2, \dots, \phi_m,$$

which contradicts our requirement.

$\square$

Definition 31. Let M be an abelian group. The **group algebra** on M is the k-algebra $k[M]$ (not a coordinate ring!) defined as follows:

$$\begin{aligned} k[M] &:= \text{the } k\text{-vectorspace with basis } M \\ &:= \left\{ \sum_{m \in M} c_m \cdot m \mid c_m \in k, \text{ almost all } c_m = 0 \right\}, \end{aligned}$$

where the multiplication on $k[M]$ extends that on M:

$$(\sum_{m \in M} c_m m)(\sum_{n \in M} d_n n) = \sum_{m,n \in M} c_m d_n mn.$$

Corollary 5. *For a diagonalizable G, we have*

$$\mathsf{O}(G) \cong k[\mathfrak{X}(G)].$$

Fact: For an abelian group M, there is exactly one Hopf algebra structure on $k[M]$ given by $\Delta(m) = m \otimes m$ for all $m \in M$.

With this definition, the above isomorphism is one of Hopf algebras.

Lemma 34. *If G, H are diagonalizable algebraic groups, then*

$$\mathsf{Hom}_{\text{alg.grp.s}}(G, H) \xrightarrow{f \mapsto f^*} \mathsf{Hom}_{\text{grp.}s}(\mathfrak{X}(H), \mathfrak{X}(G))$$

is a bijection.

Proof.

$$\begin{aligned} \mathsf{Hom}(G, H) \cong & \mathsf{Hom}_{\text{Hopf}-\text{alg.}}(\mathsf{O}(H), \mathsf{O}(G)) \\ & \cong \{\phi \in \mathsf{Hom}_{k-\text{alg.}}(\mathsf{O}(H), \mathsf{O}(G)) \mid (\phi \otimes \phi) \circ \Delta = \Delta \circ \phi\}. \end{aligned}$$

Since $\mathsf{Hom}_{k-\text{alg.}}(\mathsf{O}(H), \mathsf{O}(G)) \cong \mathsf{Hom}(k[\mathfrak{X}(H)], k[\mathfrak{X}(G)])$, this reduces to the following lemma:

Lemma 35. *Let $M_1, M2$ be two abelian groups. Then*

$$\begin{aligned} \mathsf{Hom}(M_1, M_2) & \xrightarrow{\cong} \mathsf{Hom}_{\text{Hopf}-\text{alg.}}(k[M_1], k[M_2]) \\ \phi & \longmapsto \left[\sum c_m m \mapsto \sum c_m \phi(m)\right]. \end{aligned}$$

Proof. We have to show that

$$M = \left\{x \in K[M]^\times \mid \Delta(x) = x \otimes x\right\}.$$

Then, by this, it follows for each $\phi \in \mathsf{Hom}_{\text{Hopf}-\text{alg.}}(k[M_1], k[M_2])$,

$$\phi(M_1) \subseteq M_2.$$

Ergo, $\phi|_{M_1} \in \mathsf{Hom}\,(M_1, M_2)$. Therefore, the surjectivity of the claimed bijection is shown. The injectivity is clear, since M generates $k[M]$ as a k-algebra.

To show
$$M = \left\{x \in K[M]^\times \mid \Delta(x) = x \otimes x\right\},$$
let
$$\begin{aligned} x &= \sum c_m m \in K[M]^\times \\ \Delta(x) &= \sum c_m m \otimes m \\ x \otimes x &= \sum c_m c_n m \otimes n. \end{aligned}$$
If $\Delta(x) = x \otimes x$, then it follows
$$x = m$$
for some $m \in M$.

□

□

Recall: We have seen that for diagonalizable algebraic groups G, H Lecture from 25.03.2020

$$\mathsf{Hom}\,(G,H) \cong \mathsf{Hom}\,(\mathfrak{X}(H), \mathfrak{X}(G))\,.$$

If G is diagonalizable, then

$$\mathsf{O}(G) \cong k[\mathfrak{X}(G)].$$

Theorem 14. *The functor*

$$\begin{aligned} G &\longrightarrow \mathfrak{X}(G) \\ f &\longmapsto f^* \end{aligned}$$

defines an equivalence of categories:

$\{$*diagonalizable alg. groups*$\} \cong \{$*finite-dim. abelian groups with no* $\operatorname{char}(k)$*-torsion*$\}$.

This amounts to the bijection above between Hom-spaces and the following lemma.

Lemma 36. *(i) Let G be a diagonalizable alg. group. Then, $\mathfrak{X}(G)$ is a finitely generated abelian group with no* $\operatorname{char}(k)$*-torsion.*

(ii) Let Γ be a finitely generated abelian group with no $\operatorname{char}(k)$*-torsion. Then, there is a diagonalizable algebraic group G s.t. $\mathfrak{X}(G) \cong \Gamma$.*

Proof. We will use the following facts:

- Let $n \in \mathbb{N}$. Then, $t^n - 1$ is square-free in $k[t]$ iff the ideal $(t^n - 1)$ is radical in $k[t]$ iff $t^n - 1$ has not repetitive root iff either $\operatorname{char}(k) = 0$ or $\operatorname{char}(k) = p > 0$ and $p \not| n$.

  (Proof: Galois Theory, seperable/inseperable extensions.)

