# Mitschrieb: Algebraic Groups
# SS 20

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# Vorwort

# Contents

Lecture from 03.03.2020

**Recall:** Last time we introduced the **Zariski-Topology** on $X$.
There, algebraic sets equal closed sets.
We called a set $X$ **irreducible** iff each open subset lies dense in $X$.

**Lemma 1.** *For an algebraic set $X$, the following are equivalent:*

*(1) $X$ is irreducible.*

*(2) $k[X] = k[x_1, \dots, x_n]/I(X)$ is a domain.*

*(3) $I(X)$ is a prime ideal.*

The proof of (2) $\iff$ (3) is a basic algebraic result.

**Lemma 2.** *An open base for the Zariski-Topology on an algebraic set $X$ is given by sets:*

$$D(f) := \{p \in X \mid f(p) \neq 0\}$$

*for each $f \in k[X]$. We call the $D(f)$ **basic open sets**.*

*Proof.* Suppose $U \subseteq X$ is nonempty and open. Set

$$Z := X \setminus U$$

then $Z$ is closed. Thus

$$Z = \{x \in X \mid f(x) = 0 \forall f \in I\}$$

for some ideal $I \subseteq k[X]$. Let $p \in U$, then there is an $f \in Z$ s.t.

$$f(p) \neq 0.$$

Also, $D(f) \cap Z = 0$, thus $p \in D(f) \subseteq U$. □

*Lemma 1.* It is clear that (2) is equivalent to (3).
(1) is equivalent to

$$\forall \text{ nonempty, open } U_1, U_2 \subset X : U_1 \cap U_2 \neq \emptyset$$
$$\overset{\text{Lemma 2}}{\iff} \forall \text{ nonempty, basic open } D(f_1), D(f_2) \subset X : D(f_1) \cap D(f_2) \neq \emptyset$$

Since $D(f_1) \cap D_(f_2) = D(f_1 f_2)$, this is equivalent to the statement

$$f_1, f_2 \in k[X] : f_1, f_2 \neq 0 \implies f_1 f_2 \neq 0.$$

Which states that $k[X]$ is a domain. □

**Lemma 3.** *Let $X$ be an algebraic set. We have bijections*

$$\{\textit{closed subsets } Z \subseteq X\} \leftrightarrow \{ \textit{ radical ideals } I \subset k[X]\}$$

*and*

$$\{\textit{irreducible, closed subsets } Z \subseteq X\} \leftrightarrow \{ \textit{ prime ideals } I \subset k[X]\}$$

*and*

$$\{\textit{points of } X\} \leftrightarrow \{ \textit{ maximum ideals } I \subset k[X]\}.$$

**Lemma 4** (Primary Decompositions, Atiyah, Macdonald Ch. 4)**.** *For an ideal $I$ we call $P \supseteq I$ a **minimal prime** of $I$ if $P$ is a prime ideal and we have for each prime ideal $Q$:*

$$P \supseteq Q \supseteq I \implies P = Q.$$

*Any radical ideal $I$ of $k[x_1, \dots, x_n]$ has only finitely many **minimal** primes $P_1, \dots, P_r$. Inparticular,*

$$I = \bigcap_{i=1}^{n} P_i$$

*and for each $i$*

$$P_j \not\supseteq \bigcap_{j:j\neq i} P_j.$$

**Definition 1.** An **(irreducible) component** $Z$ of $X$ is a maximal irreducible closed subset, i.e., an irreducible closed $Z \subseteq X$ s.t. there does not exist an irreducible closed $Y \subset X$ s.t. $Y \supsetneq Z$.

Then, we have the bijection

$$\{\text{irreducible components of } X\} \leftrightarrow \{ \text{ mimimal primes of } I(X)\}.$$

**Lemma 5.** *Any algebraic set $X$ has finitely many components $Z_1, \dots, Z_r$. We have*

$$X = Z_1 \cup \ldots \cup Z_r$$

*and for each $i$*

$$Z_i \not\subset \bigcup_{j:j\neq i} Z_j.$$

**Example 1.** 1. Let $X = V(x \cdot y) \subset k^2$. Then $X = Z_1 \cup Z_2$ where $Z_1 = V(x), Z_2 = V(y)$.

$X$ is connected, but not irreducible ($D(x)$ does not lie dense in $X$).

2. Let $X$ be a **finite** algebraic set. It is easy to check that every subset of $X$ is closed:
$$\{p\} = V(x_1 - p_1, \ldots, x_n - p_n)$$
for each $p \in X$. Further
$$X = \{p_1\} \cup \ldots \cup \{p_r\}.$$
Moreover: Any function $f : X \to k$ is regular (i.e. given by polynomials).

**Lemma 6.** *We call an element $e \in k[X]$* ***idempotent*** *iff $e^2 = e$.*
*Let $X$ be an algebraic set. Then*
$$\begin{aligned} X \text{ connected} &\iff \text{the only idempotents } e \in k[X] \text{ are 0 and 1} \\ &\iff k[X] \ncong A \times B \text{ for any } k\text{-algebras } A, B. \end{aligned}$$

**Lemma 7.** *Morphisms of algebraic sets are continuous.*

*Proof.* Let $\phi : X \to Y$ be a morphism. It suffices to show that for all closed $Z \subset Y$ that $\phi^{-1}(Z) \subset X$ is closed.

But, if
$$Z = V_Y(S) := \{q \in Y \mid f(q) = 0 \forall f \in S\}$$
for some ideal $S \subset k[Y]$, then
$$\phi^{-1}(Z) = V_X(\phi^*(S)) = \{\phi^*(f) = f \circ \phi \mid f \in S\}.$$
□

**Lemma 8.** *Isomorphisms of algebraic sets are homeorphisms. In particular, any isomorphism of algebraic sets $\phi : X \to X$ permutes the components $Z_1, \ldots, Z_r$ of $X$:*
$$\forall i \exists j : \phi(Z_i) = Z_j.$$

**Theorem 1.** *Let $G$ be an algebraic group.*

*(i) There is a unique component $G^0$ of $G$ with $e \in G^0$.*

*(ii) Every component $Z$ of $G$ is a coset $gG^0$ of $G$ for some $g \in Z$.*

*(iii) $G^0$ is a normal algebraic subgroup of $G$.*

*(iv) $G^0$ is of finite index, i.e.*
$$[G : G^0] = \#\left(G/G^0\right) < \infty.$$

(v) *The irreducible components are also the connected components.*

*Proof.* Let $G = Z_1 \cup \ldots \cup Z_r$ be the decomposition into components. We may assume that $e \in Z_1$.

Recall that $Z_1 \not\subset \bigcup_{j \geq 2} Z_j$. Then, there is an $x \in Z_1 \setminus \bigcup_{j \geq 2} Z_j$. Thus, for all algebraic set isomorphisms $\phi : G \to G$, we have by some previous lemma that $\phi(x)$ is likewise contained in some unique component of $G$. For example, we may take $\phi$ to be

$$\begin{aligned} \phi_g : G &\to G \\ y &\longmapsto gy \end{aligned}$$

for any $g \in G$. Then, for all $g \in G$, the element $gx = \phi_g(x)$ is contained in only one component of $G$. Ergo, each $g \in G$ is contained in exactly one component.

(i) Take $g = e$.

(iii) $G^0$ is an algebraic subset, by construction. Denote by $m : G \times G \to G$ and $i : G \to G$ the continuous multiplication and inversion map on $G$. **Why is $G^0$ a subgroup?** We need to show

$$\begin{aligned} m(G^0 \times G^0) &\subseteq G^0. \\ i(G^0) &\subseteq G^0. \end{aligned}$$

We know that $i(G^0)$ is some component of $G$, since $i$ is an isomorphism. But it contains the identity $e$, since $e^{-1} = e$. Therefore, $i(G^0) = G^0$.

If $g \in G$, then $gG^0$ is some component of $G$. Suppose $g \in G^0$. Then $gG^0 \cap G^0 \supseteq \{g\}$, therefore $gG^0 = G^0$. Ergo, $G^0$ is closed under multiplication.

**Why is $G^0$ a normal?** If $g \in G$, then $gG^0g^{-1}$ is a component that contains $e$, therefore $G^0 = gG^0g^{-1}$.

(Alternative proof that $m(G^0 \times G^0) = G^0$: Consider

- any continuous image of an irreducible set is irreducible.
- the closure of any irreducible set is irreducible.

Ergo $\overline{m(G^0 \times G^0)}$ is a closed irreducible set containing $e$. Ergo, $\overline{m(G^0 \times G^0)} = G^0$).

(ii) Let $Z \subset G$ be a component. Let $g \in Z$. Then $g \in (gG^0 \cap Z)$, so $gG^0 = Z$.

(iv) This follows from some previous lemma.

(v) This is left as a topological exercise. It is true whenever the irreducible components do not intersect.

□

It now follows:

$$\{\text{finite algebraic groups}\} \longleftrightarrow \{finitegroups\}$$

where the above arrow is an equivalence of categories.

