Lemma 1. *Let* $B = U \rtimes T$ *as above and suppose again* $\mathrm{char} k = 0$.

Then, there is an algebraic subgroup $V \subset U$ *s.t.* V *is normal in* B *and*

$$\dim(U/V) = 1.$$

Proof. U is nilpotent, since it is unipotent. Consider the chain

$$U = U_0 \supset U_1 \supset \ldots \supset U_n \supset 1$$

where

$$U_{i+1} := [U_i, U].$$

Since U is normal in B, each U_i is also normal in B. In particular, B acts on each U_i by conjugation.

Now, U/U_1 is unipotent and commutative, hence isomorphic to a vector space.

Further, T acts on U/U_1 by conjugation. Note, that T is diagonalizable, ergo reductive. Therefore, U/U_1 must be completely reducible and we can decompose it

$$U/U_1 = \bigoplus_j V_j.$$

Since T is diagonalizable and each V_j is T-invariant, each V_j must be one-dimensional. Set

$$\overline{V} := \bigoplus_{j \geq 2} V_j.$$

And now set

$$V := \pi^{-1}(\overline{V}) = \left\{ u \in U \mid uU_1 \in \overline{V} \right\}.$$

Then, we have

$$U/V = (U/U_1)/(V/U_1) = (U/U_1)/\overline{V} \cong V_1 \cong k.$$

V is normal in U, since U_1 is normal in U and T acts on V and $\overline{V}$ by conjugation. $\square$

Theorem 1 (Semisimple Elements of Solvable Groups)**.** *Let $B = U \rtimes T$ as before be a solvable connected algebraic group.*

Let $s \in B$ be semisimple. Then s is conjugated to one element in T.

Claim: Write $s = ut$ and assume that s, t do not commute. Let $\operatorname{char} k = 0$ and $\dim U = 1$. For each $h \in sU = Us$, we have for the B-conjugacy class $C(h) = \{ghg^{-1} \mid g \in B\}$

$$C(h) = sU.$$

Corollary 1. *Let G be a connected algebraic group. Then, every semisimple element of G is contained in some torus.*

Lemma 2. *Suppose* $\operatorname{char} k = 0$.

(i) Let $g \in \mathsf{GL}_n(k)$ be unipotent and set $G(g) := \overline{\langle g \rangle}$. Then, we have the following isomorphism of algebraic groups

$$G(g) = \left\{ g^t \mid t \in k \right\} \cong k$$

where

$$\begin{aligned} g^t &:= \exp(t \cdot \log(g)) \\ -\log(1 - X) &= \sum_{n=1}^{\infty} \frac{X^n}{n} \\ \exp(Y) &= \sum_{k=0}^{\infty} \frac{Y^k}{k!}. \end{aligned}$$

(ii) Any unipotent algebraic group is connected. (This does not hold if $\operatorname{char} k > 0$*.)*

(iii) Any unipotent commutative algebraic group is isomorphic to some vector space.

Proof of Claim. 1. B acts by conjugation on $Us = sU$, because G/U is a commutative group.

2. Since $\dim(U) = 1$, we have $U = \{v^k \mid k \in K\} \cong k$.

3. $h \in sU$ does not commute with u, since – otherwise – s, t would commute with u. Ergo, $h \neq u^{-1}hu$, which means $C(h) \supseteq \{h, u^{-1}hu\}$ contains at least two different elements.

4. Note, that $C(h)$ is a B-orbit and therefore connected and **locally closed** (that is a closed subset of an open subset of G). Since G/U is commutative, we have $C(h) \subset sU = hU \cong k$.

Now, the only connected, locally closed subsets of k are singletons and complements of finite sets. Since $C(h)$ is not a singleton, we have $C(h) = sU - \Sigma$ for a finite set Σ. We claim that Σ is empty. Note, that B acts by conjugation on sU and $C(h)$, ergo also on Σ. If we pick $h' \in \Sigma \subset sU$, then $C(h')$ must be finite, connected and contain two different elements. This is a contradiction. $\square$

Proof of Theorem. We only prove the theorem in case $\operatorname{char} k = 0$. Let $s \in B = U \rtimes T$ be semisimple. Since $\operatorname{char} k = 0$, U is connected. We induct on $\dim(U)$:

$\dim(U) = 0$: In this case $U = 1$ and $s \in G = T$.

$\dim(U) = 1$: Write $s = ut$ with $u \in U$ and $t \in T$. If u and t commute, then ut is a Jordan decomposition and we have $u = 1$, ergo $s \in T$. Assume therefore, that u, t don't commute. The claim now implies $t = su \in sU = C(s)$.