- Let $M := \mathbb{Z}/n\mathbb{Z}$. Then, the k-group-algebra generated by M

  $$k[M] \cong k[t]/(t^n - 1)$$

  is reduced iff either $\operatorname{char}(k) = 0$ or $\operatorname{char}(k) = p > 0, p \not| n$.

- If M_1, M_2 are abelian groups, then we have the following isomorphism of Hopf algebras

$$\begin{aligned} k[M_1] \otimes_k k[M_2] &\xrightarrow{\cong} k[M_1 \oplus M_2] \\ m_1 \otimes m_2 &\longmapsto m_1 m_2 \end{aligned}$$

where $M_1 \oplus M_2 \cong M_1 \times M_2$.

(i) Embed $G \hookrightarrow T := \mathcal{G}_m^n$ for some n. Then, we have a surjection $\mathbb{Z}^n \cong \mathfrak{X}(T) \twoheadrightarrow \Xi(G)$. Ergo, $\mathfrak{X}(G)$ is finitely generated.

Suppose $\mathrm{char}(k) = p > 0$. Let $\chi \in \mathfrak{X}(G)$ with $\chi^p = 1$. Then, for all $g \in G$, $\chi^p(g) = \chi(g^p) = 1$. The unit group $k^\times$ has not p-torsion, therefore $G \hookrightarrow T = (k^\times)^n$ has also no p-torsion. Therefore, the frobenius $g \mapsto g^p$ is an isomorphism on G. Therefore, $\chi = 1$ is a trivial character. Ergo $\mathfrak{X}(G)$ has no p-torsion.

(ii) Let Γ be a finitely generated abelian group with no $\mathrm{char}(k)$-torsion. Then,

$$\Gamma \cong \mathbb{Z}^r \oplus \mathbb{Z}/n_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/n_l\mathbb{Z}$$

where $\mathrm{char}(k) \not| n_1, \ldots, n_l$. We may reduce to the cases:

(a) $\Gamma = \mathbb{Z}$: take $G = \mathcal{G}_m$, then $\Xi(G) \cong \mathbb{Z} \cong \Gamma$.

(b) $\Gamma = \mathbb{Z}/n\mathbb{Z}$ with $\mathrm{char}(k) =: p \not| n$:
take $G := \mu_n := \{y \in k^\times \mid y^n = 1\}$. Then, since $p \not| n$, $(t^n - 1)$ is radical. So,

$$\mathsf{O}(\mu_n) \overset{\text{Nullstellensatz}}{=} k[t]/(t^n - 1) \overset{\text{as Hopf algebras}}{\cong} k[\Gamma]$$

where t gets mapped to the generator of Γ.

□

Corollary 6. *We have the bijection*

$$\{\textit{tori}\} \cong \{\textit{ finitely generated free abelian groups}(\cong \mathbb{Z}^n)\}.$$

Remark 5.

$$\{\text{algebraic group schemes}/k\} \overset{\text{not necessarily natural}}{\cong} \{\text{ f.g. Hopf algebras}\}.$$

by

$$G \mapsto \mathsf{O}(G)$$

and

$$\{\text{diagonalizable algebraic group schemes}/k\} \cong \{\text{ f.g. abelian groups}\}.$$

by

$$G \mapsto \Xi(G).$$

Where μ_p in the left hand term gets mapped to $\mathsf{O}(\mu_p) = k[t]/(t^p - 1)$ with $p = \mathrm{char} k$.

9 Trigonalization

We say a representation $r : G \to \mathsf{GL}(V)$ of a group G on a finite-dimensional k-vectorspace V is **trigonalizable** if it admits a basis with respect to which $r(V)$ is upper-triangular:

$$r(G) \subseteq \left\{ \begin{pmatrix} * & \dots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & * \end{pmatrix} \right\}$$

Definition 32. We call a subgroup $G \subseteq \mathsf{GL}(V)$ **trigonalizable**, if the identity representation is.

Lemma 37. *Let G be an algebraic group. The following are equivalent:*

(i) Every finite-dimensional representation $r : G \to \mathsf{GL}(V)$ is trigonalizable.

(ii) Every irreducible representation of G is 1-dimensional.

(iii) G is isomorphic to an algebraic subgroup of

$$B_n := \left\{ \begin{pmatrix} * & \dots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & * \end{pmatrix} \right\} \subseteq \mathsf{GL}_n(k).$$

(iv) There is a normal unipotent algebraic subgroup U of G s.t. G/U is diagonalizable.

Proof. We prove as follows:

(i) $\implies$ (ii): Let V be an irreducible representation. Then, $V \neq 0$. Choose a basis $e_1, \dots, e_n$ of V s.t.

$$r(G) \subseteq B_n.$$

Then, $r(G)e_1 \subseteq ke_1$, so $V_0 := ke_1$ is G-invariant. Ergo $V = V_0$ is 1-dimensional.

(ii) $\implies$ (i): Let V be a f.d. representation. We show by induction on $\dim(V)$ that $r : G \to \mathsf{GL}(V)$ is trigonalizable:

In the cases $\dim(V) = 0, 1$, there is nothing to show.

In the case $\dim(V) \geq 2$, assume that V is not irreducible. Then, there is a G-invariant V_0 with $0 \neq V_0 \neq V$.

