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Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 1

Monday 31 October 2022 (afternoon)

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all questions.
- Answers must be written within the answer boxes provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

23 pages

8822 – 7201
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Please **do not** write on this page.

Answers written on this page
will not be marked.

Answers must be written within the answer boxes provided. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 5]

Sergio is interested in whether an adult's favourite breakfast berry depends on their income level. He obtains the following data for 341 adults and decides to carry out a χ^2 test for independence, at the 10% significance level.

		Income level		
		Low	**Medium**	**High**
Favourite berry	**Strawberry**	21	39	30
	Blueberry	39	67	42
	Other berry	32	45	26

(a) Write down the null hypothesis. [1]

(b) Find the value of the χ^2 statistic. [2]

The critical value of this χ^2 test is 7.78.

(c) Write down Sergio's conclusion to the test in context. Justify your answer. [2]

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2. [Maximum mark: 5]

Celeste heated a cup of coffee and then let it cool to room temperature. Celeste found the coffee's temperature, T, measured in $^{\circ}\mathrm{C}$, could be modelled by the following function,

$$T(t) = 71\,\mathrm{e}^{-0.0514t} + 23\,,\ t \geq 0\,,$$

where t is the time, in minutes, after the coffee started to cool.

(a) Find the coffee's temperature 16 minutes after it started to cool. [2]

(b) Write down the room temperature. [1]

(c) Given that $T^{-1}(50) = k$, find the value of k. [2]

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3. [Maximum mark: 5]

An artificial ski slope can be modelled as a triangular prism, as shown in the diagram. Rectangle ABCD is horizontal, and rectangle CDEF is vertical.

diagram not to scale

The maximum height of the ski slope, CF, is 17 metres and the steepest angle of the ski slope, $F\hat{B}C$, is 21°.

(a) Calculate the length of [BF]. [2]

The width of the base of the ski slope, AB, is 44 metres. Mayumi skis in a straight line, starting from point E and finishing at the base of the ski slope.

(b) Find the value of the least steep angle that Mayumi can ski. [3]

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4. [Maximum mark: 7].

In a competition, a contestant has to move through a maze to find treasure. A graph of the maze is shown below, where each edge represents a corridor in the maze. The contestant starts at S and the treasure is located at T.

(a) Complete the adjacency matrix, $\boldsymbol{M}$, for the graph. [2]

$$\begin{array}{c} \\ S \\ A \\ B \\ C \\ D \\ T \end{array} \begin{array}{c} \begin{array}{cccccc} S & A & B & C & D & T \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & \square & 0 \\ 1 & 0 & 1 & 1 & \square & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ \square & \square & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

The competition rules state that the contestant can walk along a maximum of four corridors.

(b) Find the number of walks from S to T with a maximum of 4 edges. [4]

(c) Explain why the number of ways the contestant can reach the treasure is less than the answer to part (b). [1]

(This question continues on the following page)

(Question 4 continued)

5. [Maximum mark: 7]

Taizo plays a game where he throws one ball at two bottles that are sitting on a table.
The probability of knocking over bottles, in any given game, is shown in the following table.

Number of bottles knocked over	0	1	2
Probability	0.5	0.4	0.1

(a) Taizo plays two games that are independent of each other. Find the probability that Taizo knocks over a **total** of two bottles. [4]

In any given game, Taizo will win k points if he knocks over two bottles, win 4 points if he knocks over one bottle and lose 8 points if no bottles are knocked over.

(b) Find the value of k such that the game is fair. [3]

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6. [Maximum mark: 6]

A cat runs inside a circular exercise wheel, making the wheel spin at a constant rate in an anticlockwise direction. The height, h cm, of a fixed point, P, on the wheel can be modelled by $h(t) = a\sin(bt) + c$ where t is the time in seconds and a, b, $c \in \mathbb{R}^+$.

When $t = 0$, point P is at a height of 78 cm.

(a) Write down the value of c. [1]

When $t = 4$, point P first reaches its maximum height of 143 cm.

(b) Find the value of

(i) a.

(ii) b. [3]

(c) Write down the minimum height of point P. [1]

Later, the cat is tired, and it takes twice as long for point P to complete one revolution at a new constant rate.

(d) Write down the new value of b. [1]

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7. [Maximum mark: 7]

On 1 December 2022, Laviola invests 800 euros (EUR) into a savings account which pays a nominal annual interest rate of $7.5\,\%$ compounded monthly. At the end of each month, Laviola deposits an additional $\mathrm{EUR}\ 500$ into the savings account.

At the end of k months, Laviola will have saved enough money to withdraw $\mathrm{EUR}\ 10\,000$.

(a) Find the smallest possible value of k, for $k \in \mathbb{Z}^+$. [4]

(b) For this value of k, find the interest earned in the savings account. Express your answer correct to the nearest EUR. [3]

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8. [Maximum mark: 7]

Line L_1 has a vector equation $\boldsymbol{r} = \begin{pmatrix} 3p+4 \\ 2p-1 \\ p+9 \end{pmatrix}$, where $p \in \mathbb{R}$.

Line L_2 has a vector equation $\boldsymbol{r} = \begin{pmatrix} q-2 \\ 1-q \\ 2q+1 \end{pmatrix}$, where $q \in \mathbb{R}$.

The two lines intersect at point M.

(a) Find the coordinates of M. [3]

(b) Find the acute angle between the two lines. [4]

9. [Maximum mark: 8]

The transformation T is represented by the matrix $\boldsymbol{M} = \begin{pmatrix} 2 & -4 \\ 3 & 1 \end{pmatrix}$.

A pentagon with an area of $12\,\text{cm}^2$ is transformed by T.

(a) Find the area of the image of the pentagon. [2]

Under the transformation T, the image of point X has coordinates $(2t - 3,\ 6 - 5t)$, where $t \in \mathbb{R}$.

(b) Find, in terms of t, the coordinates of X. [6]

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10. [Maximum mark: 7]

Stars are classified by their brightness. The brightest stars in the sky have a magnitude of 1. The magnitude, m, of another star can be modelled as a function of its brightness, b, relative to a star of magnitude 1, as shown by the following equation.

$$m = 1 - 2.5 \log_{10}(b)$$

The star called Acubens has a brightness of 0.0525.

(a) Find the magnitude of Acubens. [2]

Ceres has a magnitude of 7 and is the least bright star visible without magnification.

(b) Find the brightness of Ceres. [2]

The star Proxima Centauri has a greater magnitude than the planet Neptune. The difference in their magnitudes is 3.2.

(c) Find how many times brighter Neptune is compared to Proxima Centauri. [3]

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Answers written on this page
will not be marked.

11. [Maximum mark: 5]

Gloria wants to model the curved edge of a butterfly wing. She inserts a photo of the wing into her graphing software and finds the coordinates of four points on the edge of the wing.

x	y
–3	2
–1	3.2
1.7	2.1
3.1	0

Gloria thinks a cubic curve will be a good model for the butterfly wing.

(a) Find the equation of the cubic regression curve for this data. [2]

For the photo of a second butterfly wing, Gloria finds the equation of the regression curve is $y = 0.0083x^3 - 0.075x^2 - 0.58x + 2.2$.

Gloria realizes that her photo of the second butterfly is an enlargement of the life-size butterfly, scale factor 2 and centred on $(0, 0)$.

(b) Find the equation of the cubic curve that models the life-size wing. [3]

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12. [Maximum mark: 5]

Four possible phase portraits for the coupled differential equations $\frac{\mathrm{d}x}{\mathrm{d}t} = ax + by$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = cx + dy$ are shown, labelled A, B, C and D.

The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has eigenvalues λ_1 and λ_2.

(a) Complete the following table by writing down the letter of the phase portrait that best matches the description. [3]

Description	Phase portrait
$\lambda_1 = 2$ with eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\lambda_2 = 3$ with eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$	
$\lambda_1 = 2$ with eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\lambda_2 = -3$ with eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$	
$\lambda_1 = -2$ with eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\lambda_2 = 3$ with eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$	

(This question continues on the following page)

(Question 12 continued)

(b) On the following axes, sketch the phase portrait that corresponds to $\lambda_1 = -2 + 3\mathrm{i}$ and $\lambda_2 = -2 - 3\mathrm{i}$, given that $\frac{\mathrm{d}y}{\mathrm{d}t} = -12$ at $(3, 0)$. [2]

13. [Maximum mark: 6]

Gen is investigating the relationship between two sets of data, labelled P and Q, that she collected. She created a scatter plot with P on the x-axis and $\log_{10} Q$ on the y-axis. Gen noticed that the points had a strong linear correlation, so she drew a line of best fit, as shown in the diagram. The line passes through the points $(0, -1.3)$ and $(6, 14.9)$.

(a) Find an equation for Q in terms of P. [3]

Gen also investigates the relationship between the same data, Q, and some new data, R. She believes that the data can be modelled by $Q = a \ln R + b$ and she decides to create a scatter plot to verify her belief.

(b) State what expression Gen should plot on each axis to verify her belief. [1]

The scatter plot has a linear relationship and Gen finds $a = 4.3$ and $b = 12.1$.

(c) Find an equation for P in terms of R. [2]

(This question continues on the following page)

(Question 13 continued)

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14. [Maximum mark: 9]

A particle moves such that its velocity, v metres per second, at time t seconds, is given by $v = t\sin(t^2)$.

(a) Find an expression for the acceleration of the particle. [2]

(b) Hence, or otherwise, find its greatest acceleration for $0 \le t \le 8$. [2]

The particle starts at the origin.

(c) Find an expression for the displacement of the particle. [3]

(d) Hence show that the particle never has a negative displacement. [2]

15. [Maximum mark: 5]

An electrical circuit contains a capacitor. The charge on the capacitor, q Coulombs, at time t seconds, satisfies the differential equation

$$\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} + 5\frac{\mathrm{d}q}{\mathrm{d}t} + 20q = 200 .$$

Initially $q = 1$ and $\frac{\mathrm{d}q}{\mathrm{d}t} = 8$.

Use Euler's method with $h = 0.1$ to estimate the maximum charge on the capacitor during the first second.

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16. [Maximum mark: 8]

The principal of a school is concerned that only 30% of her students are choosing healthy options from the school canteen. She organizes a campaign to promote healthy eating and decides to test if the campaign has increased the number of students choosing healthy options. She assumes that a student's choice is independent of other students' choices.

(a) Write down suitable hypotheses for this test. [2]

The principal decides to take a random sample of 80 students. She will reject the null hypothesis if at least 31 students choose a healthy option.

(b) Find the probability that she makes a Type I error. [3]

In fact, the campaign led to 40% of her students choosing a healthy option.

(c) Find the probability that she makes a Type II error. [3]

17. [Maximum mark: 8]

The time of sunrise, R hours after midnight, in Taipei can be modelled by

$$R = 1.08\cos(0.0165t + 0.413) + 4.94,$$

where t is the day of the year 2021 (for example, $t = 2$ represents 2 January 2021).

The time of sunset, S hours after midnight, in Taipei can be modelled by

$$S = 1.15\cos(0.0165t - 2.97) + 18.9.$$

The number of daylight hours, D, in Taipei during 2021 can be modelled by

$$D = a\cos(0.0165t + b) + c.$$

(a) Find the value of a, of b and of c. [6]

(b) Hence, or otherwise, find the largest number of daylight hours in Taipei during 2021 and the day of the year on which this occurs. [2]

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References:

11. Fleur, 2019. *photo-1560263816-d704d83cce0f.* [image online] Available at: https://unsplash.com/photos/SE2zTdS1MNo [Accessed 8 February 2022]. Source adapted.

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Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 1

30 October 2023

Zone A afternoon | **Zone B** afternoon | **Zone C** afternoon

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all questions.
- Answers must be written within the answer boxes provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

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1. [Maximum mark: 7]

The growth of a particular type of seashell is being studied by Manon. At the end of each month Manon records the increase in the width of a seashell since the end of the previous month.

She models the monthly increase in the width of the seashell by a geometric sequence with common ratio 0.8. In the first month, the width of the seashell increases by $4\,\text{mm}$.

(a) Find by how much the width of the seashell will increase during the third month, according to her model. [2]

(b) Find the total increase in the width of the seashell, predicted by Manon's model, during the first year. [2]

Manon's seashell had a width of $25\,\text{mm}$ at the beginning of the first month.

(c) Find the maximum possible width of the seashell, predicted by Manon's model. [3]

2. [Maximum mark: 7]

The graph of the function f is given in the following diagram.

(a) Write down $f(2)$. [1]

(b) On the axes, sketch $y = f^{-1}(x)$. [2]

The function g is defined as $g(x) = 3x - 1$.

(c) Find an expression for $g^{-1}(x)$. [2]

(d) Find a value of x where $f^{-1}(x) = g^{-1}(x)$. [2]

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3. [Maximum mark: 6]

The Great Pyramid of Giza is the oldest of the Seven Wonders of the Ancient World. When it was built, 4500 years ago, the measurements of the pyramid were in Royal Egyptian Cubits (REC).

[Source: Nina Aldin Thune. https://en.wikipedia.org/wiki/Great_Pyramid_of_Giza#/media/File:Kheops-Pyramid.jpg. Licensed under CC BY 2.5 https://creativecommons.org/licenses/by/2.5/#. Image adapted.]

Viktor reads online that $1\,\mathrm{REC}$ is equal to 0.52 metres, rounded to two decimal places.

(a) Write down the upper and lower bounds of $1\,\mathrm{REC}$ in metres. [2]

The Great Pyramid of Giza has a square base with side lengths of $440\,\mathrm{REC}$ and a height of $280\,\mathrm{REC}$. Viktor assumes that these two measurements are exact and that the Great Pyramid can be modelled as a square-based pyramid with smooth faces.

(b) Find the minimum possible volume of the pyramid in cubic metres. [4]

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4. [Maximum mark: 6]

Jason sails his boat from point A for a distance of $100\,\text{km}$, on a bearing of 112°, to arrive at point B. He then sails on a bearing of 041° to point C. Jason's journey is shown in the diagram.

diagram not to scale

(a) Find $A\hat{B}C$. [2]

Point C is directly east of point A.

(b) Calculate the distance that Jason sails to return directly from point C to point A. [4]

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5. [Maximum mark: 9]

A sports player on a horizontal athletic field hits a ball. The height of the ball above the field, in metres, after it is hit can be modelled using a quadratic function of the form $f(x) = ax^2 + bx + c$, where x represents the horizontal distance, in metres, that the ball has travelled from the player.

A specialized camera tracks the initial path of the ball after it is hit by the player. The camera records that the ball travels through the three points $(3, 2.36)$, $(10, 5)$ and $(17, 7.16)$, as shown in **Diagram 1**.

diagram not to scale

Diagram 1

(a) Use the coordinates $(3, 2.36)$ to write down an equation in terms of a, b, and c. [1]

(b) Use your answer to part (a) and two similar equations to find the equation of the quadratic model for the height of the ball. [3]

A 4-metre-high fence is 80 metres from where the player hit the ball, as shown in **Diagram 2**.

diagram not to scale

Diagram 2

(c) Show that the model predicts that the ball will go over the fence. [3]

(d) Find the horizontal distance that the ball will travel, from the player until it first hits the ground, according to this model. [2]

(This question continues on the following page)

(Question 5 continued)

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6. [Maximum mark: 8]

On the following Voronoi diagram, the coordinates of three farmhouses are $A(0, 3)$, $B(8, 3)$ and $C(8, 13)$, where distances are measured in kilometres. Each farmhouse owns the land that is closest to it, and their boundaries are defined by the dotted lines on **Diagram 1**.

Diagram 1

Diagram 2

To provide water to the farms it is decided to construct a well at the point where the boundaries meet on **Diagram 1**.

(a) Write down the coordinates of this point. [1]

(b) Find the equation of the perpendicular bisector of $[AC]$. [3]

An additional farmhouse $D(6, 5)$ is built on the land. The Voronoi diagram has been redrawn to show the new boundaries. The coordinates of the vertices of these boundaries are indicated on **Diagram 2**.

A wind turbine is to be built at one of the vertices.

(c) The wind turbine should be as far from the nearest farmhouses as possible.

(i) By calculating appropriate distances, find the location of the wind turbine.

(ii) Hence, write down the distance of the wind turbine to the nearest farmhouse. [4]

(This question continues on the following page)

(Question 6 continued)

7. [Maximum mark: 7]

Michèle is playing a game. In the game, she must first flip a fair coin which will result in the coin landing on heads or tails. If the coin lands on heads, then she wins a prize. If it lands on tails, then she has another chance but this time she must roll a fair six-sided dice and get a five or six in order to win a prize.

(a) Complete the tree diagram by writing in the three missing probabilities. [2]

(b) Find the probability that Michèle does **not** win a prize. [2]

(c) Given that Michèle won a prize, find the probability that the coin landed on heads. [3]

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8. [Maximum mark: 6]

Given $z = \sqrt{3} - \mathrm{i}$.

(a) Write z in the form $z = r\mathrm{e}^{\theta \mathrm{i}}$, where $r \in \mathbb{R}^+$, $-\pi < \theta \leq \pi$. [2]

Let $z_1 = \mathrm{e}^{2t\mathrm{i}}$ and $z_2 = 2\mathrm{e}^{\left(2t - \frac{\pi}{6}\right)\mathrm{i}}$.

(b) Find $\mathrm{Im}(z_1 + z_2)$ in the form $p\sin(2t + q)$, where $p > 0$, $t \in \mathbb{R}$ and $-\pi \leq q \leq \pi$. [4]

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9. [Maximum mark: 8]

Joon is a keen surfer and wants to model waves passing a particular point P, which is off the shore of his favourite beach. Joon sets up a model of the waves in terms of $h(t)$, the height of the water in metres, and t, the time in seconds from when he begins recording the height of the water at point P.

The function has the form $h(t) = p\cos\left(\frac{\pi}{6}t\right) + q$, $t \geq 0$.

(a) Find the values of p and q. [2]

(b) Find

(i) $h'(t)$.

(ii) $h''(t)$. [3]

Joon will begin to surf the wave when the rate of change of h with respect to t, at P, is at its maximum. This will first occur when $t = k$.

(c) (i) Find the value of k.

(ii) Find the height of the water at this time. [3]

(This question continues on the following page)

(Question 9 continued)

10. [Maximum mark: 7]

The decay of a chemical isotope over five years is recorded in **Table 1**. The mass of the chemical M is measured to the nearest gram at the beginning of each year t of the experiment.

