Report: Signals and Systems Lab Assignment 6

Bilkent University Electrical and Electronics Department
EE321-01

Semih Akkoç - 22103132

December 16, 2023

Part 1:

Following two figure shows the calculations for the Part 1.

$$y[n] = \sum_{i=1}^{N} a_i \, y[n-i] + \sum_{j=0}^{M} b_j \, x[n-j]$$

$$y[0] = b[0]x[0]$$

$$y[1] = a[1]y[0] + b[0]x[1] + b[1]x[0]$$
$$= a[1]b[0]x[0] + b[0]x[1] + b[1]x[0]$$

Figure 1: $y[0]$ and $y[1]$ in terms of a, b and x.

$$Y(z) = \sum_{l=1}^{N} a[l]\, Y(z)\, z^{-l} + \sum_{k=0}^{M} b[k]\, X(z)\, z^{-k}$$

$$\Rightarrow Y(z)\left[1 - \sum_{l=1}^{N} a[l] z^{-l}\right] = X(z) \sum_{k=0}^{M} b[k] z^{-k}$$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b[k] z^{-k}}{1 - \sum_{l=1}^{N} a[l] z^{-l}} = \frac{\sum_{p=0}^{P} c_n[p] z^{-p}}{\sum_{q=0}^{Q} c_d[q] z^{-q}}$$

Therefore, $P = M$ and $Q = N$

where $c_n[p] = b[k]$ and $c_d[q] = \begin{cases} 1, & q = 0 \\ -a[l], & q \neq 0 \end{cases}$

Figure 2: z-transform of the equation.

The code for the Part 1:

```
function [y]=DTLTI(a,b,x,Ny)
    y=zeros(Ny,1);

    for ii=1:Ny
        for jj=1:length(b)
            if (ii-jj)<=length(x) && (ii-jj)>=0
                y(ii)=y(ii)+b(jj)*x(ii-jj+1);
            end
        end
    end

    for ii=1:Ny
        for jj=1:length(a)
            if (ii-jj)>=0
                y(ii)=y(ii)+a(jj)*y(ii-jj+1);
            end
        end
    end
end
```

Part 2:

Part 2.a:

Following figure shows the $y[n]$ when $x[n] = \delta[n]$,

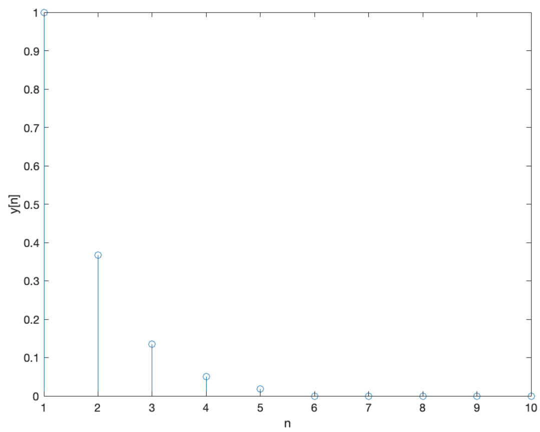

Figure 3: Plot of $y[n]$ when $x[n] = \delta[n]$.

Part 2.b:

As observed in the following figure the graph of $y[n]$ is same with the graph of $b[n]$ since $x[n] = \delta[n]$,

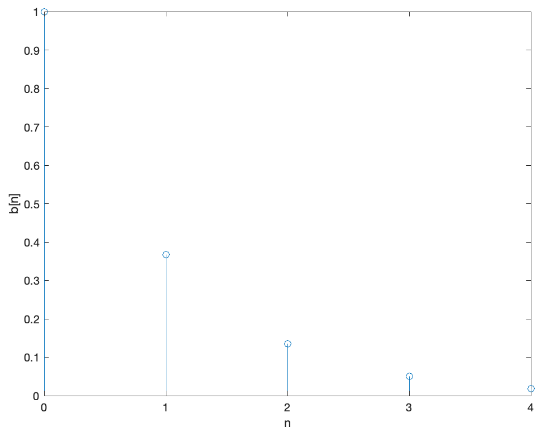

Figure 4: Plot of $b[n]$.

Part 2.c:

The system has a finite impulse response. Therefore the system is FIR and the length of the impulse response is 5. It is exactly same with M.

Part 2.d:

This part contains the analytic z-transform of the impulse response of the system.

