Robust PCA

Team 8

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Motivation

M = L + S

M - data matrix, L - low-rank matrix, S - sparse matrix

Can we hope to recover both accurately and efficiently ?

Classical Principal Component Analysis

$$M = L_0 + N_0$$

$$\begin{array}{ll} \text{minimize} & \|M - L\|_F \\ \text{subject to} & \operatorname{rank}(L) \leq k. \end{array}$$

Robust PCA - Principal Component Pursuit

But a single grossly corrupted entry in M could render the estimated L arbitrarily far from the true L0

Solution: **Robust PCA**

$$M = L_0 + S_0$$

S_0 entries can have large magnitude, support is sparse but unknown

Tracktable Convex Optimization:

$$\begin{array}{ll} \text{minimize} & \|L\|_* + \lambda \|S\|_1 \\ \text{subject to} & L + S = M \end{array}$$

$$\|M\|_* := \sum_i \sigma_i(M)$$

$$\|M\|_1 = \sum_{ij} |M_{ij}|$$

Applications

- Video Surveillance
- Face Recognition
- Latent Semantic Indexing
- Ranking and Collaborative Filtering

When Does Separation Make Sense

1) Suppos $M = e_1 e_1^*$

So, matrix is both sparse and low-rank

Assumption (1):

$$L_0 = U\Sigma V^* = \sum_{i=1}^{r} \sigma_i u_i v_i^*,\ L_0 \in \mathbb{R}^{n_1 \times n_2}$$

Incoherence condition [2]: $\max_i \|U^* e_i\|^2 \le \frac{\mu r}{n_1}, \quad \max_i \|V^* e_i\|^2 \le \frac{\mu r}{n_2}, \quad \|UV^*\|_\infty \le \sqrt{\frac{\mu r}{n_1 n_2}}.$

2) Suppose sparse matrix has low-rank

Then $dim(Im(M_0 = L_0 + S_0)) \in dim(Im(L_0))$

Assumption (2):

The sparsity pattern of the sparse component is selected uniformly at random

[2] E. J. Cand`es and B. Recht. Exact matrix completion via convex optimzation

Main Result

Below, $n_{(1)} = max(n_1, n_2)$ and $n_{(2)} = min(n_1, n_2)$

Theorem

Suppose L_0 is $n \times n$, obeys assumptions (1) and (2), and that the support of S_0 is unformly distributed among all sets of cardinality m. Then there is a numerical constant c such that with probability at least $1 - cn^{-10}$ (over the choise of support of S_0), Principal Component Pursuit with $\lambda = \frac{1}{\sqrt{n}}$ is exact, i.e. $\hat{L} = L_0$ and $\hat{S} = S_0$, provided that

$$rank(L_0) \leq \rho_r n \mu^{-1} (\log(n))^{-2},$$

$$m \leq \rho_s n^2$$

Above, $\rho_r and \rho_s$ are positive numerical constants. In the general rectangular case where L_0 is $n_1 \times n_2$, PCP with $\lambda = \frac{1}{\sqrt{n_{(1)}}}$ succeeds with probability at least $1 - cn_{(1)}^{-10}$, provided that $rank(L_0) \leq \rho_r n_{(2)} \mu^{-1} (\log(n_{(1)})^{-2})$ and $m \leq \rho_s n_1 n_2$

Algorithm

Principal Component Pursuit

$$\begin{array}{ll} \text{minimize} & \|L\|_* + \lambda \|S\|_1 \\ \text{subject to} & L + S = M \end{array}$$

Augmented Lagrange Multiplier (ALM) method [3]:

$$l(L, S, Y) = \|L\|_* + \lambda \|S\|_1 + \langle Y, M - L - S \rangle + \frac{\mu}{2} \|M - L - S\|_F^2$$

$$\arg\min_{L,S} l(L, S, Y_k)$$

$$\mathcal{S}_\tau[x] = \operatorname{sgn}(x) \max(|x| - \tau, 0) \implies \arg\min_S l(L, S, Y) = \mathcal{S}_{\lambda\mu^{-1}}(M - L + \mu^{-1} Y)$$

$$X = U \Sigma V^* \quad \mathcal{D}_\tau(X) = U \mathcal{S}_\tau(\Sigma) V^* \implies \arg\min_L l(L, S, Y) = \mathcal{D}_{\mu^{-1}}(M - S + \mu^{-1} Y)$$

[3] D.P. Bertsekas. Constrained Optimization and Lagrange Multiplier Method

Algorithm

Algorithm 1 (Principal Component Pursuit by Alternating Directions

initialize: $S_0 = Y_0 = 0, \mu > 0.$
while not converged **do**
  compute $L_{k+1} = \mathcal{D}_{\mu^{-1}}(M - S_k + \mu^{-1} Y_k);$
  compute $S_{k+1} = \mathcal{S}_{\lambda\mu^{-1}}(M - L_{k+1} + \mu^{-1} Y_k);$
  compute $Y_{k+1} = Y_k + \mu(M - L_{k+1} - S_{k+1});$
end while
output: $L, S.$

