14 DIFFERENCE EQUATIONS I

Objectives

After studying this chapter you should

- be able to detect recursive events within contextual problems;
- be able to recognise and describe associated sequences;
- be able to solve a number of first order difference equations;
- be able to apply solutions of first-order difference equations to contextual problems.

14.0 Introduction

Imagine you are to jump from an aircraft at an altitude of 1000 metres. You want to free-fall for 600 metres, knowing that in successive seconds you fall

5, 15, 25, 35, ... metres.

How many seconds do you count before you pull the rip-cord?

Developing a method for answering this type of question is an aim of this chapter. Perhaps you could attempt the problem now by studying the pattern within the sequence.

The methods employed in this chapter are widely used in applied mathematics, especially in areas such as economics, geography and biology. The above example is physical, but as you will see later there is no need to resort to physical or mechanical principles in order to solve the problem.

Activity 1   Tower of Hanoi

You may be familiar with the puzzle called the **Tower of Hanoi**, in which the object is to transfer a pile of rings from one needle to another, one ring at a time, in as few moves as possible, with never a larger ring sitting upon a smaller one.

The puzzle comes from the Far East, where in the temple of Benares a priest unceasingly moves a disc each day from an original pile of sixty four discs on one needle to another. When he has finished the world will end!
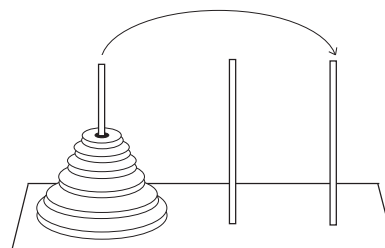
Try this game for yourself. There may be one in school or you could make one. The puzzle is commonly found among mathematics education software.

Record the number of moves required for initial piles of one, two, three rings, etc. Try to predict the number of moves required for 10 rings and 20 rings. When should the world end?

The solution to the problem involves the idea of recursion (from recur - to repeat). The next section considers a further problem through which the ideas of recursion can be explored. You will meet the Tower of Hanoi again, later on.

14.1 Recursion

Here is a simple sequence linked to a triangular dot pattern. Naturally, these are called the triangle numbers 1, 3, 6, 10, 15, ...

In order to obtain the next term (the sixth), one more row of six dots is added to the fifth term. If a term much further down the sequence were required, you could simply keep adding on 7, then 8, then 9 and so on. This process is called **recursion**.

The process can be described algebraically. Call the first term u_1, the second u_2 and the general term u_n, where n is a positive integer.

So $\qquad u_1 = 1$

$\qquad\qquad u_2 = 3$

$\qquad\qquad u_3 = 6$

$\qquad\qquad$ etc.

In order to find u_n you have to add the number n to u_{n-1}. This gives the expression

$$u_n = u_{n-1} + n$$

Expressions of this type are called **difference equations** (or **recurrence relations**).

What processes in life are recursive? How, if at all, does natural recursion differ from mathematical recursion?

In order to verify that this expression determines the sequence of triangle numbers, the term u_6 is found from using the known value of u_5 :

$$u_5 = 15$$

and $\quad u_6 = u_5 + 6$

$$= 15 + 6$$

$$= 21, \text{ as expected.}$$

Example

If $u_1 = 4$ and $u_n = 2u_{n-1} + 3n - 1$, for $n \geq 2$, find the values of u_2 and u_3.

Solution

$$u_2 = 2u_1 + 3 \times 2 - 1$$

$$= 2 \times 4 + 6 - 1$$

$$= 8 + 6 - 1$$

$$= 13$$

and $\quad u_3 = 2u_2 + 3 \times 3 - 1$

$$= 2 \times 13 + 9 - 1$$

$$= 26 + 9 - 1$$

$$= 34.$$

Activity 2   Dot patterns

Draw a number of dot patterns which increase in a systematic way, for example,

For each pattern, write down a difference equation and show that, from knowing u_1, you can use your equation to generate successive terms in your patterns.