$\dim(U) \geq 2$: Choose $V \subset U$ s.t. $\dim(U/V) = 1$ and V is normal in B. Set

$$B' = B/V \quad U' = U/V-$$

Then, B' is a connected algebraic group with

$$(B')_u = U' \quad B'/U \cong B/U = T \quad B' = U' \rtimes T.$$

Since $\dim(U') = 1$, we know that $\pi_V(s) \in B'$ is contained in a conjugacy class of T. Let $s' \in B$ be the conjugate of $s \in B$ s.t. $\pi_V(s') \in T$. Then, $s' \in TV$. But TV is a connected solvable algebraic group and we have

$$TV \cong V \rtimes T \subset U \times T.$$

Since $(TV)_u = V$ and $\dim(V) = \dim(U) - 1$, the induction hypothesis does also hold in TV. Ergo, s' is conjugated to some element in T, as we wanted. $\square$

Lemma 3 (To be Proved)**.** *Let G be a connected algebraic group with a Borel subgroup B.*

If B is nilpotent, then G is solvable i.e. $B = G$.

Theorem 2 (Low Dimensional Groups)**.** *Let G be connected with* $\dim(G) \leq 2$.

Then, G is solvable.

Example 1 (Non-Example)**.** The condition $\dim(G) \leq 2$ is necessary. Consider e.g. $G = \mathsf{SL}_2(k)$ which has a dimension of 3.

Corollary 2. *Let G be connected with* $\dim(G) = 1$. *Then, G is commutative.*

Proof. Because of the theorem, G is solvable. Therefore, $[G, G]$ is a closed proper subgroup of G. Hence, $\dim([G, G]) = 0$. Since $[G, G]$ is connected, it follows $[G, G] = 1$. □

Remark 1. If G is commutative, it decomposes nicely into semisimple and unipotent elements

$$G = G_s \times G_u.$$

So, if $\dim(G) = 1$ and if G is connected, then $G = G_s \cong \mathcal{G}_m$ is a torus, or $G = G_u \cong \mathcal{G}_a$ is unipotent.

Further, we can consider

$$G = \left\{ \begin{pmatrix} * & * \\ & 1 \end{pmatrix} \right\}.$$

G is connected and of dimension 2. It decomposes

$$G = G_s \times G_u$$

into two groups of dimension 1.

We induct on $\dim(B)$:

$\dim(B) = 0$: In this case, we have $B = 1$. $G = \bigcup_{g \in G} gBg^{-1}$, since G is connected. Since $B = 1$, it follows $G = 1$.

$\dim(B) \geq 1$: Since B is nilpotent, we have a descending chain

$$B = B_0 \supsetneq \ldots \supsetneq B_n \supsetneq 1$$

where

$$B_{i+1} = [B, B_i].$$

Note, that each B_i is connected, since B is connected. Let $Z(B) = \{b \in B \mid \forall g \in B : \; gb = bg\}$ be the center of B and let $Z := Z(B)^o$ be the component of the neutral element. Then, we have

$$B_n \subset Z.$$

Ergo, Z is not the trivial subgroup.

We want to show

$$Z \subset Z(G).$$

Let $z \in Z$ and consider the morphism

$$\begin{aligned} \phi : G/B &\longrightarrow G \\ gB &\longmapsto gzg^{-1}. \end{aligned}$$

ϕ is well-defined, because $z \in Z(B)$. Since ϕ is a morphism from a projective variety to an affine variety, ϕ must be constant. Thus,

$$Z \subset Z(G).$$

In particular, Z is normal in G. We now get an inclusion of quotient groups

$$B/Z \hookrightarrow G/Z.$$

It is clear that

$$\dim(B/Z) < \dim(B).$$

Further, B/Z is parabolic, since

$$(G/Z)/(B/Z) = G/B$$

is projective. Ergo, B/Z is Borel. By the induction hypothesis, we get

$$G/Z = B/Z.$$

Ergo, $B = G$.

Lemma 4. *Let G be a connected algebraic group, $B \subset G$ a Borel subgroup and $S \subset B$ any torus.*

Then, $Z_B(S)$ is a Borel subgroup of $Z_G(S)$.

Claim in Proof:

$$Z_G(S)B = \left\{ g \in G \mid \forall s \in S : \; g^{-1}sg \in sU \right\} =: A$$

where $U := B_u$.