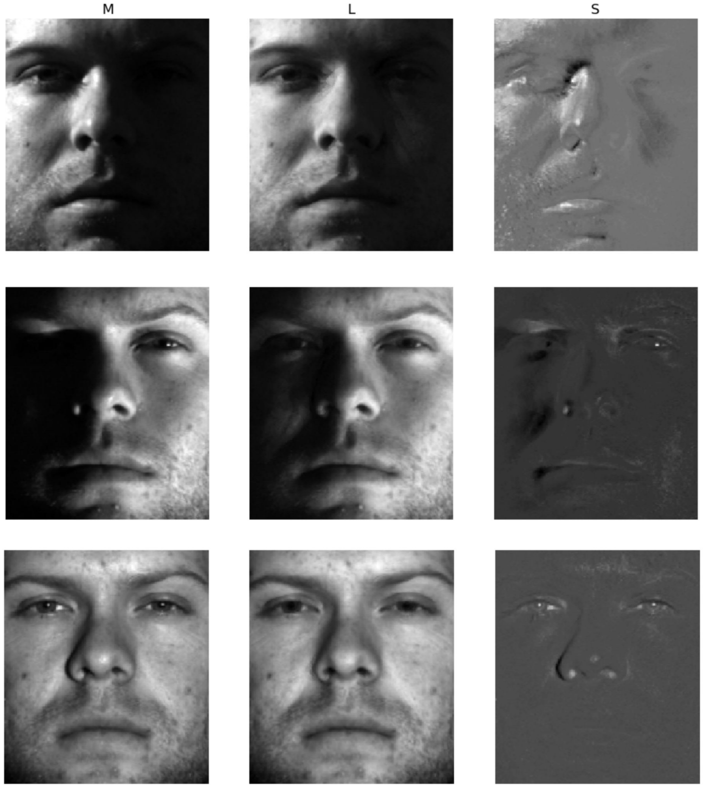
Numerical Experiments (1)

Background modeling from surveillance video

200 frames
Wall time: 6 min 14s
Rank(L) = 90

Numerical Experiments (2)

Removing shadows and specularities from face image



Numerical Experiments (2)

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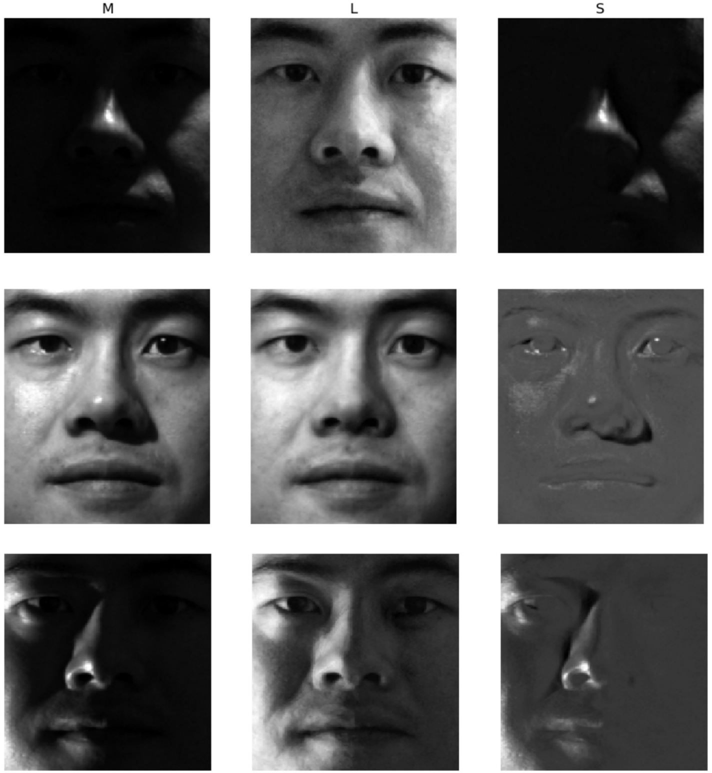

References

1) Robust Principal Component Analysis. Emmanuel J. Cand`es, Xiaodong Li, Yi Ma, and John Wright, December 17, 2009.

2) E. J. Cand`es and B. Recht. Exact matrix completion via convex optimzation. Found. of Comput. Math., 9:717–772, 2009.

3) D.P. Bertsekas. Constrained Optimization and Lagrange Multiplier Method. Academic Press, 1982.

QA

Thank you!

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