Table 1

Time t (years)	1	2	3	4	5
Mass M (grams)	1000	660	517	435	381

It is believed that the decay of the isotope can be modelled by an equation of the form $M = a \times t^b$.

(a) Use power regression on your graphic display calculator to find the value of a and the value of b. [2]

The values of t and M can be transformed such that $x = \ln t$ and $y = \ln M$. **Table 2** shows data for x and y to three decimal places.

Table 2

x	0	0.693	1.099	1.386	1.609
y	6.908	6.492	6.248	6.075	5.943

(b) Find the linear regression equation of y on x, in the form $y = cx + d$. Give the values of c and d to three decimal places. [2]

(c) Hence, show that this linear regression is equivalent to the power regression found in part (a). [3]

(This question continues on the following page)

(Question 10 continued)

11. [Maximum mark: 7]

Consider the function $f(x) = x^3 - x$, for $-1 \le x \le 1$. The shaded region, R, is bounded by the graph of $y = f(x)$ and the x-axis.

(a) (i) Write down an integral that represents the area of R.

(ii) Find the area of R. [2]

Another function, g, is defined such that $g(x) = 2f(x-1)$.

(b) On the following set of axes, the graph of $y = f(x)$ has been drawn. On the same set of axes, sketch the graph of $y = g(x)$. [2]

The region R from the original graph $y = f(x)$ is rotated through 2π radians about the x-axis to form a solid.

(c) Find the volume of the solid. [3]

(This question continues on the following page)

(Question 11 continued)

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12. [Maximum mark: 4]

Consider the differential equation $\frac{dy}{dx} = 3x - y + 1$.

(a) Find the equation of the tangent to the solution curve at the point $(-1, -1)$ in the form $ax + by + c = 0$. [2]

The slope field for this differential equation is shown in the following diagram.

(b) Sketch the solution curve that passes through the point $(-1, -1)$. [2]

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13. [Maximum mark: 8]

The velocity v of a particle at time t, as it moves along a straight line, can be modelled by the piecewise function

$$v(t) = \begin{cases} u_1(t), & 0 \le t \le T \\ u_2(t), & t \ge T \end{cases}$$

where $u_1(t) = 2t^2 - t^3$ and $u_2(t) = 8 - 4t$. It is required that $u_1(T) = u_2(T)$.

(a) Find the value of T. [2]

(b) Show that $u_1'(T) = u_2'(T)$. [2]

The displacement of the particle at time $t = 0$ is zero.

(c) Find the time when the particle returns to its initial position. [4]

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14. [Maximum mark: 7]

A straight line L has vector equation $\boldsymbol{r} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and point Q has coordinates $(11, -1, 3)$.

Point P is the point on L closest to Q.

(a) Find the coordinates of P. [4]

(b) Find a vector that is perpendicular to both L and the line passing through points P and Q. [3]

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15. [Maximum mark: 7]

The eating habits of students in a school are studied over a number of months. The focus of the study is whether non-vegetarians become vegetarians, and whether vegetarians remain vegetarians.

Each month, students choose between the vegetarian or non-vegetarian lunch options. Once they have chosen for the month, they cannot change the option until the next month.

In any month during the study, it is noticed that the probability of a non-vegetarian becoming vegetarian the following month is 0.1, and that the probability of a vegetarian remaining a vegetarian the following month is 0.8.

This situation can be represented by the transition matrix

$$\boldsymbol{T} = \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix}.$$

(a) Interpret the value 0.9 in $\boldsymbol{T}$ in terms of the changes in the eating habits of the students in the school. [1]

(b) Find the eigenvalues of matrix $\boldsymbol{T}$. [3]

One of the eigenvectors of $\boldsymbol{T}$ is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

(c) Find another, non-parallel, eigenvector and interpret it in context. [3]

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16. [Maximum mark: 6]

When Jef plays basketball, the number of shots he takes during any 6 minutes of play can be modelled by a Poisson distribution with mean 2.5.

(a) Find the probability that Jef takes less than 7 shots during any 12 minutes of play. [2]

It can be assumed that the outcomes of the shots are independent of each other, and the probability of success of a shot is constant. The probability that Jef is successful with a shot is 0.4.

It can be assumed that the probability of Jef's success with a shot is independent of the number of shots that he takes.

(b) Find the probability that during any 6 minutes of play Jef takes fewer than 4 shots and is successful at least once. [4]

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References:

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Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 1

Monday 1 November 2021 (afternoon)

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all questions.
- Answers must be written within the answer boxes provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

25 pages

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Answers must be written within the answer boxes provided. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 6]

Dilara is designing a kite ABCD on a set of coordinate axes in which one unit represents $10\,\mathrm{cm}$.

The coordinates of A, B and C are $(2,\ 0)$, $(0,\ 4)$ and $(4,\ 6)$ respectively. Point D lies on the x-axis. $[\mathrm{AC}]$ is perpendicular to $[\mathrm{BD}]$. This information is shown in the following diagram.

diagram not to scale

(a) Find the gradient of the line through A and C. [2]

(b) Write down the gradient of the line through B and D. [1]

(c) Find the equation of the line through B and D. Give your answer in the form $ax + by + d = 0$, where a, b and d are integers. [2]

(d) Write down the x-coordinate of point D. [1]

(This question continues on the following page)

(Question 1 continued)

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2. [Maximum mark: 5]

Inspectors are investigating the carbon dioxide emissions of a power plant. Let R be the rate, in tonnes per hour, at which carbon dioxide is being emitted and t be the time in hours since the inspection began.

When R is plotted against t, the total amount of carbon dioxide produced is represented by the area between the graph and the horizontal t-axis.

The rate, R, is measured over the course of two hours. The results are shown in the following table.

t	0	0.4	0.8	1.2	1.6	2
R	30	50	60	40	20	50

(a) Use the trapezoidal rule with an interval width of 0.4 to estimate the total amount of carbon dioxide emitted during these two hours. [3]

The real amount of carbon dioxide emitted during these two hours was 72 tonnes.

(b) Find the percentage error of the estimate found in part (a). [2]

3. [Maximum mark: 7]

Let the function $h(x)$ represent the height in centimetres of a cylindrical tin can with diameter $x\,\text{cm}$.

$$h(x) = \frac{640}{x^2} + 0.5 \text{ for } 4 \le x \le 14.$$

(a) Find the range of h. [3]

The function h^{-1} is the inverse function of h.

(b) (i) Find $h^{-1}(10)$.

(ii) In the context of the question, interpret your answer to part (b)(i).

(iii) Write down the range of h^{-1}. [4]

4. [Maximum mark: 6]

There are four stations used by the fire wardens in a national forest.

On the following Voronoi diagram, the coordinates of the stations are $A(6, 2)$, $B(14, 2)$, $C(18, 6)$ and $D(10.8, 11.6)$ where distances are measured in kilometres.

The dotted lines represent the boundaries of the regions patrolled by the fire warden at each station. The boundaries meet at $P(10, 6)$ and $Q(13, 7)$.

To reduce the areas of the regions that the fire wardens patrol, a new station is to be built within the quadrilateral $ABCD$. The new station will be located so that it is as far as possible from the nearest existing station.

(a) Show that the new station should be built at P. [3]

The Voronoi diagram is to be updated to include the region around the new station at P. The edges defined by the perpendicular bisectors of $[AP]$ and $[BP]$ have been added to the following diagram.

(b) (i) Write down the equation of the perpendicular bisector of $[PC]$.

(ii) Hence draw the missing boundaries of the region around P on the following diagram. [3]

(This question continues on the following page)

(Question 4 continued)

5. [Maximum mark: 9]

In this question, give all answers correct to 2 decimal places.

Raul and Rosy want to buy a new house and they need a loan of $170\,000$ Australian dollars (AUD) from a bank. The loan is for 30 years and the annual interest rate for the loan is $3.8\,\%$, compounded monthly. They will pay the loan in fixed monthly instalments at the end of each month.

(a) Find the amount they will pay the bank each month. [3]

(b) (i) Find the amount Raul and Rosy will still owe the bank at the end of the first 10 years.

(ii) Using your answers to parts (a) and (b)(i), calculate how much interest they will have paid in total during the first 10 years. [6]

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6. [Maximum mark: 5]

An infinite geometric sequence, with terms u_n, is such that $u_1 = 2$ and $\sum_{k=1}^{\infty} u_k = 10$.

(a) Find the common ratio, r, for the sequence. [2]

(b) Find the least value of n such that $u_n < \frac{1}{2}$. [3]

7. [Maximum mark: 8]

A group of 120 students sat a history exam. The cumulative frequency graph shows the scores obtained by the students.

(a) Find the median of the scores obtained. [1]

The students were awarded a grade from 1 to 5, depending on the score obtained in the exam. The number of students receiving each grade is shown in the following table.

Grade	1	2	3	4	5
Number of students	6	13	26	a	b

(b) Find an expression for a in terms of b. [2]

(This question continues on the following page)

(Question 7 continued)

(c) The mean grade for these students is 3.65.

(i) Find the number of students who obtained a grade 5.

(ii) Find the minimum score needed to obtain a grade 5. [5]

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8. [Maximum mark: 5]

Juri skis from the top of a hill to a finishing point at the bottom of the hill. She takes the shortest route, heading directly to the finishing point (F).

Let $h(x)$ define the height of the hill above F at a horizontal distance x from the starting point at the top of the hill.

The graph of the **derivative** of $h(x)$ is shown below. The graph of $h'(x)$ has local minima and maxima when x is equal to a, c and e. The graph of $h'(x)$ intersects the x-axis when x is equal to b, d, and f.

(a) (i) Identify the x value of the point where $|h'(x)|$ has its maximum value.

(ii) Interpret this point in the given context. [2]

(This question continues on the following page)

(Question 8 continued)

Juri starts at a height of 60 metres and finishes at F, where $x = f$.

(b) Sketch a possible diagram of the hill on the following pair of coordinate axes. [3]

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Answers written on this page
will not be marked.

9. [Maximum mark: 5]

Katie likes to cycle to work as much as possible. If Katie cycles to work one day then she has a probability of 0.2 of not cycling to work on the next work day. If she does not cycle to work one day then she has a probability of 0.4 of not cycling to work on the next work day.

(a) Complete the following transition diagram to represent this information. [2]

Katie works for 180 days in a year.

(b) Find the probability that Katie cycles to work on her final working day of the year. [3]

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10. [Maximum mark: 4]

The graph of $y = f(x)$ is given on the following set of axes. The graph passes through the points $(-2,\ 6)$ and $(0,\ 1)$, and has a horizontal asymptote at $y = 0$.

Let $g(x) = 2f(x-2) + 4$.

(a) Find $g(0)$. [2]

(b) On the same set of axes draw the graph of $y = g(x)$, showing any intercepts and asymptotes. [2]

(This question continues on the following page)

(Question 10 continued)

11. [Maximum mark: 6]

The following diagram shows a corner of a field bounded by two walls defined by lines L_1 and L_2. The walls meet at a point A, making an angle of 40°.

Farmer Nate has $7\,\mathrm{m}$ of fencing to make a triangular enclosure for his sheep. One end of the fence is positioned at a point B on L_2, $10\,\mathrm{m}$ from A. The other end of the fence will be positioned at some point C on L_1, as shown on the diagram.

diagram not to scale

He wants the enclosure to take up as little of the current field as possible.

Find the minimum possible area of the triangular enclosure ABC. [6]

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12. [Maximum mark: 5]

The following table shows the time, in days, from December 1st and the percentage of Christmas trees in stock at a shop on the beginning of that day.

Days since December 1st (d)	1	3	6	9	12	15	18
Percentage of Christmas trees left in stock (x)	100	51	29	21	18	16	14

The following table shows the natural logarithm of both d and x on these days to 2 decimal places.

ln (d)	0	1.10	1.79	2.20	2.48	2.71	2.89
ln (x)	4.61	3.93	3.37	3.04	2.89	2.77	2.64

(a) Use the data in the second table to find the value of m and the value of b for the regression line, $\ln x = m(\ln d) + b$. [2]

(b) Assuming that the model found in part (a) remains valid, estimate the percentage of trees in stock when $d = 25$. [3]

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13. [Maximum mark: 7]

The slope field for the differential equation $\frac{dy}{dx} = e^{-x^2} - y$ is shown in the following two graphs.

(a) Calculate the value of $\frac{dy}{dx}$ at the point $(0, 1)$. [1]

(b) Sketch, on the first graph, a curve that represents the points where $\frac{dy}{dx} = 0$. [2]

(c) On the second graph,

(i) sketch the solution curve that passes through the point $(0, 0)$.

(ii) sketch the solution curve that passes through the point $(0, 0.75)$. [4]

(This question continues on the following page)

(Question 13 continued)

14. [Maximum mark: 7]

On Paul's farm, potatoes are packed in sacks labelled $50\,\text{kg}$. The weights of the sacks of potatoes can be modelled by a normal distribution with mean weight $49.8\,\text{kg}$ and standard deviation $0.9\,\text{kg}$.

(a) Find the probability that a sack is under its labelled weight. [2]

(b) Find the lower quartile of the weights of the sacks of potatoes. [2]

The sacks of potatoes are transported in crates. There are 10 sacks in each crate and the weights of the sacks of potatoes are independent of each other.

(c) Find the probability that the total weight of the sacks of potatoes in a crate exceeds $500\,\text{kg}$. [3]

15. [Maximum mark: 9]

The following diagram shows a frame that is made from wire. The total length of wire is equal to $15\,\text{cm}$. The frame is made up of two identical sectors of a circle that are parallel to each other. The sectors have angle θ radians and radius $r\,\text{cm}$. They are connected by $1\,\text{cm}$ lengths of wire perpendicular to the sectors. This is shown in the diagram below.

(a) Show that $r = \dfrac{6}{2+\theta}$. [2]

The faces of the frame are covered by paper to enclose a volume, V.

(b) (i) Find an expression for V in terms of θ.

(ii) Find the expression $\dfrac{\mathrm{d}V}{\mathrm{d}\theta}$.

(iii) Solve algebraically $\dfrac{\mathrm{d}V}{\mathrm{d}\theta} = 0$ to find the value of θ that will maximize the volume, V. [7]

16. [Maximum mark: 9]

A ship S is travelling with a constant velocity, $\boldsymbol{v}$, measured in kilometres per hour, where

$$\boldsymbol{v} = \begin{pmatrix} -12 \\ 15 \end{pmatrix}.$$

At time $t = 0$ the ship is at a point $\mathrm{A}(300,\ 100)$ relative to an origin O, where distances are measured in kilometres.

(a) Find the position vector $\overrightarrow{\mathrm{OS}}$ of the ship at time t hours. [1]

A lighthouse is located at a point $(129,\ 283)$.

(b) Find the value of t when the ship will be closest to the lighthouse. [6]

An alarm will sound if the ship travels within 20 kilometres of the lighthouse.

(c) State whether the alarm will sound. Give a reason for your answer. [2]

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17. [Maximum mark: 7]

The sides of a bowl are formed by rotating the curve $y = 6\ln x$, $0 \le y \le 9$, about the y-axis, where x and y are measured in centimetres. The bowl contains water to a height of h cm.

(a) Show that the volume of water, V, in terms of h is $V = 3\pi\left(e^{\frac{h}{3}} - 1\right)$. [5]

(b) Hence find the maximum capacity of the bowl in cm^3. [2]

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Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 1

Friday 6 May 2022 (afternoon)

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all questions.
- Answers must be written within the answer boxes provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

20 pages

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Answers must be written within the answer boxes provided. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 5]

The height of a baseball after it is hit by a bat is modelled by the function

$$h(t) = -4.8t^2 + 21t + 1.2$$

where $h(t)$ is the height in metres above the ground and t is the time in seconds after the ball was hit.

(a) Write down the height of the ball above the ground at the instant it is hit by the bat. [1]

(b) Find the value of t when the ball hits the ground. [2]

(c) State an appropriate domain for t in this model. [2]

2. [Maximum mark: 5]

The ticket prices for a concert are shown in the following table.

Ticket Type	**Price (in Australian dollars, \$)**
Adult	15
Child	10
Student	12

- A total of 600 tickets were sold.
- The total amount of money from ticket sales was $\$7816$.
- There were twice as many adult tickets sold as child tickets.

Let the number of adult tickets sold be x, the number of child tickets sold be y, and the number of student tickets sold be z.

(a) Write down three equations that express the information given above. [3]

(b) Find the number of each type of ticket sold. [2]

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3. [Maximum mark: 7]

Leo is investigating whether a six-sided die is fair. He rolls the die 60 times and records the observed frequencies in the following table:

Number on die	1	2	3	4	5	6
Observed frequency	8	7	6	15	12	12

Leo carries out a χ^2 goodness of fit test at a 5% significance level.

(a) Write down the null and alternative hypotheses. [1]

(b) Write down the degrees of freedom. [1]

(c) Write down the expected frequency of rolling a 1. [1]

(d) Find the p-value for the test. [2]

(e) State the conclusion of the test. Give a reason for your answer. [2]

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4. [Maximum mark: 7]

Karl has three brown socks and four black socks in his drawer. He takes two socks at random from the drawer.

(a) Complete the tree diagram. [2]

(b) Find the probability that Karl takes two socks of the same colour. [2]

(c) Given that Karl has two socks of the same colour find the probability that he has two brown socks. [3]

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5. [Maximum mark: 7]

A ball is dropped from a height of 1.8 metres and bounces on the ground. The maximum height reached by the ball, after each bounce, is 85% of the previous maximum height.

diagram not to scale

(a) Show that the maximum height reached by the ball after it has bounced for the sixth time is $68\,\text{cm}$, to the nearest cm. [2]

(b) Find the number of times, after the first bounce, that the maximum height reached is greater than $10\,\text{cm}$. [2]

(c) Find the total **vertical** distance travelled by the ball from the point at which it is dropped until the fourth bounce. [3]

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6. [Maximum mark: 8]

A vertical pole stands on a sloped platform. The bottom of the pole is used as the origin, O, of a coordinate system in which the top, F, of the pole has coordinates $(0, 0, 5.8)$. All units are in metres.

diagram not to scale

The pole is held in place by ropes attached at F.

