Limiting absorption principle and mixed variational formulation for resonant Maxwell's equations in cold magnetized plasma

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Summary

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1. Electromagnetic waves in magnetized plasma
2. The model
   - Simplified model and frameworks
   - Singular solutions
   - Mixed variational formulation
   - Numerical simulations
3. Conclusion

Waves in plasma

The electromagnetic waves in magnetized plasma follows the following Maxwell system:

$$\begin{cases} \boldsymbol{curl}\,\boldsymbol{E} = \boldsymbol{B}, \\ \boldsymbol{curl}\,\boldsymbol{B} = \mathbb{\varepsilon}\boldsymbol{E}, \end{cases} \tag{1.1}$$

with $\boldsymbol{E} = (E_x, E_y, E_z)^\top$, $\boldsymbol{B} = (B_x, B_y, B_z)^\top$,

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with $\boldsymbol{E} = (E_x, E_y, E_z)^\top$, $\boldsymbol{B} = (B_x, B_y, B_z)^\top$, and the classical plasma dielectric tensor [Stix, 1992]:

$$\epsilon = \begin{pmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{pmatrix}, \tag{1.2}$$

where the coefficient S, D and P may vary in the plasma.

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To make it short, we assume the following:

- we consider a resonance in the plasma : the coefficient S changes of sign continuously and the coefficient D stays positively bounded;
- we restrict ourselves to the transverse electric problem $E_z = 0$ and $B_x = B_y = 0$.

Resonance with transverse electric field

This leads us to the one-unknowns equations

$$curl_\perp \left(\varepsilon_\perp^{-1} \mathbf{curl}_\perp B_z\right) = B_z, \tag{1.3}$$

$$\mathbf{curl}_\perp \left(curl_\perp \mathbf{E}\right) = \varepsilon_\perp \mathbf{E}. \tag{1.4}$$

with $\varepsilon_\perp = \begin{pmatrix} S & -iD \\ iD & S \end{pmatrix} = \begin{pmatrix} \alpha & i\delta \\ -i\delta & \alpha \end{pmatrix}$ and $D = -\delta$. Notice that $\mathbf{E} = (E_x, E_y)^\top$.
The differential operators are:

$$\mathbf{curl}_\perp B_z = \begin{pmatrix} \partial_y B_z \\ -\partial_x B_z \end{pmatrix} \qquad \textit{and} \qquad curl_\perp \mathbf{E} = \partial_x E_y - \partial_y E_x.$$

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The first equations can be rewritten as [Nicolopoulos et al., 2020]

$$\mathrm{div}_\perp\left[\mathbb{a}\nabla_\perp B_z\right] = B_z, \tag{1.5}$$

with

$$\mathbb{a} = \frac{1}{\delta^2 - \alpha^2}\begin{pmatrix} \alpha & -i\delta \\ i\delta & \alpha \end{pmatrix}.$$

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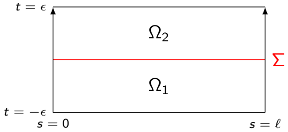
The simplified model

- $\Omega \subset \mathbb{R}^2$ bounded open set, Γ its boundary,
- $\alpha : \Omega \to \mathbb{R}$ with $\Omega_1 := \{\alpha > 0\}$, $\Gamma_1 = \Gamma \cap \partial\Omega_1$ and $\Omega_2 := \{\alpha < 0\}$, $\Gamma_2 = \Gamma \cap \partial\Omega_2$,
- $\Sigma := \partial\Omega_1 \cap \partial\Omega_2 = \{\alpha = 0\}$ is a closed curve inside Ω



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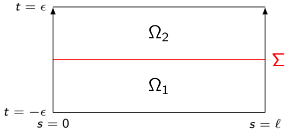

We assume that $\alpha \sim_\Gamma \mathrm{dist}_\Sigma$.