By the induction hypothesis, V_0 and V/V_0 are trigonalizable. Ergo, V is trigonalizable.

(**Recall:** we used this criterion above in the proof that unipotent groups are trigonalizable by showing that every ??? of each G is trivial.)

(i) $\implies$ (iii): Choose a faithful representation V of G. Then, $G \cong r(G)$. Since r is trigonalizable, there is a basis of V s.t.

$$r(G) \subseteq B_n \subseteq \mathsf{GL}_n(k).$$

(iii) $\implies$ (ii): Suppose $G \subseteq B_n \subseteq \mathsf{GL}_n(k)$. Set

$$\begin{aligned} A_n &:= \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & * \end{pmatrix} \right\} \subseteq \mathsf{GL}_n(k), \\ U_n &:= \left\{ \begin{pmatrix} 1 & \dots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & 1 \end{pmatrix} \right\} \subseteq \mathsf{GL}_n(k) \text{ normal algebraic subgroup of } B_n, \\ U &:= G \cap U_n \text{ normal unipotent algebraic subgroup of } G. \end{aligned}$$

Let V be an irreducible representation of G, then V is not zero. Consider the subspace of V fixed by U

$$V^U := \{v \in V \mid r(u)v = v \forall u \in U\}.$$

Then, we get a representation

$$r|_U : U \longrightarrow \mathsf{GL}(V).$$

Then, $r(U)$ is a unipotent algebraic group of $\mathsf{GL}(V)$. Ergo,

$$r(U) \subseteq \left\{ \begin{pmatrix} 1 & \dots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Ergo, $V^U \neq 0$. Since U is normal in G, the subspace V^U of V is G-invariant: if $v \in V^U, g \in G$, then for all $u \in U$ we have

$$r(u)r(g)v = r(g)r(g^{-1}ug)v = r(g)v$$

since $v \in V^U$. Ergo $r(g)v \in V^U$.

Since V is irreducible, $V = V^U$, i.e. U acts trivially on V. Ergo, r descends to a representation of the group G/U.

But $G/U \hookrightarrow B_n/U_n \cong A_n$. Therefore, G/U and $r(G)$ are commutative. Moreover, for all $g \in G$, $r(g) \in \mathsf{GL}(V)$ is semisimple:

if $g = g_s g_u$, then $g_u \in U$, because U_n is the group of unipotent elements of B_n.

Hence, $r(g) = r(g_s)r(g_u) = r_(g_s)$ is semisimple.

It follows that $r(G)$ is commutative and consists of semisimple elements. By some HW: $r(G)$ is trigonalizable. It is easy to show now that V is one-dimensional. (Since V is irreducible and ke_1 is G-invariant.)

□

Definition 33. G is **trigonalizable**, if it satisfies one of the above equivalent conditions.

Later, we will see, that if G is connected, then being trigonalizable implies being solvable.

10 Commutative Groups

Let G be an algebraic group. Denote by G_s resp. G_u the subsets of semisimple resp. unipotent elements of G.

Then, G_u is always algebraical i.e. closed: if $G \hookrightarrow \mathsf{GL}_n(k)$, then $G_u = \{g \mid (g-1)^n = 0\}$. G_u does not need to be closed under multiplication (for example, take $G = \mathsf{SL}_2(k)$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$).

G_s needs not to be algebraic: for example, take $G = \mathsf{SL}_2(k)$ and if G_s were algebraic, then

$$\left\{\lambda \in k^\times \mid \begin{pmatrix} \lambda & 1 \\ 0 & \lambda^{-1} \end{pmatrix} \in G_s \right\} = \left\{\lambda \mid \lambda \neq \lambda^{-1}\right\}$$

but the last set is not algebraic. Also, G_s does not need to be a group.

We have the a surjective map of sets

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$$\begin{aligned} G_s \times G_u &\longrightarrow G \\ (g_1, g_2) &\longmapsto g_1 g_2. \end{aligned}$$

Example 16 (Non-Example)**.** Take generic $g \in G_s, h \in G_u$ for $G = \mathsf{SL}_2(k)$. Then, g, h do not commute and we have

$$((gh)_s, (gh_u)) \neq (g, h)$$

because Jordan components don't commute.

Theorem 15. *Let G be a commutative algebraic group. Then:*

(i) G_s, G_u are closed subgroups and the multiplicative map $G_s \times G_u \to G$ is an isomorphism of algebraic groups.

(ii) G is trigonalizable. Moreover, for each finite dimensional representation $r : G \to \mathsf{GL}(V)$ there is a basis s.t.

$$r(G_s) \subseteq \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & * \end{pmatrix} \right\} \qquad r(G_u) \subseteq \left\{ \begin{pmatrix} 1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

(iii) G_s is diagonalizable.

Proof. (ii) Let V be any irreducible representation of G. We have seen that commuting semisimple operators may be simultanously diagonalizable, then

$$V = \bigoplus_{\chi : G_s \to \mathcal{G}_m} V_\chi$$

where
$$V_\chi = \{ v \in V \mid r(h)v = \chi(h)v \ \forall h \in G_s \}.$$

Since G is commutative, each subspace V_χ is G-invariant $(r(h)r(g)v = r(g)r(h)v = r(g)\chi(h)v = \chi(h)r(g)v)$.

Since V is irreducible, we must have $V = V_\chi$ for some χ.