One of these ropes is attached to the platform at point $\mathrm{A}(3.2, 4.5, -0.3)$. The rope forms a straight line from A to F.

(a) Find $\overrightarrow{\mathrm{AF}}$. [1]

(b) Find the length of the rope. [2]

(c) Find $\mathrm{F\hat{A}O}$, the angle the rope makes with the platform. [5]

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7. [Maximum mark: 4]

A slope field for the differential equation $\frac{dy}{dx} = x^2 + \frac{y}{2}$ is shown.

Some of the solutions to the differential equation have a local maximum point and a local minimum point.

(a) (i) Write down the equation of the curve on which all these maximum and minimum points lie.

(ii) Sketch this curve on the slope field. [2]

The solution to the differential equation that passes through the point $(0, -2)$ has both a local maximum point and a local minimum point.

(b) On the slope field, sketch the solution to the differential equation that passes through $(0, -2)$. [2]

(This question continues on the following page)

(Question 7 continued)

8. [Maximum mark: 6]

Consider the curve $y = 2x(4 - e^x)$.

(a) Find

(i) $\frac{dy}{dx}$.

(ii) $\frac{d^2y}{dx^2}$. [4]

The curve has a point of inflexion at (a, b).

(b) Find the value of a. [2]

9. [Maximum mark: 6]

A company produces bags of sugar with a labelled weight of $1\,\text{kg}$. The weights of the bags are normally distributed with a mean of $1\,\text{kg}$ and a standard deviation of $100\,\text{g}$. In an inspection, if the weight of a randomly chosen bag is less than $950\,\text{g}$ then the company fails the inspection.

(a) Find the probability that the company fails the inspection. [2]

A statistician in the company suggests it would be fairer if the company passes the inspection when the mean weight of five randomly chosen bags is greater than $950\,\text{g}$.

(b) Find the probability of passing the inspection if the statistician's suggestion is followed. [4]

10. [Maximum mark: 7]

The following Argand diagram shows a circle centre 0 with a radius of 4 units.

A set of points, $\{z_\theta\}$, on the Argand plane are defined by the equation

$$z_\theta = \frac{1}{2}\theta e^{\theta i}, \text{ where } \theta \geq 0.$$

(a) Plot on the Argand diagram the points corresponding to

(i) $\theta = \frac{\pi}{2}$.

(ii) $\theta = \pi$.

(iii) $\theta = \frac{3\pi}{2}$. [3]

Consider the case where $|z_\theta| = 4$.

(b) (i) Find this value of θ.

(ii) For this value of θ, plot the approximate position of z_θ on the Argand diagram. [4]

(This question continues on the following page)

(Question 10 continued)

11. [Maximum mark: 5]

The matrix $\boldsymbol{M} = \begin{pmatrix} 0.2 & 0.7 \\ 0.8 & 0.3 \end{pmatrix}$ has eigenvalues -0.5 and 1.

(a) Find an eigenvector corresponding to the eigenvalue of 1. Give your answer in the form $\begin{pmatrix} a \\ b \end{pmatrix}$, where $a, b \in \mathbb{Z}$. [3]

A switch has two states, A and B. Each second it either remains in the same state or moves according to the following rule: If it is in state A it will move to state B with a probability of 0.8 and if it is in state B it will move to state A with a probability of 0.7.

(b) Using your answer to (a), or otherwise, find the long-term probability of the switch being in state A. Give your answer in the form $\frac{c}{d}$, where $c, d \in \mathbb{Z}^+$. [2]

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12. [Maximum mark: 8]

The strength of earthquakes is measured on the Richter magnitude scale, with values typically between 0 and 8 where 8 is the most severe.

The Gutenberg–Richter equation gives the average number of earthquakes per year, N, which have a magnitude of at least M. For a particular region the equation is

$$\log_{10} N = a - M, \text{ for some } a \in \mathbb{R}.$$

This region has an average of 100 earthquakes per year with a magnitude of at least 3.

(a) Find the value of a. [2]

The equation for this region can also be written as $N = \frac{b}{10^M}$.

(b) Find the value of b. [2]

Within this region the most severe earthquake recorded had a magnitude of 7.2.

(c) Find the average number of earthquakes in a year with a magnitude of at least 7.2. [1]

The number of earthquakes in a given year with a magnitude of at least 7.2 can be modelled by a Poisson distribution, with mean N. The number of earthquakes in one year is independent of the number of earthquakes in any other year.

Let Y be the number of years between the earthquake of magnitude 7.2 and the next earthquake of at least this magnitude.

(d) Find $\mathrm{P}(Y > 100)$. [3]

13. [Maximum mark: 5]

At $1{:}00\ \text{pm}$ a ship is $1\,\text{km}$ east and $4\,\text{km}$ north of a harbour. A coordinate system is defined with the harbour at the origin. The position vector of the ship at $1{:}00\ \text{pm}$ is given by $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$.

The ship has a constant velocity of $\begin{pmatrix} 1.2 \\ -0.6 \end{pmatrix}$ kilometres per hour (km h^{-1}).

(a) Write down an expression for the position vector $\boldsymbol{r}$ of the ship, t hours after $1{:}00\ \text{pm}$. [1]

(b) Find the time at which the bearing of the ship from the harbour is 045°. [4]

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14. [Maximum mark: 8]

(a) (i) Expand $\left(\frac{1}{u}+1\right)^2$.

(ii) Find $\int\left(\frac{1}{(x+2)}+1\right)^2 \mathrm{d}x$.

[4]

The region bounded by $y=\frac{1}{(x+2)}+1$, $x=0$, $x=2$ and the x-axis is rotated through 2π about the x-axis to form a solid.

(b) Find the volume of the solid formed. Give your answer in the form $\frac{\pi}{4}(a+b\ln(c))$, where a, b, $c\in\mathbb{Z}$.

[4]

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15. [Maximum mark: 7]

The number of cars arriving at a junction in a particular town in any given minute between $9{:}00\ \text{am}$ and $10{:}00\ \text{am}$ is historically known to follow a Poisson distribution with a mean of 5.4 cars per minute.

A new road is built near the town. It is claimed that the new road has decreased the number of cars arriving at the junction.

To test the claim, the number of cars, X, arriving at the junction between $9{:}00\ \text{am}$ and $10{:}00\ \text{am}$ on a particular day will be recorded. The test will have the following hypotheses:

H_0: the mean number of cars arriving at the junction has not changed,
H_1: the mean number of cars arriving at the junction has decreased.

The alternative hypothesis will be accepted if $X \leq 300$.

(a) Assuming the null hypothesis to be true, state the distribution of X. [1]

(b) Find the probability of a Type I error. [2]

(c) Find the probability of a Type II error, if the number of cars now follows a Poisson distribution with a mean of 4.5 cars per minute. [4]

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16. [Maximum mark: 7]

The wind chill index W is a measure of the temperature, in $^\circ\mathrm{C}$, felt when taking into account the effect of the wind.

When Frieda arrives at the top of a hill, the relationship between the wind chill index and the speed of the wind v in kilometres per hour $(\mathrm{km\,h^{-1}})$ is given by the equation

$$W = 19.34 - 7.405v^{0.16}$$

(a) Find an expression for $\frac{\mathrm{d}W}{\mathrm{d}v}$. [2]

When Frieda arrives at the top of a hill, the speed of the wind is 10 kilometres per hour and increasing at a rate of $5\,\mathrm{km\,h^{-1}\,minute^{-1}}$.

(b) Find the rate of change of W at this time. [5]

17. [Maximum mark: 8]

A function f is of the form $f(t) = pe^{q\cos(rt)}$, p, q, $r \in \mathbb{R}^+$. Part of the graph of f is shown.

The points A and B have coordinates A(0, 6.5) and B(5.2, 0.2), and lie on f.

The point A is a local maximum and the point B is a local minimum.

Find the value of p, of q and of r.

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Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 1

8 May 2023

Zone A afternoon | **Zone B** morning | **Zone C** afternoon

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all questions.
- Answers must be written within the answer boxes provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

23 pages

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Answers must be written within the answer boxes provided. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 5]

On 1 January 2022, Mina deposited $\$1000$ into a bank account with an annual interest rate of 4%, compounded monthly. At the end of January, and the end of every month after that, she deposits $\$100$ into the same account.

(a) Calculate the amount of money in her account at the start of 2024. Give your answer to two decimal places. [3]

(b) Find how many complete months, counted from 1 January 2022, it will take for Mina to have more than $\$5000$ in her account. [2]

2. [Maximum mark: 6]

Carys believes that, on a memory retention test, the mean score of bilingual people (μ_b) will be higher than the mean score of monolingual people (μ_m). Carys gave a memory retention test to a random sample of students in her class. The results are shown in the two tables.

	Scores									
Bilingual	100	94	100	90	100	94	98	98	98	98

	Scores							
Monolingual	97	92	88	98	88	94	100	100

Carys performs a one-tailed t-test at a 5% level of significance. It is assumed that the scores are normally distributed and the samples have equal variances.

(a) State the null and alternative hypotheses. [2]

(b) Calculate the p-value for this test. [2]

(c) State the conclusion of the test in the context of the question. Justify your answer. [2]

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3. [Maximum mark: 5]

Three towns have positions $A(35, 26)$, $B(11, 24)$, and $C(28, 7)$ according to the coordinate system shown where distances are measured in miles.

Dominique's farm is located at the position $D(24, 19)$.

(a) Find AD. [2]

On a particular day, the mean temperatures recorded in each of towns A, B and C are $34\,°C$, $29\,°C$ and $30\,°C$ respectively.

(b) Use nearest neighbour interpolation to estimate the temperature at Dominique's farm on that particular day. [3]

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4. [Maximum mark: 8]

The cross section of a scale model of a hill is modelled by the following graph.

The heights of the model are measured at horizontal intervals and are given in the table.

Horizontal distance, x cm	0	10	20	30	40
Vertical distance, y cm	0	3	8	9	0

(a) Use the trapezoidal rule with $h = 10$ to find an approximation for the cross-sectional area of the model. [2]

It is given that the equation of the curve is $y = 0.04x^2 - 0.001x^3$, $0 \le x \le 40$.

(b) (i) Write down an integral to find the exact cross-sectional area.

(ii) Calculate the value of the cross-sectional area to two decimal places. [4]

(c) Find the percentage error in the area found using the trapezoidal rule. [2]

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5. [Maximum mark: 5]

A boat travels $8\,\text{km}$ on a bearing of $315°$ and then a further $6\,\text{km}$ on a bearing of $045°$.
Find the bearing on which the boat should travel to return directly to the starting point.

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6. [Maximum mark: 7]

In a baseball game, Sakura is the batter standing beside home plate. The ball is thrown towards home plate along a path that can be modelled by the following function.

$$y = -0.045x + 2$$

In this model, x is the horizontal distance of the ball from the point the ball is thrown and y is the vertical height of the ball above the ground. Both measured in metres.

The outcome of the throw is called a strike if the height of the ball is between $0.53\,\text{m}$ and $1.24\,\text{m}$ at some point while it travels over home plate. The length of home plate is $0.43\,\text{m}$.

diagram not to scale

When the ball reaches the front of home plate, the height of the ball above the ground is $1.25\,\text{m}$. The height of the ball changes by a metres as the ball travels over the length of home plate.

(a) (i) Find the value of a.

(ii) Justify why this throw is a strike. [4]

On the next throw, Sakura hits the ball towards a wall that is 5 metres high. The horizontal distance of the wall from the point where the ball was hit is 96 metres. The path of the ball after it is hit can be modelled by the function $h(d)$.

$$h(d) = -0.01d^2 + 1.04d + 0.66, \text{ for } h,\ d > 0$$

In this model, h is the height of the ball above the ground and d is the horizontal distance of the ball from the point where it was hit. Both h and d are measured in metres.

(b) Determine whether the ball will go over the wall. Justify your answer. [3]

(This question continues on the following page)

(Question 6 continued)

7. [Maximum mark: 9]

A triangular cover is positioned over a walled garden to provide shade. It is anchored at points A and C, located at the top of a $2\,\mathrm{m}$ wall, and at a point B, located at the top of a $1\,\mathrm{m}$ vertical pole fixed to a top corner of the wall.

The three edges of the cover can be represented by the vectors

$\overrightarrow{\mathrm{AB}} = \begin{pmatrix} 0 \\ 6 \\ 1 \end{pmatrix}$, $\overrightarrow{\mathrm{AC}} = \begin{pmatrix} 7 \\ 3 \\ 0 \end{pmatrix}$ and $\overrightarrow{\mathrm{BC}} = \begin{pmatrix} 7 \\ -3 \\ -1 \end{pmatrix}$, where distances are measured in metres.

(a) Calculate the vector product $\overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{AC}}$. [2]

(b) Hence find the area of the triangular cover. [2]

The point X on $[\mathrm{AC}]$ is such that $[\mathrm{BX}]$ is perpendicular to $[\mathrm{AC}]$.

(c) Use your answer to part (b) to find the distance BX. [3]

(d) Find the angle the cover makes with the horizontal plane. [2]

8. [Maximum mark: 7]

The random variables (X, Y) follow a bivariate normal distribution with product-moment correlation coefficient ρ. The values of six random observations of (X, Y) are shown in the table.

x	6.3	4.1	5.6	9.2	7.8	8.2
y	9.2	4.9	8.9	10.3	8.9	9.8

(a) State null and alternative hypotheses which could be used to test whether there is a linear correlation between X and Y. [2]

(b) Determine the value of

(i) the product-moment correlation coefficient, r, of the sample.

(ii) the corresponding p-value. [3]

(c) State whether your result from part (b)(ii) indicates there is sufficient evidence to claim that, at the 5% significance level, X and Y are not linearly correlated.

Give a reason for your answer. [2]

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9. [Maximum mark: 5]

On a specific day, the speed of cars as they pass a speed camera can be modelled by a normal distribution with a mean of $67.3\,\mathrm{km\,h^{-1}}$.

A speed of $75.7\,\mathrm{km\,h^{-1}}$ is two standard deviations from the mean.

(a) Find the standard deviation for the speed of the cars. [2]

It is found that 82% of cars on this road travel at speeds between $p\,\mathrm{km\,h^{-1}}$ and $q\,\mathrm{km\,h^{-1}}$, where $p < q$. This interval includes cars travelling at a speed of $74\,\mathrm{km\,h^{-1}}$.

(b) Show that the region of the normal distribution between p and q is **not** symmetrical about the mean. [3]

10. [Maximum mark: 9]

A decorative hook can be modelled by the curve with equation $y = f(x)$. The graph of $y = f(x)$ is shown and consists of a line segment from $(0, 0)$ to $(3, 3)$ and two sections formed by quadratic curves.

(a) Write down the equation of the line segment for $0 \le x \le 3$. [1]

The quadratic curve, with endpoints $(-2, 0)$ and $(0, 0)$, has the same gradient at $(0, 0)$ as the line segment.

(b) Find the equation of the curve between $(-2, 0)$ and $(0, 0)$. [3]

The second quadratic curve, with endpoints $(3, 3)$ and $(6, 2)$, has the same gradient at $(3, 3)$ as the line segment.

(c) Find the equation of this curve. [4]

(d) Write down f as a piecewise function. [1]

(This question continues on the following page)

(Question 10 continued)

11. [Maximum mark: 6]

A shop sells oranges and lemons. The weights of the oranges are assumed to be normally distributed with mean 205 grams and standard deviation 5 grams. The weights of the lemons are assumed to be normally distributed with mean 105 grams and standard deviation 3 grams.

Nelia selects 1 orange and 2 lemons at random and independent of each other. Calculate the probability that the weight of her orange exceeds the combined weight of her lemons.

12. [Maximum mark: 5]

Two AC (alternating current) electrical sources with the same frequencies are combined. The voltages from these sources can be expressed as $V_1 = 6\sin(at + 30°)$ and $V_2 = 6\sin(at + 90°)$.

The combined total voltage can be expressed in the form $V_1 + V_2 = V\sin(at + \theta°)$.

Determine the value of V and the value of θ.

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13. [Maximum mark: 6]

The displacement, $x\,(\mathrm{cm})$, of the end of a spring, at time t (seconds), is given by

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+2\frac{\mathrm{d}x}{\mathrm{d}t}+10x=0\,.$$

At $t=0$, $x=0.75$ and $\frac{\mathrm{d}x}{\mathrm{d}t}=0$.

Use Euler's method, with a step length 0.1 seconds, to estimate the value of x when $t=0.5$.

14. [Maximum mark: 6]

Petra examines two quantities, x and y, and plots data points on a log-log graph.

She observes that on this graph the data points follow a perfect straight line. Given that the line passes through the points $(2, 13.1951)$ and $(4, 34.822)$, find the equation of the relationship connecting x and y. Your final answer should not include logarithms.

15. [Maximum mark: 6]

The solid shown is formed by rotating the hexagon with vertices $(2, 1)$, $(0, 1)$, $(0, 5)$, $(2, 5)$, $(4, 4)$ and $(4, 2)$ about the y-axis.

Solid

Hexagon

Find the volume of this solid.

16. [Maximum mark: 6]

The relationship between the intensity, I, of a light source and the distance, d, from the light source can be modelled by $I = \frac{k}{d^2}$.

Pablo measures the intensity of a light source at different distances. The data collected is shown in the table.

d (m)	1	2	5
I (lm)	42	11	1.5

Pablo finds the sum of square residuals in the form $1.0641k^2 - 89.62k + c$.

(a) Find the exact value of c. [4]

(b) Hence find the least squares regression curve of the form $I = \frac{k}{d^2}$. [2]

17. [Maximum mark: 9]

An airplane, P, is flying at a constant altitude of $3000\,\mathrm{m}$ at a speed of $250\,\mathrm{m\,s^{-1}}$. Its path passes over a tracking station, S, at ground level. Let Q be the point $3000\,\mathrm{m}$ directly above the tracking station.

At a particular time, T, as the airplane is flying towards Q, the angle of elevation, θ, of the airplane from S is increasing at a rate of 0.075 radians per second. The distance from Q to P is given by x.