Since the system is $y[n] = \sum_{k=0}^{M} b[k]\, x[n-k]$

$$\Rightarrow Y(z) = \sum_{k=0}^{M} b[k]\, X(z)\, z^{-k} \Rightarrow H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^{M} b[k]\, z^{-k}$$

If $z = e^{j\omega}$ then we transform z-transform to DTFT,

$$\Rightarrow H(e^{j\omega}) = \sum_{k=0}^{M} b[k]\, e^{-j\omega k}$$

we used $x[n-k] \xrightarrow{z\text{-tran}} X(z) = \sum_{n=-\infty}^{\infty} x[n-k]\, z^{-n} \underset{m=n-k}{=} \underbrace{\sum_{m=-\infty} x[m]\, z^{-m}}_{X(z)} \cdot z^{-k} = X(z)\, z^{-k}$

$$H(e^{j\omega}) = \sum_{k=0}^{M} b[k]\, e^{-j\omega k} \underset{\text{for } b[k]=e^{-k}}{=} \sum_{k=0}^{M} e^{-k} e^{-j\omega k} = \sum_{k=0}^{M} e^{-k(1+j\omega)}$$

$$= \frac{1 - e^{-(M+1)(1+j\omega)}}{1 - e^{-(1+j\omega)}}$$

Figure 5: z-transform of the impulse response.

Part 2.e:

This is a low-pass filter. The lowpass filter minimizes its response at high frequencies. Within a lowpass filter (LPF), the 3dB bandwidth represents the scope where the signal experiences attenuation of up to 3 dB. The frequency at which this attenuation occurs is known as the cutoff frequency. At this cutoff frequency, the peak magnitude measures 1.578, therefore, $1.578\frac{1}{\sqrt{2}} \approx 1.116$ is the frequency where 3dB occurs. Additionally, the cutoff frequency signifies the point where the signal's power is halved.

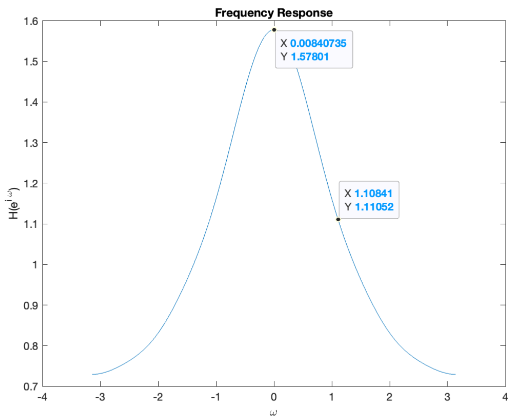

Figure 6: z-transform of the impulse response.

Part 2.f:

During this step, the objective involves providing the system with a sweep function with frequencies from 0 to 700 Hz. This methodology holds significance as it allows observation of the system's behavior across various frequency ranges.

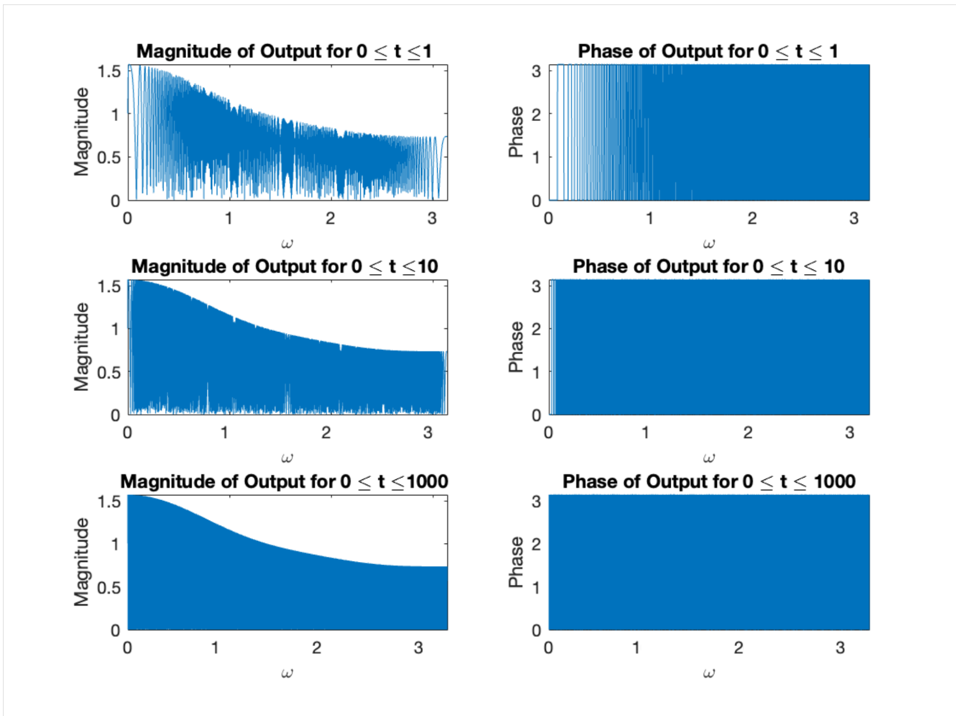

Figure 7: Frequency response of the system for linear chirp signal.