**Example 2.**

- Let $G = \{g_1, \ldots, g_r\}$ be a finite algebraic group. Then,

$$G^0 = \{e\}.$$

- Without proofs:

$$G \in \{\mathsf{GL}_n(k), \mathsf{SO}_n(k), \mathsf{SL}_n(k)\} \implies G^0 = G.$$

Further,

$$G = O_n(k) \implies G^0 = \mathsf{SO}_n(k)$$

(but only if $-1 = 1$ i.e. $\mathsf{char}k = 2$. Otherwise $[G : G^0] = 2$.)

## 0.1 Jordan Decomposition

As usual, $k = \overline{k}$ is an algebraically closed field.

**Definition 2.** Let $V$ be a finite-dimensional vector space.

An element $x \in \mathsf{End}(V)$ is **semisimple**, if it is diagonalizable, i.e. it has a basis of eigenvectors, or equivalently, if the minimal polynomial of $x$ is square-free.

Then, there is a decomposition $V = \bigoplus_{i=1}^r V_i$ and distinct elements $\lambda_1, \ldots, \lambda_n \in k$ s.t.

$$x|_{V_i} = \lambda_i.$$

If $\dim(V_i) = n_i$, then

$$\text{char polynomial of } x = \prod_{i=1}^r (T_i - \lambda_i)_i^n \in k[T]$$

and

$$\text{minimal polynomial of } x = \prod_{i=1}^{r}(T_i - \lambda_i) \in k[T].$$

(Where the minimal polynomial of $x$ is defined as the least degree monic $m$ s.t. $m(x) = 0$ and Cayley-Hamilton $m|c$. )

**Definition 3.** $x \in End(V)$ is **nilpotent** if $x^n = 0$ for some $n$. (Equivalent to: characteristic polynomial of $x$ is $T^{\dim(V)}$.)

$x$ is **unipotent**, if $x - 1$ is nilpotent.

**Lemma 9.** *If $x$ is semisimple and nilpotent, then $x = 0$.*

*If $x$ is semisimple and unipotent, then $x = 1$.*

**Lemma 10.** *If $x, y$ are commuting elements, thar are semisimple resp. unipotent or nilpotent, then so is $xy$.*

**Theorem 2** (Goal)**.** *For all algebraic groups $G$ and for all $g \in G$, there exist unique group elements $g_s, g_u \in G$ s.t.*

$$g = g_s g_u = g_u g_s$$

*and for all finite-dimensional representations $\rho : G \to \mathsf{GL}(V)$, $\rho(g_s)$ is semisimple and $\rho(g_u)$ is unipotent.*

**Example 3.** If $g = \begin{pmatrix} \lambda & 1 & 0 \\ & \lambda & 1 \\ & & \lambda \end{pmatrix} \in G = \mathsf{GL}_3(k)$, then $g_s = \begin{pmatrix} \lambda & 0 & 0 \\ & \lambda & 0 \\ & & \lambda \end{pmatrix}$, $g_u = \begin{pmatrix} 1 & \lambda^{-1} & 0 \\ & 1 & \lambda^{-1} \\ & & 1 \end{pmatrix}$.

Lecture from 09.03.2020

**Theorem 3** (Goal Theorem). *Let $G$ an algebraic group. For all $g \in G$ there is exactly one pair $g_s, g_u \in G$ s.t.*

$$g = g_s g_u = g_u g_s$$

*and for all finite-dimensional representations $r : G \to \mathsf{GL}_n(V)$, the element $r(g_s)$ resp. $r(g_u)$ is semisimple resp. unipotent.*

Last time, we saw:

**Lemma 11.**
- *If $g, h$ are commuting and semisimple resp. commuting and unipotent then so is $gh$.*
- *If $g$ is semisimple and unipotent, then $g = 1$.*

**Proposition 1.** *Let $V$ be a finite-dimensional vector space and $g \in \mathsf{GL}(V)$. There exist unique elements $g_s, g_u \in \mathsf{GL}(V)$ s.t.*

$$g = g_s g_u = g_u g_s$$

*and $g_s$ is semisimple and $g_u$ is unipotent.*

*Moreover, $g_s, g_u \in k[g] = \{\sum_{i=1}^m a_i g^i \mid a_i \in k\} \subseteq \mathsf{End}(V)$.*

*Proof.* **Existence** (Sketch): Say

$$g = \begin{pmatrix} \lambda & 1 & 0 \\ & \lambda & 1 \\ & & \lambda \end{pmatrix}$$

then take

$$g_s = \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{pmatrix}, \quad g_u = \begin{pmatrix} 1 & \lambda^{-1} & 0 \\ & 1 & \lambda^{-1} \\ & & 1 \end{pmatrix}.$$

For $\lambda \in k$, define the **generalized $\lambda$-eigenspace** of $g$ by

$$V_\lambda := \{v \in V \mid \exists n \in \mathbb{N}_0 : \ (g - \lambda)^n v = 0\}.$$

Then

$$V = \bigoplus_{\lambda \in k} V_\lambda.$$

Here $V_\lambda$ = sum of domains of all Jordan blocks with $\lambda$s on the diagonal. (It follows from the Jordan Decomposition for matrices that such a decomposition exist.)

Let's define $g_s \in \mathsf{GL}(V)$ by

$$g_s|_{V_\lambda} = \lambda \cdot \mathrm{Id}.$$

Note that $gV_\lambda \subset V_\lambda$, hence $g$ commutes with $g_s$, hence $g, g_s$ commutes with $g_u := gg_s^{-1}$. Then, $g = g_s g_u = g_u g_s$.

Write $\det(T-g) = \prod_\lambda (T-\lambda)^{n(\lambda)}$, $n(\lambda) = \dim(V_\lambda)$. Since the polynomials $T-\lambda$ for $\lambda \in k$ are coprime, the chinese remainder theorem implies that there is a $Q \in k[T]$ s.t.

$$Q \equiv \lambda \mod (T-\lambda)^{n(\lambda)}$$

for each $\lambda \in k$.

We claim that

$$Q(g) = g_s.$$

Indeed, since $gV_\lambda \subseteq V_\lambda$, we have

$$Q(g)V_\lambda \subseteq V_\lambda.$$

So, it suffices to show for all $v \in V_\lambda$

$$Q(g)v = g_s v = \lambda v.$$

Note that, by Cayley-Hamilton,

$$V_\lambda = \left\{ v \in V \mid (g-\lambda)^{n(\lambda)} v = 0. \right\}$$

Write

$$Q = \lambda + R \cdot (T-\lambda)^{n(\lambda)}$$

for some $R \in k[T]$. Since $(g-\lambda)^{n(\lambda)}v = 0$, deduce that $Q(g)v = \lambda v$, as required.

If $Q' \equiv T(T-\lambda)^{n(\lambda)}$, then

$$Q'(g) = g_u.$$

If $Q'' \equiv \lambda^{-1}(T-\lambda)^{n(\lambda)}$, then $Q''(g) = g_s^{-1}$. check corresponding stuff for $g_u$.

**Uniqueness**: Suppose given some other decomposition

$$g = g_s' g_u' = g_u' g_s'$$

with $g_s'$ semisimple and $g_u'$ unipotent. Then $g_s'$ commutes with $g_s'$ and $g_u'$, hence with $g$, hence also with any element in $k[g]$. Ergo, $g_s'$ commutes with $g_s$ and $g_u$. Similarly, $g_u'$ commutes with $g_s$ and $g_u$.

Consider

$$h := g'_s g_s^{-1} = g'_s g'_u (g'_u)^{-1} g_s^{-1} = g(g'_u)^{-1} g_s^{-1} = g_u (g'_u)^{-1}.$$

Then $h = g'_s g_s^{-1}$ is a product of semisimple elements and $h = g_u(g'_u)^{-1}$ is a product of unipotent elements. By proceeding lemmas, $h$ is semisimple and unipotent, ergo trivial. It follows $g'_s = g_s$ and $g'_u = g_u$. □

**Corollary 1.** *Let $g \in \mathsf{GL}(V)$, let $W \subset V$ be any $g$-**invariant subspace**, i.e. $gW \subseteq W$.*

*Then, $W$ is $g_s$-invariant and $g_u$-invariant.*

*Proof.* This is clear, since $g_s$ and $g_u$ are algebraically generated by $g$ over $g$. □

**Lemma 12.** *Let $\phi : V \to W$ be a linear map between finite-dimensional vector spaces.*

*Let $\alpha \in \mathsf{GL}(W)$ and $\beta \in \mathsf{GL}(W)$ s.t.*

$$\begin{array}{ccc} V & \xrightarrow{\alpha} & V \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} \\ W & \xrightarrow{\beta} & W, \end{array}$$

*i.e. $\phi \circ \alpha = \beta \circ \phi$.*

*Then,*

$$\phi \circ \alpha_s = \beta_s \circ \phi,$$
$$\phi \circ \alpha_u = \beta_u \circ \phi.$$

*Proof.* Write $V = \bigoplus_{\lambda \in k} V_\lambda$, $W = \bigoplus_{\lambda \in k} W_\lambda$ where $V_\lambda$ are the generalized $\alpha$-eigenspaces and $W_\lambda$ are the generalized $\beta$-eigenspaces.

We claim that

$$\phi(V_\lambda) \subset W_\lambda.$$

Indeed, let $v \in V_\lambda$, then

$$(\beta - \lambda)^n \phi(v) = \phi((\alpha - \lambda)^n v) = 0.$$

Since $(\alpha - \lambda)^n v = 0$, the claim follows.