What other patterns occur in life? Can they be described using numbers?

Exercise 14A

1. For each equation you are given the first term of a sequence. Find the 4th term in each case :

   (a) $u_1 = 2$ and $u_n = u_{n-1} + 3$, $n \geq 2$

   (b) $u_1 = 1$ and $u_n = 3u_{n-1} + n$, $n \geq 2$

   (c) $u_1 = 0$ and $u_n - u_{n-1} = n + 1$, $n \geq 2$.

2. For each sequence write down a difference equation which describes it:

   (a) 3 5 7 9 11

   (b) 2 5 11 23 47

   (c) 1 2 5 14 41.

3. A vacuum pump removes one third of the remaining air in a cylinder with each stroke. Form an equation to represent this situation. After how many strokes is just 1 / 1 000 000 of the initial air remaining?

4. Write down the first four terms of each of these sequences and the associated difference equation.

   (a) $u_n = \sum_{r=1}^{n} (2r - 1)$ (b) $u_n = \sum_{r=1}^{n} (10 - r)$

   (c) $u_n = \sum_{r=1}^{n} 3(2r + 1)$

5. Write a simple computer program (say in Basic) which calculates successive terms of a sequence from a difference equation you have met. Here is one for the triangle numbers to help you.

   ```
   1O  REM"SEQUENCE OF TRIANGLE NUMBERS"
   20  INPUT"NUMBER OF TERMS REQUIRED";N
   30  CLS
   40  U=0:PRINT"FIRST ";N;" TRIANGLE
       NUMBERS ARE"
   50  FOR i=1 TO N:U=U+i:PRINTU:NEXTi
   60  STOP
   ```

14.2 Iteration

Consider again the Tower of Hanoi.

You should have found a sequence of minimum moves as follows :

Number of rings	1	2	3	4	5	6	...
Number of moves	1	3	7	15	31	63	...

Successive terms are easily found by doubling and adding one to the previous term, but it takes quite a long time to reach the sixty-fourth term, which by the way is about 1.85×10^{19} or 18.5 million million million!

A different approach is to try to work 'backwards' from the nth term u_n, rather than starting at u_1, and building up to it. In this case :

$$u_n = 2u_{n-1} + 1, \ \ n \geq 2 \qquad\qquad (1)$$

For example,

$$u_6 = 2u_5 + 1$$

$$= 2 \times 15 + 1$$

$$= 31.$$

In a similar way, u_{n-1} can be written in terms of u_{n-2} as

$$u_{n-1} = 2u_{n-2} + 1. \tag{2}$$

Then equation (2) can be substituted into equation (1) to give

$$u_n = 2(2u_{n-2} + 1) + 1$$

$$= 4u_{n-2} + 2 + 1.$$

Repeating this process using $u_{n-2} = 2u_{n-3} + 1$ gives

$$u_n = 4(2u_{n-3} + 1) + 2 + 1$$

$$= 8u_{n-3} + 4 + 2 + 1$$

$$\Rightarrow \quad u_n = 2^3 u_{n-3} + 2^2 + 2^1 + 2^0.$$

You can see a pattern developing. Continuing until u_n is expressed in terms of u_1, gives

$$u_n = 2^{n-1} u_1 + 2^{n-2} + 2^{n-3} + \ldots + 2^2 + 2^1 + 2^0$$

$$= 2^{n-1} + 2^{n-2} + \ldots + 2^2 + 2^1 + 2^0, \text{ (as } u_1 = 1\text{)}. \tag{3}$$

You should recognise (3) as a geometric progression (GP) with first term 1 and common ratio 2. Using the formula for the sum to n terms of a GP,

$$u_n = \frac{1(2^n - 1)}{2 - 1} = 2^n - 1, \quad n \geq 1.$$

This is the solution to the differential equation $u_n = 2u_{n-1} + 1$. This process involved the repeated use of a formula and is known as **iteration**.

You can now see how easy it is to calculate a value for u_n.