Recall that $G \cong G_s \times G_u$ as abstract groups. We have seen that $r(G_s) \subseteq \mathcal{G}_m^n$. We proved a while ago that any unipotent group, such as G_u, is trigonalizable. Ergo, V is trigonalizable. Since V is irreducible, we have $\dim V = 1$.

If we apply the same argument without assuming that V is irreducible, then we see that V is the coproduct of V_χ's as above and that each V_χ admits a basis s.t.

$$r(G_s)|_{V_\chi} \subseteq \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & * \end{pmatrix} \right\} \qquad r(G_u)|_{V_\chi} \subseteq \left\{ \begin{pmatrix} 1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

This yields the same conclusion for V.

(i) We have to show that G_s and G_u are closed and $j : G_s \times G_u \to G$ is an isomorphism of groups. Take any faithful representation

$$G \xrightarrow{\cong, r} r(G) \subseteq \mathsf{GL}(V)$$

and apply (ii). Then we have

$$r(G) \subseteq \left\{ \begin{pmatrix} * & * & * \\ 0 & \ddots & * \\ 0 & 0 & * \end{pmatrix} \right\} =: B$$

$$B_u = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & 1 \end{pmatrix} \right\},$$

$$r(G_s) \subseteq \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & * \end{pmatrix} \right\} =: A.$$

In fact, $r(G_s) = r(G) \cap A$, because if $g \in G$ with $r(g) \in A$, then $r(g)$ is semisimple, so $g \in G_s$.

Therefore, G_s is closed in G. Ergo, G_s and G_u are closed subgroups.

Then, the map j is a morphism of algebraic groups.

We need to show that j^{-1} is a morphism of algebraic groups. For this, it suffices to verify that the projection $G \to G_s$ is a morphism. But this map is given under r by the morphism:

$$t := \begin{pmatrix} a_1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & a_n \end{pmatrix} \longmapsto \begin{pmatrix} a_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_n \end{pmatrix} =: t_s.$$

This suffices because if $g = g_s g_u$, then $g_u = g_s^{-1} g$, so if the map $g \mapsto g_s$ is a morphism, so is $g \mapsto g g_s^{-1} = g_u$, hence so is $g \mapsto (g_s, g_u)$.

(iii) We have seen that G_s is a closed subgroup. Hence G_s is a commutative algebraic group where elements are semisimple. Ergo, G_s is diagonalizable.

□

11 Connected Solvable Groups

Theorem 16 (Lie-Kolchin). *Let G be a connected solvable algebraic group. Then G is trigonalizable.*

(By comparison, recall that we have seen so far that, if G is commutative or unipotent, then G is trigonalizable.) We can reformulate this theorem as: Any connected solvable subgroup of $\mathsf{GL}(V)$ stabilizes some complete flag $\mathcal{F} = (V_0 \subsetneq \ldots \subsetneq V_n)$.

Generalization (Borel's Fixed Point Theorem): Any connected algebraic group G acting on a projective variety X has a fixed point in X.

We get a relation between complete flags and projective varieties.

Proof. Induct on the number n s.t. $G^{(n)} = 1$.

For $n = 0$, there is nothing to show.

If $n = 1$, $(G, G) = 1$, then G is commutative, ergo trigonalizable.

Let $n \geq 2$. Then, we have $G' := (G, G) \neq 1$. We will show the following lemma:

□

Lemma 38. *Let $G \subseteq \mathsf{GL}(V)$ be a subgroup.*

If G is connected, then the group G' with the induced topology is connected ($\iff$ the Zariski Closure of G' is connected).

Proof. We have the following facts:

- An increasing union of connected spaces is connected.
- A continuous image of a connected space is connected.

We have

$$\begin{aligned} G' &= \left\langle (g,h) := ghg^{-1}h^{-1} \mid g, h \in G \right\rangle \\ &= \bigcup_{j \geq 0} \bigcup_{g_1, h_1, \ldots, g_j, h_j \in G} \{(g_1, h_1) \cdots (g_j, h_j)\}. \end{aligned}$$

Since

$$\bigcup_{g_1,h_1,\dots,g_j,h_j\in G} \{(g_1,h_1)\cdots(g_j,h_j)\} = \mathsf{Img}\phi_j$$

for some continuous map $\phi_j : G^{2j} \to G$, the claim follows. Ergo, G' is connected. $\square$

Remark 6. It is equivalent to show that $(*)$ any subgroup G of $\mathsf{GL}(V)$ – s.t. G is connected and solvable – is trigonalizable in $\mathsf{GL}(V)$.

Indeed, the theorem implies $(*)$: the Zariski closure of G is a connected algebraic group that is solvable (which extends by continuity). If $Zcl(G)$ is trigonalizable, then also G is trigonalizable.

On the other hand: $(*)$ implies the theorem, since if G is given as in the theorem, apply $(*)$ to $r(G) \subseteq \mathsf{GL}(V)$.

Proof of Theorem. If $G^{(n)} = 1$, then $(G')^{(n-1)} = G^{(n)} = 1$. By induction, we may assume that G' satisfies the following:

For all finite dimensional representations $r : G \to \mathsf{GL}(V)$, $r(G')$ is trigonalizable.

Our aim is to show that any irreducible representation V of G has dimension 1.