Since, $\alpha_s|_{V_\lambda} = \lambda \mathrm{Id}$ and $\beta_s|_{W_\lambda} = \lambda \mathrm{Id}$, deduce that

$$\phi \circ \alpha_s = \beta_s \circ \phi.$$

Indeed, both sides are given on $V_\lambda$ by $\lambda \cdot \phi$. Thus

$$\begin{aligned}\phi \circ \alpha_u &= \phi \circ \alpha\alpha_s^{-1} \\ &= \beta\beta_s^{-1} \circ \phi \\ &= \beta_u \circ \phi.\end{aligned}$$

□

**Lemma 13.** *Let $\alpha \in GL(V)$, $\beta \in GL(W)$. Then the **tensor** $\alpha \otimes \beta \in GL(V \otimes W)$ is defined by*

$$(\alpha \otimes \beta)(u \otimes v) = \alpha(u) \otimes \beta(v).$$

*Then, we have*

$$(\alpha \otimes \beta)_s \overset{(1)}{=} \alpha_s \otimes \beta_s$$

$$(\alpha \otimes \beta)_u \overset{(2)}{=} \alpha_u \otimes \beta_u.$$

*Proof.* It suffices to prove (1), since

$$\begin{aligned}(\alpha \otimes \beta)_u &= (\alpha \otimes \beta) \circ (\alpha \otimes \beta)_s^{-1} \\ &\overset{(1)}{=} (\alpha \otimes \beta) \circ (\alpha_s \otimes \beta_s)^{-1} \\ &= \alpha\alpha_s^{-1} \otimes \beta\beta_s^{-1} \\ &= \alpha_u^{-1} \otimes \beta_u^{-1}\end{aligned}$$

For (1), consider

$$V = \bigoplus_{\lambda \in k} V_\lambda,$$

$$W = \bigoplus_{\lambda \in k} W_\lambda.$$

It follows

$$V \otimes W = \bigoplus_{\lambda, \mu \in k} V_\lambda \otimes W_\mu.$$

Now,

$$\alpha_s \otimes \beta_s|_{V_\lambda \otimes W_\mu} = \lambda\mu \cdot \mathrm{Id}.$$

Ergo, $\alpha_s \otimes \beta_s$ is semisimple. By Proposition, we reduce to checking that $\alpha_u \otimes \beta_u$ is unipotent. Indeed,

$$\alpha_u \otimes \beta_u - 1 = (\alpha_u - 1) \otimes (\beta_u - 1) + 1 \otimes (\beta_u - 1) + (\alpha_u - 1) \otimes 1$$

is nilpotent (You can also check that $\alpha_u \otimes \beta_u = (\alpha_u \otimes 1) \circ (1 \otimes \beta_u)$ is unipotent.) □

**Example 4.** Let $1 \in \mathsf{GL}(V)$. Then $1_s = 1$ and $1_u = 1$.

**Summary** : Let $G$ be an algebraic group. Let $r_V : G \to \mathsf{GL}(V)$ be a finite-dimensional representation. Also, fix $g \in G$.

Let $\lambda_V := r_V(g)_s$ (or $r_V(g)_u$).

We get a family of operators $\lambda_V \in \mathsf{End}(V)$ with the following properties:

(i) if $V = k$ and $r_V(g') = 1$ for all $g' \in G$, then $\lambda_V = 1$.

(ii) for any two representations in $V$ and $W$, we have

$$\lambda_{V\otimes W} = \lambda_V \otimes \lambda_W.$$

(iii) for all $G$-equivariant $\phi : V \to W$ we have

$$\phi \circ \lambda_V = \lambda_W \circ \phi.$$

**Theorem 4.** *Let $G$ be an algebraic group. Let $\lambda_V \in \mathsf{End}(V)$ (i.e. $V = (r_V, V)$ is a finite-dim. representation of $G$) be a family of operations satisfying (i), (ii), (iii).*

*Then, there is exactly one $g \in G$ s.t. $\lambda_V = r_V(g)$ for all $V$.*

Note, that this theorem implies our goal theorem.

Applying the theorem to $\lambda_V = r_V(g_s)$ implies

$$\exists g_s \in G : r_V(g_s) = r_V(g)_s$$

and

$$\exists g_u \in G : r_V(g_u) = r_V(g)_u.$$

*Proof of Goal Theorem.* There exist unique $g_s, g_u \in G$ s.t.

$$g \overset{(*)}{=} g_u g_s = g_s g_u,$$

Then, $r_V(g) = r_V(g_s)r_V(g_u) = r_V(g_u)r_V(g_s)$.

Since $r_V(g_u)$ is unipotent and $r_V(g_s)$ is semisimple, it follows $r_V(g_u) = r_V(g)_u$ and $r_V(g_s) = r_V(g)_s$.

To deduce $(*)$, take any $r_V : G \hookrightarrow \mathsf{GL}(V)$. We know for each $V$

$$r_V(g) = r_V(g_s)r_V(g_u) = r_V(g_u)r_V(g_s).$$

$\square$

*Proof of Theorem.* We first extend the assignment

$$V \mapsto \lambda_V$$

to all representations of $G$.

Say $V = \bigcup_j W_j$ where each $W_j$ is a finite-dimensional $G$-invariant subspace. Try to define $\lambda_V \in \mathsf{End}(V)$ by

$$\lambda_V|_{W_j} := \lambda_{W_j}.$$

For this to be well-defined, we need to show for each $i, j$

$$\lambda_{W_i}|_{W_i \cap W_j} \overset{(*)}{=} \lambda_{W_j}|_{W_i \cap W_j}.$$

**Proof of** $(*)$**:** Apply assumption (iii) to the $G$-equivariant linear maps

$$W_i \cap W_j \overset{\phi}{\hookrightarrow} W_i,$$
$$W_i \cap W_j \overset{\phi'}{\hookrightarrow} W_j.$$

Then,

$$\begin{aligned} \lambda_{W_i}|_{W_i \cap W_j} &= \lambda_{W_i} \circ \phi \\ &\overset{(iii)}{=} \phi \circ \lambda_{W_i \cap W_j} \\ &= \phi' \circ \lambda_{W_i \cap W_j} \end{aligned}$$

and

$$\lambda_{W_j}|_{W_i \cap W_j} = \lambda_{W_i} \circ \phi' = \phi' \circ \lambda_{W_i \cap W_j}.$$

Recall here that any finite-dimensional $G$-invariant $W \subset V$ is a representation. $\square$

[0]Not necessarily finite-dimensional, but may be written as a filtered union of finite-dimensional $G$-invariant subspaces of $W$.

Lecture from 11.03.2020

Let $G$ be an algebraic group.

**Easy Exercise** : If $V_1, V_2$ are representations $r_1, r_2$ of $G$, then $V_1 \otimes V_2$ is also a representation with

$$r = r_1 \otimes r_2 : G \to \mathsf{GL}(V_1 \otimes V_2)$$

given by

$$r(g)(v_1 \otimes v_2) = (r_1(g)v_1) \otimes (r_2(g)v_2).$$

*Proof.* Given $\Delta_j : V_j \to V_j \otimes k[G]$, define

$$\Delta : V_1 \otimes V_2 \longrightarrow V_1 \otimes V_2 \otimes k[G]$$

by: if

$$\Delta_1 u = \sum u_i \otimes f_i, \quad \Delta_2 v = \sum v_j \otimes h_j,$$

then

$$\Delta(u \otimes v) \sum \sum u_i \otimes v_j \otimes f_i h_j.$$

Set $A := k[G]$, then

$$r_A := \text{right regular representation with } r_A(g)f(x) = f(xg).$$

The map

$$\begin{aligned} A \otimes A &\xrightarrow{m} A \\ f_1 \otimes f_2 &\longmapsto f_1 f_2 \end{aligned}$$

defines a morphism of representations

$$(A, r_A) \otimes (A, r_A) \to (A, r_A).$$

Indeed,

$$\begin{aligned} m((r_A \otimes r_A)(g)(f_1 \otimes f_2))(x) &= f_1(xg)f_2(xg), \\ &= f_1 f_2(xg) = r_A(g)(m(f_1 \otimes f_2))(x), \end{aligned}$$

since $f_1(\_g) \otimes f_2(\_g) = (r_A \otimes r_A)(g)(f_1 \otimes f_2)$.

Ergo $m \circ (r_A \otimes r_A)(g) = r_A(g) \circ m$. $\square$

Recall: We stated the following theorem

**Theorem 5.** *Let $\lambda_V \in \mathsf{End}(V)$ be given s.t. for all finite-dim. rep.s $V$ of $G$ s.t.:*

*(i)* $\lambda_k = 1$

*(ii)* $\lambda_{V\otimes W} = \lambda_V \otimes \lambda_W$

*(iii) for all morphisms of rep.s $\phi : V \to W$ we have*

$$\phi \circ \lambda_V = \lambda_W \circ \phi.$$

*Then, there is exactly one $g \in G$ s.t. $\lambda_V = r_V(g)$ for all $V$.*

*Proof.* Last time, we saw that any such family $V \mapsto \lambda_V$ extends to **all** rep.s $V$ of $G$.