Introducing a chirp signal to a system can prompt sudden changes in the output, hinting at resonant frequencies or nonlinear behaviors within the system. Linear systems are expected to transition smoothly as the chirp signal's frequency rises, but abrupt shifts in output suggest sensitivity to specific frequencies, often linked to system poles. These poles represent frequencies where the system's response approaches infinity, causing significant peaks in output at these frequencies. Sampling issues, like aliasing, could also cause sudden changes if the chirp signal's frequency nears the sampling frequency. To investigate, comparing the chirp and sampling frequencies reveals no aliasing concerns due to the sufficient sampling rate. Therefore, abrupt changes likely result from the system's poles, influencing magnitude responses at specific frequencies. Overall, the signal's magnitude decreases over time, peaking at the midpoint of the time interval along with the chirp signal's frequency.

The code for Part 2.f:

```
figure;

D=mod(22103132,4);
M=5+D;
a=zeros(2,1);
b=zeros(M,1);

for ii=0:M-1
    b(ii+1)=exp(-ii);
end


% for 0<t<1
fs=1400;
t=0:1/fs:1;
x=cos(pi*(700*t.^2));

y1=DTLTI(a,b,x,length(x));

subplot(3,2,1)
plot(t*pi,abs(y1))
xlim([0,pi]);
title("Magnitude of Output for 0 \leq t \leq 1 ")
xlabel("\omega")
ylabel("Magnitude")
```

```
subplot(3,2,2)
plot(t*pi,angle(y1))
xlim([0,pi]);
title("Phase of Output for 0 \leq t \leq 1")
xlabel("\omega")
ylabel("Phase")


% for 0<t<10
t=0:1/fs:10;
x=cos(pi*(70*t.^2));
y2=DTLTI(a,b,x,length(t));

subplot(3,2,3)
plot(t*pi,abs(y2))
xlim([0,pi]);
title("Magnitude of Output for 0 \leq t \leq 10 ")
xlabel("\omega")
ylabel("Magnitude")

subplot(3,2,4)
plot(t*pi,angle(y2))
xlim([0,pi]);
title("Phase of Output for 0 \leq t \leq 10")
xlabel("\omega")
ylabel("Phase")



% for 0<t<1000
t=0:1/fs:1000;
x=cos(2*pi*(0.7*t.^2/2));

y3=DTLTI(a,b,x,length(x));

subplot(3,2,5)
plot(t*pi,abs(y3))
xlim([0,pi]);
title("Magnitude of Output for 0 \leq t \leq 1000 ")
xlabel("\omega")
ylabel("Magnitude")
```

```
subplot(3,2,6)
plot(t*pi,angle(y3))
xlim([0,pi]);
title("Phase of Output for 0 \leq t \leq 1000")
xlabel("\omega")
ylabel("Phase")
```

Part 3:

After calculating the z_1, p_1 and p_2 are found as following,

- $z_1 = j$
- $p_1 = \frac{2}{3}(1 + j)$
- $p_2 = 0.3045 + 0.9045j$

Part 3.a:

The z-transform of the filter can be found in following relation.

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z - z_1}{(z - p_1)(z - p_2)} = \frac{p_1 - z_1}{(z - p_1)(p_1 - p_2)} + \frac{p_2 - z_1}{(z - p_2)(p_2 - p_1)}$$

Part 3.b:

The filter can be expressed in as Eq1. in the following form,

$$\frac{Y(z)}{X(z)} = \frac{z - z_1}{(z - p_1)(z - p_2)}$$

$$\Rightarrow Y(z)(z - p_1)(z - p_2) = X(z)(z - z_1)$$

$$\Rightarrow Y(z)(z^2 - (p_1 + p_2)z + p_1p_2) = X(z)(z - z_1)$$

Apply inverse z-transform then,

$$\Rightarrow y[n - 2] - (p_1 + p_2)y[n - 1] + p_1p_2y[n] = x[n - 1] - z_1x[n]$$

$$\Rightarrow y[n] = \frac{(p_1 + p_2)}{p_1p_2}y[n - 1] - \frac{1}{p_1p_2}y[n - 2] + \frac{1}{p_1p_2}x[n - 1] - \frac{z_1}{p_1p_2}x[n]$$