(a) Use related rates to show that, at time T, $\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = -\dfrac{10\,000}{3}$. [2]

(b) Find $x(\theta)$, x as a function of θ. [1]

(c) Find an expression for $\dfrac{\mathrm{d}x}{\mathrm{d}\theta}$ in terms of $\sin\theta$. [3]

(d) Hence find the horizontal distance from the station to the plane at time T. [3]

(This question continues on the following page)

(Question 17 continued)

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References:

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Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 1

Thursday 6 May 2021 (afternoon)

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all questions.
- Answers must be written within the answer boxes provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

24 pages

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Answers must be written within the answer boxes provided. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 4]

George goes fishing. From experience he knows that the mean number of fish he catches per hour is 1.1. It is assumed that the number of fish he catches can be modelled by a Poisson distribution.

On a day in which George spends 8 hours fishing, find the probability that he will catch more than 9 fish.

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2. [Maximum mark: 6]

The Voronoi diagram below shows three identical cellular phone towers, $T1$, $T2$ and $T3$. A fourth cellular phone tower, $T4$ is located in the shaded region. The dashed lines in the diagram below represent the edges in the Voronoi diagram.

Horizontal scale: 1 unit represents $1\,\text{km}$.
Vertical scale: 1 unit represents $1\,\text{km}$.

Tim stands inside the shaded region.

(a) Explain why Tim will receive the strongest signal from tower $T4$. [1]

Tower $T2$ has coordinates $(-9, 5)$ and the edge connecting vertices A and B has equation $y = 3$.

(b) Write down the coordinates of tower $T4$. [2]

Tower $T1$ has coordinates $(-13, 3)$.

(c) Find the gradient of the edge of the Voronoi diagram between towers $T1$ and $T2$. [3]

(This question continues on the following page)

(Question 2 continued)

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3. [Maximum mark: 8]

Charlie and Daniella each began a fitness programme. On day one, they both ran $500\,\mathrm{m}$. On each subsequent day, Charlie ran $100\,\mathrm{m}$ more than the previous day whereas Daniella increased her distance by $2\,\%$ of the distance ran on the previous day.

(a) Calculate how far

(i) Charlie ran on day 20 of his fitness programme.

(ii) Daniella ran on day 20 of her fitness programme. [5]

On day n of the fitness programmes Daniella runs more than Charlie for the first time.

(b) Find the value of n. [3]

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4. [Maximum mark: 6]

Professor Wei observed that students have difficulty remembering the information presented in his lectures.

He modelled the percentage of information retained, R, by the function $R(t) = 100\,\mathrm{e}^{-pt}$, $t \geq 0$, where t is the number of days after the lecture.

He found that 1 day after a lecture, students had forgotten $50\,\%$ of the information presented.

(a) Find the value of p. [2]

(b) Use this model to find the percentage of information retained by his students 36 hours after Professor Wei's lecture. [2]

Based on his model, Professor Wei believes that his students will always retain some information from his lecture.

(c) State a mathematical reason why Professor Wei might believe this. [1]

(d) Write down one possible limitation of the **domain** of the model. [1]

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5. [Maximum mark: 6]

A garden has a triangular sunshade suspended from three points $A(2, 0, 2)$, $B(8, 0, 2)$ and $C(5, 4, 3)$, relative to an origin in the corner of the garden. All distances are measured in metres.

(a) (i) Find $\overrightarrow{CA}$.

(ii) Find $\overrightarrow{CB}$. [2]

(b) Find $\overrightarrow{CA} \times \overrightarrow{CB}$. [2]

(c) Hence find the area of the triangle ABC. [2]

6. [Maximum mark: 5]

A triangular field ABC is such that $AB = 56\,\text{m}$ and $BC = 82\,\text{m}$, each measured correct to the nearest metre, and the angle at B is equal to 105°, measured correct to the nearest 5°.

diagram not to scale

Calculate the maximum possible area of the field.

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7. [Maximum mark: 8]

Ellis designs a gift box. The top of the gift box is in the shape of a right-angled triangle GIK.

A rectangular section HIJL is inscribed inside this triangle. The lengths of GH, JK, HL, and LJ are p cm, q cm, 8 cm and 6 cm respectively.

diagram not to scale

The area of the top of the gift box is $A\,\text{cm}^2$.

(a) (i) Find A in terms of p and q.

(ii) Show that $A = \frac{192}{q} + 3q + 48$. [4]

(b) Find $\frac{\mathrm{d}A}{\mathrm{d}q}$. [2]

Ellis wishes to find the value of q that will minimize the area of the top of the gift box.

(c) (i) Write down an equation Ellis could solve to find this value of q.

(ii) Hence, or otherwise, find this value of q. [2]

(This question continues on the following page)

(Question 7 continued)

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8. [Maximum mark: 7]

A game is played where two unbiased dice are rolled and the score in the game is the greater of the two numbers shown. If the two numbers are the same, then the score in the game is the number shown on one of the dice. A diagram showing the possible outcomes is given below.

Let T be the random variable "the score in a game".

(a) Complete the table to show the probability distribution of T. [2]

t	1	2	3	4	5	6
$\mathrm{P}(T = t)$						

(b) Find the probability that

(i) a player scores at least 3 in a game.

(ii) a player scores 6, given that they scored at least 3. [3]

(c) Find the expected score of a game. [2]

(This question continues on the following page).

(Question 8 continued)

9. [Maximum mark: 8]

Consider $w = \mathrm{i}z + 1$, where $w, z \in \mathbb{C}$.

(a) Find w when

(i) $z = 2\mathrm{i}$.

(ii) $z = 1 + \mathrm{i}$. [3]

Point z on the Argand diagram can be transformed to point w by two transformations.

(b) Describe these two transformations and give the order in which they are applied. [3]

(c) Hence, or otherwise, find the value of z when $w = 2 - \mathrm{i}$. [2]

10. [Maximum mark: 7]

An engineer plans to visit six oil rigs (A–F) in the Gulf of Mexico, starting and finishing at A. The travelling time, in minutes, between each of the rigs is shown in the table.

	A	**B**	**C**	**D**	**E**	**F**
A		55	63	79	87	93
B	55		46	58	88	92
C	63	46		87	77	66
D	79	58	87		23	70
E	87	88	77	23		47
F	93	92	66	70	47	

The data above can be represented by a graph G.

(a) (i) Use Prim's algorithm to find the weight of the minimum spanning tree of the subgraph of G obtained by deleting A and starting at B. List the order in which the edges are selected.

(ii) Hence find a lower bound for the travelling time needed to visit all the oil rigs. [6]

(b) Describe how an improved lower bound might be found. [1]

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11. [Maximum mark: 7]

A factory, producing plastic gifts for a fast food restaurant's Jolly meals, claims that just 1% of the toys produced are faulty.

A restaurant manager wants to test this claim. A box of 200 toys is delivered to the restaurant. The manager checks all the toys in this box and four toys are found to be faulty.

(a) Identify the type of sampling used by the restaurant manager. [1]

The restaurant manager performs a one-tailed hypothesis test, at the 10% significance level, to determine whether the factory's claim is reasonable. It is known that faults in the toys occur independently.

(b) Write down the null and alternative hypotheses. [2]

(c) Find the p-value for the test. [2]

(d) State the conclusion of the test. Give a reason for your answer. [2]

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12. [Maximum mark: 8]

A tank of water initially contains 400 litres. Water is leaking from the tank such that after 10 minutes there are 324 litres remaining in the tank.

The volume of water, V litres, remaining in the tank after t minutes, can be modelled by the differential equation

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -k\sqrt{V}, \text{ where } k \text{ is a constant.}$$

(a) Show that $V = \left(20 - \frac{t}{5}\right)^2$. [6]

(b) Find the time taken for the tank to empty. [2]

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13. [Maximum mark: 7]

A submarine is located in a sea at coordinates $(0.8, 1.3, -0.3)$ relative to a ship positioned at the origin O. The x direction is due east, the y direction is due north and the z direction is vertically upwards.

All distances are measured in kilometres.

The submarine travels with direction vector $\begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}$.

(a) Assuming the submarine travels in a straight line, write down an equation for the line along which it travels. [2]

The submarine reaches the surface of the sea at the point P.

(b) (i) Find the coordinates of P.

(ii) Find OP. [5]

14. [Maximum mark: 6]

The weights of apples from Tony's farm follow a normal distribution with mean $158\,\text{g}$ and standard deviation $13\,\text{g}$. The apples are sold in bags that contain six apples.

(a) Find the mean weight of a bag of apples. [2]

(b) Find the standard deviation of the weights of these bags of apples. [2]

(c) Find the probability that a bag selected at random weighs more than $1\,\text{kg}$. [2]

15. [Maximum mark: 5]

The diagram shows the slope field for the differential equation

$$\frac{dy}{dx} = \sin(x + y),\ -4 \le x \le 5,\ 0 \le y \le 5.$$

The graphs of the two solutions to the differential equation that pass through points $(0, 1)$ and $(0, 3)$ are shown.

For the two solutions given, the local minimum points lie on the straight line L_1.

(a) Find the equation of L_1, giving your answer in the form $y = mx + c$. [3]

For the two solutions given, the local maximum points lie on the straight line L_2.

(b) Find the equation of L_2. [2]

(This question continues on the following page)

(Question 15 continued)

16. [Maximum mark: 5]

An ant is walking along the edges of a wire frame in the shape of a triangular prism.

The vertices and edges of this frame can be represented by the graph below.

(a) Write down the adjacency matrix, $\boldsymbol{M}$, for this graph. [3]

(b) Find the number of ways that the ant can start at the vertex A, and walk along exactly 6 edges to return to A. [2]

(This question continues on the following page)

(Question 16 continued)

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17. [Maximum mark: 7]

The graph of the function $f(x) = \ln x$ is translated by $\begin{pmatrix} a \\ b \end{pmatrix}$ so that it then passes through the points $(0, 1)$ and $(e^3, 1 + \ln 2)$.

Find the value of a and the value of b.

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References:

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Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 1

Friday 6 May 2022 (afternoon)

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all questions.
- Answers must be written within the answer boxes provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

25 pages

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Answers must be written within the answer boxes provided. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 6]

A group of 130 applicants applied for admission into either the Arts programme or the Sciences programme at a university. The outcomes of their applications are shown in the following table.

	Accepted	**Rejected**
Arts programme	17	24
Sciences programme	25	64

(a) Find the probability that a randomly chosen applicant from this group was accepted by the university. [1]

An applicant is chosen at random from this group. It is found that they were accepted into the programme of their choice.

(b) Find the probability that the applicant applied for the Arts programme. [2]

Two different applicants are chosen at random from the original group.

(c) Find the probability that both applicants applied to the Arts programme. [3]

(This question continues on the following page)

(Question 1 continued)

2. [Maximum mark: 8]

The owner of a convenience store installs two security cameras, represented by points $C1$ and $C2$. Both cameras point towards the centre of the store's cash register, represented by the point R.

The following diagram shows this information on a cross-section of the store.

diagram not to scale

The cameras are positioned at a height of $3.1\,\text{m}$, and the horizontal distance between the cameras is $6.4\,\text{m}$. The cash register is sitting on a counter so that its centre, R, is $1.0\,\text{m}$ above the floor.

The distance from Camera 1 to the centre of the cash register is $2.8\,\text{m}$.

(a) Determine the angle of depression from Camera 1 to the centre of the cash register. Give your answer in degrees. [2]

(b) Calculate the distance from Camera 2 to the centre of the cash register. [4]

(c) Without further calculation, determine which camera has the largest angle of depression to the centre of the cash register. Justify your response. [2]

(This question continues on the following page)

(Question 2 continued)

Please **do not** write on this page.

Answers written on this page
will not be marked.

3. [Maximum mark: 7]

A polygraph test is used to determine whether people are telling the truth or not, but it is not completely accurate. When a person tells the truth, they have a 20% chance of failing the test. Each test outcome is independent of any previous test outcome.

10 people take a polygraph test and all 10 tell the truth.

(a) Calculate the expected number of people who will pass this polygraph test. [2]

(b) Calculate the probability that exactly 4 people will fail this polygraph test. [2]

(c) Determine the probability that fewer than 7 people will pass this polygraph test. [3]

4. [Maximum mark: 7]

The graphs of $y = 6 - x$ and $y = 1.5x^2 - 2.5x + 3$ intersect at $(2\,,\ 4)$ and $(-1\,,\ 7)$, as shown in the following diagrams.

In **diagram 1**, the region enclosed by the lines $y = 6 - x$, $x = -1$, $x = 2$ and the x-axis has been shaded.

diagram not to scale

Diagram 1

(a) Calculate the area of the shaded region in **diagram 1**. [2]

In **diagram 2**, the region enclosed by the curve $y = 1.5x^2 - 2.5x + 3$, and the lines $x = -1$, $x = 2$ and the x-axis has been shaded.

diagram not to scale

Diagram 2

(b) (i) Write down an integral for the area of the shaded region in **diagram 2**.

(ii) Calculate the area of this region. [3]

(c) Hence, determine the area enclosed between $y = 6 - x$ and $y = 1.5x^2 - 2.5x + 3$. [2]

(This question continues on the following page)

(Question 4 continued)

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5. [Maximum mark: 8]

The graph below shows the average savings, S thousand dollars, of a group of university graduates as a function of t, the number of years after graduating from university.

(a) Write down one feature of this graph which suggests a cubic function might be appropriate to model this scenario. [1]

The equation of the model can be expressed in the form $S = at^3 + bt^2 + ct + d$, where a, b, c and d are real constants.

The graph of the model must pass through the following four points.

t	0	1	2	3
S	–5	3	–1	–5

(b) (i) Write down the value of d.

(ii) Write down three simultaneous equations for a, b and c.

(iii) Hence, or otherwise, find the values of a, b and c. [4]

A negative value of S indicates that a graduate is expected to be in debt.

(c) Use the model to determine the total length of time, in years, for which a graduate is expected to be in debt after graduating from university. [3]

(This question continues on the following page)

(Question 5 continued)

6. [Maximum mark: 5]

Consider the following directed network.

(a) Write down the adjacency matrix for this network. [2]

(b) Determine the number of different walks of length 5 that start and end at the same vertex. [3]

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7. [Maximum mark: 5]

The sum of an infinite geometric sequence is 9.

The first term is 4 more than the second term.

Find the third term. Justify your answer.

8. [Maximum mark: 8]

The diagram shows a sector, OAB, of a circle with centre O and radius r, such that $\mathrm{A\hat{O}B} = \theta$.

Sam measured the value of r to be $2\,\mathrm{cm}$ and the value of θ to be 30°.

(a) Use Sam's measurements to calculate the area of the sector. Give your answer to four significant figures. [2]

It is found that Sam's measurements are accurate to only one significant figure.

(b) Find the upper bound and lower bound of the area of the sector. [3]

(c) Find, with justification, the largest possible percentage error if the answer to part (a) is recorded as the area of the sector. [3]

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9. [Maximum mark: 8]

A psychologist records the number of digits (d) of π that a sample of IB Mathematics higher level candidates could recall.

d	2	3	4	5	6	7
Frequency	2	6	24	21	11	3

(a) Find an unbiased estimate of the population mean of d. [1]

(b) Find an unbiased estimate of the population variance of d. [2]

The psychologist has read that in the general population people can remember an average of 4.4 digits of π. The psychologist wants to perform a statistical test to see if IB Mathematics higher level candidates can remember more digits than the general population.

(c) $\mathrm{H}_0 : \mu = 4.4$ is the null hypothesis for this test.

(i) State the alternative hypothesis.

(ii) Given that all assumptions for this test are satisfied, carry out an appropriate hypothesis test. State and justify your conclusion. Use a 5% significance level. [5]

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10. [Maximum mark: 5]

The function $f(x) = \ln\left(\frac{1}{x-2}\right)$ is defined for $x > 2$, $x \in \mathbb{R}$.

(a) Find an expression for $f^{-1}(x)$. You are not required to state a domain. [3]

(b) Solve $f(x) = f^{-1}(x)$. [2]

11. [Maximum mark: 6]

Juliana plans to invest money for 10 years in an account paying $3.5\,\%$ interest, compounded annually. She expects the annual inflation rate to be $2\,\%$ per year throughout the 10-year period.

Juliana would like her investment to be worth a real value of $\$4000$, compared to current values, at the end of the 10-year period. She is considering two options.

Option 1: Make a one-time investment at the start of the 10-year period.

Option 2: Invest $\$1000$ at the start of the 10-year period and then invest $\$x$ into the account at the end of each year (including the first and last years).

(a) For option 1, determine the minimum amount Juliana would need to invest. Give your answer to the nearest dollar. [3]

(b) For option 2, find the minimum value of x that Juliana would need to invest each year. Give your answer to the nearest dollar. [3]

12. [Maximum mark: 6]

The sex of cuttlefish is difficult to determine visually, so it is often found by weighing the cuttlefish.

The weights of adult male cuttlefish are known to be normally distributed with mean $10\,\text{kg}$ and standard deviation $0.5\,\text{kg}$.

The weights of adult female cuttlefish are known to be normally distributed with mean $12\,\text{kg}$ and standard deviation $1\,\text{kg}$.

A zoologist uses the null hypothesis that in the absence of information a cuttlefish is male.

If the weight is found to be above $11.5\,\text{kg}$ the cuttlefish is classified as female.

(a) Find the probability of making a Type I error when weighing a male cuttlefish. [2]

(b) Find the probability of making a Type II error when weighing a female cuttlefish. [2]

$90\,\%$ of adult cuttlefish are male.

(c) Find the probability of making an error using the zoologist's method. [2]

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13. [Maximum mark: 5]

An electric circuit has two power sources. The voltage, V_1, provided by the first power source, at time t, is modelled by

$$V_1 = \mathrm{Re}\left(2\mathrm{e}^{3t\mathrm{i}}\right).$$

The voltage, V_2, provided by the second power source is modelled by

$$V_2 = \mathrm{Re}\left(5\mathrm{e}^{(3t+4)\mathrm{i}}\right).$$

The total voltage in the circuit, V_T, is given by

$$V_T = V_1 + V_2.$$

(a) Find an expression for V_T in the form $A\cos(Bt + C)$, where A, B and C are real constants. [4]

(b) Hence write down the maximum voltage in the circuit. [1]

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14. [Maximum mark: 4]

The shape of a vase is formed by rotating a curve about the y-axis.

The vase is $10\,\text{cm}$ high. The internal radius of the vase is measured at $2\,\text{cm}$ intervals along the height:

Height (cm)	**Radius (cm)**
0	4
2	6
4	8
6	7
8	3
10	5

Use the trapezoidal rule to estimate the volume of water that the vase can hold.