In this domain, we study the degenerate elliptic equation:

$$\begin{cases} -\mathrm{div}(\alpha \nabla u) - u = 0 \quad \text{in } \Omega, \\ \end{cases}$$

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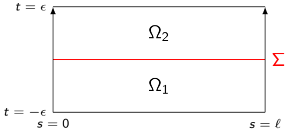

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In this domain, we study the degenerate elliptic equation with $f \in L^2(\Gamma)$:

$$\begin{cases} -\operatorname{div}(\alpha \nabla u) - u = 0 & \text{in } \Omega, \\ \alpha \partial_n u + i\lambda u = f & \text{on } \Gamma. \end{cases} \tag{2.1}$$

Some frameworks

We can add some absorption ν:

$$\begin{cases} -\operatorname{div}((\alpha+i\nu)\nabla u) - u = 0 & \text{in } \Omega, \\ (\alpha+i\nu)\partial_n u + i\lambda u = f & \text{on } \Gamma. \end{cases} \tag{2.2}$$

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Theorem

The problem with absorption is well-posed in $H^1(\Omega)$.

The problem with absorption coincide with the plasma waves model with collision[Després and Weder, 2016].

Some frameworks

Definition

The weighted Sobolev space adapted to obtain regular solutions of (2.1) is:

$$H^1_{1/2}(\Omega_j) := \left\{ f \in L^2(\Omega_j), \quad |\alpha|^{1/2}\nabla f \in L^2(\Omega_j) \right\}, \tag{2.3}$$

with its natural scalar product

$$(u, v)_{H^1_{1/2}(\Omega_j)} = \int_{\Omega_j} (|\alpha|^{1/2}\nabla u \cdot \overline{|\alpha|^{1/2}\nabla v} + u\overline{v}) d\mathbf{x}.$$

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Lemma ([Nicolopoulos et al., 2020])

$H^1_{1/2}(\Omega_j)$ is compactly embedded in $L^2(\Omega_j)$.

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Lemma ([Nicolopoulos et al., 2020])

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The problem without absorption is well-posed in $H^1_{1/2}(\Omega_1) \times H^1_{1/2}(\Omega_2)$.

Limits of the framework

We have

$$\begin{aligned} |\alpha|^{1/2}\nabla u \in L^2(\Omega_j) &\Rightarrow \alpha\nabla u \in L^2(\Omega_j) \\ &\Rightarrow \alpha\nabla u \in H(\mathrm{div},\Omega_j) \\ &\Rightarrow (\alpha\nabla u\cdot\boldsymbol{\nu})_{|\Sigma} \in H^{-1/2}(\Sigma). \end{aligned}$$

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But

Corollary

Let $u_j \in H^1_{1/2}(\Omega_j)$. Then $\alpha\nabla u_j\cdot\nu_{|\Sigma} = 0$.

$\Rightarrow$ We potentially miss solutions.

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- solutions of $\mathrm{div}(x\nabla u) = 0$ with periodical conditions on the rectangle:

$$u^+(x,y) = \begin{cases} A + B\left(\ln|x| - \frac{i\pi}{2}\,\mathrm{sign}(x)\right), & \text{if } k = 0, \\ \left[A\ I_0\left(2k\pi x\right) + B\ K_0\left(2k\pi x + i0^+\right)\right] e^{2ik\pi y}, & \text{for } k \neq 0, \end{cases} \tag{2.4}$$

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The logarithm is singular since

- $|x|^{1/2}\frac{d}{dx}\ln x \notin L^2$
- $x\frac{d}{dx}\ln x = 1$

Summary

Singular solutions

Let $r = \partial_\nu \alpha_{|\Sigma}$ and $\mathrm{s}_g^\nu(x, y) := \frac{g(y)}{r(y)} \left[\log(rx + i\nu) + i\frac{\pi}{2}\right]$ where $g(s)$ is the behavior of the singularity.

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$$\mathrm{s}_g^+(x,y) := \frac{g(y)}{r(y)} \left(\log(|rx|) - \mathrm{sign}(rx)\frac{i\pi}{2}\right). \tag{2.5}$$

Notice that $(\alpha \nabla \mathrm{s}_g^+ \cdot \nu)_{|\Sigma} = g$.