Let's note also that, if $(V_0, r_0)$ is any representation of $G$ with trivial action, i.e. $r(g) = 1$ for all $g$, then $\lambda_{V_0} = 1$. Indeed, let $v \in V_0$. We must check that $\lambda_{V_0} v = v$. Since the action is trivial, any subsapce of $V_0$ is $G$-invariant.

Consider the map

$$\begin{aligned} \phi : k &\longrightarrow V_0 \\ \alpha &\longmapsto \alpha v \end{aligned}$$

where $v = \phi(1)$. Then, $\phi$ is a morphism of rep.s because the action is trivial.

Thus,

$$\lambda_V v = (\lambda_V \circ \phi)(1) \overset{(iii)}{=} (\phi \circ \lambda_k)(1) \overset{(i)}{=} \phi(1) = v.$$

Consider $\lambda_A \in \mathsf{End}(A)$. Then,

$$\lambda_{A\otimes A} = \lambda_A \otimes \lambda_A.$$

It is an easy exercise to see that $m : (A, r_A) \otimes (A, r_A) \to (A, r_A)$ is a morphism of rep.s.

By (iii) it follows, $m \circ (\lambda_A \otimes \lambda_A) = \lambda_A \circ m$, i.e.

$$\lambda_A(f_1 f_2) = \lambda_A(f_1)\lambda_A(f_2)$$

for all $f_1, f_2 \in A$. Thus, $\lambda_A$ is an algebra morhism (check, using the morphism $k \hookrightarrow A$, that $\lambda_A(1) = 1$).

Thus, $\lambda_A = \phi^*$ for some unique morphism $\phi$ of algebraic sets $\phi : G \to G$.

We claim that $\phi$ commutes left multiplication i.e.

$$\phi(hx) = h\phi(x)$$

for all $h, x \in G$. Indeed, let's consider the map

$$\begin{aligned} A &\longrightarrow A \\ f &\longmapsto f(h \cdot \_). \end{aligned}$$

This induces a morphism

$$(A, r_A) \xrightarrow{\psi} (A, r_A).$$

By (ii), $\psi \circ \lambda_A = \lambda_A \circ \psi$.

Since $\lambda_A = \phi^*$, this implies the claim.

Now, set $g := \phi(e)$. Then for all $h \in G$,

$$\phi(h) = \phi(he) = hg.$$

Thus, $\lambda_A = \phi^* = r_A(g)$.

(It remains to show that

$$\lambda_V = r_V(g)$$

for each finite-dim. rep. $V$.)

Let $V = (V, r)$ be any rep. This induces a map

$$\Delta : V \longrightarrow V \otimes A.$$

If $\Delta v = \sum v_i \otimes f_i$, then

$$hv = \sum f_i(h_i) \otimes v_i.$$

Let

$$\begin{aligned} \varepsilon : V \otimes A &\longrightarrow V \\ v \otimes f &\longmapsto f(1)v. \end{aligned}$$

It follows $\varepsilon \circ \Delta : V \to V$ is the identity map.

Let $(V_0, r_0)$ be the representation of $G$ with $V_0 := V$ and $r_0$ the trivial action. Then, $\Delta : V \to V_0 \otimes A$ is a morphism of representations.

(Indeed, if $\Delta v = \sum v_i \otimes f_i$, then

$$\Delta(r(h)v) \overset{?}{=} (r_0(h) \otimes r_A(h))\Delta v$$

since

$$\begin{aligned} \Delta v &= \sum v_i \otimes f_i \\ \iff xv &= \sum f_i(x_i) v_i \ \forall x \in G \\ \iff xhv &= \sum f_i(xh) v_i \ \forall x, h \in G. \end{aligned}$$

Since $r(h)v = hv$, it follows

$$\Delta(hv) = \sum v_i \otimes f_i(\cdot h) \implies (?).)$$

We want to show

$$\lambda_V = r_V(g).$$

We have

$$\begin{aligned} \Delta \circ \lambda_V &\overset{(iii)}{=} \lambda_{V_0 \otimes A} \circ \Delta \\ &\overset{(ii)}{=} \lambda_{V_0} \otimes \lambda_A \\ &= 1 \otimes \lambda_A = 1 \otimes r_A(g). \end{aligned}$$

This implies

$$\Delta \circ \lambda_V = (1 \otimes r_A(g)) \circ \Delta$$

but also

$$\Delta \circ r_V(g) = (1 \otimes r_A(g)) \circ \Delta.$$

Because of the injectivity of $\Delta$ it now follows

$$\lambda_V = r_V(g).$$

□

**Corollary 2.** *Let $\phi : G \to H$ be any morphism of algebraic groups. Then, for all $g \in G$*

$$\begin{aligned} \phi(g)_s &= \phi(g_s) \\ \phi(g)_u &= \phi(g_u). \end{aligned}$$

*Proof.* Let $V$ be any **faithful** representation of $H$, i.e. $r_V : H \to \mathsf{GL}(V)$ is injective, (for a finite-dim. $V$).

Then, $r_V \circ \phi$ is a rep. of $G$. To prove (i), it suffices to show

$$r_V(\phi(g)_s) = r_V(\phi(g_s))$$

since $H$ operates faithfully on $V$.

We know that

$$r_V(\phi(g)_s) = r_V(\phi(g))_s$$

(characterizing property of $h_s$ for $h \in H$). On the other hand,

$$r_V(\phi(g_s)) = (r_V \circ \phi)(g_s) = r_V(\phi(g))_s.$$

Therefore, claim (i) follows. (ii) works analogously. □

**Definition 4.** Let $g \in G$ where $G$ is an algebraic group. We call $g$ **semisimple**, if $g = g_s$.

We call $g$ **unipotent**, if $g = g_u$.

**Lemma 14.** *For $g \in G$, the following are equivalent:*

*(i) $g$ is semisimple.*

*(ii) $r_V(g)$ is semisimple for all finite-dim. rep. $V$.*

*(iii) $r_V(g)$ is semisimple for at least one faithful f.d. rep. $V$ of $G$.*

We get an analogous lemma for unipotent group elements.

*Proof.* We have

$$\begin{aligned} (i) &\iff g = g_s \\ &\overset{\text{Def. of } g_s \text{ by goal thm.}}{\iff} r_V(g) = r_V(g)_s \forall \text{ f.d. } V \\ &\iff r_V(g) \text{ is semisimple} \\ &\iff (ii) \implies (iii). \end{aligned}$$

On the other hand,

$$(iii) \implies \exists \text{ faithful f.d. } V \text{ s.t. } r_V(g) = r_V(g)_s = r_V(g_s) \implies g = g_s.$$

□

## 0.2 Non-Commutative Algebra

**Definition 5.** A **ring** $R$ (for now) is unital, associative but not necessarily commutative.

**Example 5.** The ring of matrices over some field or ring.

**Definition 6.** A **left ideal** $I \subset R$ is a subset that is an abelian subgroup of $(R, +)$ s.t.$ra \in I$ for all $r \in R, a \in I$.

A **right ideal** $I \subset R$ is a subset that is an abelian subgroup with

$$IR \subset I.$$

A two-sided ideal $I$ is a subset that is a left and a right ideal of $R$.

It is easy to check that for any homomorphism of rings $\phi : R \to S$, $\mathsf{Kern}\phi$ is a two-sided ideal. Also, if $J \subset R$ is any two-sided ideal, then there exists a unique ring structure on $R/J$ s.t. the projection $R \to R/J$ is a ring homomorphism.

**Definition 7.** A **left module** $M$ for $R$ is an abelian group equipped with a ring homomorphism

$$R \xrightarrow{\alpha} \mathsf{End}(M)$$

where $\mathsf{End}(M)$ acts on the left of $M$. We write

$$rm := \alpha(r)m.$$

We have

$$(r_1 r_2)(m) = r_1(r_2(m)).$$

If $R$ **acts** on $M$ by the right, we write

$$R \curvearrowright M.$$

**Example 6.** $M_n(k) \curvearrowright k^n$ where $k^n$ is the space of column vectors.

If $k^n$ denotes the space of row vectors, we have $k^n \curvearrowleft M_n(k)$.

**Definition 8.** A **(left) submodule** $N \subset M$ is an algebraic subgroup s.t.

$$RN \subset N.$$

It follows that $N$ is itself is a left module.

**Definition 9.** A (left) module $M$ of $R$ is **simple** (or irreducible) if it has exactly the two submodules: $0 = \{0\}$ and $M$.

**Definition 10.** A ring $R$ is a **division ring** if it satisfies any of the following equivalent requirements:

(i) $R^\times = R \setminus \{0\}$ where $R^\times = \{r \in R \mid \exists a, b \in R : ar = rb = 1.\}$

(ii) $R$ has no nontrivials left or right ideals.

**Definition 11.** If $R \curvearrowright M$, then we can define

$$\mathsf{End}_R(M) := \{\phi \in \mathsf{End}(M) \mid \phi(rm) = r\phi(m)\ \forall r \in R, m \in M\}.$$

Note, that $\mathsf{End}_R(M)$ is a ring.