Example 2.

$$\begin{aligned}
G &= \mathsf{GL}_5(k) \\
B &= \left\{ \begin{pmatrix} * & * & * & * & * \\ & * & * & * & * \\ & & * & * & * \\ & & & * & * \\ & & & & * \end{pmatrix} \right\} \\
S &= \left\{ \begin{pmatrix} t_1 & & & & \\ & t_1 & & & \\ & & t_1 & & \\ & & & t_2 & \\ & & & & t_2 \end{pmatrix} \mid t_1, t_2 \in k^\times \right\} \\
Z_G(S) &= \left\{ \begin{pmatrix} * & * & * & & \\ * & * & * & & \\ * & * & * & & \\ & & & * & * \\ & & & * & * \end{pmatrix} \right\} = \mathsf{GL}_3(k) \times \mathsf{GL}_2(k) \\
Z_G(S) &= \left\{ \begin{pmatrix} * & * & * & & \\ & * & * & & \\ & & * & & \\ & & & * & * \\ & & & & * \end{pmatrix} \right\}.
\end{aligned}$$

We showed before, that $Z_G(S)$ is connected, if G is connected and S a torus.

Proof of Claim: It is easy to see, that

$$Z_G(S) \subset A.$$

For $b \in B$, we have

$$b^{-1}sb \in sU,$$

since B/U is commutative.

Now, let $g \in A$. Then,

$$g^{-1}Sg \subset SU \subset B.$$

One can extend S to a maximal torus T of B. Then,

$$B = U \rtimes T \supset SU = U \rtimes S.$$

Since S is closed in T, SU is closed in B. Further, $g^{-1}Sg$ and S are maximal tori in SU. Then, there is a $b \in B$ s.t.

$$b(g^{-1}Sg)b^{-1} = S.$$

Set

$$z := gb^{-1}.$$

We need to show, that z lies in $Z_G(S)$.

Since B/U is commutative, we have for each $s \in S$

$$z^{-1}sz = b(g^{-1}sg)b^{-1} \in g^{-1}sgU = sU,$$

since $g \in A$. Now, we have for each $s \in S$

$$z^{-1}sz \in sU \cap S = \{s\}.$$

Ergo, $z \in Z_G(S)$.

Proof of Lemma: We showed that

$$Z_G(S)B = \left\{g \in G \mid \forall s \in S: \ g^{-1}sg \in sU\right\} = \{g \in G \mid \forall s \in S : [g, s] \in U\}.$$

Then, $Z_G(S)B$ is closed. Since

$$\pi : G \twoheadrightarrow G/B$$

is an open and surjective map, it is easy to see that

$$Z_G(S)/Z_B(S) \cong \pi(Z_G(S)B)$$

is closed. Since $Z_B(S)$ is closed, $Z_B(S)$ is a parabolic subgroup of $Z_G(S)$. Since $Z_B(S)$ is contained in B, it is solvable, hence a Borel subgroup.

Theorem 3 (Normalizers of Borel Subgroups)**.** *Let G be a connected algebraic group with a Borel subgroup $B \subset G$. Then,*

$$N_G(B) = B.$$

Corollary 3. *We have a bijection:*

$$\begin{aligned} G/B &\longrightarrow \{\textit{Borel Subgroups of } G\} \\ gB &\longmapsto gBg^{-1}. \end{aligned}$$

We induct on $\dim(G)$:
$\dim(G) \leq 1$: B is nilpotent, ergo $G = B$.

$\dim(G) \geq 2$: Let T be a maximal torus in B. Let $x \in N_G(B)$. Then, xTx^{-1} is again a maximal torus in B. Since all maximal tori in B are related via B-conjugation, there is $b \in B$ s.t. $xTx^{-1} = bTb^{-1}$. We therefore replace x by $b^{-1}x$ to achieve $xTx^{-1} = T$.

Now, consider the map $\rho : T \to T, t \mapsto txt^{-1}x^{-1}$.

If ρ is **not surjective**, then we have, since all tori are irreducible, $\dim(\mathsf{Img}(\rho)) < \dim(T)$ and $\dim(\mathsf{Kern}(\rho)^o) > 0$. If we set $S := \mathsf{Kern}(\rho)^o$, then S is a non-trivial torus in T. Since $S \subset \mathsf{Kern}(\rho)$, x centralizes S and normalizes B. Hence, x normalizes $Z_B(S)$. Because of the previous lemma, $Z_B(S)$ is Borel subgroup of $Z_G(S)$. If $Z_G(S) \neq G$, then the induction hypothesis implies $x \in N_{Z_G(S)}(Z_B(S)) = Z_B(S) \subset B$.

Otherwise, if $Z_G(S) = G$, then B/S is a Borel subgroup of G/S. So, the induction hypothesis implies $xS \in N_{G/S}(B/S) = B/S$, ergo $x \in B$.

ρ is **surjective**: Then, $T = \mathsf{Img}\rho \subset [N_G(B), N_G(B)]$. Set $H := N_G(B)$ and choose a finite-dimensional representation $G \longrightarrow \mathsf{GL}(V)$ and a line $L \subset V$ s.t. $H = \{g \in G \mid gL = L\}$.