Then if this equation were to be equated with Eq1. then it yields,

$y[n] = \sum_{i=1}^{2} a_iy[n - i] + \sum_{j=0}^{1} b_jx[n - j]$ where $a_1 = \frac{(p_1 + p_2)}{p_1p_2}$ and $a_1 = \frac{-1}{p_1p_2}$ $b_0 = \frac{-z_1}{p_1p_2}$ and $b_0 = \frac{1}{p_1p_2}$.

Part 3.c:

Since the system is causal it will be right-sided therefore region of convergence will be outside of the magnitude wise greatest pole. So the impulse response can be shown as using the Part 3.a,

$$\begin{aligned} H(z) &= \frac{p_1 - z_1}{(z - p_1)(p_1 - p_2)} + \frac{p_2 - z_1}{(z - p_2)(p_2 - p_1)} \\ &= \frac{p_1 - z_1}{(1 - \frac{z}{p_1})(p_1 p_2 - p_1^2)} + \frac{p_2 - z_1}{(1 - \frac{z}{p_2})(p_1 p_2 - p_2^2)} \end{aligned}$$

Apply inverse z-transform then,

$$h[n] = \frac{p_1 - z_1}{(p_1 p_2 - p_1^2)} p_1^n + \frac{p_2 - z_1}{(p_1 p_2 - p_2^2)} p_2^n$$

Part 3.d:

The region of convergence and plot of the magnitude response of the system can be found in the following figure,

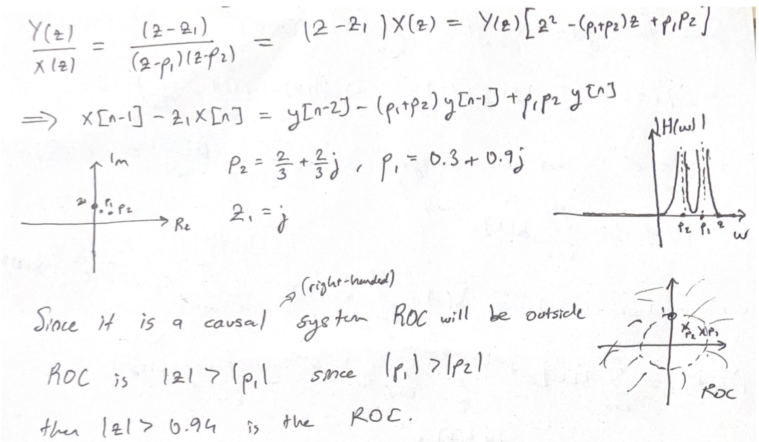

Figure 8: Magnitude response of the system and ROC.

Part 3.e:

Since the region of convergence contains unit circle this means system has DTFT, therefore, system is stable.

Part 3.f:

Since system has poles system is infinite impulse response (IIR).

Part 3.g:

It is some kind of a band-pass filter. This has been inferred by looking at the shape of the filter.