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Ansatz of solution

The solution u^+ is decomposed as

$$u = \left| \begin{array}{ll} u_1 + \mathrm{s}_g^+ & \text{in } \Omega_1 \\ u_2 + \mathrm{s}_g^+ & \text{in } \Omega_2 \end{array} \right. , \tag{2.6}$$

with the parameter $g \in H^1(\Sigma)$ (H^2 in [Nicolopoulos et al., 2020]) and $(u_1, u_2) \in H^1_{1/2}(\Omega_1) \times H^1_{1/2}(\Omega_1)$.

Variational formulation

Problem on the regular part

Given $g \in H^1(\Sigma)$, find $\mathbf{u} = (u_1, u_2) \in Q \coloneqq H^1_{1/2}(\Omega_1) \times H^1_{1/2}(\Omega_1)$ such that for any $\mathbf{v} \in Q$:

$$b(\mathbf{u}, \mathbf{v}) = -\ell_g(\mathbf{v}) + \ell(\mathbf{v}),$$

with

$$b(\mathbf{u}, \mathbf{v}) \coloneqq \sum_{j=1,2} \int_{\Omega_j} \left(\alpha^{1/2} \nabla u_j \cdot \overline{|\alpha|^{1/2} \nabla v_j} - u_j \overline{v_j} \right) d\mathbf{x} + \int_{\Gamma_j} i\lambda u_j \overline{v_j} ds, \quad (2.7)$$

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$$\ell_g(\mathbf{v}) := \sum_{j=1,2} \int_{\Omega_j} \left(\alpha^{1/2} \partial_y \mathrm{s}_g^+ \overline{|\alpha|^{1/2} \partial_y v_j} + \mathrm{d}_g^+ \overline{v_j}\right) d\mathbf{x} + \int_{\Gamma_j} \mathrm{b}_g^+ \overline{v_j} ds, \quad (2.8)$$

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with $\mathrm{d}_g^+ = -\partial_x(\alpha \partial_x \mathrm{s}_g^+) - \mathrm{s}_g^+$ and $\mathrm{b}_g^+ = \alpha \partial_n \mathrm{s}_g^+ + i\lambda \mathrm{s}_g^+$.

Summary

Mixed variational formulation

Using energy estimate, we obtain the mixed variational formulation:

Find $(\mathbf{u}, g, h) \in V$ and $\boldsymbol{\lambda} \in Q$ such that

$$\begin{cases} a^+\left((\mathbf{u}, g, h), (\mathbf{v}, k, l)\right) - \overline{b^+\left((\mathbf{v}, k, l), \boldsymbol{\lambda}\right)} = 0, & \forall (\mathbf{v}, k, l) \in V, \\ b^+\left((\mathbf{u}, g, h), \boldsymbol{\mu}\right) = \ell(\boldsymbol{\mu}), & \forall \boldsymbol{\mu} \in Q. \end{cases}$$

with $V = Q \times H^1(\Sigma) \times H^1(\Sigma)$.

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$b^+\left((\mathbf{u}, g, h), \mathbf{v}\right) = b(\mathbf{u}, \mathbf{v}) + \ell_g(\mathbf{v})$ as a constraint

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and as the differential of the functional to minimize:

$$\begin{aligned} a^+\left((\mathbf{u}, g, h), (\mathbf{v}, k, l)\right) &= \ell_h((\mathbf{v} + \mathrm{s}^+_{k-l})\varphi) - \overline{\ell_l((\mathbf{u} + \mathrm{s}^+_{g-h})\varphi)} \\ &+ F(\mathbf{u} + \mathrm{s}^+_{g-h}, \mathbf{v} + \mathrm{s}^+_{k-l}) - \overline{F(\mathbf{v} + \mathrm{s}^+_{k-l}, \mathbf{u} + \mathrm{s}^+_{g-h})} \end{aligned} \quad (2.9)$$

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with $F(\mathbf{u}, \mathbf{v}) \coloneqq \sum_j \int_{\Omega_j} \alpha^{1/2} u_j \overline{|\alpha|^{1/2} \partial_x v_j} \partial_x \varphi d\mathbf{x}$ and φ a cut-off function varying only in normal direction.