**Lemma 15** (Schur's Lemma)**.** *If $M$ is simple, then $\mathsf{End}_R(M)$ is a division ring.*

**Lemma 16.** *Let $k$ be a field. Then, $M_n(k)$ has no nontrivial twosided ideals.*

**Theorem 6** (Jacobson Density Theorem (Double Commutant Theorem))**.** *Suppose $M$ is a simple left module which is finitely generated as a right $D$-module for $D = \mathsf{End}_R(M)$.*

*Assume that $R$ acts faithfully on $M$, i.e. $R \to \mathsf{End}_R(M)$ is injective.*

*Then, the map $R \to \mathsf{End}_D(M)$ is an isomorphism.*

[0]If $ar = rb = 1$, then $a = arb = b$.

Lecture from 16.03.2020 (Corona-Madness started here...)

**Recap:**

– Basics: definitions, Hopf-algebras, ...

– Jordan decomposition

– Primer on non-commutative algebra

- Jacobson density theorem

– Unipotent groups

– Tori

### 0.2.1 Jacobson Density Theorem

We had last week

$$\mathsf{End}_D(M) := \{\phi \in \mathsf{End}(M) \mid \phi \circ d = d \circ \phi \forall d \in D\}.$$

Let $k$ be an algebraically cloesd field, $V$ a non-trivial finite-dimensional $k$-vector space and let $G$ be a subgroup of $\mathsf{GL}(V)$ that acts **irreducibly** on $V$, i.e., $V$ is $G$**-irreducible**, i.e., the only $G$-invariant subspaces of $V$ are 0 and $V$.

Set

$$D := \{d \in \mathsf{End}_k(V) \mid dg = gd \forall g \in G\} = \operatorname{span}(G) = \left\{ \sum_{i=1}^{n} c_i g_i \mid c_i \in k, g_i \in G, n \in \mathbb{N}_0 \right\}.$$

Then,

$$D = \mathsf{End}_R(V)$$

where $R$ is the $k$-subalgebra of $\mathsf{End}(V)$ that is generated by $G$.

**Lemma 17** (Schur's Lemma)**.** *We understand* $k \overset{End}{\hookrightarrow} (V)$ *as the inclusion of operations which operate by scalar multiplication*

$$k \xrightarrow{\cong} \{\phi : V \to V \mid \phi : v \mapsto t \cdot v \quad \textit{for some } t \in k\}.$$

*Then, we have*

$$D \cong k.$$

*Proof.* Let $d \in D$. Since $V \neq 0$, there is an eigenspace $V_\lambda \neq 0$ for $d$. Observe that $V_\lambda$ has to be $G$-invariant:
if $g \in G$ and $v \in V_\lambda$, then $gv \in V_\lambda$, since

$$dgv = gdv = g(\lambda v) = \lambda gv.$$

Since $V_\lambda$ is a non-trivial $G$-invariant subspace and $V$ is irreducible under $G$, we have

$$V_\lambda = V.$$

Ergo $d = \lambda$ in the sense of $k \hookrightarrow \mathsf{End}(V)$. □

**Consequence of the Jacobson Density Theorem:** $R = \mathsf{End}_k(V)$, i.e., $G$ generates all linear operations on $V$, if $V$ is $G$-irreducible.
We will prove this after a lemma.

**Lemma 18.** *Let $n \in \mathbb{N}$. Set*

$$V^n := V \oplus V \oplus \ldots \oplus V = V_1 \oplus \ldots \oplus V_n$$

*where each $V_i = V$.*
*Let $v = (v_1, \ldots, v_n) \in V^n$ and set*

$$Rv := \{(rv_1, \ldots, rv_n) \mid r \in R\} = \operatorname{span}\{(gv_1, \ldots, gv_n) \mid g \in G\}.$$

*Then, $Rv \neq V^n$ iff the $v_j$ are linearly dependent over $k$.*

**Consequence:** Take $n := \dim(V)$. Let $\{e_1, \ldots, e_n\}$ be a basisi of $V$ and set

$$e := (e_1, \ldots, e_n) \in V^n.$$

Since the $(e_i)_i$ are linearly independent, the lemma states that $Re = V^n$.
Now, let $x \in \mathsf{End}_k(V)$. Choose $r \in R$ s.t.

$$re = (xe_1, \ldots, xe_n).$$

Then $re_i = xe_i$ for all $i$, thus $x = r$. Hence, $R = \mathsf{End}_k(V)$.

*Proof.* Choose $J \in \{1, \ldots, n\}$ as large as possible with

$$Rv + V_1 + V_2 + \ldots + V_{J-1} =: U \neq V^n$$

. Such an $J$ does exist, since we know that $Rv \neq V^n$.

Then, $V_J \nsubseteq U$, otherwise we may increase $J$. Also, $U$ is invariant by the diagonal action of $G$ on $V^n$. Thus, $V_J \cap U \subsetneq V_J$ is a proper $G$-invariant subspace of the $G$-irreducible $V_J \cong V$. Therefore, $V_J \cap U = 0$.

On the other hand, by maximality of $J$, we have

$$U \oplus V_J = V^n.$$

Ergo, the map (composition)

$$V \cong V_J \hookrightarrow V^n \twoheadrightarrow V^n/U$$

is a $G$-equivariant isomorphism, since $U$ is $G$-invariant.

Let $z : V^n/U \xrightarrow{\cong} V$ be the inverse isomorphism. Let $l$ be the $G$-equivariant map given by

$$\begin{tikzcd} V^n \arrow[r, "l"] \arrow[d] & V \\ V^n/U \arrow[ur, "z"] & \end{tikzcd}$$

and let $l_j$ be the $G$-equivariant maps by restricting $l$ on $V_j$. Then $l_j \in D \cong k$.

Say $l_j = t_j \in k$. Then,

$$l(w) = t_1 w_1 + \dots t_n w_n.$$

Since $z$ is an isomorphism, $l$ is nonzero and $(t_1, \dots, t_n) \neq (0, \dots, 0)$.

Since $l|_U = 0$, we can deduce for all $u \in U$

$$t_1 u_1 + \dots + t_n u_n = 0.$$

But $v \in Rv \subseteq U$, so we may conclude – as required – that the $(v_i)_i$ are linearly dependent ($l(v) = 0$). $\square$

### 0.2.2 Unipotent Groups

Let $G$ be a subgroup of $\mathsf{GL}(V)$ where $V$ is a finite-dimensional vector space and $k$ an algebrically closed field.

**Definition 12.** We say that $G$ is **unipotent** if one of the following equivalent conditions hold:

- each $g \in G$ is unipotent (i.e. $(g-1)^n = 0$ for some $n \in \mathbb{N}$).
- all eigenvalues of $g$ are 1.
- $g$ is conjugate to $\begin{pmatrix} 1 & \star & \star \\ 0 & \ddots & \star \\ 0 & 0 & 1 \end{pmatrix}$.

**Theorem 7.** *Any unipotent subgroup of $\mathsf{GL}_n(k)$ is conjugate to a subgroup of*

$$U_n := \left\{ \begin{pmatrix} 1 & \star & \star \\ 0 & \ddots & \star \\ 0 & 0 & 1 \end{pmatrix} \right\} = \left\{ U \in M_n(k) \mid U_{i,j} = \begin{matrix} 0, & \text{if } i > j \\ 1, & \text{if } i = j \\ \text{arbitrary}, & \text{otherwise.} \end{matrix} \right\}.$$

**Definition 13.** For two subgroups $G, H$ of some common supergroup, define their **commutator** by

$$[G,H] := \left\langle ghg^{-1}h^{-1} \mid g \in G, h \in H \right\rangle.$$

A group $G$ is called **nilpotent**, if one of its commutators is trivial, i.e. if we set

$$G_0 := G \text{ and } G_{i+1} := [G_i, G],$$

then $G$ is called nilpotent iff there is an $j \in \mathbb{N}$ with $G_j = 1$.

**Corollary 3.** *Any unipotent subgroup of $\mathsf{GL}(V)$ is nilpotent.*

**Definition 14.** A group $G$ is called **solvable**, if $G^{(n)} = 1$ for some $n$ where

$$G^{(0)} := G,$$
$$G^{(i+1)} := [G^{(i)}, G^{(i)}].$$

**Notation 1.** In the following, we will write $G' := [G,G]$.

**Definition 15.** Let $n := \dim(V)$. A **complete flag** is a maximal strictly increasing chain of subspaces

$$0 = V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_n = V.$$

Any complete flag is of the form

$$V_j := \mathrm{span}\{e_1, \dots, e_j\}$$

for some basis $e_1, \dots, e_n$ of V.

Let $B$ be the basis of some flag $0 = V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_n = V$. For $x \in \mathsf{End}(V)$, we have that $x$ is upper-triangle with respect to $B$ iff $x$ leaves each member $V_i$ of the flag invariant, i.e. $xV_i \subseteq V_i$.

**Proposition 2** (Key Proposition)**.** *Let $G$ be a unipotent subgroup of $\mathsf{GL}(V)$. Then there is a complete flag $V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_n$ consisting of $G$-invariant subspaces, i.e., each $V_i$ is $G$-invariant.*

*Proof.* Recall, that $G$ is a unipotent subgroup of $\mathsf{GL}_n(V)$. We will give an induction on $n = \dim V$.