The induction hypothesis implies that $r(G')$ is trigonalizable. In particular, there exists an eigenspace $V_\chi \subseteq V$ for G' for some character $\chi : G' \to k^\times$. Since G' is normal in G we know that G acts from the left on

$$\{\text{eigenspaces } V_\chi \text{ in } V \text{ for } G'\}.$$

Ergo, $\bigoplus_{\chi:G'\to k^\times} V_\chi$ is G-invariant. Since V is G-irreducible, we have

$$V = \bigoplus_{\chi:G'\to k^\times} V_\chi = \bigoplus_{\chi\in\mathfrak{X}'} V_\chi$$

for some finite subset $\mathfrak{X}' = \{\chi \mid V_\chi \neq 0\}$ of $\mathsf{Hom}\,(G', \mathcal{G}_m)$, since V is finite dimensional.

Claim: Let $h \in G'$. Then, the map

$$\begin{aligned} G &\longrightarrow \mathsf{GL}(V) \\ g &\longmapsto r(ghg^{-1}) \end{aligned}$$

has a finite image.

Proof. Denote by $\chi \mapsto \chi^g$ the action of $g \in G$ in $\mathsf{Hom}\,(G', \mathcal{G}_m)$ given by $\chi^g(h) := \chi(ghg^{-1})$. This is an action, since G' is normal.

Note, that $\mathfrak{X}' \subseteq \mathsf{Hom}\,(G', \mathcal{G}_m)$ is a finite subset.

Also note, that the action $\chi \mapsto \chi^g$ descends to an action $G \curvearrowright \mathfrak{X}'$.

Now, let $\mathfrak{X}' = \{\chi_1, \dots, \chi_r\}$. The matrix $r(h)$ is totally determined by the values $\chi_1(h), \dots, \chi_r(h)$. Then, the element $r(ghg^{-1})$ is totally determined by the values $\chi_1^g(h), \dots, \chi_r^g(h)$. It follows

$$\#\left\{r(ghg^{-1}) \mid g \in G\right\} \leq r!.$$

□

The following lemma is easy to show:

Lemma 39. *Let G be an algebraic set. Then, G is connected iff for each finite algebraic set X, and for each morphism $f : G \to X$ of algebraic sets, we have that f is constant.*

Claim with the Lemma implies that the map $g \mapsto t(ghg^{-1})$ is constant. This implies that $r(ghg^{-1}) = r(h)$ for all $g \in G, h \in G'$. Ergo, G stabilizes each eigenspace V_χ for G'. Ergo, $V = V_{\chi_0}$, since V is irreducible. □

Lemma 40. *Let G be any group with a finite dimensional representation $r : G \to \mathsf{GL}(V)$. Then, the subspaces V_χ for $\chi \in \mathsf{Hom}\,(G, k^\times)$ are linearly independent, i.e., the map*

$$\oplus V_\chi \longrightarrow V$$

is injective.

Proof. The spaces V_χ are G-invariant. Suppose, there exist distinct $\chi_1, \dots, \chi_n$ of non-zero $v_j \in V_{\chi_j}$ s.t. $\sum_j v_j = 0$.

We may assume that n, the number of v_j, is minimal. W.l.o.g., $n \geq 2$.

Choose $g \in G$ s.t. $\chi_1(g) \neq \chi_2(g)$. Use that $0 = g\sum_j v_j = \sum_j gv_j$ and take the linear combination as in the proof of linear indpendence of characters to contradict the minimality of n.

($g - \chi_1(g)$ is not zero, but reduces $\sum_j v_j$ by one summand.) □

Finishing Proof of Theorem. Since $G' = \langle ghg^{-1}h^{-1} \mid g, h \in G \rangle$, so $\det(r(G')) = 1$. Lecture from 01.04.2020

On the other hand, for each $g \in G'$, we have

$$r(g) = \begin{pmatrix} \chi_0(g) & & \\ & \ddots & \\ & & \chi_0(g) \end{pmatrix}$$

since $V = V_{\chi_0}$. This implies

$$1 = \det(r(g)) = \chi_0(g)^d.$$

Ergo, χ_0 defines a morphism

$$\chi_0 : G' \longrightarrow \mu_d \subseteq \mathcal{G}_m.$$

But G' is connected and μ_d is finite. Since χ_0 is a morphism, χ_0 must be constant, ergo the trivial character.

As a consequence, we get $r(G') = 1$ on $V = V_{\chi_0}$.

Lemma 41. *Let G be an algebraic group, $r : G \to \mathsf{GL}(V)$ a representation. $v \in V$ shall be a simultaneous non-zero eigenvector for $r(G)$.*

Then, for each $g \in G$, there is a value $\chi(g) \in k^\times$ s.t.

$$r(g)v =: \chi(g)v.$$

Then, the mapping $\chi : G \to \mathcal{G}_m$ is a morphism of algebraic groups.

Therefore, r descends to a representation of the commutative group

$$\overline{r} : G/G' \longrightarrow \mathsf{GL}(V).$$

Ergo, $r(G/G') = r(G)$ is commutative and therefore trigonalizable (because of irreducibility).

□

Example 17 (Non-Example)**.**
- Take $G = D_4 \hookrightarrow \mathsf{GL}_2(\mathbb{C})$ which is solvable and has an irreducible and faithful representation over $\mathbb{C}^2$.