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15. [Maximum mark: 7]

The equation of the line $y = mx + c$ can be expressed in vector form $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$.

(a) Find the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in terms of m and/or c. [2]

The matrix $\boldsymbol{M}$ is defined by $\begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix}$.

(b) Find the value of $\det \boldsymbol{M}$. [1]

The line $y = mx + c$ (where $m \neq -2$) is transformed into a new line using the transformation described by matrix $\boldsymbol{M}$.

(c) Show that the equation of the resulting line does not depend on m or c. [4]

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16. [Maximum mark: 7]

The position vector of a particle, P, relative to a fixed origin O at time t is given by

$$\overrightarrow{\mathrm{OP}} = \begin{pmatrix} \sin(t^2) \\ \cos(t^2) \end{pmatrix}.$$

(a) Find the velocity vector of P. [2]

(b) Show that the acceleration vector of P is never parallel to the position vector of P. [5]

17. [Maximum mark: 8]

The cross-section of a beach is modelled by the equation $y = 0.02x^2$ for $0 \le x \le 10$ where y is the height of the beach (in metres) at a horizontal distance x metres from an origin. t is the time in hours after low tide.

At $t = 0$ the water is at the point $(0, 0)$. The height of the water rises at a rate of 0.2 metres per hour. The point $\mathrm{W}(x(t), y(t))$ indicates where the water level meets the beach at time t.

(a) When W has an x-coordinate equal to 1, find the horizontal component of the velocity of W. [3]

A snail is modelled as a single point. At $t = 0$ it is positioned at $(1, 0.02)$. The snail travels away from the incoming water at a speed of 1 metre per hour in the direction along the curve of the cross-section of the beach. The following diagram shows this for a value of t, such that $t > 0$.

(b) (i) Find the time taken for the snail to reach the point $(10, 2)$.

(ii) Hence show that the snail reaches the point $(10, 2)$ before the water does. [5]

(This question continues on the following page)

(Question 17 continued)

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Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 1

8 May 2023

Zone A afternoon | **Zone B** morning | **Zone C** afternoon

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all questions.
- Answers must be written within the answer boxes provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

21 pages

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1. [Maximum mark: 7]

A player throws a basketball. The height of the basketball is modelled by

$$h(t) = -4.75t^2 + 8.75t + 1.5\,,\ t \geq 0\,,$$

where h is the height of the basketball above the ground, in metres, and t is the time, in seconds, after it was thrown.

(a) Find how long it takes for the basketball to reach its maximum height. [2]

(b) Assuming that no player catches the basketball, find how long it would take for the basketball to hit the ground. [2]

Another player catches the basketball when it is at a height of 1.2 metres.

(c) Find the value of t when this player catches the basketball. [2]

(d) Write down one limitation of using $h(t)$ to model the height of the basketball. [1]

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2. [Maximum mark: 4]

A company that owns many restaurants wants to determine if there are differences in the quality of the food cooked for three different meals: breakfast, lunch and dinner.

Their quality assurance team randomly selects 500 items of food to inspect. The quality of this food is classified as perfect, satisfactory, or poor. The data is summarized in the following table.

		Quality			
		Perfect	**Satisfactory**	**Poor**	**Total**
Meal	**Breakfast**	101	124	7	**232**
	Lunch	68	81	5	**154**
	Dinner	35	69	10	**114**
	Total	**204**	**274**	**22**	**500**

A χ^2 test at the 5% significance level is carried out to determine if there is significant evidence of a difference in the quality of the food cooked for the three meals.

The critical value for this test is 9.488.

The hypotheses for this test are:

H_0: The quality of the food and the type of meal are independent.
H_1: The quality of the food and the type of meal are not independent.

(a) Find the χ^2 statistic. [2]

(b) State, with justification, the conclusion for this test. [2]

(This question continues on the following page)

(Question 2 continued)

3. [Maximum mark: 7]

The following Venn diagram shows two independent events, R and S. The values in the diagram represent probabilities.

(a) Find the value of x. [3]

(b) Find the value of y. [2]

(c) Find $\mathrm{P}(R' \mid S')$. [2]

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4. [Maximum mark: 6]

Angel has $\$520$ in his savings account. Angel considers investing the money for 5 years with a bank. The bank offers an annual interest rate of 1.2% compounded quarterly.

(a) Calculate the amount of money Angel would have at the end of 5 years with the bank. Give your answer correct to two decimal places. [3]

Instead of investing the money, Angel decides to buy a phone that costs $\$520$. At the end of 5 years, the phone will have a value of $\$30$. It may be assumed that the depreciation rate per year is constant.

(b) Calculate the annual depreciation rate of the phone. [3]

5. [Maximum mark: 6]

The lengths of the seeds from a particular mango tree are approximated by a normal distribution with a mean of $4\,\text{cm}$ and a standard deviation of $0.25\,\text{cm}$.

A seed from this mango tree is chosen at random.

(a) Calculate the probability that the length of the seed is less than $3.7\,\text{cm}$. [2]

It is known that $30\,\%$ of the seeds have a length greater than $k\,\text{cm}$.

(b) Find the value of k. [2]

For a seed of length $d\,\text{cm}$, chosen at random, $\text{P}(4 - m < d < 4 + m) = 0.6$.

(c) Find the value of m. [2]

6. [Maximum mark: 5]

Ruhi buys a scoop of ice cream in the shape of a sphere with a radius of $3.4\,\text{cm}$. The ice cream is served in a cone, and it may be assumed that $\frac{1}{5}$ of the volume of the ice cream is inside the cone. This is shown in the following diagram.

diagram not to scale

(a) Calculate the volume of ice cream that is not inside the cone. [3]

The cone has a slant height of $11\,\text{cm}$ and a radius of $3\,\text{cm}$.

The outside of the cone is covered with chocolate.

(b) Calculate the surface area of the cone that is covered with chocolate. Give your answer correct to the nearest cm^2. [2]

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7. [Maximum mark: 6]

Akar starts a new job in Australia and needs to travel daily from Wollongong to Sydney and back. He travels to work for 28 consecutive days and therefore makes 56 single journeys. Akar makes all journeys by bus.

The probability that he is successful in getting a seat on the bus for any single journey is 0.86.

(a) Determine the expected number of these 56 journeys for which Akar gets a seat on the bus. [1]

(b) Find the probability that Akar gets a seat on at least 50 journeys during these 28 days. [3]

The probability that Akar gets a seat on at most n journeys is at least 0.25.

(c) Find the smallest possible value of n. [2]

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8. [Maximum mark: 7]

The following directed, unweighted, graph shows a simplified road network on an island, connecting five small villages marked A to E.

(a) Construct the adjacency matrix $\boldsymbol{M}$ for this network. [3]

Beatriz the bus driver starts at village E and drives to seven villages, such that the seventh village is A.

(b) (i) Determine how many possible routes Beatriz could have taken, to travel from E to A.

(ii) Describe one possible route taken by Beatriz, by listing the villages visited in order. [4]

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9. [Maximum mark: 9]

At a running club, Sung-Jin conducts a test to determine if there is any association between an athlete's age and their best time taken to run $100\,\text{m}$. Eight athletes are chosen at random, and their details are shown below.

Athlete	A	B	C	D	E	F	G	H
Age (years)	13	17	22	18	19	25	11	36
Time (seconds)	13.4	14.6	13.4	12.9	12.0	11.8	17.0	13.1

Sung-Jin decides to calculate the Spearman's rank correlation coefficient for his set of data.

(a) Complete the table of ranks. [2]

Athlete	A	B	C	D	E	F	G	H
Age rank			3					
Time rank							1	

(b) Calculate the Spearman's rank correlation coefficient, r_s. [2]

(c) Interpret this value of r_s in the context of the question. [1]

(d) Suggest a mathematical reason why Sung-Jin may have decided not to use Pearson's product-moment correlation coefficient with his data from the original table. [1]

(e) (i) Find the coefficient of determination for the data from the original table.

(ii) Interpret this value in the context of the question. [3]

(This question continues on the following page)

(Question 9 continued)

10. [Maximum mark: 6]

A chocolate company plans to produce chocolate bars with special flavours. They survey 246 people to determine if there is any particular preference for one of the flavours.

The table below shows the information collected.

Hot chilli	Almond crunch	Spiced Chai	Ginger'n'lime
75	59	46	66

A χ^2 goodness of fit test at the 5% significance level is carried out on the data.

The critical value for the test is 7.82.

(a) State the null and alternative hypotheses for this test. [2]

(b) Perform the test and give your conclusion in context. [4]

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11. [Maximum mark: 6]

Two AC (alternating current) electrical sources of equal frequencies are combined.

The voltage of the first source is modelled by the equation $V = 30\sin(t + 60^\circ)$.

The voltage of the second source is modelled by the equation $V = 60\sin(t + 10^\circ)$.

(a) Determine the maximum voltage of the combined sources. [2]

(b) Using your graphic display calculator, find a suitable equation for the combined voltages, giving your answer in the form $V = V_0\sin(at + b)$, where a, b and V_0 are constants, $a > 0$ and $0^\circ \le b < 180^\circ$. [4]

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12. [Maximum mark: 5]

A spherical balloon is being inflated such that its volume is increasing at a rate of $15\,\text{cm}^3\,\text{s}^{-1}$.

(a) Find the radius of the balloon when its volume is $288\pi\,\text{cm}^3$. [2]

(b) Hence or otherwise, find the rate of change of the radius at this instant. [3]

13. [Maximum mark: 7]

The matrices $\boldsymbol{P} = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ and $\boldsymbol{Q} = \begin{pmatrix} -4 & 1 \\ 1 & 3 \end{pmatrix}$ represent two transformations.

A triangle T is transformed by $\boldsymbol{P}$, and this image is then transformed by $\boldsymbol{Q}$ to form a new triangle, T'.

(a) Find the single matrix that represents the transformation $T' \to T$, which will undo the transformation described above. [4]

The area of T' is $273\,\text{cm}^2$.

(b) Using your answer to part (a), or otherwise, determine the area of T. [3]

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14. [Maximum mark: 8]

In this question, $\boldsymbol{i}$ denotes a unit vector due east, and $\boldsymbol{j}$ denotes a unit vector due north.

Two ships, A and B, are each moving with constant velocities.

The position vector of ship A, at time t hours, is given as $\boldsymbol{r}_A = (1 + 2t)\boldsymbol{i} + (3 - 3t)\boldsymbol{j}$.

The position vector of ship B, at time t hours, is given as $\boldsymbol{r}_B = (-2 + 4t)\boldsymbol{i} + (-4 + t)\boldsymbol{j}$.

(a) Find the bearing on which ship A is sailing. [3]

(b) Find the value of t when ship B is directly south of ship A. [2]

(c) Find the value of t when ship B is directly south-east of ship A. [3]

15. [Maximum mark: 6]

A random sample of eight packets of Apollo coffee granules are selected from a supermarket shelf.

The weights of the coffee granules present in each packet are as follows:

$222\,\text{g}$ $226\,\text{g}$ $221\,\text{g}$ $228\,\text{g}$ $227\,\text{g}$ $225\,\text{g}$ $222\,\text{g}$ $223\,\text{g}$

(a) (i) Find an unbiased estimate for the mean weight of coffee granules in a packet of Apollo coffee.

(ii) Calculate a $95\,\%$ confidence interval for the population mean. Give your answer to four significant figures. [3]

(b) State one assumption you have made in order for your interval to be valid. [1]

(c) The label of each packet has a description which includes the phrase: "contains $226\,\text{g}$ of coffee granules".

Using your answer to part (a)(ii), briefly comment on the claim on the label. [2]

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16. [Maximum mark: 9]

The following diagram shows parts of the curves of $y = \cos x$ and $y = \frac{\sqrt{x}}{2}$.

P is the point of intersection of the two curves.

(a) Use your graphic display calculator to find the coordinates of P. [2]

The shaded region is rotated $360°$ **about the y-axis** to form a volume of revolution V.

(b) Express V as the sum of two definite integrals. [5]

(c) Hence find the value of V. [2]

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17. [Maximum mark: 6]

Consider the differential equation

$$\left(x^2+1\right)\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x}{2y-2}\text{, for } x\geq 0,\ y\geq 1,$$

where $y=1$ when $x=0$.

(a) Explain why Euler's method cannot be used to find an approximate value for y when $x=0.1$. [1]

(b) By solving the differential equation, show that $y=1+\sqrt{\frac{\ln\left(x^2+1\right)}{2}}$. [4]

(c) Hence deduce the value of y when $x=0.1$. [1]

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References:

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Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 1

Thursday 6 May 2021 (afternoon)

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all questions.
- Answers must be written within the answer boxes provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

20 pages

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Answers must be written within the answer boxes provided. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 7]

Two schools are represented by points $\mathrm{A}(2,\ 20)$ and $\mathrm{B}(14,\ 24)$ on the graph below. A road, represented by the line R with equation $-x+y=4$, passes near the schools. An architect is asked to determine the location of a new bus stop on the road such that it is the same distance from the two schools.

(a) Find the equation of the perpendicular bisector of $[\mathrm{AB}]$. Give your equation in the form $y=mx+c$. [5]

(b) Determine the coordinates of the point on R where the bus stop should be located. [2]

(This question continues on the following page)

(Question 1 continued)

2. [Maximum mark: 7]

A function is defined by $f(x) = 2 - \frac{12}{x+5}$ for $-7 \le x \le 7$, $x \ne -5$.

(a) Find the range of f. [3]

(b) Find an expression for the inverse function $f^{-1}(x)$. The domain is not required. [3]

(c) Write down the range of $f^{-1}(x)$. [1]

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3. [Maximum mark: 6]

At Springfield University, the weights, in kg, of 10 chinchilla rabbits and 10 sable rabbits were recorded. The aim was to find out whether chinchilla rabbits are generally heavier than sable rabbits. The results obtained are summarized in the following table.

Weight of chinchilla rabbits, kg	4.9	4.2	4.1	4.4	4.3	4.6	4.0	4.7	4.5	4.4
Weight of sable rabbits, kg	4.2	4.1	4.1	4.2	4.5	4.4	4.5	3.9	4.2	4.0

A t-test is to be performed at the 5 % significance level.

(a) Write down the null and alternative hypotheses. [2]

(b) Find the p-value for this test. [2]

(c) Write down the conclusion to the test. Give a reason for your answer. [2]

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4. [Maximum mark: 6]

The diagram below shows a helicopter hovering at point H, $380\,\mathrm{m}$ vertically above a lake. Point A is the point on the surface of the lake, directly below the helicopter.

diagram not to scale

Minta is swimming at a constant speed in the direction of point A. Minta observes the helicopter from point C as she looks upward at an angle of 25°. After 15 minutes, Minta is at point B and she observes the same helicopter at an angle of 40°.

(a) Find the distance from A to C. [2]

(b) Find the distance from B to C. [3]

(c) Find Minta's speed, in metres per hour. [1]

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5. [Maximum mark: 5]

Roger buys a new laptop for himself at a cost of £495. At the same time, he buys his daughter Chloe a higher specification laptop at a cost of £2200.

It is anticipated that Roger's laptop will depreciate at a rate of 10% per year, whereas Chloe's laptop will depreciate at a rate of 15% per year.

(a) Estimate the value of Roger's laptop after 5 years. [2]

Roger and Chloe's laptops will have the same value k years after they were purchased.

(b) Find the value of k. [2]

(c) Comment on the validity of your answer to part (b). [1]

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6. [Maximum mark: 7]

Consider the function $f(x) = ax^2 + bx + c$. The graph of $y = f(x)$ is shown in the diagram. The vertex of the graph has coordinates $(0.5, -12.5)$. The graph intersects the x-axis at two points, $(-2, 0)$ and $(p, 0)$.

diagram not to scale

(a) Find the value of p. [1]

(b) Find the value of

(i) a.

(ii) b.

(iii) c. [5]

(c) Write down the equation of the axis of symmetry of the graph. [1]

(This question continues on the following page)

(Question 6 continued)

7. [Maximum mark: 6]

A meteorologist models the height of a hot air balloon launched from the ground. The model assumes the balloon travels vertically upwards and travels 450 metres in the first minute.

Due to the decrease in temperature as the balloon rises, the balloon will continually slow down. The model suggests that each minute the balloon will travel only 82% of the distance travelled in the previous minute.

(a) Find how high the balloon will travel in the first 10 minutes after it is launched. [3]

(b) The balloon is required to reach a height of at least 2520 metres.

Determine whether it will reach this height. [2]

(c) Suggest a limitation of the given model. [1]

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8. [Maximum mark: 7]

Two lines L_1 and L_2 are given by the following equations, where $p \in \mathbb{R}$.

$$L_1 : r = \begin{pmatrix} 2 \\ p+9 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} p \\ 2p \\ 4 \end{pmatrix}$$

$$L_2 : r = \begin{pmatrix} 14 \\ 7 \\ p+12 \end{pmatrix} + \mu \begin{pmatrix} p+4 \\ 4 \\ -7 \end{pmatrix}$$

It is known that L_1 and L_2 are perpendicular.

(a) Find the possible value(s) for p. [3]

(b) In the case that $p < 0$, determine whether the lines intersect. [4]

9. [Maximum mark: 8]

A newspaper vendor in Singapore is trying to predict how many copies of *The Straits Times* they will sell. The vendor forms a model to predict the number of copies sold each weekday. According to this model, they expect the same number of copies will be sold each day.

To test the model, they record the number of copies sold each weekday during a particular week. This data is shown in the table.

Day	Monday	Tuesday	Wednesday	Thursday	Friday
Number of copies sold	74	97	91	86	112

A goodness of fit test at the 5% significance level is used on this data to determine whether the vendor's model is suitable. The critical value for the test is 9.49.

(a) Find an estimate for how many copies the vendor expects to sell each day. [1]

(b) (i) State the null and alternative hypotheses for this test.

(ii) Write down the degrees of freedom for this test.

(iii) Write down the conclusion to the test. Give a reason for your answer. [7]

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10. [Maximum mark: 6]

A manufacturer of chocolates produces them in individual packets, claiming to have an average of 85 chocolates per packet.

Talha bought 30 of these packets in order to check the manufacturer's claim.

Given that the number of individual chocolates is x, Talha found that, from his 30 packets, $\sum x = 2506$ and $\sum x^2 = 209\,738$.