For example,

$$u_{100} = 2^{100} - 1$$

$$\approx 1.27 \times 10^{30}.$$

Activity 3

You invest £500 in a building society for a number of years at a rate of 10% interest per annum. Find out how much will be in the

bank after 1, 2 or 3 years. Try to write down the difference equation which describes the relationship between the amount in the bank, u_n, at the end of the nth year with the amount u_{n-1} at the end of the previous year. Solve your equation by iteration in the way shown for the Tower of Hanoi problem.

Use your solution to find the amount accrued after 10 years.

How long does it take for your money to double?

Exercise 14B

1. Solve by iteration, giving u_n in terms of u_1.

(a) $u_n = u_{n-1} + 2, \ n \geq 2$

(b) $u_n = 4u_{n-1} - 1, \ n \geq 2$

(c) $u_n = 3u_{n-1} + 2, \ n \geq 2$

2. A population is increasing at a rate of 25 per thousand per year. Define a difference equation which describes this situation. Solve it and find the population in 20 years' time, assuming the population is now 500 million. How long will it take the population to reach 750 million?

3. Calculate the monthly repayment on a £500 loan over 2 years at an interest rate of $1\frac{1}{2}$% per month.

14.3 First order difference equations

Equations of the type $u_n = ku_{n-1} + c$, where k, c are constants, are called **first order linear difference equations** with constant coefficients. All of the equations you have met so far in this chapter have been of this type, except for the one associated with the triangle numbers in Section 14.1

For the triangle numbers $u_n = u_{n-1} + n$, and since n is not constant, this is not a linear difference equation with constant coefficients.

If the equation is of the type

$$u_n = ku_{n-1},$$

then the solution can be found quite simply. You know that $u_n = ku_{n-1}, \ u_{n-1} = ku_{n-2},$ and so on, so that the difference equation becomes

$$u_n = ku_{n-1}$$
$$= k(ku_{n-2})$$
$$= k^2 u_{n-2}$$
$$= k^2(ku_{n-3})$$
$$= k^3 u_{n-3}$$

etc.

$$\Rightarrow \quad u_n = k^{n-1}u_1.$$

So if $u_n = ku_{n-1}$, k constant, $n \geq 2$, then its solution is
$u_n = k^{n-1}u_1$.

Example

Solve $u_n = 5u_{n-1}$, where $u_1 = 2$, and find u_5.

Solution

$$u_n = 5^{n-1}u_1$$
$$= 2 \times 5^{n-1}$$
$$\Rightarrow \quad u_5 = 2 \times 5^4$$
$$= 1250.$$

If the equation is of the type $u_n = ku_{n-1} + c$ then a general
solution can be found as follows :

$$u_n = ku_{n-1} + c$$
$$= k(ku_{n-2} + c) + c$$
$$= k^2 u_{n-2} + kc + c$$
$$= k^2(ku_{n-3} + c) + kc + c$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$= k^{n-1}u_1 + k^{n-2}c + k^{n-3}c + \ldots + kc + c$$
$$= k^{n-1}u_1 + c(1 + k + k^2 + \ldots + k^{n-2}).$$

Again, there is a GP, $1 + k + k^2 + \ldots + k^{n-2}$, to be summed. This

has a first term of 1, $n-1$ terms and a common ratio of k. So, provided that $k \neq 1$, the sum is

$$\frac{1\left(k^{n-1}-1\right)}{k-1}.$$

So if the difference equation is of the form $u_n = ku_{n-1} + c$, where k and c are constant, and $k \neq 1$, then it has solution

$$u_n = k^{n-1}u_1 + \frac{c\left(k^{n-1}-1\right)}{k-1}$$

The case $k = 1$ will be dealt with in a moment.

Example

Solve $u_n = 2u_{n-1} - 3$, $n \geq 2$, given $u_1 = 4$.