- Consider the solvable group

$$G = \left\langle \begin{pmatrix} \pm 1 & \\ & \pm 1 \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \right\rangle$$

which is a finite subgroup of $\mathsf{GL}_2(\mathbb{C})$, s.t. $\mathbb{C}^2$ defines an irreducible representation of G.

Lemma 42 (Form of Schur's Lemma)**.** *Let S be any commutative subset of $\mathsf{GL}(V)$ for a finite-dimensional $0 \neq V$ over an algebrically closed field k. Let V be S-irreducible.*

Then, $\dim V = 1$.

Proof. There is nothing to show if S is empty.

Let $s \in S$ and denote by $V_\lambda \subseteq V$ the λ-eigenspace for s. Then, since S is commutative, V_λ is S-invariant. Therefore, $V = V_\lambda$ for one $\lambda \in k^\times$.

Thus, every $s \in S$ acts by scaling, therefore every subspace of V is S-invariant. Since V is invariant, we get $\dim V = 1$. □

Corollary 7. *Let G be a connected algebraic group. Then, G is solvable iff G is trigonalizable.*

Proposition 3. *If G is trigonalizable, then G_u is a normal algebraic subgroup.*

Proof. We have

$$G \hookrightarrow B := \left\{ \begin{pmatrix} * & \dots & * \\ & \ddots & \vdots \\ & & * \end{pmatrix} \right\} \subseteq \mathsf{GL}_n(k).$$

B has the normal subgroup $U := \left\{ \begin{pmatrix} 1 & \dots & * \\ & \ddots & \vdots \\ & & 1 \end{pmatrix} \right\}$ and we have $G_u = G \cap U$. Now, U is the kernel of the multiplicative morphism

$$\begin{pmatrix} a_1 & \dots & * \\ & \ddots & \vdots \\ & & a_n \end{pmatrix} \longmapsto \begin{pmatrix} a_1 & \\ & a_n \end{pmatrix}.$$

□

Corollary 8. *If G is connected and solvable, then G_u is a normal algebraic subgroup.*

12 Semisimple Elements of nilpotent Groups

Theorem 17. *Let G be a connected nilpotent algebraic group. Then, we have*

$$G_s \subseteq Z(G)$$

where $Z(G)$ denotes the ***center*** *of G, i.e.*

$$Z(G) = \{g \in G \mid \forall h \in G : \ gh = hg\}.$$

Theorem 18 (Lie-algebraic Analogue)**.** *Let V be a finite-dimensional vectorspace. Let $\mathfrak{g}$ be the Lie-Subalgebra of $\mathsf{End}(V)$, i.e. $\mathfrak{g}$ is a subspace s.t. we have for each $x, y \in \mathfrak{g}$*

$$[x, y] := xy - yx \in \mathfrak{g}.$$

Assume that $\mathfrak{g}$ is nilpotent, i.e. there is an $n \in \mathbb{N}_0$ s.t.

$$[x_1, [x_2, [\ldots, [x_{n-1}, x_n]]]] = 0$$

for all $x_1, \ldots, x_n \in \mathfrak{g}$.

Then, any semisimple (semisimple in $\mathsf{End}(V)$ that is) $x \in \mathfrak{g}$ is ***central*** *in $\mathfrak{g}$, i.e. $[x, y] = 0$ for each $y \in \mathfrak{g}$.*

Remark 7. The Lie-algebraic Analogue implies the general theorem if – for example – $k = \mathbb{C}$.

Proof. Let $g \in G_s$. We want to show $Z_G(g) = G$.

Fact from the theory of Lie-Algebras: For the Lie-Algebra $\mathrm{Lie}Z_G(g)$ we have

$$\mathrm{Lie}Z_G(g) = \ker(\mathsf{Ad}(g))$$

where Ad is the map

$$\begin{aligned} \mathsf{Ad} : G &\longrightarrow \mathsf{GL}(\mathfrak{g}) \\ x &\longmapsto gxg^{-1}. \end{aligned}$$

Since G is connected, it suffices to verify

$$\ker(\mathsf{Ad}(g)) = \mathfrak{g}$$

i.e. $\mathsf{Ad}(g) = 1$.

Since g is semisimple, we have for suitable basis

$$g = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$$

with $a_j \in \mathbb{C}^\times$. This is $\exp(x)$ for a suitable diagonal matrix $x \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} \in$ $\mathsf{GL}_n(\mathbb{C})$.

Fact: We may assume that $x \in \mathfrak{g} := \mathrm{Lie}(G)$.

Since G is nilpotent, it can be shown that $\mathfrak{g}$ is nilpotent.

By the theorem, x is central in $\mathfrak{g}$. By the properties of exp we have

$$\mathsf{Ad}(g) = \exp(\mathrm{ad}(g)) = 1$$

ergo $\mathrm{ad}(x) = 0$ where $\mathrm{ad} : \mathfrak{g} \to \mathfrak{g}$ is defined by

$$\mathrm{ad}(x) \cdot y := [x, y].$$

□

Proof. If $\mathfrak{g}$ is nilpotent, then $\mathrm{ad}(x) \in \mathsf{End}(\mathfrak{g})$ is nilpotent.

