Mitschrieb: Algebraic Groups
SS 20

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Vorwort

Contents

Lecture from 03.02.2020

Recall: Last time we introduced the **Zariski-Topology** on X.
There, algebraic sets equal closed sets.
We called a set X **irreducible** iff each open subset lies dense in X.

Lemma 1. *For an algebraic set X, the following are equivalent:*

(1) X is irreducible.

(2) $k[X] = k[x_1, \ldots, x_n]/I(X)$ is a domain.

(3) $I(X)$ is a prime ideal.

The proof of (2) $\iff$ (3) is a basic algebraic result.

Lemma 2. *An open base for the Zariski-Topology on an algebraic set X is given by sets:*

$$D(f) := \{p \in X \mid f(p) \neq 0\}$$

for each $f \in k[X]$. We call the $D(f)$ ***basic open sets****.*

Proof. Suppose $U \subseteq X$ is nonempty and open. Set

$$Z := X \setminus U$$

then Z is closed. Thus

$$Z = \{x \in X \mid f(x) = 0 \forall f \in I\}$$

for some ideal $I \subseteq k[X]$. Let $p \in U$, then there is an $f \in Z$ s.t.

$$f(p) \neq 0.$$

Also, $D(f) \cap Z = 0$, thus $p \in D(f) \subseteq U$. $\square$

Lemma 1. It is clear that (2) is equivalent to (3).
(1) is equivalent to

$$\forall \text{ nonempty, open } U_1, U_2 \subset X : U_1 \cap U_2 \neq \emptyset$$
$$\overset{\text{Lemma 2}}{\iff} \forall \text{ nonempty, basic open } D(f_1), D(f_2) \subset X : D(f_1) \cap D(f_2) \neq \emptyset$$

Since $D(f_1) \cap D_(f_2) = D(f_1 f_2)$, this is equivalent to the statement

$$f_1, f_2 \in k[X] : f_1, f_2 \neq 0 \implies f_1 f_2 \neq 0.$$

Which states that $k[X]$ is a domain. $\square$

Lemma 3. *Let X be an algebraic set. We have bijections*

$$\{\textit{closed subsets } Z \subseteq X\} \leftrightarrow \{ \textit{ radical ideals } I \subset k[X]\}$$

and

$$\{\textit{irreducible, closed subsets } Z \subseteq X\} \leftrightarrow \{ \textit{ prime ideals } I \subset k[X]\}$$

and

$$\{\textit{points of } X\} \leftrightarrow \{ \textit{ maximum ideals } I \subset k[X]\}.$$

Lemma 4 (Primary Decompositions, Atiyah, Macdonald Ch. 4)**.** *For an ideal I we call $P \supseteq I$ a* ***minimal prime*** *of I if P is a prime ideal and we have for each prime ideal Q:*

$$P \supseteq Q \supseteq I \implies P = Q.$$

Any radical ideal I of $k[x_1, \dots, x_n]$ has only finitely many ***minimal*** *primes $P_1, \dots, P_r$. Inparticular,*

$$I = \bigcap_{i=1}^{n} P_i$$

and for each i

$$P_j \not\supseteq \bigcap_{j:j\neq i} P_j.$$

Definition 1. An **(irreducible) component** Z of X is a maximal irreducible closed subset, i.e., an irreducible closed $Z \subseteq X$ s.t. there does not exist an irreducible closed $Y \subset X$ s.t. $Y \supsetneq Z$.

Then, we have the bijection

$$\{\text{irreducible components of } X\} \leftrightarrow \{ \text{ mimimal primes of } I(X)\}.$$

Lemma 5. *Any algebraic set X has finitely many components $Z_1, \dots, Z_r$. We have*

$$X = Z_1 \cup \ldots \cup Z_r$$

and for each i

$$Z_i \not\subset \bigcup_{j:j\neq i} Z_j.$$

Example 1.

1. Let $X = V(x \cdot y) \subset k^2$. Then $X = Z_1 \cup Z_2$ where $Z_1 = V(x), Z_2 = V(y)$.

   X is connected, but not irreducible ($D(x)$ does not lie dense in X).