If $n = 0$, there is nothing to show.

Let $n \geq 1$. We may assume that $V$ is $G$-irreducible. Because, if not, there is a $G$-invariant subspace $0 \neq W \subset V$ s.t. $W$ and $V/W$ have dimension $< n$. Then there exist complete $G$-invariant flags in $W$ and $V/W$ and the claim – that there is a complete $G$-invariant flag in $V$ – follows by the induction hypothesis.

By Jacobson Density Theorem, we have

$$R := \mathrm{span}(G) = \mathsf{End}(V) := \mathsf{End}_k(V).$$

Since $G$ is unipotent, we have for each $g \in G$

$$\mathrm{trace}(g) = n.$$

Ergo, for $g, h \in G$

$$\mathrm{trace}(gh) = \mathrm{trace}(h)$$

and

$$\mathrm{trace}((g-1)h) = \mathrm{trace}(gh) - \mathrm{trace}(h) = 0.$$

Since $\mathrm{span}(G) = \mathsf{End}(V)$, it now in particularly follows for all $g \in G, \phi \in \mathsf{End}(V)$

$$\mathrm{trace}((g-1)\phi) = 0.$$

Since the above holds for all $\phi in \mathsf{End}(V)$, it must hold

$$g - 1 = 0$$

for all $g \in G$ (take for example the elementary matrices $\phi = E_{i,j}$). Ergo, $G$ is trivial. Then, any complete flag is trivially $G$-invariant. $\square$

*Remark* 1. This gives the group analogue of Engel's Theorem.

*Proof Goal Theorem.* Let $B$ be a basis of $V$ s.t. $G$ leaves each subspace in the corresponding flag invariant. Then, $G$ is upper-triangle with respect to this basis.

On the other hand, each $g \in G$ us unipotent, hence its diagonal (i.e. eigenvalues) are all 1. Thus, with respect to $B$

$$G \subseteq \left\{ \begin{pmatrix} 1 & \star & \star \\ 0 & \ddots & \star \\ 0 & 0 & 1 \end{pmatrix} \right\} = U_n.$$

□

*Remark* 2. Tori are of the form $(k^\times)^n$. In the case $k = \mathbb{C}$, $(\mathbb{C}^\times)^n$ are the complexification of $U(1)^n$. This equals tori in top. sense.

$$\begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix} \subseteq \mathsf{GL}_2(\mathbb{C})$$

is a non-algebraic unipotent group.

**Exercise.** (to be discussed next time)

it would have sufficed to prove the Goal theorem in the special case that $G$ is algebraic.

**Corollary of Proof:** If $G \subset \mathsf{GL}(V)$ (with $V \neq 0$) is unipotent and acts irreducibly (?), then $G = 1$, $\dim V = 1$.

Lecture from 18.03.2020

**Answert to last Exercise:** Recall that the main point was to show that any unipotent subgroup $G \subseteq \mathsf{GL}(V)$ leaves invariant some complete flag $\mathcal{F} = (V_0 \subset V_1 \ldots)$. But by some homework (problem 1), the group

$$\mathsf{GL}(V)_{\mathcal{F}} := \{g \in \mathsf{GL}(V) \mid g\mathcal{F} = \mathcal{F}\}$$

is algebraic.

**Proof:** If $\mathcal{F}$ is the standard flag with $V_i = \operatorname{span}(e_1, \ldots, e_i)$ for the standard basis $\{e_1, \ldots, e_n\}$, then

$$\mathsf{GL}(V)_{\mathcal{F}} = \{A \in \mathsf{GL}(V) \mid A \text{ is upper-triangle}\}.$$

The condition that $A$ is upper triangle can be realized by polynomials. $\square$

Thus,

$$\begin{aligned} & G \text{ fixes } \mathcal{F} \\ \iff & G \subseteq \mathsf{GL}(V)_{\mathcal{F}} \\ \overset{\mathsf{GL}(V)_{\mathcal{F}} \text{ is algebraic}}{\iff} & \overline{G} \subseteq \mathsf{GL}(V)_{\mathcal{F}} \\ \iff & \overline{G} \text{ fixes } \mathcal{F}. \end{aligned}$$

Now, the Zariski-Closure $\overline{G}$ of any group $G$ is an algebraic group (shown in some homework).

Further, if $G$ is unipotent, then $\overline{G}$ is unipotent.

## 0.3 Tori

**Definition 16.** A **torus** is an algebraic group that is isomorphic to $\mathcal{G}_m^n$ for some $n \in \mathbb{N}_0$ where $\mathcal{G}_m = k^\times = \mathsf{GL}_1(k)$ is the unit group of $k$.

We think of $\mathcal{G}_m^n \subseteq \mathsf{GL}_n(k)$ as the subgroup of diagonal matrices.

**Lemma 19.** *Let $G$ be a commutative algebraic group. Then the following are equivalent:*

*(i) each $g \in G$ is semisimple.*

*(ii) for each finite-dimensional representation $V$ of $G$ and for each $g \in G$, the operator $r_V(g)$ is diagonalizable.*

*(iii) for all finite-dimensional representations $V$ of $G$, there is a basis of common eigenvectors for $r_V(G)$, i.e. a basis s.t.*

$$r_V(G) \subseteq \mathcal{G}_m^n.$$

*(iv) $G$ is isomorphic to an algebraic subgroup of a torus.*

(i) *Proof.* $\implies$ (ii): This follows from the Jordan decomposition and definition of semisimple.

(ii) $\implies$ (iii) : This is homework. Note that any commutative subset $S$ of $\mathsf{GL}(V)$ consisting of semisimple operators may be diagonalized simultaneously.

(iii) $\implies$ (iv) : Take any faithful representation $V$ of $G$ and diagonalize it simultaneously. Then, $G \cong r_V(G) \subseteq \mathcal{G}_m^n$.

(iv) $\implies$ (i) : Any diagonal matrix is semisimple.

□

**Definition 17.** A commutative algebraic group $G$ is called **diagonalizable**, if it satisfies one of the above equivalent conditions.

**Definition 18.** A character $\chi$ of any algebraic group $F$ is an element $\chi \in \mathsf{Hom}_{\text{alg.grp.}}(G, k^\times)$, i.e., a homomorphism $\chi : G \to k^\times$ of algebraic groups.

**Notation 2.** For an algebraic group $G$, set $\Xi(G) := \mathsf{Hom}_{\text{alg.grp.}}(G, k^\times)$.

Also denote now by $\mathcal{O}(X) := k[T]/I(X)$ the coordinate ring of an algebraic set $X$ (rather than $k[X]$).

**Lemma 20.** *There is a bijection*

$$\Xi(G) = \{\text{characters } \chi \text{ of } G\} \longleftrightarrow \{x \in \mathcal{O}(G)^\times \mid \Delta(x) = x \otimes x\}.$$

*Proof.* Note, that any $x \in O(G)^\times$ can be thought of as a map $x : G \to k^\times \subset k$.

We have

$$\begin{aligned}\mathsf{Hom}_{\text{alg.grp.}}(G, \mathcal{G}_m) &= \{\phi \in \mathsf{Hom}_{\text{alg.sets}}(G, \mathcal{G}_m) \mid \phi(gh) = \phi(g)\phi(h)\ \forall g, h\} \\ &= \{\phi \in \mathsf{Hom}_{k-\text{alg.}}(\mathcal{O}(\mathcal{G}_m), \mathcal{O}(G)) \mid (\phi \otimes \phi) \circ \Delta = \Delta \circ \phi\}.\end{aligned}$$

**Recall:** $\mathcal{O}(\mathcal{G}_m) \cong k[t, \frac{1}{t}]$ with $\Delta(t) = t \otimes t$.

Thus for any $k$-algebra $A$, $\mathsf{Hom}_{k-\text{alg.}}(\mathcal{O}(\mathcal{G}_m), A) \overset{A^\times}{\cong}$ via

$$[t \mapsto a, (t^{-1} \mapsto a^{-1})] \longleftrightarrow a.$$

Thus,

$$\mathsf{Hom}_{\text{alg.grp.}}(G, \mathcal{G}_m) \cong \left\{ a \in \mathcal{O}(G)^\times \mid a \otimes a = \Delta(a) \right\}.$$

Therefore, it suffices to test the condition $(\phi \otimes \phi) \circ \Delta = \Delta \circ \phi$ on the generators $t, t^{-1}$ of $\mathcal{O}(\mathcal{G}_m)$. Now, the above isomorphism is given by

$$\phi \mapsto a = \phi(t)$$

which is equivalent or regarding $\chi : G \to \mathcal{G}_m$ as a map $\chi : G \to k$. $\square$

**Example 7.** Let $G = \mathcal{G}_m$, then $\mathcal{O}(G) = k[t, \frac{1}{t}]$. Which $x = \sum_{m \in \mathbb{Z}} c_m t^m \in \mathcal{O}(G)$, almost all $c_m = 0$, but not all of them, have the property

$$\Delta(x) = x \otimes x.$$

We have

$$x \otimes x = \sum_{m,n \in \mathbb{Z}} c_m c_n t^m \otimes t^n,$$
$$\Delta(x) = \sum_{m \in \mathbb{Z}} c_m t^m \otimes t^m.$$

Those sums equal, if

$$c_m c_n = o \text{ for all } m \neq n,$$
$$c_m^2 = c_m \text{ for all m.}$$

By those conditions, it follows

$$x = t^m.$$

Therefore

$$\Xi(G) = \{\chi_m \mid m \in \mathbb{Z}\} \cong \mathbb{Z}$$

with

$$\chi_m(y) = y^m.$$

**Example 8.** Let $T \cong \mathcal{G}_m^n$ be a torus. Then,

$$\Xi(T) = \{\chi_m \mid m \in \mathbb{Z}^n\} \cong \mathbb{Z}^n$$

where $\chi_m(y) = y^m = y_1^{m_1} \cdots y_n^{m_n}$.