Solution

Using the formula above,

$$u_n = 2^{n-1} \times 4 - \frac{3\left(2^{n-1}-1\right)}{2-1}$$

$$= 4 \times 2^{n-1} - 3 \times 2^{n-1} + 3$$

$$= 2^{n-1} + 3.$$

You can see this formula works by finding, say, the value of u_2 from the difference equation as well.

Using the difference equation,

$$u_2 = 2u_1 - 3$$

$$= 2 \times 4 - 3$$

$$= 8 - 3$$

$$= 5.$$

Using the formula,

$$u_2 = 2^1 + 3$$

$$= 5.$$

Special case

In the formula for u_n, k cannot equal one. In this case, when $k = 1$, the difference equation is of the type

$$u_n = u_{n-1} + c, \quad n \geq 2.$$

This has the simple solution

$$\boxed{u_n = u_1 + (n-1)c}$$

This is the type of sequence in the parachute jump problem of Section 14.0. If, in that example, you let u_n be the number of metres fallen after n seconds, then

$$u_n = u_{n-1} + 10, \quad u_1 = 5$$

$$\Rightarrow \quad u_n = 10(n-1) + 5$$

For example,

$$u_4 = 10 \times 3 + 5$$

$$= 30 + 5$$

$$= 35.$$

You can now use the formula to solve the original problem of how long it takes to fall 600 metres.

Notation

In some cases, it is convenient to number the terms of a sequence,

$$u_0, u_1, u_2 \ldots$$

rather than

$$u_1, u_2, u_3, \ldots$$

This will often be the case in the next chapter. The different numbering affects both the difference equation and its solution.

For example, look again at the triangle numbers

$$1, 3, 6, 10, 15, \ldots$$

If these are denoted by u_1, u_2, u_3, ..., then, for example, $u_3 = 6$, $u_4 = 10$, $u_4 = u_3 + 4$, and in general $u_n = u_{n-1} + n$. The solution turns out to be

$$u_n = \tfrac{1}{2} n(n+1).$$

If, however, you start again and instead denote that sequence of triangle numbers by u_0, u_1, u_2, ..., then $u_2 = 6$, $u_3 = 10$, $u_3 = u_2 + 4$, in general $u_n = u_{n-1} + (n+1)$. In this case the solution is given by

$$u_n = \tfrac{1}{2}(n+1)(n+2)$$

(which, of course, could be obtained from the previous general solution by replacing n by $n+1$).

If you choose to write sequences beginning with the term u_0, then the solutions to difference equations of the form

$$u_n = ku_{n-1} + c$$

alter somewhat as shown below.

If $\qquad u_n = ku_{n-1} + c, \quad n \geq 1$,

then $\qquad \boxed{u_n = k^n u_0 + \dfrac{c(k^n - 1)}{k-1}, \quad k \neq 1}$

or, if $u_n = u_{n-1} + c$, then $u_n = u_0 + nc$.

Example

Find the solution of the equation $u_n = 3u_{n-1} + 4$, given $u_0 = 2$.

Solution

$$\begin{aligned}
u_n &= 3^n u_0 + \frac{4(3^n - 1)}{3-1} \\
&= 3^n \times 2 + 2(3^n - 1) \\
&= 2 \times 3^n + 2 \times 3^n - 2 \\
&= 4 \times 3^n - 2.
\end{aligned}$$

In the example above the **particular solution** to the difference equation $u_n = 3u_{n-1} + 4$ when $u_0 = 2$ has been found.

If you had not substituted for the value of u_0 (perhaps not knowing u_0) then a **general solution** could have been given as

$$u_n = 3^n u_0 + 2(3^n - 1).$$

This solution is valid for all sequences which have the same difference equation whatever the initial term u_0.