2. Let X be a **finite** algebraic set. It is easy to check that every subset of X is closed:
$$\{p\} = V(x_1 - p_1, \ldots, x_n - p_n)$$
for each $p \in X$. Further
$$X = \{p_1\} \cup \ldots \cup \{p_r\}.$$
Moreover: Any function $f : X \to k$ is regular (i.e. given by polynomials).

Lemma 6. *We call an element $e \in k[X]$* ***idempotent*** *iff $e^2 = e$.*
Let X be an algebraic set. Then

$$\begin{aligned} X \textit{ connected} &\iff \textit{the only idempotents } e \in k[X] \textit{ are 0 and 1} \\ &\iff k[X] \ncong A \times B \textit{ for any } k\textit{-algebras } A, B. \end{aligned}$$

Lemma 7. *Morphisms of algebraic sets are continuous.*

Proof. Let $\phi : X \to Y$ be a morphism. It suffices to show that for all closed $Z \subset Y$ that $\phi^{-1}(Z) \subset X$ is closed.

But, if
$$Z = V_Y(S) := \{q \in Y \mid f(q) = 0 \forall f \in S\}$$
for some ideal $S \subset k[Y]$, then
$$\phi^{-1}(Z) = V_X(\phi^*(S)) = \{\phi^*(f) = f \circ \phi \mid f \in S\}.$$
□

Lemma 8. *Isomorphisms of algebraic sets are homeorphisms. In particular, any isomorphism of algebraic sets $\phi : X \to X$ permutes the components $Z_1, \ldots, Z_r$ of X:*
$$\forall i \exists j : \phi(Z_i) = Z_j.$$

Theorem 1. *Let G be an algebraic group.*

(i) There is a unique component G^0 of G with $e \in G^0$.

(ii) Every component Z of G is a coset gG^0 of G for some $g \in Z$.

(iii) G^0 is a normal algebraic subgroup of G.

(iv) G^0 is of finite index, i.e.
$$[G : G^0] = \#\left(G/G^0\right) < \infty.$$

(v) *The irreducible components are also the connected components.*

Proof. Let $G = Z_1 \cup \ldots \cup Z_r$ be the decomposition into components. We may assume that $e \in Z_1$.

Recall that $Z_1 \not\subset \bigcup_{j\geq 2} Z_j$. Then, there is an $x \in Z_1 \setminus \bigcup_{j\geq 2} Z_j$. Thus, for all algebraic set isomorphisms $\phi : G \to G$, we have by some previous lemma that $\phi(x)$ is likewise contained in some unique component of G. For example, we may take ϕ to be

$$\begin{aligned} \phi_g : G &\to G \\ y &\longmapsto gy \end{aligned}$$

for any $g \in G$. Then, for all $g \in G$, the element $gx = \phi_g(x)$ is contained in only one component of G. Ergo, each $g \in G$ is contained in exactly one component.

(i) Take $g = e$.

(iii) G^0 is an algebraic subset, by construction. Denote by $m : G \times G \to G$ and $i : G \to G$ the continuous multiplication and inversion map on G. **Why is G^0 a subgroup?** We need to show

$$\begin{aligned} m(G^0 \times G^0) &\subseteq G^0. \\ i(G^0) &\subseteq G^0. \end{aligned}$$

We know that $i(G^0)$ is some component of G, since i is an isomorphism. But it contains the identity e, since $e^{-1} = e$. Therefore, $i(G^0) = G^0$.

If $g \in G$, then gG^0 is some component of G. Suppose $g \in G^0$. Then $gG^0 \cap G^0 \supseteq \{g\}$, therefore $gG^0 = G^0$. Ergo, G^0 is closed under multiplication.

Why is G^0 a normal? If $g \in G$, then gG^0g^{-1} is a component that contains e, therefore $G^0 = gG^0g^{-1}$.

(Alternative proof that $m(G^0 \times G^0) = G^0$: Consider

- any continuous image of an irreducible set is irreducible.
- the closure of any irreducible set is irreducible.

