ABOUT LOW DIMENSION BESSEL SDES AND ASSOCIATED SEMILINEAR PDES

Abstract

One of the main objectives of our work was to study (to implement a meaning for existence and uniqueness of solution) formal SDEs of type

$$dX_t = [\beta(X_t) + \Gamma(X^t)]dt + \sigma(X_t)dW_t, X_0 = x_0 \in \mathbb{R}, t \in [0,T], \tag{0.1}$$

where β is not a regular continuous function (a Schwartz distribution for example), Γ is a locally bounded functional and X^t represents the whole trajectory of X up to t.

We analyzed particularly the case where $\Gamma = 0$, $\sigma = 1$ and β is the derivative of

$$b(x) = \begin{cases} \frac{\delta-1}{2}\log|x|, x \in \mathbb{R}^*, & | \quad 0 \le \delta < 1 \\ H(x), x \in \mathbb{R}, & | \quad \delta = 1, \end{cases} \tag{0.2}$$

We call this case the $\delta-$dimensional Bessel process case and it is formally represented by

$$dX_t = \frac{\delta-1}{2}\frac{1}{X_t}dt + dW_t, X_0 = x > 0,\ t \in [0,T]. \tag{0.3}$$

Inspired by the classical Stroock-Varadhan theory we developed new concept of martingale problem and through this concept we investigated (0.3). As in the classical time-homogeneous framework , where the martingale problem is associated to a second order operator defined on C^2, we also provided a second order operator L^δ on a subset $D_{L^\delta} := \{\varphi \in C^2(\mathbb{R}); \partial_x\varphi(0) = 0\}$ of C^2.

$$L^\delta f(x) = \begin{cases} \frac{f''(x)}{2} + \dfrac{(\delta-1)f'(x)}{2x} & : \quad x \neq 0 \\ \frac{f''(0)}{2} & : \quad x = 0. \end{cases} \tag{0.4}$$

Once (0.3) has a precise stochastic sense we could advance and investigate a deterministic approach to it. In the classical framework (Lipschitz continuity and linear growth of the coefficients) the so called Feynman-Kac representation and Kolmogorov equations are examples of this approach. We consider the parabolic semilinear Kolmogorov type PDE

$$\begin{cases} (\partial_t + L^\delta)u + f(\cdot,\cdot,u,\partial_x u) = 0 \\ u(T) = g, \end{cases} \tag{0.5}$$

where f, g are continuous f has linear growth and is Lipschitz in the third argument and $g \in C^2$ has linear growth. We proved existence of solution (mild and weak) and uniqueness. In order to do that we made extensive use of Semigroup theory as well as other functional analysis tools. In both problems (0.2) and (0.5) one of the crucial elements is the operator L^δ (and its domain) which, we emphasize, is a creation of ours.