**Note:** For each algebraic group $G$, $\Xi(G)$ is naturally an abelian group:

$$(\chi_1 \cdot \chi_2)(g) := \chi_1(g) \cdot \chi_2(g).$$

Given a morphism of algebraic groups $f : G \to H$, we get a morphism of abelian groups

$$\begin{aligned} f^* : \Xi(H) &\longrightarrow \Xi(G) \\ \chi &\longmapsto \chi \circ f =: f^*(\chi). \end{aligned}$$

This induces a contravariant functor from the category of algebraic groups to the category of abelian groups.

**Lemma 21.** *Let $G$ be a diagonalizable algebraic group. Then, $\Xi(G)$ is a $k$-basis for $\mathcal{O}(G)$.*

**Example 9.** Let $G = \mathcal{G}_m^n$ be a torus. Then, we have the embedding

$$\begin{aligned} \Xi(G) &\hookrightarrow \mathcal{O}(G) \\ \chi_m &\longmapsto t^m. \end{aligned}$$

The lemma is obvious in this case: each elment of $\mathcal{O}(G) = k[t_1, \ldots, t_m, t_1^{-1}, \ldots, t_m^{-1}]$ can be written uniquely as a linear combination of monomials.

*Proof.* (i) $\Xi(G)$ spans $\mathcal{O}(G)$:
Choose an embedding $G \subset \mathcal{G}_m^n$ of algebraic groups. Then, by restriction, we get

$$\mathcal{O}(\mathcal{G}_m^n) \twoheadrightarrow \mathcal{O}(G).$$

Since the $\chi_m, m \in \mathbb{Z}^n$, span $\mathcal{O}(\mathcal{G}_m^n)$, their images $\chi_m|_G \in \Xi(G)$ span $\mathcal{O}(G)$.

(ii) $\Xi(G)$ is linearly independent:
Suppose otherwise and let $\phi_1, \ldots, \phi_m$ be a linearly dependent subset of $\Xi(G)$ with $m \geq 1$ chosen minimally, with $c_1, \ldots, c_m \in k^\times$ s.t.

$$\sum_{i=1}^m c_i \phi_i = 0.$$

We distinguish the following cases:

$m = 1$: In this case, we have $\phi_1 = 0$, but $\phi_1(1) = 1$, a contradiction.

$m > 1$: We can assume $\phi_1 \neq \phi_2$, so there is an $h \in G$ s.t. $\phi_1(h) \neq \phi_2(h)$. Then,

$$\phi_1(h) \sum_{i=1}^{m} c_i \phi_i = 0,$$

but also for all $h, g \in G$

$$\sum_{i=1}^{m} c_i \phi_i(hg) = \sum_{i=1}^{m} c_i \phi_i(h)\phi_i(g) = 0.$$

This implies

$$\sum_{i=1}^{m} c_i \phi_i(h)\phi = 0.$$

Ergo

$$\sum_{i=1}^{m} c_j(\phi_i(h) - \phi_1(h))\phi_i = \sum_{i=2}^{m} c_j(\phi_i(h) - \phi_1(h))\phi_i = 0.$$

Now, $\phi_i(h) - \phi_1(h)$ is zero if $i = 1$ and non-zero, if $i = 2$. Therefore, this yields a shorter linear dependency for the elements

$$\phi_2, \dots, \phi_m,$$

which contradicts our requirement.

□

**Definition 19.** Let $M$ be an abelian group. The **group algebra** on $M$ is the $k$-algebra $k[M]$ (not a coordinate ring!) defined as follows:

$$\begin{aligned} k[M] &:= \text{the } k\text{-vectorspace with basis } M \\ &:= \left\{ \sum_{m \in M} c_m \cdot m \mid c_m \in k, \text{ almost all } c_m = 0 \right\}, \end{aligned}$$

where the multiplication on $k[M]$ extends that on $M$:

$$(\sum_{m \in M} c_m m)(\sum_{n \in M} d_n n) = \sum_{m,n \in M} c_m d_n mn.$$

**Corollary 4.** *For a diagonalizable $G$, we have*

$$\mathcal{O}(G) \cong k[\Xi(G)].$$

**Fact:** For an abelian group $M$, there is exactly one Hopf algebra structure on $k[M]$ given by $\Delta(m) = m \otimes m$ for all $m \in M$.

With this definition, the above isomorphism is one of Hopf algebras.

**Lemma 22.** *If $G, H$ are diagonalizable algebraic groups, then*

$$\mathsf{Hom}_{\text{alg.grp.s}}(G, H) \xrightarrow{f \mapsto f^*} \mathsf{Hom}_{\text{grp.}s}(\Xi(H), \Xi(G))$$

*is a bijection.*

*Proof.*

$$\begin{aligned}\mathsf{Hom}(G, H) &\cong \mathsf{Hom}_{\text{Hopf}-\text{alg.}}(\mathcal{O}(H), \mathcal{O}(G)) \\ &\cong \{\phi \in \mathsf{Hom}_{k-\text{alg.}}(\mathcal{O}(H), \mathcal{O}(G)) \mid (\phi \otimes \phi) \circ \Delta = \Delta \circ \phi\}.\end{aligned}$$

Since $\mathsf{Hom}_{k-\text{alg.}}(\mathcal{O}(H), \mathcal{O}(G)) \cong \mathsf{Hom}(k[\Xi(H)], k[\Xi(G)])$, this reduces to the following lemma:

**Lemma 23.** *Let $M_1, M2$ be two abelian groups. Then*

$$\begin{aligned}\mathsf{Hom}(M_1, M_2) &\xrightarrow{\cong} \mathsf{Hom}_{\text{Hopf}-\text{alg.}}(k[M_1], k[M_2]) \\ \phi &\longmapsto \left[\sum c_m m \mapsto \sum c_m \phi(m)\right].\end{aligned}$$

*Proof.* We have to show that

$$M = \left\{x \in K[M]^\times \mid \Delta(x) = x \otimes x\right\}.$$

Then, by this, it follows for each $\phi \in \mathsf{Hom}_{\text{Hopf}-\text{alg.}}(k[M_1], k[M_2])$,

$$\phi(M_1) \subseteq M_2.$$

Ergo, $\phi|_{M_1} \in \mathsf{Hom}(M_1, M_2)$. Therefore, the surjectivity of the claimed bijection is shown. The injectivity is clear, since $M$ generates $k[M]$ as a $k$-algebra.

To show

$$M = \left\{x \in K[M]^\times \mid \Delta(x) = x \otimes x\right\},$$

let

$$\begin{aligned}x &= \sum c_m m \in K[M]^\times \\ \Delta(x) &= \sum c_m m \otimes m \\ x \otimes x &= \sum c_m c_n m \otimes n.\end{aligned}$$

If $\Delta(x) = x \otimes x$, then it follows

$$x = m$$

for some $m \in M$.

$\square$

$\square$

Lecture from 25.03.2020

**Recall:** We have seen that for diagonalizable algebraic groups $G, H$

$$\mathsf{Hom}\,(G,H) \cong \mathsf{Hom}\,(\Xi(H),\Xi(G))\,.$$

If $G$ is diagonalizable, then

$$\mathcal{O}(G) \cong k[\Xi(G)].$$

**Theorem 8.** *The functor*

$$\begin{aligned} G &\longrightarrow \Xi(G) \\ f &\longmapsto f^* \end{aligned}$$

*defines an equivalence of categories:*

$\{$*diagonalizable alg. groups*$\} \cong \{$*finite-dim. abelian groups with no* $\mathrm{char}(k)$*-torsion*$\}$.

This amounts to the bijection above between Hom-spaces and the following lemma.

**Lemma 24.** *(i) Let $G$ be a diagonalizable alg. group. Then, $\Xi(G)$ is a finitely generated abelian group with no* $\mathrm{char}(k)$*-torsion.*

*(ii) Let $\Gamma$ be a finitely generated abelian group with no* $\mathrm{char}(k)$*-torsion. Then, there is a diagonalizable algebraic group $G$ s.t. $\Xi(G) \cong \Gamma$.*

*Proof.* We will use the following facts:

- Let $n \in \mathbb{N}$. Then, $t^n - 1$ is square-free in $k[t]$ iff the ideal $(t^n-1)$ is radical in $k[t]$ iff $t^n - 1$ has not repetitive root iff either $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) = p > 0$ and $p \not| n$.

  (Proof: Galois Theory, seperable/inseperable extensions.)