Ergo $\overline{m(G^0 \times G^0)}$ is a closed irreducible set containing e. Ergo, $\overline{m(G^0 \times G^0)} = G^0$).

(ii) Let $Z \subset G$ be a component. Let $g \in Z$. Then $g \in (gG^0 \cap Z)$, so $gG^0 = Z$.

(iv) This follows from some previous lemma.

(v) This is left as a topological exercise. It is true whenever the irreducible components do not intersect.

□

It now follows:

$$\{\text{finite algebraic groups}\} \longleftrightarrow \{finitegroups\}$$

where the above arrow is an equivalence of categories.

Example 2.

- Let $G = \{g_1, \ldots, g_r\}$ be a finite algebraic group. Then,
$$G^0 = \{e\}.$$
- Without proofs:
$$G \in \{\mathsf{GL}_n(k), \mathsf{SO}_n(k), \mathsf{SL}_n(k)\} \implies G^0 = G.$$
Further,
$$G = O_n(k) \implies G^0 = \mathsf{SO}_n(k)$$
(but only if $-1 = 1$ i.e. $\mathsf{char} k = 2$. Otherwise $[G : G^0] = 2$.)

0.1 Jordan Decomposition

As usual, $k = \overline{k}$ is an algebraically closed field.

Definition 2. Let V be a finite-dimensional vector space.

An element $x \in \mathsf{End}(V)$ is **semisimple**, if it is diagonalizable, i.e. it has a basis of eigenvectors, or equivalently, if the minimal polynomial of x is square-free.

Then, there is a decomposition $V = \bigoplus_{i=1}^r V_i$ and distinct elements $\lambda_1, \ldots, \lambda_n \in k$ s.t.

$$x|_{V_i} = \lambda_i.$$

If $\dim(V_i) = n_i$, then

$$\text{char polynomial of } x = \prod_{i=1}^{r} (T_i - \lambda_i)_i^n \in k[T]$$

and

$$\text{minimal polynomial of } x = \prod_{i=1}^{r}(T_i - \lambda_i) \in k[T].$$

(Where the minimal polynomial of x is defined as the least degree monic m s.t. $m(x) = 0$ and Cayley-Hamilton $m|c$.)

Definition 3. $x \in End(V)$ is **nilpotent** if $x^n = 0$ for some n. (Equivalent to: characteristic polynomial of x is $T^{\dim(V)}$.)

x is **unipotent**, if $x - 1$ is nilpotent.

Lemma 9. *If x is semisimple and nilpotent, then $x = 0$.*

If x is semisimple and unipotent, then $x = 1$.

Lemma 10. *If x, y are commuting elements, thar are semisimple resp. unipotent or nilpotent, then so is xy.*

Theorem 2 (Goal)**.** *For all algebraic groups G and for all $g \in G$, there exist unique group elements $g_s, g_u \in G$ s.t.*

$$g = g_s g_u = g_u g_s$$

and for all finite-dimensional representations $\rho : G \to \mathsf{GL}(V)$, $\rho(g_s)$ is semisimple and $\rho(g_u)$ is unipotent.

Example 3. If $g = \begin{pmatrix} \lambda & 1 & 0 \\ & \lambda & 1 \\ & & \lambda \end{pmatrix} \in G = \mathsf{GL}_3(k)$, then $g_s = \begin{pmatrix} \lambda & 0 & 0 \\ & \lambda & 0 \\ & & \lambda \end{pmatrix}$, $g_u = \begin{pmatrix} 1 & \lambda^{-1} & 0 \\ & 1 & \lambda^{-1} \\ & & 1 \end{pmatrix}$.

Lecture from 09.02.2020

Theorem 3 (Goal Theorem)**.** *Let G an algebraic group. For all $g \in G$ there is exactly one pair $g_s, g_u \in G$ s.t.*

$$g = g_s g_u = g_u g_s$$

and for all finite-dimensional representations $r : G \to \mathsf{GL}_n(V)$, the element $r(g_s)$ resp. $r(g_u)$ is semisimple resp. unipotent.