Since x is semisimple, $\mathrm{ad}(x)$ is semisimple, because $\mathrm{ad}(x)$ is the restriction to $\mathfrak{g}$ of the map

$$\begin{aligned} \mathsf{End}(V) &\longrightarrow \mathsf{End}(V) \\ y &\longmapsto [x, y] \end{aligned}$$

and, if $e_1, \ldots, e_n$ are a basis of eigenvectors for x, then $E_{i,j}$ is a basis of eigenvectors for ℓ.

So, $\mathrm{ad}(x)$ is nilpotent and semisimple, therefore $\mathrm{ad}(x) = 0$. □

Proof Theorem. Let G be a connected nilpotent algebraic group, $G \overset{\mathsf{GL}}{\hookrightarrow} (V)$.

Let $g \in G_s$, we want to show that $g \in Z(G)$.

Assume otherwise, then we have a $h \in G$ s.t. $(g, h) = ghg^{-1}h^{-1} \neq 1$.

Since G is connected and nilpotent (ergo solvable), we know by Lie-Kolchin that G stabilizes some complete flag $V_0 \subset \ldots \subset V_n$.

We have $g|_{V_i}, h|_{V_i} \in \mathsf{GL}(V_i)$. They commute, if $i = 0$, but not if $i = n$.

So, there is an i s.t. $g|_{V_i}, h|_{V_i}$ commute but $g|_{V_{i+1}}, h|_{V_{i+1}}$ don't commute. W.l.o.g. $V = V_{i+1}, g = g|_{V_{i+1}}, h = h|_{V_{i+1}}$. Set $a := g|_{V_i}, b := h|_{V_i} \in \mathsf{GL}(V_i)$. a will be semisimple, since g is.

Since g is semisimple, there is an eigenvector $v \in V_{i+1}$ for g s.t.

$$V_{i+1} = V_i \oplus \langle v \rangle .$$

We have an isomorphism of vector spaces

$$\mathsf{End}(V_{i+1}) \cong \mathsf{End}(V_i) \oplus \mathsf{Hom}\left(\langle v \rangle, V_i\right) \oplus \mathsf{Hom}\left(V_i, \langle v \rangle\right) \oplus \mathsf{End}(\langle v \rangle)$$

with

$$\mathsf{End}(\langle v \rangle) \cong k \text{ and } \mathsf{Hom}\left(\langle v \rangle, V_i\right) \cong V_i.$$

So, we can write $g|_{V_{i+1}}, h|_{V_{i+1}}$ write as

$$g = \begin{pmatrix} a & \\ & * \in k \end{pmatrix} \text{ and } h = \begin{pmatrix} b & c \in V_i \\ & * \end{pmatrix}.$$

We may replace g, h with scalar multiples to reduce to the case that $* = 1$. Then, So, we can write $g|_{V_{i+1}}, h|_{V_{i+1}}$ write as

$$g = \begin{pmatrix} a & \\ & 1 \end{pmatrix} \text{ and } h = \begin{pmatrix} b & c \\ & 1 \end{pmatrix}.$$

Then,

$$h \neq ghg^{-1} = \begin{pmatrix} b & ac \\ & 1 \end{pmatrix}.$$

Ergo, $c \neq ac$, i.e. $c \notin \ker(a-1)$. Define

$$h_1 := h^{-1}ghg^{-1} = \begin{pmatrix} 1 & b^{-1}(a-1)c \\ & 1 \end{pmatrix}.$$

We claim that h_1 does not commute with g. This claim implies the theorem, since we can iterate the claim to obtain elements h_i by $h_{i+1} := h_i^{-1} g h_i g^{-1}$. Then, h_i does not commute with g. But G is nilpotent, therefore $h_i = 1$ for some large enough i.

We can prove the claim as follows: By some calculation as for h and g, we see, that h_1 and g don't commute iff $b^{-1}(a-1)c \notin \ker(a-1)$. This is equivalent to

$$\begin{aligned} &\iff (a-1)b^{-1}(a-1)c \neq 0 \\ &\iff b^{-1}(a-1)^2 c \neq 0 \\ &\iff (a-1)^2 c \neq 0 \\ &\iff c \notin \ker((a-1)^2). \end{aligned}$$

But a being semisimple implies $a - 1$ being semisimple, therefore

$$\ker((a-1)^2) = \ker(a-1).$$

So h_1, g don't commute iff $c \in \ker(a-1)$ iff h, g don't commute. $\square$

13 Projective Space

Lecture from 06.04.2020

Let V be a finite-dimensional vector space. Then $\mathcal{G}_m = k^\times$ acts on V by scalar multiplication. $\{0\}$ is a $\mathcal{G}_m$-invariant subspace of V. We are interested on the orbits of $\mathcal{G}_m$ on $V \setminus \{0\}$.

Define the **projective space** over V by

$$\mathbb{P}V := \mathcal{G}_m \backslash (V - 0) = (V - 0)/\sim \cong \{\text{lines in } V\}$$

where for $a, b \in V - 0$ we set

$$a \sim b :\Longleftrightarrow \exists \lambda \in k^\times : \lambda a = b.$$

If $V = k^{n+1}$, we denote the n-dimensional projective space by $\mathbb{P}^n := \mathbb{P}V$.