Then, we have a morphism of algebraic groups $\gamma : H \longrightarrow \mathsf{GL}(L) = \mathcal{G}_m(k)$. Since the right side is a torus, we have $\gamma_{|H_u} \equiv 1$ and $\gamma_{|[H,H]} \equiv 1$. Ergo, $\gamma(T) = 1$ and, since $B = B_u \rtimes T$, $\gamma(B) = 1$.

Choose a non-zero element $v \in L$ and consider $\phi : G/B \to V, gB \mapsto gv.$. Since G/B is a projective variety, while V is an affine variety, ϕ must be constant. Therefore, we have $gv \in L$ for each $g \in G$. Ergo, $G = H$ and B is normal in G. But, now $G = \bigcup_{g \in G} gBg^{-1} = B$. Ergo $H = B$.

Let G be a connected algebraic group with a maximal torus T. Set

$$\mathcal{B}^T := \{B \subset G \text{ Borel} \mid T \subset B\}.$$

Then, $N_G(T)$ acts on $\mathcal{B}^T$ by conjugation.

Example 3. Let $G = \mathsf{GL}_2(k)$ with $T = \left\{\begin{pmatrix} * & \\ & * \end{pmatrix}\right\}$. Then,

$$\mathcal{B}^T = \left\{\begin{pmatrix} * & * \\ & * \end{pmatrix}, \begin{pmatrix} * & \\ * & * \end{pmatrix}\right\}.$$

Lemma 5. *The action of $Z_G(T)$ on $\mathcal{B}^T$ by conjugation is trivial.*
Equivalently (since $B = N_G(B)$), $Z_G(T) \subset B$ for each $B \in B^T$.

Corollary 4. *The action $N_G(T) \curvearrowright \mathcal{B}^T$ induces am action by the Weyl group $W(G,T) = N_G(T)/Z_G(T)$ on $\mathcal{B}^T$.*

Corollary 5. *In the proof, we could see that $N_G(T)$ and $W(G,T)$ act transitively on $\mathcal{B}^T$.*

Corollary 6.

$$\#\mathcal{B}^T \leq \#W < \infty.$$

Proof. We know, that $Z_G(T)$ is connected, since T is a torus. Further, since $T \subset Z_G(T)$ is central and a maximal torus, is must be the unique maximal torus in $Z_G(T)$. We showed before, that this is equivalent to $Z_G(T)$ being nilpotent. Thus, $Z_G(T)$ is contained in some Borel group $B_0 \in \mathcal{B}^T$.

Let $B \in \mathcal{B}^T$ and choose $g \in G$ s.t.

$$B = gB_0g^{-1}.$$

Since maximal tori in B are B-conjugated, we can choose $g \in G$ s.t. $g \in N_G(T)$. (Otherwise, we can replace g bx bg s.t. $bgTg^{-1}b^{-1} = T$.)

One can show that

$$g \in N_G(T) \implies g \in N_G(Z_G(T)).$$

Thus

$$g^{-1}Z_G(T)g = Z_G(T) \subset B_0$$

which implies

$$Z_G(T) \subset gB_0g^{-1} = B.$$

$\square$

Theorem 4 (Borel Subgroups Containing a Given Torus)**.** *W acts **simply-transitively** on $\mathcal{B}^T$, i.e., for each $B_1, B_2 \in \mathcal{B}^T$ there is exactly* one *$g \in G$ s.t.*

$$gB_1g^{-1} = B_2.$$

In particular,

$$\#\mathcal{B}^T = \#W.$$

Corollary 7. *Since $N_B(T) \subset Z_G(T)$ we have for each Borel group B and maximal torus T of B*

$$W(B,T) = 1.$$

In particular,

$$\mathcal{B}^T = \{B\}.$$

Proposition 1. *Let G be a connected non-solvable algebraic group (this implies $\dim G \geq 3$). Let B be a Borel subgroup with a maximal torus T. Then,*

$$\#W(G,T) \geq 2.$$

Moreover,

$$\#W = 2 \iff \dim(G/B) = 1.$$

Proof. Let $B \in \mathcal{B}^T$. We need to show

$$N_G(T) \cap N_G(B) \subset Z_G(T).$$

Note, that

$$N_G(T) \cap N_G(B) = N_G(T) \cap B = N_B(T).$$

Set $U := B_u$, then $B = U \rtimes T$.

Choose $b \in N_B(T)$ with $b = ut$, $u \in U, t \in T$. Then,

$$T = bTb^{-1} = uTu^{-1}.$$

Since $t \in Z_G(T)$, it suffices to show that $u \in Z_G(T)$.

Let $t \in T$ and set $t' = utu^{-1} \in T$. Since, we have an isomorphism

$$T \hookrightarrow B \twoheadrightarrow B/U$$

and B/U is commutative, t and t' must be equal in T. Ergo, $u \in Z_G(T)$. $\square$