Last time, we saw:

Lemma 11.
- *If g, h are commuting and semisimple resp. commuting and unipotent then so is gh.*
- *If g is semisimple and unipotent, then $g = 1$.*

Proposition 1. *Let V be a finite-dimensional vector space and $g \in \mathsf{GL}(V)$. There exist unique elements $g_s, g_u \in \mathsf{GL}(V)$ s.t.*

$$g = g_s g_u = g_u g_s$$

and g_s is semisimple and g_u is unipotent.

Moreover, $g_s, g_u \in k[g] = \{\sum_{i=1}^m a_i g^i \mid a_i \in k\} \subseteq \mathsf{End}(V)$.

Proof. **Existence** (Sketch): Say

$$g = \begin{pmatrix} \lambda & 1 & 0 \\ & \lambda & 1 \\ & & \lambda \end{pmatrix}$$

then take

$$g_s = \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{pmatrix}, \quad g_u = \begin{pmatrix} 1 & \lambda^{-1} & 0 \\ & 1 & \lambda^{-1} \\ & & 1 \end{pmatrix}.$$

For $\lambda \in k$, define the **generalized λ-eigenspace** of g by

$$V_\lambda := \{v \in V \mid \exists n \in \mathbb{N}_0 : \ (g - \lambda)^n v = 0\}.$$

Then

$$V = \bigoplus_{\lambda \in k} V_\lambda.$$

Here V_λ = sum of domains of all Jordan blocks with λs on the diagonal. (It follows from the Jordan Decomposition for matrices that such a decomposition exist.)

Let's define $g_s \in \mathsf{GL}(V)$ by

$$g_s|_{V_\lambda} = \lambda \cdot \mathrm{Id}.$$

Note that $gV_\lambda \subset V_\lambda$, hence g commutes with g_s, hence g, g_s commutes with $g_u := gg_s^{-1}$. Then, $g = g_s g_u = g_u g_s$.

Write $\det(T - g) = \prod_\lambda (T - \lambda)^{n(\lambda)}$, $n(\lambda) = \dim(V_\lambda)$. Since the polynomials $T - \lambda$ for $\lambda \in k$ are coprime, the chinese remainder theorem implies that there is a $Q \in k[T]$ s.t.

$$Q \equiv \lambda \mod (T - \lambda)^{n(\lambda)}$$

for each $\lambda \in k$.

We claim that

$$Q(g) = g_s.$$

Indeed, since $gV_\lambda \subseteq V_\lambda$, we have

$$Q(g)V_\lambda \subseteq V_\lambda.$$

So, it suffices to show for all $v \in V_\lambda$

$$Q(g)v = g_s v = \lambda v.$$

Note that, by Cayley-Hamilton,

$$V_\lambda = \left\{ v \in V \mid (g - \lambda)^{n(\lambda)} v = 0. \right\}$$

Write

$$Q = \lambda + R \cdot (T - \lambda)^{n(\lambda)}$$

for some $R \in k[T]$. Since $(g - \lambda)^{n(\lambda)} v = 0$, deduce that $Q(g)v = \lambda v$, as required.

If $Q' \equiv T(T - \lambda)^{n(\lambda)}$, then

$$Q'(g) = g_u.$$

If $Q'' \equiv \lambda^{-1}(T - \lambda)^{n(\lambda)}$, then $Q''(g) = g_s^{-1}$. check corresponding stuff for g_u.

Uniqueness: Suppose given some other decomposition

$$g = g'_s g'_u = g'_u g'_s$$

with g'_s semisimple and g'_u unipotent. Then g'_s commutes with g'_s and g'_u, hence with g, hence also with any element in $k[g]$. Ergo, g'_s commutes with g_s and g_u. Similarly, g'_u commutes with g_s and g_u.

Consider

$$h := g'_s g_s^{-1} = g'_s g'_u (g'_u)^{-1} g_s^{-1} = g(g'_u)^{-1} g_s^{-1} = g_u (g'_u)^{-1}.$$

Then $h = g'_s g_s^{-1}$ is a product of semisimple elements and $h = g_u(g'_u)^{-1}$ is a product of unipotent elements. By proceeding lemmas, h is semisimple and unipotent, ergo trivial. It follows $g'_s = g_s$ and $g'_u = g_u$. $\square$

Corollary 1. *Let $g \in \mathsf{GL}(V)$, let $W \subset V$ be any g-**invariant subspace**, i.e. $gW \subseteq W$.*

Then, W is g_s-invariant and g_u-invariant.