Example

Find the general solution of the difference equation
$$u_n = u_{n-1} + 4, \quad n \geq 1.$$

Solution

$$u_n = u_0 + 4n.$$

Exercise 14C

1. Write down the general solutions of :

    (a) $u_n = 4u_{n-1} + 2, \quad n \geq 2$

    (b) $u_n = 4u_{n-1} + 2, \quad n \geq 1$

    (c) $u_n = 3u_{n-1} - 5, \quad n \geq 1$

    (d) $u_{n+1} = u_n + 6, \quad n \geq 0$

    (e) $u_n = u_{n-1} - 8, \quad n \geq 2$

    (f) $u_n = -2u_{n-1} + 4, \quad n \geq 1$

    (g) $u_n + 3u_{n-1} - 2 = 0, \quad n \geq 1$

    (h) $u_n + 4u_{n-1} + 3 = 0, \quad n \geq 1$

    (i) $u_n = 4u_{n-1}, \quad n \geq 2$

    (j) $u_{n+1} = 4u_n - 5, \quad n \geq 0$

2. Find the particular solutions of these equations :

    (a) $u_0 = 1$ and $u_n = 3u_{n-1} + 5, \quad n \geq 1$

    (b) $u_1 = 3$ and $u_n = -2u_{n-1} + 6, \quad n \geq 2$

    (c) $u_0 = 4$ and $u_n = u_{n-1} - 3, \quad n \geq 1$

    (d) $u_1 = 0$ and $u_{n+1} = 5u_n + 3, \quad n \geq 1$

    (e) $u_0 = 3$ and $u_{n+1} = u_n + 7, \quad n \geq 0$

    (f) $u_0 = 1$ and $u_n + 3u_{n-1} = 1, \quad n \geq 1$.
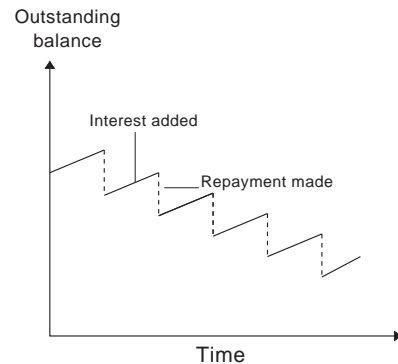
14.4 Loans

Activity 4

Find out about the repayments on a loan from a bank, building society or other lending agency. You will need to know the rate of interest per annum, the term of the loan and the frequency of the repayment.

Use these variables in order to set up a difference equation :

u_n = amount in £ owing after n repayments

k = interest multiplier

c = repayment in £

It will be of the type described in Section 14.2.

Outstanding balance

Interest added

Repayment made

Time

Solve your equation and then evaluate c, the repayment. Remember that there is nothing owing after the final repayment has been made.

How does your result compare with the repayments specified by the lending agency?

___

The activity is quite difficult. So if you need help an example of a similar type follows.

Example

Find the monthly repayment on a £500 loan at an interest rate of 24% p.a. over 18 months.

Solution

Let u_n be the amount owing after n months, so $u_0 = £500$ and c is the repayment each month. You are given that the interest rate per annum is 24%, so the interest rate per month is assumed to be 2%. (Can you see why this is an approximation?) Then

$$u_n = u_{n-1} + (2\% \text{ of } u_{n-1}) - c.$$

Here, the interest has been added to the previous outstanding loan, and one month's repayment subtracted.

$$\Rightarrow \quad u_n = u_{n-1} + 0.02u_{n-1} - c$$

$$u_n = 1.02\,u_{n-1} - c.$$

Remembering that you started with u_0, and noting the switch to a negative c, the general solution of this type of equation is

$$u_n = k^n u_0 + \frac{c\left(k^n - 1\right)}{(k-1)}$$

where in this case $k = 1.02$.

So
$$u_n = (1.02)^n u_0 - \frac{c\left(1.02^n - 1\right)}{1.02 - 1}$$

$$= 1.02^n u_0 - 50c\left(1.02^n - 1\right)$$

or
$$u_n = 1.02^n u_0 - 50c\left(1.02^n - 1\right).$$

At the end of the term $n = 18$, so

$$u_{18} = 1.02^{18} u_0 - 50c\left(1.02^{18} - 1\right).$$

But $u_0 = 500$ and $u_{18} = 0$ (the loan has been paid off)

$$\Rightarrow \quad 0 = 500 \times 1.02^{18} - 50c\left(1.02^{18} - 1\right)$$

$$\Rightarrow \quad c = \frac{500 \times 1.02^{18}}{50\left(1.02^{18} - 1\right)}$$

$$\approx \frac{714.123}{21.4123}$$

$$\approx £33.35.$$

Repayments are made at the rate of £33.35 per month.