(a) Find an unbiased estimate for the mean number (μ) of chocolates per packet. [1]

(b) Use the formula $s_{n-1}^2 = \dfrac{\sum x^2 - \frac{(\sum x)^2}{n}}{n-1}$ to determine an unbiased estimate for the variance of the number of chocolates per packet. [2]

(c) Find a $95\,\%$ confidence interval for μ. You may assume that all conditions for a confidence interval have been met. [2]

(d) Suggest, with justification, a valid conclusion that Talha could make. [1]

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11. [Maximum mark: 7]

The diagram below shows a network of roads in a small village with the weights indicating the distance of each road, in metres, and junctions indicated with letters.

Musab is required to deliver leaflets to every house on each road. He wishes to minimize his total distance.

(a) Musab starts and finishes from the village bus-stop at A. Determine the total distance Musab will need to walk. [5]

Instead of having to catch the bus to the village, Musab's sister offers to drop him off at any junction and pick him up at any other junction of his choice.

(b) Explain which junctions Musab should choose as his starting and finishing points. [2]

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12. [Maximum mark: 8]

It is given that $z_1 = 3\,\text{cis}\left(\frac{3\pi}{4}\right)$ and $z_2 = 2\,\text{cis}\left(\frac{n\pi}{16}\right)$, $n \in \mathbb{Z}^+$.

(a) In parts (a)(i) and (a)(ii), give your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, $r \geq 0$, $-\pi < \theta \leq \pi$.

(i) Find the value of $z_1^{\,3}$.

(ii) Find the value of $\left(\frac{z_1}{z_2}\right)^4$ for $n = 2$. [5]

(b) Find the least value of n such that $z_1 z_2 \in \mathbb{R}^+$. [3]

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13. [Maximum mark: 8]

The graph below shows a small maze, in the form of a network of directed routes. The vertices A to F show junctions in the maze and the edges show the possible paths available from one vertex to another.

A mouse is placed at vertex A and left to wander the maze freely. The routes shown by dashed lines indicate paths sprinkled with sugar.

When the mouse reaches any junction, she rests for a constant time before continuing.

At any junction, it may also be assumed that
- the mouse chooses any available normal path with equal probability
- if the junction includes a path sprinkled with sugar, the probability of choosing this path is twice that of a normal path.

(a) Determine the transition matrix for this graph. [3]

(b) If the mouse was left to wander indefinitely, use your graphic display calculator to estimate the percentage of time that the mouse would spend at point F. [3]

(c) Comment on your answer to part (b), referring to at least one limitation of the model. [2]

(This question continues on the following page)

(Question 13 continued)

14. [Maximum mark: 7]

A geometric transformation $T: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x' \\ y' \end{pmatrix}$ is defined by

$$T: \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 7 & -10 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -5 \\ 4 \end{pmatrix}.$$

(a) Find the coordinates of the image of the point $(6, -2)$. [2]

(b) Given that $T: \begin{pmatrix} p \\ q \end{pmatrix} \mapsto 2\begin{pmatrix} p \\ q \end{pmatrix}$, find the value of p and the value of q. [3]

(c) A triangle L with vertices lying on the xy plane is transformed by T.

Explain why both L and its image will have exactly the same area. [2]

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15. [Maximum mark: 7]

The number of coffees sold per hour at an independent coffee shop is modelled by a Poisson distribution with a mean of 22 coffees per hour.

Sheila, the shop's owner wants to increase the number of coffees sold in the shop. She decides to offer a discount to customers who buy more than one coffee.

To test how successful this strategy is, Sheila records the number of coffees sold over a single 5-hour period. Sheila decides to use a 5% level of significance in her test.

(a) State the null and alternative hypotheses for the test. [1]

(b) Find the probability that Sheila will make a type I error in her test conclusion. [4]

Sheila finds 126 coffees were sold during the 5-hour period.

(c) State Sheila's conclusion to the test. Justify your answer. [2]

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16. [Maximum mark: 8]

A particle P moves in a straight line, such that its displacement x at time $t\ (t \geq 0)$ is defined by the differential equation $\dot{x} = x\cos t\left(\mathrm{e}^{-\sin t}\right)$. At time $t = 0$, $x = \frac{1}{\mathrm{e}}$.

(a) By using Euler's method with a step length of 0.1, find an approximate value for x when $t = 0.3$. [3]

(b) By solving the differential equation, find the percentage error in your approximation for x when $t = 0.3$. [5]

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References:

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Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 2

Tuesday 1 November 2022 (morning)

2 hours

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

8822 – 7202

12 pages

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Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 15]

At Mirabooka Primary School, a survey found that 68% of students have a dog and 36% of students have a cat. 14% of students have both a dog and a cat.

This information can be represented in the following Venn diagram, where m, n, p and q represent the percentage of students within each region.

(a) Find the value of

(i) m.

(ii) n.

(iii) p.

(iv) q. [4]

(b) Find the probability that a randomly chosen student

(i) has a dog but does not have a cat.

(ii) has a dog given that they do not have a cat. [3]

(This question continues on the following page)

(Question 1 continued)

Each year, one student is chosen randomly to be the school captain of Mirabooka Primary School.

Tim is using a binomial distribution to make predictions about how many of the next 10 school captains will own a dog. He assumes that the percentages found in the survey will remain constant for future years and that the events “being a school captain” and “having a dog” are independent.

Use Tim’s model to find the probability that in the next 10 years

(c) (i) 5 school captains have a dog.

(ii) more than 3 school captains have a dog.

(iii) exactly 9 school captains in succession have a dog. [7]

John randomly chooses 10 students from the survey.

(d) State why John should not use the binomial distribution to find the probability that 5 of these students have a dog. [1]

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2. [Maximum mark: 13]

Six restaurant locations (labelled A, B, C, D, E and F) are shown, together with their Voronoi diagram. All distances are measured in kilometres.

(a) Elena wants to eat at the closest restaurant to her. Write down the restaurant she should go to, if she is at

(i) $(2, 7)$.

(ii) $(0, 1)$, when restaurant A is closed. [2]

Restaurant C is at $(7, 8)$ and restaurant D is at $(7, 5)$.

(b) Find the equation of the perpendicular bisector of CD. [2]

Restaurant B is at $(3, 6)$.

(c) Find the equation of the perpendicular bisector of BC. [5]

(d) Hence find

(i) the coordinates of the point which is of equal distance from B, C and D.

(ii) the distance of this point from D. [4]

3. [Maximum mark: 17]

Linda owns a field, represented by the shaded region R. The plan view of the field is shown in the following diagram, where both axes represent distance and are measured in metres.

The segments [AB], [CD] and [AD] respectively represent the western, eastern and southern boundaries of the field. The function, $f(x)$, models the northern boundary of the field between points B and C and is given by

$$f(x) = \frac{-x^2}{50} + 2x + 30, \text{ for } 0 \le x \le 70.$$

(a) (i) Find $f'(x)$.

(ii) Hence find the coordinates of the point on the field that is furthest north. [5]

Point A has coordinates $(0, 0)$, point B has coordinates $(0, 30)$, point C has coordinates $(70, 72)$ and point D has coordinates $(70, 0)$.

(b) (i) Write down the integral which can be used to find the area of the shaded region R.

(ii) Find the area of Linda's field. [4]

(This question continues on the following page)

(Question 3 continued)

Linda used the trapezoidal rule with ten intervals to estimate the area. This calculation underestimated the area by $11.4\,\text{m}^2$.

(c) (i) Calculate the percentage error in Linda's estimate.

(ii) Suggest how Linda might be able to reduce the error whilst still using the trapezoidal rule. [3]

Linda would like to construct a building on her field. The **square** foundation of the building, EFGH, will be located such that [EH] is on the southern boundary and point F is on the northern boundary of the property. A possible location of the foundation of the building is shown in the following diagram.

The area of the square foundation will be largest when [GH] lies on [CD].

(d) (i) Find the x-coordinate of point E for the largest area of the square foundation of building EFGH.

(ii) Find the largest area of the foundation. [5]

4. [Maximum mark: 14]

A company has six offices, A, B, C, D, E and F. One of the company managers, Nanako, needs to visit the offices. She creates the following graph that shows the distances, in kilometres, between some of the offices.

diagram not to scale

(a) Write down a Hamiltonian cycle for this graph. [1]

(b) State, with a reason, whether the graph contains an Eulerian circuit. [1]

Nanako wishes to find the shortest cycle to visit all the offices. She decides to complete a weighted adjacency table, showing the least distance between each pair of offices.

	A	B	C	D	E	F
A		27	52	p	35	18
B			25	26	12	q
C				17	28	r
D					14	34
E						22
F						

(c) Write down the value of

(i) p.

(ii) q.

(iii) r. [3]

(This question continues on the following page)

(Question 4 continued)

(d) Starting at vertex E, use the nearest neighbour algorithm to find an upper bound for Nanako's cycle. [3]

(e) By deleting vertex F, find a lower bound for Nanako's cycle. [4]

(f) Explain, with a reason, why the answer to part (e) might not be the best lower bound. [2]

5. [Maximum mark: 13]

Adesh is designing a glass. The glass has an inner surface and an outer surface. Part of the cross section of his design is shown in the following graph, where the shaded region represents the glass. The two surfaces meet at the top of the glass. 1 unit represents $1\,\text{cm}$.

The inner surface is modelled by $f(x) = \frac{1}{2}x^3 + 1$ for $0 \le x \le p$.

The outer surface is modelled by $g(x) = \begin{cases} 0 & \text{for } 0 \le x < 1 \\ (x-1)^4 & \text{for } 1 \le x \le p \end{cases}$.

(a) Find the value of p. [2]

The glass design is finished by rotating the shaded region in the diagram through 360° about the y-axis.

(b) Find the volume of liquid that can be contained inside the finished glass. [5]

(c) Find the volume of the region between the two surfaces of the finished glass. [6]

6. [Maximum mark: 18]

A company makes doors for kitchen cupboards from two layers. The inside layer is wood, and its thickness is normally distributed with mean $7\,\text{mm}$ and standard deviation $0.3\,\text{mm}$. The outside layer is plastic, and its thickness is normally distributed with mean $3\,\text{mm}$ and standard deviation $0.16\,\text{mm}$. The thickness of the plastic is independent of the thickness of the wood.

(a) Find the probability that a randomly chosen door has a total thickness of less than $9.5\,\text{mm}$. [5]

Eight doors are to be packed into a box to send to a customer. The width of the box is $82\,\text{mm}$. The thickness of each door is independent.

(b) Find the probability that the total thickness of the eight doors is greater than the width of the box. [4]

The company buys two new machines, A and B, to make the wooden layers. An employee claims that the layers from machine B are thinner than the layers from machine A. In order to test this claim, a random sample is taken from each machine.

The seven layers in the sample from machine A have a thickness, in mm, of

$$6.23,\ 7.04,\ 7.31,\ 6.79,\ 6.91,\ 6.79,\ 7.47.$$

(c) Find the

(i) mean.

(ii) unbiased estimate of the population variance. [3]

The eight layers in the sample from machine B have a mean thickness of $6.89\,\text{mm}$ and $s_{\text{n}-1} = 0.31$.

(d) Perform a suitable test, at the $5\,\%$ significance level, to test the employee's claim. You may assume the thickness of the wooden layers from each machine are normally distributed with equal population variance. [6]

7. [Maximum mark: 20]

The position vector of a particle at time t is given by $\boldsymbol{r} = 3\cos(3t)\boldsymbol{i} + 4\sin(3t)\boldsymbol{j}$.
Displacement is measured in metres and time is measured in seconds.

(a) (i) Find an expression for the velocity of the particle at time t.

(ii) Hence find the speed when $t = 3$. [4]

(b) (i) Find an expression for the acceleration of the particle at time t.

(ii) Hence show that the acceleration is always directed towards the origin. [4]

The position vector of a second particle is given by $\boldsymbol{r} = -4\sin(4t)\boldsymbol{i} + 3\cos(4t)\boldsymbol{j}$.

(c) For $0 \le t \le 10$, find the time when the two particles are closest to each other. [5]

At time k, where $0 < k < 1.5$, the second particle is moving parallel to the first particle.

(d) (i) Find the value of k.

(ii) At time k, show that the two particles are moving in the opposite direction. [7]

References:

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Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 2

31 October 2023

Zone A afternoon | **Zone B** afternoon | **Zone C** afternoon

2 hours

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

11 pages

8823 – 7202
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Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 15]

Madhu is designing a jogging track for the campus of her school. The following diagram shows an incomplete portion of the track.

Madhu wants to design the track such that the inner edge is a smooth curve from point A to point B, and the other edge is a smooth curve from point C to point D. The distance between points A and B is 50 metres.

diagram not to scale

To create a smooth curve, Madhu first walks to M, the midpoint of $[\mathrm{AB}]$.

(a) Write down the length of $[\mathrm{BM}]$. [1]

(This question continues on the following page)

(Question 1 continued)

Madhu then walks 20 metres in a direction perpendicular to $[AB]$ to get from point M to point F. Point F is the centre of a circle whose arc will form the smooth curve between points A and B on the track, as shown in the following diagram.

diagram not to scale

(b) (i) Find the length of $[BF]$.

(ii) Find $B\hat{F}M$. [4]

(c) Hence, find the length of arc AB. [3]

The outer edge of the track, from C to D, is also a circular arc with centre F, such that the track is 2 metres wide.

(d) Calculate the area of the curved portion of the track, $ABDC$. [4]

The base of the track will be made of concrete that is $12\,\text{cm}$ deep.

(e) Calculate the volume of concrete needed to create the curved portion of the track. [3]

2. [Maximum mark: 18]

The heights, h, of 200 university students are recorded in the following table.

Height (cm)	**Frequency**
$140 \le h < 160$	11
$160 \le h < 170$	51
$170 \le h < 180$	68
$180 \le h < 190$	47
$190 \le h < 210$	23

(a) (i) Write down the mid-interval value of $140 \le h < 160$.

(ii) Calculate an estimate of the mean height of the 200 students. [3]

This table is used to create the following cumulative frequency graph.

(b) Use the cumulative frequency curve to estimate the interquartile range. [2]

Laszlo is a student in the data set and his height is $204\,\text{cm}$.

(c) Use your answer to part (b) to estimate whether Laszlo's height is an outlier for this data. Justify your answer. [3]

(This question continues on the following page)

(Question 2 continued)

It is believed that the heights of university students follow a normal distribution with mean $176\,\text{cm}$ and standard deviation $13.5\,\text{cm}$.

It is decided to perform a χ^2 goodness of fit test on the data to determine whether this sample of 200 students could have plausibly been drawn from an underlying distribution $\mathrm{N}(176,\ 13.5^2)$.

(d) Write down the null and the alternative hypotheses for the test. [2]

As part of the test, the following table is created.

Height of student (cm)	**Observed frequency**	**Expected frequency**
$h < 160$	11	23.6
$160 \le h < 170$	51	42.1
$170 \le h < 180$	68	a
$180 \le h < 190$	47	46.7
$190 \le h$	23	b

(e) (i) Find the value of a and the value of b.

(ii) Hence, perform the test to a $5\,\%$ significance level, clearly stating the conclusion in context. [8]

3. [Maximum mark: 16]

Tiffany wants to buy a house for a price of $285\,000$ US Dollars (USD). She goes to a bank to get a loan to buy the house. To be eligible for the loan, Tiffany must make an initial down payment equal to $15\,\%$ of the price of the house.

The bank offers her a 30-year loan for the remaining balance, with a $4\,\%$ nominal interest rate per annum, compounded monthly. Tiffany will pay the loan in fixed payments at the end of each month.

(a) (i) Find the original amount of the loan after the down payment is paid. Give the exact answer.

(ii) Calculate Tiffany's monthly payment for this loan, to two decimal places. [5]

(b) Using your answer from part (a)(ii), calculate the total amount Tiffany will pay over the life of the loan, to the nearest dollar. Do **not** include the initial down payment. [2]

Tiffany would like to repay the loan faster and increases her payments such that she pays $1300\,\text{USD}$ each month.

(c) Find the total number of monthly payments she will need to make to pay off the loan. [2]

This strategy will result in Tiffany's final payment being less than $1300\,\text{USD}$.

(d) Determine the amount of Tiffany's final payment, to two decimal places. [4]

(e) Hence, determine the total amount Tiffany will save, to the nearest dollar, by making the higher monthly payments. [3]

4. [Maximum mark: 12]

A plane takes off from a horizontal runway. Let point O be the point where the plane begins to leave the runway and x be the horizontal distance, in km, of the plane from O. The function h models the vertical height, in km, of the nose of the plane from the horizontal runway, and is defined by

$$h(x) = \frac{10}{1+150\mathrm{e}^{-0.07x}} - 0.06,\ x \geq 0.$$

diagram not to scale

(a) (i) Find $h(0)$.

(ii) Interpret this value in terms of the context. [2]

(b) (i) Find the horizontal asymptote of the graph of $y = h(x)$.

(ii) Interpret this value in terms of the context. [2]

(c) Find $h'(x)$ in terms of x. [4]

A safety regulation recommends that $h'(x)$ never exceed 0.2.

(d) Given that this plane flies a distance of at least $200\,\mathrm{km}$ horizontally from point O, determine whether the plane is following this safety regulation. [4]

5. [Maximum mark: 18]

The following diagram is a map of a group of four islands and the closest mainland. Travel from the mainland and between the islands is by boat. The scheduled boat routes between the ports A, B, C, D and E are shown as dotted lines on the map.

Let the undirected graph G represent the boat routes between the ports A, B, C, D and E.

(a) Draw graph G. [1]

(b) Graph G can be represented by an adjacency matrix $\boldsymbol{P}$, where the rows and columns represent the ports in alphabetical order.

(i) Given that $\boldsymbol{P}^3 = \begin{pmatrix} 0 & 1 & 2 & 4 & 1 \\ 1 & 2 & 5 & a & 2 \\ 2 & 5 & 4 & 6 & 5 \\ 4 & a & 6 & 4 & 6 \\ 1 & 2 & 5 & 6 & 2 \end{pmatrix}$, find the value of a.

(ii) Hence, write down the number of different ways that someone could start at port B and end at port C, using three boat route journeys. [3]

(c) Find a possible Eulerian trail in G, starting at port A. [2]

(This question continues on the following page)

(Question 5 continued)

The cost of a journey on the different boat routes is given in the following table; all prices are given in USD. The cost of a journey is the same in either direction between two ports.