Proof. This is clear, since g_s and g_u are algebraically generated by g over g. $\square$

Lemma 12. *Let $\phi : V \to W$ be a linear map between finite-dimensional vector spaces.*

Let $\alpha \in \mathsf{GL}(W)$ and $\beta \in \mathsf{GL}(W)$ s.t.

$$\begin{tikzcd} V \arrow[r, "\alpha"] \arrow[d, "\phi"] & V \arrow[d, "\phi"] \\ W \arrow[r, "\beta"] & W, \end{tikzcd}$$

i.e. $\phi \circ \alpha = \beta \circ \phi$.

Then,

$$\phi \circ \alpha_s = \beta_s \circ \phi,$$
$$\phi \circ \alpha_u = \beta_u \circ \phi.$$

Proof. Write $V = \bigoplus_{\lambda \in k} V_\lambda$, $W = \bigoplus_{\lambda \in k} W_\lambda$ where V_λ are the generalized α-eigenspaces and W_λ are the generalized β-eigenspaces.

We claim that

$$\phi(V_\lambda) \subset W_\lambda.$$

Indeed, let $v \in V_\lambda$, then

$$(\beta - \lambda)^n \phi(v) = \phi((\alpha - \lambda)^n v) = 0.$$

Since $(\alpha - \lambda)^n v = 0$, the claim follows.

Since, $\alpha_s|_{V_\lambda} = \lambda \mathrm{Id}$ and $\beta_s|_{W_\lambda} = \lambda \mathrm{Id}$, deduce that

$$\phi \circ \alpha_s = \beta_s \circ \phi.$$

Indeed, both sides are given on V_λ by $\lambda \cdot \phi$. Thus

$$\begin{aligned} \phi \circ \alpha_u &= \phi \circ \alpha\alpha_s^{-1} \\ &= \beta\beta_s^{-1} \circ \phi \\ &= \beta_u \circ \phi. \end{aligned}$$ □

Lemma 13. *Let $\alpha \in GL(V)$, $\beta \in GL(W)$. Then the* ***tensor*** *$\alpha \otimes \beta \in GL(V \otimes W)$ is defined by*

$$(\alpha \otimes \beta)(u \otimes v) = \alpha(u) \otimes \beta(v).$$

Then, we have

$$(\alpha \otimes \beta)_s \overset{(1)}{=} \alpha_s \otimes \beta_s$$

$$(\alpha \otimes \beta)_u \overset{(2)}{=} \alpha_u \otimes \beta_u.$$

Proof. It suffices to prove (1), since

$$\begin{aligned} (\alpha \otimes \beta)_u &= (\alpha \otimes \beta) \circ (\alpha \otimes \beta)_s^{-1} \\ &\overset{(1)}{=} (\alpha \otimes \beta) \circ (\alpha_s \otimes \beta_s)^{-1} \\ &= \alpha\alpha_s^{-1} \otimes \beta\beta_s^{-1} \\ &= \alpha_u^{-1} \otimes \beta_u^{-1} \end{aligned}$$

For (1), consider

$$\begin{aligned} V &= \bigoplus_{\lambda \in k} V_\lambda, \\ W &= \bigoplus_{\lambda \in k} W_\lambda. \end{aligned}$$

It follows

$$V \otimes W = \bigoplus_{\lambda, \mu \in k} V_\lambda \otimes W_\mu.$$

Now,

$$\alpha_s \otimes \beta_s|_{V_\lambda \otimes W_\mu} = \lambda\mu \cdot \mathrm{Id}.$$

Ergo, $\alpha_s \otimes \beta_s$ is semisimple. By Proposition, we reduce to checking that $\alpha_u \otimes \beta_u$ is unipotent. Indeed,

$$\alpha_u \otimes \beta_u - 1 = (\alpha_u - 1) \otimes (\beta_u - 1) + 1 \otimes (\beta_u - 1) + (\alpha_u - 1) \otimes 1$$

is nilpotent (You can also check that $\alpha_u \otimes \beta_u = (\alpha_u \otimes 1) \circ (1 \otimes \beta_u)$ is unipotent.) □

Example 4. Let $1 \in \mathsf{GL}(V)$. Then $1_s = 1$ and $1_u = 1$.