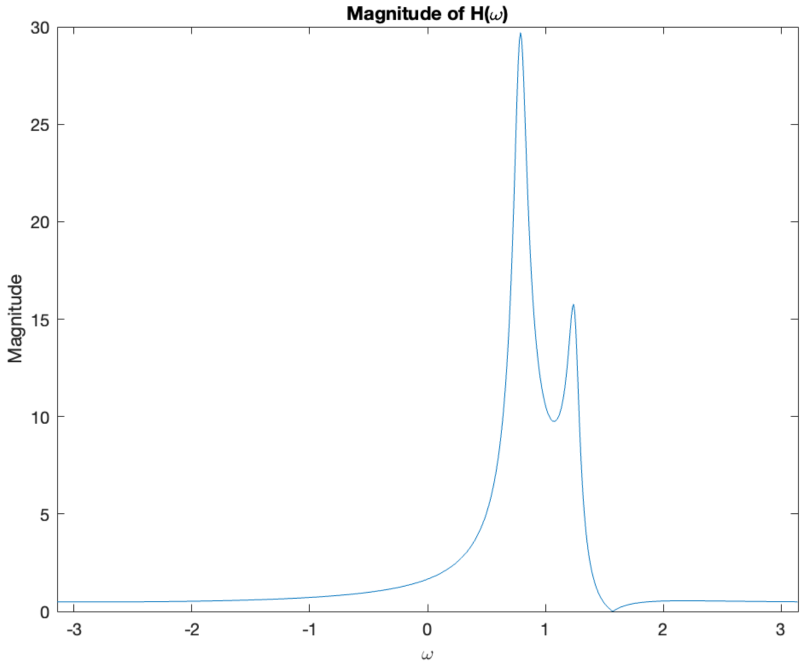

Figure 9: Magnitude response of the system.

Part 3.h:

The symmetry of a magnitude response in an LTI system reflects its real-valued nature. A real-valued system showcases symmetry around the origin due to the Hermitian symmetry in the Fourier transform of real functions. However, for a complex-valued system, this symmetry might not be clear. Given our system's complex-valued nature, it's not expected to exhibit symmetry around the origin.

Regarding the sampling frequency and the chirp signal, capturing the full behavior of a system ranging from -600Hz to 800Hz requires satisfying the Nyquist-Shannon sampling theorem. According to this theorem, the sampling frequency (f_s) should be at least double the maximum signal frequency. In our case, with a maximum frequency of 800Hz, the necessary sampling frequency would be at least 1600Hz. However, our sampling frequency stands at 1400Hz, indicating potential aliasing that could lead to loss of information.

The code for the Part 3.h:

```
id=[2,2,1,0,3,1,3,2];
n=zeros(8,1);
for ii=1:8
    if ii+2<=8
        n(ii)=id(ii+2);
    else
        n(ii)=id(ii+2-8);
    end
end

z1 = (n(2)+1i*n(3))/sqrt(n(2)^2+n(3)^2);
p1 = (n(1)+1i*n(5))/sqrt(1+n(1)^2+n(5)^2);
p2 = (n(8)+1i*n(6))/sqrt(1+n(6)^2+n(8)^2);

w=-pi:0.01:pi;
Hw=(exp(1i.*w)-z1)./((exp(1i.*w)-p1).*(exp(1i.*w)-p2));

% a=[(p1+p2)/(p1*p2), -1/(p1*p2)];
% b=[-z1/(p1*p2), 1/(p1*p2)];

a=[p1+p2,p1*p2];
b=[1,-z1];


plot(w,abs(Hw))
xlim([-pi,pi]);
title("Magnitude of H(\omega)")
xlabel("\omega")
ylabel("Magnitude")


% for 0<t<1
t=0:1/1400:1;
x=exp(1i*2*pi*(-700*t+(700)*t.^2));

y1=(DTLTI(a,b,x,length(x)));

figure;
subplot(3,2,1)
plot(t*pi,abs(y1))
xlim([-pi,pi]);
```

```
title("Magnitude of Output for 0 \leq t \leq1 ")
xlabel("\omega")
ylabel("Magnitude")

subplot(3,2,2)
plot(t*pi,angle(y1))
xlim([-pi,pi]);
title("Phase of Output for 0 \leq t \leq 1")
xlabel("\omega")
ylabel("Phase")


% for 0<t<10
t=0:1/1400:10;
x=exp(1i*2*pi*(-700*t+(70)*t.^2));

y2=(DTLTI(a,b,x,length(x)));

subplot(3,2,3)
plot(t*pi,abs(y2))
xlim([-pi,pi]);
title("Magnitude of Output for 0 \leq t \leq10 ")
xlabel("\omega")
ylabel("Magnitude")

subplot(3,2,4)
plot(t*pi,angle(y2))
xlim([-pi,pi]);
title("Phase of Output for 0 \leq t \leq 10")
xlabel("\omega")
ylabel("Phase")



% for 0<t<1000
t=0:1/1400:1000;
x=exp(1i*2*pi*(-700*t+(0.7)*t.^2));


y3=DTLTI(a,b,x,length(x));

subplot(3,2,5)
plot(t*pi,abs(y3))
```

```
xlim([-pi,pi]);
title("Magnitude of Output for 0 \leq t \leq1000 ")
xlabel("\omega")
ylabel("Magnitude")

subplot(3,2,6)
plot(t*pi,angle(y3))
xlim([-pi,pi]);
title("Phase of Output for 0 \leq t \leq 1000")
xlabel("\omega")
ylabel("Phase")
```