In the example on the previous page, as in many situations involving loans, the monthly rate is taken as one twelfth of the annual rate. Is the rate of 24% p.a. justifiable as a measure of the interest paid? What is the significance of an APR?

Exercise 14D

1. (a) Find the general solution of
   $u_n = 3u_{n-1} + 4, \, n \geq 2$

   (b) Find the general solution of
   $u_n = \frac{1}{2}u_{n-1} + 2, \, n \geq 2$

   (c) Solve $q_n = q_{n-1} + 3$, given $q_3 = 7$, $n \geq 2$

   (d) Solve $a_n - 2a_{n-1} = 0$, given $a_1 = 4$, $n \geq 2$

   (e) Solve $b_n = 4b_{n-1} + 5$, given $b_1 = 2$, $n \geq 2$

2. Form and solve the difference equation associated with the sequence
   7, 17, 37, 77, 157 ...

3. Find the monthly repayment on a £400 loan over a period of $1\frac{1}{2}$ years at an interest rate of 15% p.a.

4. Find the monthly repayment on a £2000 loan over a term of 3 years at an interest rate of 21% p.a.

5. Write a computer program to solve the type of problem in Question 3.

6. A steel works is increasing production by 1% per month from a rate of 2000 tonnes per month. Orders (usage of the steel) remain at 1600 tonnes per month. How much steel will be stock-piled after periods of 12 months and 2 years?

7. A loan of £1000 is taken out at an interest rate of 24% p.a. How long would it take to repay the loan at a rate of £50 per month.

   [**Note**: this problem is similar to Question 3, except that the value of *n* cannot easily be found after you have solved the difference equation. A search technique on your calculator or a simple computer program will be necessary.]

14.5 Non-homogeneous linear equations

So far you have met equations of the form

$$u_n = ku_{n-1}, \qquad k \text{ constant}$$

This is called a **homogeneou**s equation, involving terms in u_n, and u_{n-1} only.

You have also solved equations of the form

$$u_n = ku_{n-1} + c, \quad k, c \text{ constant}$$

This is a **non-homogeneous** equation, due to the extra term c, but k and c are still constant.

Earlier you met the difference equation associated with the triangle numbers

$$u_n = u_{n-1} + n$$

This is again a non-homogeneous equation, but the term n is not constant.

In general, equations of the type $u_n = ku_{n-1} + f(n)$ are difficult to solve. Expanding u_n as a series is occasionally successful - it depends usually on how complicated $f(n)$ is. In the next chapter you will meet two methods for solving these equations, and also second order equations of a similar type (these involve terms in u_n, u_{n-1} and u_{n-2}). One method is based on trial and error and the other on the use of **generating functions**. Simpler equations such as the one for the triangle numbers can be solved quite quickly by expanding u_n as a series. Thus

$$u_n = (u_{n-2} + n - 1) + n$$

$$= u_{n-2} + (n-1) + n$$

$$= (u_{n-3} + n - 2) + (n-1) + n$$

$$= \dots \quad \dots \quad \dots \dots \dots$$

$$= 1 + 2 + 3 + \dots + (n-1) + n.$$

This is an **arithmetic progression** (AP). Summing gives

$$u_n = \frac{n}{2}(n+1).$$

You can check this result by evaluating, say, u_5.

Activity 5

Try out this method on one of the 'dot' patterns which you designed in Activity 2.

The problems in the following exercise should all yield to a method similar to the one above. The function $f(n)$ will be restricted to the form n, n^2 or k^n, where k is a constant. This will also be true of later difference equations in the next chapter.