	A	B	C	D	E
A				10	
B			20	25	
C		20		50	45
D	10	25	50		30
E			45	30	

Sofia wants to make a trip where she travels on each of the boat routes at least once, beginning and ending at port A.

(d) Find the minimum cost of Sofia's trip. [3]

The boat company decides to add an additional boat route to make it possible to travel on each boat route **exactly** once, starting and ending at the same port.

(e) (i) Identify between which two ports the additional boat route should be added.

(ii) Determine the cost of the additional boat route such that the overall cost of the trip is the same as your answer to part (d). [2]

The boat company plans to redesign which ports are connected by boat routes. Their aim is to have a single boat trip that visits all the islands and minimizes the total distance travelled, starting and finishing at the mainland, A.

The following table shows the distances in kilometres between the ports A, B, C, D and E.

	A	B	C	D	E
A		80	90	50	60
B	80		30	70	120
C	90	30		45	100
D	50	70	45		55
E	60	120	100	55	

(f) (i) Use the nearest neighbour algorithm to find an upper bound for the minimum total distance.

(ii) Use the deleted vertex algorithm on port A to find a lower bound for the minimum total distance. [7]

6. [Maximum mark: 14]

François is a video game designer. He designs his games to take place in two dimensions, relative to an origin O. In one of his games, an object travels on a straight line L_1 with vector equation

$$\boldsymbol{r} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

(a) Write down L_1 in the form $x = x_0 + \lambda l$ and $y = y_0 + \lambda m$, where $l,\ m \in \mathbb{Z}$. [1]

François uses the matrix $\boldsymbol{T} = \begin{pmatrix} 1 & 7 \\ 7 & -1 \end{pmatrix}$ to transform L_1 into a new straight line L_2. The object will then travel along L_2.

(b) Find the vector equation of L_2. [4]

François knows that the transformation given by matrix $\boldsymbol{T}$ is made up of the following three separate transformations (in the order listed):

- A rotation of $\frac{\pi}{4}$, anticlockwise (counter-clockwise) about the origin O
- An enlargement of scale factor $5\sqrt{2}$, centred at O
- A reflection in the straight line $y = mx$, where $m = \tan\alpha,\ 0 \le \alpha < \pi$

(c) Write down the matrix that represents

(i) the rotation.

(ii) the enlargement. [2]

(d) The matrix $\boldsymbol{R}$ represents the reflection. Write down $\boldsymbol{R}$ in terms of α. [1]

(e) Given that $\boldsymbol{T} = \boldsymbol{RX}$,

(i) use your answers to part (c) to find matrix $\boldsymbol{X}$.

(ii) hence, find the value of α. [6]

7. [Maximum mark: 17]

The city of Melba has an adult population of four million people. It is assumed that the weights of adults in Melba can be modelled by a normal distribution with mean $72\,\text{kg}$ and standard deviation $10\,\text{kg}$.

(a) If 10 adults in Melba are chosen independently and at random, find the probability that more than 3 of them have a weight greater than $85\,\text{kg}$. [4]

Laetitia runs a travel agency in Melba. The elevator to her office is designed to hold a maximum of 8 people.

(b) Write down a probability distribution that models the total weight of 8 adults chosen independently and at random from Melba. [3]

The total weight of 8 adults exceeds w on less than $1\,\%$ of all occasions that 8 adults enter the elevator.

(c) Find the value of w. [2]

A newspaper claims that $42\,\%$ of the adults in Melba who go on holiday choose to go abroad. Laetitia believes that this is an overestimation of the true number. During the past month, Laetitia found that 67 of her clients chose a holiday abroad, and 133 chose a holiday that was not abroad.

(d) Laetitia decides to perform a test using the binomial distribution on her data for the population proportion, p, that go on holiday abroad.

(i) State **two** assumptions that Laetitia makes in order to conduct the test.

(ii) Write down the null and the alternative hypotheses for Laetitia's test, in terms of p.

(iii) Using the data from Laetitia's sample, perform the test at a $5\,\%$ significance level to determine whether Laetitia's belief is reasonable. [8]

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Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 2

Tuesday 2 November 2021 (morning)

2 hours

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

10 pages

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Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 14]

A large water reservoir is built in the form of part of an upside-down right pyramid with a horizontal square base of length 80 metres. The point C is the centre of the square base and point V is the vertex of the pyramid.

diagram not to scale

The bottom of the reservoir is a square of length 60 metres that is parallel to the base of the pyramid, such that the depth of the reservoir is 6 metres as shown in the diagram.

(This question continues on the following page)

(Question 1 continued)

The second diagram shows a vertical cross section, MNOPC, of the reservoir.

diagram not to scale

(a) Find the angle of depression from M to N. [2]

(b) (i) Find CV.

(ii) Hence or otherwise, show that the volume of the reservoir is $29\,600\,\text{m}^3$. [5]

Every day $80\,\text{m}^3$ of water from the reservoir is used for irrigation.

Joshua states that, if no other water enters or leaves the reservoir, then when it is full there is enough irrigation water for at least one year.

(c) By finding an appropriate value, determine whether Joshua is correct. [2]

To avoid water leaking into the ground, the five interior sides of the reservoir have been painted with a watertight material.

(d) Find the area that was painted. [5]

2. [Maximum mark: 20]

A wind turbine is designed so that the rotation of the blades generates electricity. The turbine is built on horizontal ground and is made up of a vertical tower and three blades.

The point A is on the base of the tower directly below point B at the top of the tower. The height of the tower, AB, is $90\,\mathrm{m}$. The blades of the turbine are centred at B and are each of length $40\,\mathrm{m}$. This is shown in the following diagram.

diagram not to scale

The end of one of the blades of the turbine is represented by point C on the diagram. Let h be the height of C above the ground, measured in metres, where h varies as the blade rotates.

(a) Find the

(i) maximum value of h.

(ii) minimum value of h. [2]

The blades of the turbine complete 12 rotations per minute under normal conditions, moving at a constant rate.

(b) (i) Find the time, in seconds, it takes for the blade $[\mathrm{BC}]$ to make one complete rotation under these conditions.

(ii) Calculate the angle, in degrees, that the blade $[\mathrm{BC}]$ turns through in one second. [3]

(This question continues on the following page)

(Question 2 continued)

The height, h, of point C can be modelled by the following function. Time, t, is measured from the instant when the blade $[\mathrm{BC}]$ first passes $[\mathrm{AB}]$ and is measured in seconds.

$$h(t) = 90 - 40\cos(72t^{\circ}),\ t \geq 0$$

(c) (i) Write down the amplitude of the function.

(ii) Find the period of the function. [2]

(d) Sketch the function $h(t)$ for $0 \leq t \leq 5$, clearly labelling the coordinates of the maximum and minimum points. [3]

(e) (i) Find the height of C above the ground when $t = 2$.

(ii) Find the time, in seconds, that point C is above a height of $100\,\mathrm{m}$, during each complete rotation. [5]

The wind speed increases and the blades rotate faster, but still at a constant rate.

(f) Given that point C is now higher than $110\,\mathrm{m}$ for 1 second during each complete rotation, find the time for one complete rotation. [5]

3. [Maximum mark: 16]

Arianne plays a game of darts.

The distance that her darts land from the centre, O, of the board can be modelled by a normal distribution with mean $10\,\mathrm{cm}$ and standard deviation $3\,\mathrm{cm}$.

(a) Find the probability that

(i) a dart lands less than $13\,\mathrm{cm}$ from O.

(ii) a dart lands more than $15\,\mathrm{cm}$ from O. [3]

Each of Arianne's throws is independent of her previous throws.

(b) Find the probability that Arianne throws two consecutive darts that land more than $15\,\mathrm{cm}$ from O. [2]

In a competition a player has three darts to throw on each turn. A point is scored if a player throws **all** three darts to land within a central area around O. When Arianne throws a dart the probability that it lands within this area is 0.8143.

(c) Find the probability that Arianne does **not** score a point on a turn of three darts. [2]

In the competition Arianne has ten turns, each with three darts.

(d) (i) Find Arianne's expected score in the competition.

(ii) Find the probability that Arianne scores at least 5 points in the competition.

(iii) Find the probability that Arianne scores at least 5 points and less than 8 points.

(iv) Given that Arianne scores at least 5 points, find the probability that Arianne scores less than 8 points. [9]

4. [Maximum mark: 18]

A flying drone is programmed to complete a series of movements in a horizontal plane relative to an origin O and a set of x-y-axes.

In each case, the drone moves to a new position represented by the following transformations:

- a rotation anticlockwise of $\frac{\pi}{6}$ radians about O
- a reflection in the line $y = \frac{x}{\sqrt{3}}$.
- a rotation clockwise of $\frac{\pi}{3}$ radians about O.

All the movements are performed in the listed order.

(a) (i) Write down each of the transformations in matrix form, clearly stating which matrix represents each transformation.

(ii) Find a single matrix $\boldsymbol{P}$ that defines a transformation that represents the overall change in position.

(iii) Find $\boldsymbol{P}^2$.

(iv) Hence state what the value of $\boldsymbol{P}^2$ indicates for the possible movement of the drone. [12]

(b) Three drones are initially positioned at the points A, B and C. After performing the movements listed above, the drones are positioned at points A′, B′ and C′ respectively.

Show that the area of triangle ABC is equal to the area of triangle A′B′C′. [2]

(c) Find a single transformation that is equivalent to the three transformations represented by matrix $\boldsymbol{P}$. [4]

5. [Maximum mark: 13]

(a) Let $z = 1 - \mathrm{i}$.

(i) Plot the position of z on an Argand Diagram.

(ii) Express z in the form $z = a\mathrm{e}^{\mathrm{i}b}$, where a, $b \in \mathbb{R}$, giving the exact value of a and the exact value of b. [3]

(b) Let $w_1 = \mathrm{e}^{\mathrm{i}x}$ and $w_2 = \mathrm{e}^{\mathrm{i}\left(x - \frac{\pi}{2}\right)}$, where $x \in \mathbb{R}$.

(i) Find $w_1 + w_2$ in the form $\mathrm{e}^{\mathrm{i}x}(c + \mathrm{i}d)$.

(ii) Hence find $\mathrm{Re}(w_1 + w_2)$ in the form $A\cos(x - \alpha)$, where $A > 0$ and $0 < \alpha \leq \frac{\pi}{2}$. [6]

The current, I, in an AC circuit can be modelled by the equation $I = a\cos(bt - c)$ where b is the frequency and c is the phase shift.

Two AC voltage sources of the same frequency are independently connected to the same circuit. If connected to the circuit alone they generate currents I_{A} and I_{B}. The maximum value and the phase shift of each current is shown in the following table.

Current	**Maximum value**	**Phase shift**
I_{A}	12 amps	0
I_{B}	12 amps	$\frac{\pi}{2}$

When the two voltage sources are connected to the circuit at the same time, the total current I_{T} can be expressed as $I_{\mathrm{A}} + I_{\mathrm{B}}$.

(c) (i) Find the maximum value of I_{T}.

(ii) Find the phase shift of I_{T}. [4]

6. [Maximum mark: 15]

A shock absorber on a car contains a spring surrounded by a fluid. When the car travels over uneven ground the spring is compressed and then returns to an equilibrium position.

The displacement, x, of the spring is measured, in centimetres, from the equilibrium position of $x = 0$. The value of x can be modelled by the following second order differential equation, where t is the time, measured in seconds, after the initial displacement.

$$\ddot{x} + 3\dot{x} + 1.25x = 0$$

(a) Given that $y = \dot{x}$, show that $\dot{y} = -1.25x - 3y$. [2]

The differential equation can be expressed in the form $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \boldsymbol{A}\begin{pmatrix} x \\ y \end{pmatrix}$, where $\boldsymbol{A}$ is a 2×2 matrix.

(b) Write down the matrix $\boldsymbol{A}$. [1]

(c) (i) Find the eigenvalues of matrix $\boldsymbol{A}$.

(ii) Find the eigenvectors of matrix $\boldsymbol{A}$. [6]

(d) Given that when $t = 0$ the shock absorber is displaced $8\,\text{cm}$ and its velocity is zero, find an expression for x in terms of t. [6]

7. [Maximum mark: 14]

Loreto is a manager at the Da Vinci health centre. If the mean rate of patients arriving at the health centre exceeds 1.5 per minute then Loreto will employ extra staff. It is assumed that the number of patients arriving in any given time period follows a Poisson distribution.

Loreto performs a hypothesis test to determine whether she should employ extra staff. She finds that 320 patients arrived during a randomly selected 3-hour clinic.

(a) (i) Write down null and alternative hypotheses for Loreto's test.

(ii) Using the data from Loreto's sample, perform the hypothesis test at a 5% significance level to determine if Loreto should employ extra staff. [7]

Loreto is also concerned about the average waiting time for patients to see a nurse. The health centre aims for at least 95% of patients to see a nurse in under 20 minutes.

Loreto assumes that the waiting times for patients are independent of each other and decides to perform a hypothesis test at a 10% significance level to determine whether the health centre is meeting its target.

Loreto surveys 150 patients and finds that 11 of them waited more than 20 minutes.

(b) (i) Write down null and alternative hypotheses for this test.

(ii) Perform the test, clearly stating the conclusion in context. [7]

References:

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Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 2

Monday 9 May 2022 (morning)

2 hours

Instructions to candidates

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- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

12 pages

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1. [Maximum mark: 16]

The scores of the eight highest scoring countries in the 2019 Eurovision song contest are shown in the following table.

	Eurovision score
Netherlands	498
Italy	472
Russia	370
Switzerland	364
Sweden	334
Norway	331
North Macedonia	305
Azerbaijan	302

(a) For this data, find

(i) the upper quartile.

(ii) the interquartile range. [4]

(b) Determine if the Netherlands' score is an outlier for this data. Justify your answer. [3]

(This question continues on the following page)

(Question 1 continued)

Chester is investigating the relationship between the highest-scoring countries' Eurovision score and their population size to determine whether population size can reasonably be used to predict a country's score.

The populations of the countries, to the nearest million, are shown in the table.

	Population (x) (millions)	**Eurovision score (y)**
Netherlands	17	498
Italy	60	472
Russia	145	370
Switzerland	9	364
Sweden	10	334
Norway	5	331
North Macedonia	2	305
Azerbaijan	10	302

Chester finds that, for this data, the Pearson's product moment correlation coefficient is $r = 0.249$.

(c) State whether it would be appropriate for Chester to use the equation of a regression line for y on x to predict a country's Eurovision score. Justify your answer. [2]

Chester then decides to find the Spearman's rank correlation coefficient for this data, and creates a table of ranks.

	Population rank (to the nearest million)	**Eurovision score rank**
Netherlands	3	1
Italy	2	2
Russia	1	3
Switzerland	a	4
Sweden	b	5
Norway	7	6
North Macedonia	8	7
Azerbaijan	c	8

(This question continues on the following page)

(Question 1 continued)

(d) Write down the value of:

(i) a,

(ii) b,

(iii) c. [3]

(e) (i) Find the value of the Spearman's rank correlation coefficient r_s.

(ii) Interpret the value obtained for r_s. [3]

(f) When calculating the ranks, Chester incorrectly read the Netherlands' score as 478. Explain why the value of the Spearman's rank correlation r_s does not change despite this error. [1]

2. [Maximum mark: 17]

A sector of a circle, centre O and radius $4.5\,\text{m}$, is shown in the following diagram.

diagram not to scale

(a) (i) Find the angle $\text{A}\hat{\text{O}}\text{B}$.

(ii) Find the area of the shaded segment. [8]

A square field with side $8\,\text{m}$ has a goat tied to a post in the centre by a rope such that the goat can reach all parts of the field up to $4.5\,\text{m}$ from the post.

diagram not to scale

(b) Find the area of the field that can be reached by the goat. [4]

Let V be the volume of grass eaten by the goat, in cubic metres, and t be the length of time, in hours, that the goat has been in the field.

The goat eats grass at the rate of $\frac{\text{d}V}{\text{d}t} = 0.3\,t\text{e}^{-t}$.

(c) Find the value of t at which the goat is eating grass at the greatest rate. [2]

The goat is tied in the field for 8 hours.

(d) Find the total volume of grass eaten by the goat during this time. [3]

3. [Maximum mark: 13]

A Principal would like to compare the students in his school with a national standard. He decides to give a test to eight students made up of four boys and four girls. One of the teachers offers to find the volunteers from his class.

(a) Name the type of sampling that best describes the method used by the Principal. [1]

The marks out of 40, for the students who took the test, are:

$$25,\ 29,\ 38,\ 37,\ 12,\ 18,\ 27,\ 31.$$

(b) For the eight students find

(i) the mean mark.

(ii) the standard deviation of the marks. [3]

The national standard mark is 25.2 out of 40.

(c) Perform an appropriate test at the 5% significance level to see if the mean marks achieved by the students in the school are higher than the national standard. It can be assumed that the marks come from a normal population. [5]

(d) State one reason why the test might not be valid. [1]

Two additional students take the test at a later date and the mean mark for all ten students is 28.1 and the standard deviation is 8.4.

For further analysis, a standardized score out of 100 for the ten students is obtained by multiplying the scores by 2 and adding 20.

(e) For the ten students, find

(i) their mean standardized score.

(ii) the standard deviation of their standardized score. [3]

4. [Maximum mark: 13]

A particle moves such that its displacement, x metres, from a point O at time t seconds is given by the differential equation

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0$$

(a) (i) Use the substitution $y = \frac{dx}{dt}$ to show that this equation can be written as

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

(ii) Find the eigenvalues for the matrix $\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}$.

(iii) Hence state the long-term velocity of the particle. [5]

The equation for the motion of the particle is amended to

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 3t + 4.$$

(b) (i) Use the substitution $y = \frac{dx}{dt}$ to write the differential equation as a system of coupled, first order differential equations.

When $t = 0$ the particle is stationary at O.

(ii) Use Euler's method with a step length of 0.1 to find the displacement of the particle when $t = 1$.

(iii) Find the long-term velocity of the particle. [8]

5. [Maximum mark: 15]

The aircraft for a particular flight has 72 seats. The airline's records show that historically for this flight only 90% of the people who purchase a ticket arrive to board the flight. They assume this trend will continue and decide to sell extra tickets and hope that no more than 72 passengers will arrive.