- Let $M := \mathbb{Z}/n\mathbb{Z}$. Then, the $k$-group-algebra generated by $M$

  $$k[M] \cong k[t]/(t^n - 1)$$

  is reduced iff either $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) = p > 0, p \not| n$.

- If $M_1, M_2$ are abelian groups, then we have the following isomorphism of Hopf algebras

$$\begin{aligned} k[M_1] \otimes_k k[M_2] &\xrightarrow{\cong} k[M_1 \oplus M_2] \\ m_1 \otimes m_2 &\longmapsto m_1 m_2 \end{aligned}$$

  where $M_1 \oplus M_2 \cong M_1 \times M_2$.

(i) Embed $G \hookrightarrow T := \mathcal{G}_m^n$ for some $n$. Then, we have a surjection $\mathbb{Z}^n \cong \Xi(T) \twoheadrightarrow \Xi(G)$. Ergo, $\Xi(G)$ is finitely generated.

Suppose $\mathrm{char}(k) = p > 0$. Let $\chi \in \Xi(G)$ with $\chi^p = 1$. Then, for all $g \in G$, $\chi^p(g) = \chi(g^p) = 1$. The unit group $k^\times$ has not $p$-torsion, therefore $G \hookrightarrow T = (k^\times)^n$ has also no $p$-torsion. Therefore, the frobenius $g \mapsto g^p$ is an isomorphism on $G$. Therefore, $\chi = 1$ is a trivial character. Ergo $\Xi(G)$ has no $p$-torsion.

(ii) Let $\Gamma$ be a finitely generated abelian group with no $\mathrm{char}(k)$-torsion. Then,

$$\Gamma \cong \mathbb{Z}^r \oplus \mathbb{Z}/n_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/n_l\mathbb{Z}$$

where $\mathrm{char}(k) \not| n_1, \ldots, n_l$. We may reduce to the cases:

(a) $\Gamma = \mathbb{Z}$: take $G = \mathcal{G}_m$, then $\Xi(G) \cong \mathbb{Z} \cong \Gamma$.

(b) $\Gamma = \mathbb{Z}/n\mathbb{Z}$ with $\mathrm{char}(k) =: p \not| n$:
take $G := \mu_n := \{y \in k^\times \mid y^n = 1\}$. Then, since $p \not| n$, $(t^n - 1)$ is radical. So,

$$\mathcal{O}(\mu_n) \overset{Nullstellensatz}{=} k[t]/(t^n - 1) \overset{as Hopf algebras}{\cong} k[\Gamma]$$

where $t$ gets mapped to the generator of $\Gamma$.

□

**Corollary 5.** *We have the bijection*

$$\{\textit{tori}\} \cong \{\textit{ finitely generated free abelian groups}(\cong \mathbb{Z}^n)\}.$$

*Remark* 3.

$$\{\text{algebraic group schemes}/k\} \overset{\text{not necessarily natural}}{\cong} \{\text{ f.g. Hopf algebras}\}.$$

by

$$G \mapsto \mathcal{O}(G)$$

and

$$\{\text{diagonalizable algebraic group schemes}/k\} \cong \{\text{ f.g. abelian groups}\}.$$

by

$$G \mapsto \Xi(G).$$

Where $\mu_p$ in the left hand term gets mapped to $\mathcal{O}(\mu_p) = k[t]/(t^p - 1)$ with $p = \operatorname{char} k$.

### 0.3.1 Trigonalization

We say a representation $r : G \to \mathsf{GL}(V)$ of a group $G$ on a finite-dimensional $k$-vectorspace $V$ is **trigonalizable** if it admits a basis with respect to which $r(V)$ is upper-triangular:

$$r(G) \subseteq \left\{ \begin{pmatrix} * & \dots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & * \end{pmatrix} \right\}$$

**Definition 20.** We call a subgroup $G \subseteq \mathsf{GL}(V)$ **trigonalizable**, if the identity representation is.

**Lemma 25.** *Let $G$ be an algebraic group. The following are equivalent:*

*(i) Every finite-dimensional representation $r : G \to \mathsf{GL}(V)$ is trigonalizable.*

*(ii) Every irreducible representation of $G$ is 1-dimensional.*

*(iii) $G$ is isomorphic to an algebraic subgroup of*

$$B_n := \left\{ \begin{pmatrix} * & \dots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & * \end{pmatrix} \right\} \subseteq \mathsf{GL}_n(k).$$

*(iv) There is a normal unipotent algebraic subgroup $U$ of $G$ s.t. $G/U$ is diagonalizable.*

*Proof.* We prove as follows:

(i) $\implies$ (ii): Let $V$ be an irreducible representation. Then, $V \neq 0$. Choose a basis $e_1, \dots, e_n$ of $V$ s.t.

$$r(G) \subseteq B_n.$$

Then, $r(G)e_1 \subseteq ke_1$, so $V_0 := ke_1$ is $G$-invariant. Ergo $V = V_0$ is 1-dimensional.

(ii) $\implies$ (i): Let $V$ be a f.d. representation. We show by induction on $\dim(V)$ that $r : G \to \mathsf{GL}(V)$ is trigonalizable:

In the cases $\dim(V) = 0, 1$, there is nothing to show.

In the case $\dim(V) \geq 2$, assume that $V$ is not irreducible. Then, there is a $G$-invariant $V_0$ with $0 \neq V_0 \neq V$.

By the induction hypothesis, $V_0$ and $V/V_0$ are trigonalizable. Ergo, $V$ is trigonalizable.

(**Recall:** we used this criterion above in the proof that unipotent groups are trigonalizable by showing that every ??? of each $G$ is trivial.)

(i) $\implies$ (iii): Choose a faithful representation $V$ of $G$. Then, $G \cong r(G)$. Since $r$ is trigonalizable, there is a basis of $V$ s.t.

$$r(G) \subseteq B_n \subseteq \mathsf{GL}_n(k).$$

(iii) $\implies$ (ii): Suppose $G \subseteq B_n \subseteq \mathsf{GL}_n(k)$. Set

$$A_n := \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & * \end{pmatrix} \right\} \subseteq \mathsf{GL}_n(k),$$

$$U_n := \left\{ \begin{pmatrix} 1 & \dots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & 1 \end{pmatrix} \right\} \subseteq \mathsf{GL}_n(k) \text{ normal algebraic subgroup of } B_n,$$

$$U := G \cap U_n \text{ normal unipotent algebraic subgroup of } G.$$

Let $V$ be an irreducible representation of $G$, then $V$ is not zero. Consider the subspace of $V$ fixed by $U$

$$V^U := \{ v \in V \mid r(u)v = v \forall u \in U \}.$$

Then, we get a representation

$$r|_U : U \longrightarrow \mathsf{GL}(V).$$

Then, $r(U)$ is a unipotent algebraic group of $\mathsf{GL}(V)$. Ergo,

$$r(U) \subseteq \left\{ \begin{pmatrix} 1 & \dots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Ergo, $V^U \neq 0$. Since $U$ is normal in $G$, the subspace $V^U$ of $V$ is $G$-invariant: if $v \in V^U, g \in G$, then for all $u \in U$ we have

$$r(u)r(g)v = r(g)r(g^{-1}ug)v = r(g)v$$

since $v \in V^U$. Ergo $r(g)v \in V^U$.

Since $V$ is irreducible, $V = V^U$, i.e. $U$ acts trivially on $V$. Ergo, $r$ descends to a representation of the group $G/U$.

But $G/U \hookrightarrow B_n/U_n \cong A_n$. Therefore, $G/U$ and $r(G)$ are commutative. Moreover, for all $g \in G$, $r(g) \in \mathsf{GL}(V)$ is semisimple:

if $g = g_s g_u$, then $g_u \in U$, because $U_n$ is the group of unipotent elements of $B_n$.

Hence, $r(g) = r(g_s)r(g_u) = r_(g_s)$ is semisimple.

It follows that $r(G)$ is commutative and consists of semisimple elements. By some HW: $r(G)$ is trigonalizable. It is easy to show now that $V$ is one-dimensional. (Since $V$ is irreducible and $ke_1$ is $G$-invariant.)

□

**Definition 21.** $G$ is **trigonalizable**, if it satisfies one of the above equivalent conditions.

Later, we will see, that if $G$ is connected, then being trigonalizable implies being solvable.

### 0.3.2 Commutative Groups

Let $G$ be an algebraic group. Denote by $G_s$ resp. $G_u$ the subsets of semisimple resp. unipotent elements of $G$.

Then, $G_u$ is always algebraical i.e. closed: if $G \hookrightarrow \mathsf{GL}_n(k)$, then $G_u = \{g \mid (g-1)^n = 0\}$. $G_u$ does not need to be closed under multiplication (for example, take $G = \mathsf{SL}_2(k)$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$).

$G_s$ needs not to be algebraic: for example, take $G = \mathsf{SL}_2(k)$ and if $G_s$ were algebraic, then

$$\left\{\lambda \in k^\times \mid \begin{pmatrix} \lambda & 1 \\ 0 & \lambda^{-1} \end{pmatrix} \in G_s\right\} = \{\lambda \mid \lambda \neq \lambda^{-1}\}$$

but the last set is not algebraic. Also, $G_s$ does not need to be a subgroup.