Exercise 14E

1. Solve $u_n = u_{n-1} + n$ if $u_1 = 5$.

2. Find the general solution in terms of u_1 :

   (a) $u_n = u_{n-1} + n^2$ (b) $u_n = u_{n-1} + 2^n$

   *(c) $u_n = 2u_{n-1} + n$

3. If $u_n = ku_{n-1} + 5$ and $u_1 = 4$, $u_2 = 17$ find the values of k and u_6.

4. The productivity of an orchard of 2000 trees increases by 5% each year due to improved farming techniques. The farmer also plants a further 100 trees per year. Estimate the percentage improvement in productivity during the next 10 years.

14.6 A population problem

Changes in population can often be modelled using difference equations. The underlying problems are similar to some of the financial problems you have met in this section.

What factors influence demographic change?

Activity 6

The birth and death rates in a country are 40 per thousand and 15 per thousand per year respectively. The initial population is 50 million.

Form a difference equation which gives the population at the end of a year in relation to that at the end of the previous year. Solve the equation and estimate the population in 10 years time.

If, due to the high birth rate, emigration takes place at a rate of 10 000 per year, how will this change your results?

Population analysis can be refined by taking into account many more factors than in the above problem. In particular, the population pyramid shows that different age group sizes affect the whole population in different ways.

Suppose the population of a country is split into two age groups:

> **Group 1**   consisting of the 0 - 12 year olds, and

> **Group 2**   consisting of the rest,

and assume that births only occur in Group 2. Each group will have its own death rate.

Define $p_1(t)$ as the population of the 0 - 12 group in year t

$p_2(t)$ as the population of the 13+ group in year t

b as the birth rate

d_1 as the death rate in the 0 - 12 group

d_2 as the death rate in the 13+ group.

One further assumption made is that in each year one twelfth of the survivors from Group 1 progress to Group 2.

For **Group 1**

$$p_1(t+1) \;=\; \underbrace{b\,p_2(t)}_{\substack{\text{those bor n} \\ \text{to people in} \\ \text{Group 2}}} \;+\; \underbrace{\tfrac{11}{12}p_1(t)\left(1-d_1\right)}_{\substack{\text{of those who survive,} \\ \tfrac{11}{12}\text{ remain in Group 1}}}$$

Remember that $\frac{1}{12}$ of the survivors of Group 1 transfer to Group 2 each year.

For **Group 2**

$$p_2(t+1) = \tfrac{1}{12}p_1(t)(1-d_1) + p_2(t)(1-d_2)$$

Using matrices, both equations can be more simply written as

$$\begin{bmatrix} p_1(t+1) \\ p_2(t+1) \end{bmatrix} = \begin{bmatrix} \tfrac{11}{12}(1-d_1) & b \\ \tfrac{1}{12}(1-d_1) & 1-d_2 \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix}$$

Let $P_t \equiv \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix}$

and
$$A \;=\; \begin{bmatrix} \frac{11}{12}(1-d_1) & b \\ \frac{1}{12}(1-d_1) & 1-d_2 \end{bmatrix}$$

then
$$P_{t+1} = AP_t.$$

This is a difference equation using matrices! Using the earlier solution to this type of equation you can see that

$$P_t \;=\; A^t P_0$$

where P_0 is the initial population .

Suppose that $p_1(0) = 5$ million, $p_2(0) = 15$ million, and that the population parameters are

$$b \;=\; 0.04$$

$$d_1 \;=\; 0.016$$

$$d_2 \;=\; 0.03$$

then the above solution for P_t becomes

$$P_t \;=\; \begin{bmatrix} \frac{11}{12}(1-0.016) & 0.04 \\ \frac{1}{12}(1-0.016) & 1-0.03 \end{bmatrix}^t \begin{bmatrix} 5 \\ 15 \end{bmatrix}$$

$$=\; \begin{bmatrix} 0.902 & 0.04 \\ 0.082 & 0.97 \end{bmatrix}^t \begin{bmatrix} 5 \\ 15 \end{bmatrix}$$

For example, the population after one year can be calculated by simple matrix multiplication :

$$P_1 \;=\; \begin{bmatrix} 0.902 & 0.04 \\ 0.082 & 0.97 \end{bmatrix} \begin{bmatrix} 5 \\ 15 \end{bmatrix}$$

$$=\; \begin{bmatrix} 5.11 \\ 14.96 \end{bmatrix}$$

The total population is therefore 20.07 million.