Summary : Let G be an algebraic group. Let $r_V : G \to \mathsf{GL}(V)$ be a finite-dimensional representation. Also, fix $g \in G$.

Let $\lambda_V := r_V(g)_s$ (or $r_V(g)_u$).

We get a family of operators $\lambda_V \in \mathsf{End}(V)$ with the following properties:

(i) if $V = k$ and $r_V(g') = 1$ for all $g' \in G$, then $\lambda_V = 1$.

(ii) for any two representations in V and W, we have

$$\lambda_{V \otimes W} = \lambda_V \otimes \lambda_W.$$

(iii) for all G-equivariant $\phi : V \to W$ we have

$$\phi \circ \lambda_V = \lambda_W \circ \phi.$$

Theorem 4. *Let G be an algebraic group. Let $\lambda_V \in \mathsf{End}(V)$ (i.e. $V = (r_V, V)$ is a finite-dim. representation of G) be a family of operations satisfying (i), (ii), (iii).*

Then, there is exactly one $g \in G$ s.t. $\lambda_V = r_V(g)$ for all V.

Note, that this theorem implies our goal theorem.

Applying the theorem to $\lambda_V = r_V(g_s)$ implies

$$\exists g_s \in G : r_V(g_s) = r_V(g)_s$$

and

$$\exists g_u \in G : r_V(g_u) = r_V(g)_u.$$

Proof of Goal Theorem. There exist unique $g_s, g_u \in G$ s.t.

$$g \overset{(*)}{=} g_u g_s = g_s g_u,$$

Then, $r_V(g) = r_V(g_s) r_V(g_u) = r_V(g_u) r_V(g_s)$.

Since $r_V(g_u)$ is unipotent and $r_V(g_s)$ is semisimple, it follows $r_V(g_u) = r_V(g)_u$ and $r_V(g_s) = r_V(g)_s$.

To deduce $(*)$, take any $r_V : G \hookrightarrow \mathsf{GL}(V)$. We know for each V

$$r_V(g) = r_V(g_s) r_V(g_u) = r_V(g_u) r_V(g_s).$$

□

Proof of Theorem. We first extend the assignment

$$V \mapsto \lambda_V$$

to all representations of G.

Say $V = \bigcup_j W_j$ where each W_j is a finite-dimensional G-invariant subspace. Try to define $\lambda_V \in \mathsf{End}(V)$ by

$$\lambda_V|_{W_j} := \lambda_{W_j}.$$

For this to be well-defined, we need to show for each i, j

$$\lambda_{W_i}|_{W_i \cap W_j} \overset{(*)}{=} \lambda_{W_j}|_{W_i \cap W_j}.$$

Proof of $(*)$**:** Apply assumption (iii) to the G-equivariant linear maps

$$W_i \cap W_j \overset{\phi}{\hookrightarrow} W_i,$$
$$W_i \cap W_j \overset{\phi'}{\hookrightarrow} W_j.$$

Then,

$$\begin{aligned} \lambda_{W_i}|_{W_i \cap W_j} &= \lambda_{W_i} \circ \phi \\ &\overset{(iii)}{=} \phi \circ \lambda_{W_i \cap W_j} \\ &= \phi' \circ \lambda_{W_i \cap W_j} \end{aligned}$$

and

$$\lambda_{W_j}|_{W_i \cap W_j} = \lambda_{W_i} \circ \phi' = \phi' \circ \lambda_{W_i \cap W_j}.$$

Recall here that any finite-dimensional G-invariant $W \subset V$ is a representation. $\square$

[0]Not necessarily finite-dimensional, but may be written as a filtered union of finite-dimensional G-invariant subspaces of W.