Well-posedness

To understand the well-posedness theorem, we need to introduce some notations. We define:

$B^{+} : V \to Q'$ the linear continuous operator associated with b^{+} such that for any $(\mathbf{v}, k, l) \in V$ and $\boldsymbol{\mu} \in Q$,

$$\left\langle B^{+}(\mathbf{v}, k, l), \boldsymbol{\mu} \right\rangle_{Q', Q} = b^{+}\left((\mathbf{v}, k, l), \boldsymbol{\mu}\right), \tag{2.10}$$

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$A^+ : V \to V'$ the bilinear continuous operator associated with a^+ such that for any $(\mathbf{u}, g, h) \in V$ and $(\mathbf{v}, k, l) \in V$,

$$\left\langle A^+(\mathbf{u}, g, h), (\mathbf{v}, k, l) \right\rangle_{V',V} = a^+\left((\mathbf{u}, g, h), (\mathbf{v}, k, l)\right), \tag{2.11}$$

and $A^+_{KK'} : K \to K'$ its restriction to K the kernel of B^+.

Well-posedness

Using these two operators, the mixed problem can be written as

Find
$(\mathbf{u}, g, h) \in V = Q \times H^1(\Sigma) \times H^1(\Sigma)$, $\boldsymbol{\mu} \in Q = H^1_{1/2}(\Omega_1) \times H^1_{1/2}(\Omega_2)$ s.t.

$$\begin{cases} A(\mathbf{u}, g, h) - {B^+}^\dagger(\boldsymbol{\lambda}) = 0 & \text{in } V', \\ B^+(\mathbf{u}, g, h) = f & \text{in } Q', \end{cases} \tag{2.12}$$

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Theorem ([Boffi et al., 2013, Assous et al., 2017])

The system has a unique solution if and only if

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Well-posedness

Using these two operators, the mixed problem can be written as

Find
$(\mathbf{u}, g, h) \in V = Q \times H^1(\Sigma) \times H^1(\Sigma)$, $\boldsymbol{\mu} \in Q = H^1_{1/2}(\Omega_1) \times H^1_{1/2}(\Omega_2)$ s.t.

$$\begin{cases} A(\mathbf{u}, g, h) - {B^+}^\dagger(\boldsymbol{\lambda}) = 0 & \text{in } V', \\ B^+(\mathbf{u}, g, h) = f & \text{in } Q', \end{cases} \tag{2.12}$$

where ${B^+}^\dagger : Q \to V'$ is the hermitian transpose of B^+.

Theorem ([Boffi et al., 2013, Assous et al., 2017])

The system has a unique solution if and only if

1. $A^+_{KK'}$ *is an isomorphism,*
2. $\mathrm{Im} B^+ = Q'$.

Well-posedness

Definition (Regularized mixed formulation)

Let $a_r^+((\mathbf{u}, g, h), (\mathbf{v}, k, l)) :=$
$a^+((\mathbf{u}, g, h), (\mathbf{v}, k, l)) + i\left(-\rho(g, k)_{H^1(\Sigma)} + \mu(h', l')_{L^2(\Sigma)}\right).$

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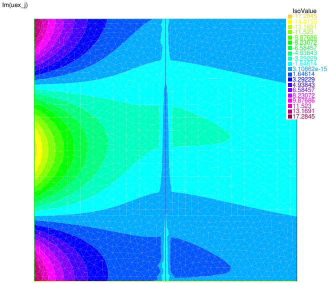
Theorem

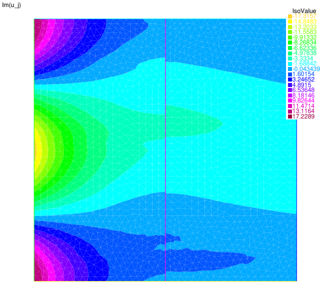
The regularized mixed formulation is well-posed.

Summary

Numerical simulations

Simulation using Xlife++ then Freefem++.





The simulation are currently not stable asymptotically.

Summary

Conclusion

What have been done ?

- Weakening of the regularity of $g \in H^1(\Sigma)$.
- Simulation with P1-Lagrange finite elements.

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What to be done ?

- Weakening of the regularity of $g \in H^1(\Sigma)$.
- Find how stabilize the simulations.
- Study with δ varying.

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