Given $a = (a_0, a_1 \ldots, a_n) \in k^{n+1} - 0$, we denote the $\sim$-class of a by

$$[a] = [a_0, \ldots a_n] \in \mathbb{P}^n.$$

Define S to be the graded algebra of polynomials in k

$$S := k[x_0, \ldots, x_n] = \bigoplus_{d \geq 0} S_d$$

where each S_d is the space of homogenous polynomials of degree d, i.e.

$$S_d = \bigoplus_{i_1, \ldots, i_d \in \{0, \ldots, n\}} k \cdot x_{i_1} \cdots x_{i_d}.$$

We identify k with the space of constant polynomials $S_0 \subseteq S$.

We have

$$S_d = \left\{ f \in S \mid f(\lambda X) = \lambda^d f(X) \; \forall \lambda \in k^\times \right\}.$$

Given $f \in S_d$, the set

$$\left\{ a \in k^{n+1} \mid f(a) = 0 \right\}$$

is $\mathcal{G}_m$-invariant. In other words, given $a \in \mathbb{P}^n$ and $f \in S^d$, it is well-defined to state $f(a) = 0$ and $f(a) \neq 0$.

Definition 34. A **projective** algebraic subset $X \subseteq \mathbb{P}^n$ is a set of the form

$$X = V(\Sigma) := V_{\mathbb{P}^n}(\Sigma)$$

where Σ is a collection of homogenous elements of S, where

$$V_{\mathbb{P}^n}(\Sigma) := \{ a \in \mathbb{P}^n \mid f(a) = 0 \; \forall f \in \Sigma \}.$$

Facts:

- Hilbert basis theorem states

  $$V(\Sigma) = V(f_1, \ldots, f_m)$$

  for some finite collection $f_1, \ldots, f_m \in \Sigma$.

- It is useful to extend the meaning of "$f(a) = 0$" for $a \in \mathbb{P}^n$ to *general* elements $f \in S$ by requiring that $f(a') = 0$ for each $a' \in [a]$.

  If we write $f = \sum_{d \geq 0} f_d$, $f_d \in S_d$, then we have

  $$f(a) = 0 \iff f_d(a) = 0 \ \forall d \geq 0.$$

  Therefore, we can extend the definition of $V(\Sigma)$ to any $\Sigma \subseteq S$.

- We have $V(\Sigma) = V((\Sigma))$ where (Σ) is the ideal generated by some finite subset of Σ.

- We call an ideal $I \subseteq S$ **homogenous** if it is the direct sum of its d-homogeneous components, i.e.

  $$I = \sum_{d \geq 0} I_d$$

  where $I_d = \{f \in I \mid f \text{ is homogenous of degree } d\}$.

  I is homogeneous iff it is generated by homogeneous elements.

- We have the following *Nullstellensatz*:

  For any $X \subseteq \mathbb{P}^n$, set $I(X)$ to be the ideal generated by all homogeneous polynomials of S vanishing on X. Then, we have

  $$I(V_{\mathbb{P}^n}(I)) = I$$

  for each homogeneous ideal $I \subseteq S$ for which we have:

  1. I is radical.
  2. I is not $(x_0, \ldots, x_n)$.

  Example 18 (Anti-example)**.** The second property is necessary:

  Set $I = (x_0, \ldots, x_n)$. Then $V_{k^{n+1}}(I) = 0$. Therefore, $V_{\mathbb{P}^n}(I) = \emptyset$. However,

  $$I(V_{\mathbb{P}^n}(I)) = S.$$

- The above point induces a bijection between algebraic subsets of $\mathbb{P}^n$ and radical ideals $I \subset S$ which are not $(x_0, \ldots, x_n)$.

For $i = 0, \ldots, n$, set $D(x_i) := \{a \in \mathbb{P}^n \mid a_i \neq 0\}$. $D(x_i)$ is an open set homeomorphic to k^n by mapping

$$\phi_i : a \longmapsto (\frac{a_0}{a_i}, \ldots, \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i}, \ldots, \frac{a_n}{a_i}).$$

The $D(x_i)$ cover $\mathbb{P}^n = \bigcup_i D(x_i)$.

Given a projective algebraic subset $X \subset \mathbb{P}^n$, define $X^{(i)} \subset k^n$ by

$$X^{(i)} := \phi_i(X \cap D(x_i)).$$

If $X = V_{\mathbb{P}^n}(I)$, then

$$X^{(i)} = V_{k^n}(I^{(i)})$$

where

$$I^{(i)} := \left\{ f^{(i)} \mid f \in I \right\}$$

where $f^{(i)}(t_1, \ldots, t_n) := f(t_1, \ldots, t_{i-1}, 1, t_i, \ldots, t_n)$. Thus, $X^{(i)}$ is an algebraic subset of k^n.

Definition 35. The **Zariski topology** on $\mathbb{P}^n$ is defined by setting the set of closed sets to be the set of projective algebraic sets.

Facts:

- $D(x_i)$ is open in $\mathbb{P}^n$, since $D(x_i) = \mathbb{P}^n - V(x_i)$.
- The bijections $D(x_i) \cong k^n$ are homeomorphims.

Definition 36. A **quasi-projective** algebraic set Y is an open subset of a projective algebraic set $X \subseteq \mathbb{P}^n$.

Example 19. Any algebraic set in k^n is quasi-projective.

Definition 37. A quasi-projective variety is an irreducible quasi-projective algebraic set.