If the population in 10 years is required, then it is fairly straightforward to evaluate A^{10} using a simple program for matrix multiplication with a computer or a graphic/ programmable calculator.

Activity 7

Produce a simple program which will multiply matrices as required in the above example. Use it to evaluate the population in 10 years time.

Activity 8

Find a population pyramid with associated birth and death rates for a country of your choice and model the population growth (or decay) as in the above example.

You may wish to consider more than two age groupings. This may prove quite difficult and you will need to adapt your program to multiply larger matrices.

14.7 Miscellaneous Exercises

1. Find the general solution to these difference equations in terms of u_1:

   (a) $u_n = 2u_{n-1}$ (b) $u_n - 3u_{n-1} = 3$

   (c) $u_n - 3u_{n-1} = n$

2. By writing down a difference equation and solving it, find the tenth term of this sequence:

   $$2 \quad 4 \quad 10 \quad 28 \quad 82 \quad 244.$$

3. Find the monthly repayment on a loan of £600 over a period of 12 months at a rate of interest of 3% per month.

4. The population of a country is $12\frac{1}{2}$ million. The birth rate is 0.04, the death rate is 0.03 and 50 000 immigrants arrive in the country each year. Estimate the population in 20 years' time.

5. In a round robin tournament every person (or team) plays each of the others. If there are n players, how many more games are needed if one more player is included? Use this result to set up a difference equation, solve it, and then evaluate the number of games needed for 20 players (this confirms the answer found to Question 4(b) in Exercise 6A).

6. If $u_n = pu_{n-1} + q$, $n \geq 1$, and $u_1 = 2$, $u_2 = 3$, $u_3 = 7$, find the value of u_6.

7. Compare the monthly repayment on a mortgage of £30 000 at an interest rate of 12% p.a. over 25 years for the two standard types of mortgage :

   (a) a **repayment** mortgage, where you pay a fixed amount each month, the interest is calculated each month on the remaining debt, and the amount of repayment is calculated so that the debt is paid off after 25 years;

   (b) an **endowment** mortgage where the loan stays at £30 000 for the whole of the 25 years, you pay the interest on that and an additional £40 a month for a type of insurance policy known as an endowment policy. At the end of the 25 years the insurance policy matures and the insurance firm pays off the debt.

   (You will find that the latter costs more, but in practice when the insurance policy matures it pays well over the £30 000, thus giving you an additional lump sum back.)

8. A population of 100 million can be divided into age groups. **Group 1**, 0-16 years, has a death rate of 0.025 (no birth rate) and a population of 60 million. **Group 2**, 17+ years, has a birth rate of 0.04 and a death rate of 0.03. Investigate the growth/decay of the population. Make a prediction for the population size in 3 years' time.

9. A person is repaying a loan of £5000 at £200 per month. The interest rate is 3% per month. How long will it take to repay the loan?

10. Within a population of wild animals the birth rate is 0.2, while the death rate is 0.4. Zoos worldwide are attempting to reintroduce animals. At present the population is 5000 and 100 animals per year are being introduced. The rate of increase of introduction is 20% per year. Will the animals survive in the long run?

11. Investigate the problem of the Tower of Hanoi with four needles for the rings.

12. Use iteration to solve the recurrence relation

$$u_n = \frac{u_{n-1}}{(n+2)}, \ n \geq 2$$

subject to the initial condition $u_1 = \frac{1}{6}$.