The number of passengers that arrive to board this flight is assumed to follow a binomial distribution with a probability of 0.9.

(a) The airline sells 74 tickets for this flight. Find the probability that more than 72 passengers arrive to board the flight. [3]

(b) (i) Write down the expected number of passengers who will arrive to board the flight if 72 tickets are sold. [2]

(ii) Find the maximum number of tickets that could be sold if the expected number of passengers who arrive to board the flight must be less than or equal to 72. [2]

Each passenger pays $\$150$ for a ticket. If too many passengers arrive, then the airline will give $\$300$ in compensation to each passenger that cannot board.

(c) Find, to the nearest integer, the expected increase or decrease in the money made by the airline if they decide to sell 74 tickets rather than 72. [8]

6. [Maximum mark: 18]

Consider the curve $y = \sqrt{x}$.

(a) (i) Find $\frac{dy}{dx}$.

(ii) Hence show that the equation of the tangent to the curve at the point $(0.16,\ 0.4)$ is $y = 1.25x + 0.2$. [4]

The shape of a piece of metal can be modelled by the region bounded by the functions f, g, the x-axis and the line segment [AB], as shown in the following diagram. The units on the x and y axes are measured in metres.

The piecewise function f is defined by

$$f(x) = \begin{cases} \sqrt{x} & 0 \le x \le 0.16 \\ 1.25x + 0.2 & 0.16 < x \le 0.5 \end{cases}$$

The graph of g is obtained from the graph of f by:

- a stretch scale factor of $\frac{1}{2}$ in the x direction,
- followed by a stretch scale factor $\frac{1}{2}$ in the y direction,
- followed by a translation of 0.2 units to the right.

Point A lies on the graph of f and has coordinates $(0.5,\ 0.825)$. Point B is the image of A under the given transformations and has coordinates $(p,\ q)$.

(b) Find the value of p and the value of q. [2]

(This question continues on the following page)

(Question 6 continued)

The piecewise function g is given by

$$g(x) = \begin{cases} h(x) & 0.2 \le x \le a \\ 1.25x + b & a < x \le p \end{cases}$$

(c) Find

(i) an expression for $h(x)$.

(ii) the value of a.

(iii) the value of b. [5]

(d) (i) Find the area enclosed by $y = f(x)$, the x-axis and the line $x = 0.5$.

The area enclosed by $y = g(x)$, the x-axis and the line $x = p$ is $0.0627292\,\text{m}^2$ correct to six significant figures.

(ii) Find the area of the shaded region on the diagram. [7]

7. [Maximum mark: 18]

A transformation, T, of a plane is represented by $\boldsymbol{r}' = \boldsymbol{P}\boldsymbol{r} + \boldsymbol{q}$, where $\boldsymbol{P}$ is a 2×2 matrix, $\boldsymbol{q}$ is a 2×1 vector, $\boldsymbol{r}$ is the position vector of a point in the plane and $\boldsymbol{r}'$ the position vector of its image under T.

The triangle OAB has coordinates $(0, 0)$, $(0, 1)$ and $(1, 0)$. Under T, these points are transformed to $(0, 1)$, $\left(\frac{1}{4}, 1 + \frac{\sqrt{3}}{4}\right)$ and $\left(\frac{\sqrt{3}}{4}, \frac{3}{4}\right)$ respectively.

(a) (i) By considering the image of $(0, 0)$, find $\boldsymbol{q}$.

(ii) By considering the image of $(1, 0)$ and $(0, 1)$, show that [6]

$$\boldsymbol{P} = \begin{pmatrix} \frac{\sqrt{3}}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{\sqrt{3}}{4} \end{pmatrix}.$$

$\boldsymbol{P}$ can be written as $\boldsymbol{P} = \boldsymbol{R}\boldsymbol{S}$, where $\boldsymbol{S}$ and $\boldsymbol{R}$ are matrices.

$\boldsymbol{S}$ represents an enlargement with scale factor 0.5, centre $(0, 0)$.

$\boldsymbol{R}$ represents a rotation about $(0, 0)$.

(b) Write down the matrix $\boldsymbol{S}$. [1]

(c) (i) Use $\boldsymbol{P} = \boldsymbol{R}\boldsymbol{S}$ to find the matrix $\boldsymbol{R}$.

(ii) Hence find the angle and direction of the rotation represented by $\boldsymbol{R}$. [7]

The transformation T can also be described by an enlargement scale factor $\frac{1}{2}$, centre (a, b), followed by a rotation about the same centre (a, b).

(d) (i) Write down an equation satisfied by $\begin{pmatrix} a \\ b \end{pmatrix}$.

(ii) Find the value of a and the value of b. [4]

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Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 2

9 May 2023

Zone A afternoon | **Zone B** morning | **Zone C** afternoon

2 hours

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

11 pages

2223 – 7207
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Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 13]

The mean annual temperatures for Earth, recorded at fifty-year intervals, are shown in the table.

Year (x)	1708	1758	1808	1858	1908	1958	2008
Temperature °C (y)	8.73	9.22	9.10	9.12	9.13	9.45	9.76

Tami creates a linear model for this data by finding the equation of the straight line passing through the points with coordinates $(1708, 8.73)$ and $(1958, 9.45)$.

(a) Calculate the gradient of the straight line that passes through these two points. [2]

(b) (i) Interpret the meaning of the gradient in the context of the question.

(ii) State appropriate units for the gradient. [2]

(c) Find the equation of this line giving your answer in the form $y = mx + c$. [2]

(d) Use Tami's model to estimate the mean annual temperature in the year 2000. [2]

Thandizo uses linear regression to obtain a model for the data.

(e) (i) Find the equation of the regression line y on x.

(ii) Find the value of r, the Pearson's product-moment correlation coefficient. [3]

(f) Use Thandizo's model to estimate the mean annual temperature in the year 2000. [2]

2. [Maximum mark: 15]

The depth of water, w metres, in a particular harbour can be modelled by the function $w(t) = a\cos(bt^\circ) + d$ where t is the length of time, in minutes, after $06{:}00$.

On 20 January, the first high tide occurs at $06{:}00$, at which time the depth of water is $18\,\text{m}$. The following low tide occurs at $12{:}15$ when the depth of water is $4\,\text{m}$. This is shown in the diagram.

(a) Find the value of a. [2]

(b) Find the value of d. [2]

(c) Find the period of the function in minutes. [3]

(d) Find the value of b. [2]

Naomi is sailing to the harbour on the morning of 20 January. Boats can enter or leave the harbour only when the depth of water is at least $6\,\text{m}$.

(e) Find the latest time before $12{:}00$, to the nearest minute, that Naomi can enter the harbour. [4]

(f) Find the length of time (in minutes) between $06{:}00$ and $15{:}00$ on 20 January during which Naomi **cannot** enter or leave the harbour. [2]

3. [Maximum mark: 17]

A large international sports tournament tests their athletes for banned substances.
They interpret a positive test result as meaning that the athlete uses banned substances.
A negative result means that they do not.

The probability that an athlete uses banned substances is estimated to be 0.06.

If an athlete **uses** banned substances, the probability that they will test positive is 0.71.

If an athlete does **not use** banned substances, the probability that they will test negative is 0.98.

(a) Using the information given, **copy** (into your answer booklet) and complete the following tree diagram. [2]

(b) (i) Determine the probability that a randomly selected athlete does not use banned substances and tests negative.

(ii) If two athletes are selected at random, calculate the probability that both athletes do not use banned substances and both test negative. [4]

(c) (i) Calculate the probability that a randomly selected athlete will receive an **incorrect** test result.

(ii) A random sample of 1300 athletes at the tournament are selected for testing. Calculate the expected number of athletes in the sample that will receive an incorrect test result. [5]

Team X are competing in the tournament. There are 20 athletes in this team. It is known that none of the athletes in Team X use banned substances.

(d) Calculate the probability that none of the athletes in Team X will test positive. [4]

(e) Determine the probability that more than 2 athletes in Team X will test positive. [2]

4. [Maximum mark: 17]

The vertices in the following graph represent seven towns. The edges represent their connecting roads. The weight on each edge represents the distance, in kilometres, between the two connected towns.

(a) Determine whether it is possible to complete a journey that starts and finishes at different towns that also uses each of the roads exactly once. Give a reason for your answer. [2]

The shortest distance, in kilometres, between any two towns is given in the table.

	A	B	C	D	E	F	G
A		6	8	5	11	9	19
B	6		12	5	7	3	13
C	8	12		7	7	***a***	***b***
D	5	5	7		6	5	***c***
E	11	7	7	6		4	11
F	9	3	***a***	5	4		***d***
G	19	13	***b***	***c***	11	***d***	

(This question continues on the following page)

(Question 4 continued)

(b) Find the value of

(i) a;

(ii) b;

(iii) c;

(iv) d. [2]

(c) Use the nearest neighbour algorithm, starting at vertex G, to find an upper bound for the travelling salesman problem. [3]

(d) (i) Sketch a minimum spanning tree for the subgraph with vertices $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}, \mathrm{F}$.

(ii) Write down the total weight of the minimum spanning tree. [4]

(e) Hence find a lower bound for the travelling salesman problem. [2]

(f) Explain one way in which an improved lower bound could be found. [1]

It is found that the optimum solution starting at A is actually $\mathrm{A}–\mathrm{C}–\mathrm{E}–\mathrm{G}–\mathrm{B}–\mathrm{F}–\mathrm{D}–\mathrm{A}$.

(g) Given that the length of each road shown on the graph is given to the nearest kilometre, find the lower bound for the total distance in the optimal solution. [3]

5. [Maximum mark: 15]

Goran is interested in the number of sightings of a particular bird each week in the 50 weeks following the first day of September. He collects some data which is shown in the table.

Number of sightings	0	1	2	3	4	5	More than 5
Number of weeks	8	16	13	8	3	2	0

The sample mean number of sightings per week for this data is 1.76.

(a) Calculate the unbiased estimate of the population variance of sightings per week. [3]

Goran believes that the data follows a Poisson distribution.

(b) State why your answer to part (a) supports Goran's belief. [1]

Goran decides to test at the 5% significance level to see if his belief is correct.

His null hypothesis is $X \sim \mathrm{Po}(m)$, where the random variable, X, is defined as the number of sightings per week.

Goran estimates parameter m to be the mean of the sample, 1.76. He calculates the expected frequencies for sightings per week in the 50 weeks after the first day of September. These are shown to two decimal places in the following table.

Number of sightings	0	1	2	3	4	5 or more
Expected frequencies	8.60	15.14	13.32	7.82	j	k

(c) Find the value of

(i) j;

(ii) k. [5]

(d) State a reason why Goran should combine groups to conduct his significance test. [1]

(e) Write down the degrees of freedom for the test. [1]

(f) Find the p-value for the test. [2]

(g) State the conclusion of the test. Justify your answer. [2]

6. [Maximum mark: 15]

A model speedboat has its position, at time t seconds $t \geq 0$, defined by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 5y - 0.05x\,,\ \frac{\mathrm{d}y}{\mathrm{d}t} = -5x - 0.05y\,,$$

where x metres is the distance east and y metres is the distance north of a fixed point O.

(a) Find the eigenvalues of $\boldsymbol{A} = \begin{pmatrix} -0.05 & 5 \\ -5 & -0.05 \end{pmatrix}$, giving your answers in the form $a + b\mathrm{i}$, where $a \neq 0$, $b \neq 0$. [4]

(b) (i) State what $a \neq 0$ indicates about the path of the speedboat.

(ii) State what the sign of a indicates about the path of the speedboat. [2]

At time $t = 0$, the speedboat has position $(20, 0)$.

(c) At time $t = 0$, find the value of

(i) $\dfrac{\mathrm{d}y}{\mathrm{d}t}$.

(ii) $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. [5]

(d) Use your answers to parts (b) and (c) to sketch the path of the model speedboat. [4]

7. [Maximum mark: 18]

A trapezoid, Q, has vertices $(0, -1)$, $(0, -2)$, $(\sin 15°, -3-\cos 15°)$, $(\sin 15°, -1-\cos 15°)$ as shown.

(a) Show that the area of the trapezoid is $\frac{3}{2}\sin 15°$. [2]

A design is created with 24 elements. Each element is obtained by transforming the trapezoid Q. These elements are shaded in the following diagram such that the y-axis is a line of symmetry.

(This question continues on the following page)

(Question 7 continued)

The transformation that produces each of the elements on the **right side** of the design can be represented by a matrix of the form

$$\boldsymbol{M}_k = \begin{pmatrix} \left(1-\frac{k}{12}\right)\cos\left(k\times 15^\circ\right) & -\left(1-\frac{k}{12}\right)\sin\left(k\times 15^\circ\right) \\ \left(1-\frac{k}{12}\right)\sin\left(k\times 15^\circ\right) & \left(1-\frac{k}{12}\right)\cos\left(k\times 15^\circ\right) \end{pmatrix}$$

where $k = 0, 1, 2, 3, \ldots, 11$.

(b) (i) Find the matrix $\boldsymbol{M}_6$. Give your answer in the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a, b, c, d \in \mathbb{Q}$.

(ii) Hence find the coordinates of the image of the vertex $(0, -1)$ after it is transformed by the matrix $\boldsymbol{M}_6$. [4]

The matrix $\boldsymbol{M}_k$ can be expressed as the product of a rotation matrix and an enlargement matrix.

(c) Write down, in terms of k,

(i) the rotation matrix;

(ii) the enlargement matrix;

(iii) the angle of the rotation;

(iv) the scale factor of the enlargement. [4]

(d) Using your answer to part (c)(iv), or otherwise, find the determinant of the matrix $\boldsymbol{M}_k$ in terms of k. [2]

(e) Hence, or otherwise, find the total area of the elements in the **whole** design. [4]

Each element on the **left side** of the design can be obtained through a transformation of the trapezoid Q by applying the matrix $\boldsymbol{N}_k$, where $k = 0, 1, 2, 3, \ldots, 11$.

(f) Write down the matrix $\boldsymbol{N}_k$ as a product of two matrices. [2]

References:

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Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 2

2 May 2024

Zone A morning | **Zone B** morning | **Zone C** morning

2 hours

Instructions to candidates

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- A graphic display calculator is required for this paper.
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- A clean copy of the **mathematics: applications and interpretation HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

13 pages

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1. [Maximum mark: 15]

The company Fred Express delivers packages. From past experience, the time taken, T, to deliver a package follows a normal distribution with mean 64 hours and standard deviation 12 hours.

(a) State $\mathrm{P}(T < 64)$. [1]

(b) Find $\mathrm{P}(44 < T < 64)$. [2]

30% of packages are delivered in less than k hours.

(c) (i) Sketch a diagram of this normal distribution, shading the region that represents $\mathrm{P}(T < k)$.

(ii) Find the value of k. [4]

For quality control, the manager randomly selects five outgoing packages. These selections are independent.

(d) Find the probability that exactly two of these packages are delivered in less than k hours. [3]

Fred Express charges a fixed amount of $\$4.50$ for any package weighing $1\,\mathrm{kg}$ or less. Heavier packages are charged an additional fee of $\$2.00$ per kg. This fee is applied for any weight **in excess of** $1\,\mathrm{kg}$. For example, a $1.5\,\mathrm{kg}$ package is charged an additional $\$1.00$.

(e) Write down an expression for the amount charged to deliver a package of weight $x\,\mathrm{kg}$, where $x > 1$. [2]

(f) Find the amount Fred Express charges for a $5.3\,\mathrm{kg}$ package. [1]

Meiling is charged $\$7.20$ for the delivery of a package.

(g) Find the weight of Meiling's package. [2]

2. [Maximum mark: 12]

A national park contains three mountains whose summits are at points A, B and C.

According to a coordinate system, the position of A is $(0, 0, 2.8)$ and the position of B is $(7.2, 5.1, 2.4)$. All the values are in kilometres.

diagram not to scale

(a) (i) Find the vector $\overrightarrow{AB}$.

(ii) Hence find AB, the distance between A and B. [3]

The vector $\overrightarrow{AC}$ is parallel to the vector $\begin{pmatrix} 1.1 \\ 8.4 \\ 0.2 \end{pmatrix}$.

(b) Find the angle between $\begin{pmatrix} 1.1 \\ 8.4 \\ 0.2 \end{pmatrix}$ and $\overrightarrow{AB}$. [5]

The angle between $\overrightarrow{BA}$ and $\overrightarrow{BC}$ is $55.2°$.

(c) Use the sine rule to find AC. [4]

3. [Maximum mark: 13]

A shop uses the following model to estimate n, the number of smoothies sold per day, in terms of x, the price of a single smoothie in pesos.

$$n = \frac{40\,000}{x^2}$$

The maximum number of smoothies the shop can make in a day is 400.

(a) Find the maximum price they could charge per smoothie for the shop to sell 400 in one day. [2]

(b) On a day when the shop sells smoothies at 50 pesos each, use the model to find

(i) the number of smoothies sold.

(ii) the total income from the smoothies sold. [2]

The cost of making each smoothie is 20 pesos. The profit per day (P) is the total income from the sale of smoothies that day minus the cost of making them.

(c) (i) Show that, according to the model, $P = \frac{40\,000}{x} - \frac{800\,000}{x^2}$.

(ii) Find $\frac{\mathrm{d}P}{\mathrm{d}x}$.

(iii) Find the value of x for which $\frac{\mathrm{d}P}{\mathrm{d}x} = 0$.

(iv) Find the number of smoothies sold when the profit is maximized. [9]

4. [Maximum mark: 12]

A type of generator will only function if a particular switch is working. The generator has a main switch, A, and a 'back up' switch, B.

The manufacturer claims the probability of switch A failing within one month of being fitted is 0.1 and the probability of the cheaper switch B failing within one month is 0.3. Whether or not a switch fails is independent of the state of the other switch.

If both switches fail, the generator needs to shut down to replace the switches. Both switches are replaced after a month of use (whether they have failed or not) or whenever the generator needs to be shut down.

The following tree diagram shows the probabilities of a switch failing within one month of them both being replaced, assuming the manufacturer's claim is correct.

(a) Write down the values of

(i) a.

(ii) b.

(iii) c. [2]

(b) Hence find the probability that the generator needs to shut down within one month of the switches being replaced. [1]

(This question continues on the following page)

(Question 4 continued)

The owner of the generator is suspicious of the switch manufacturer's claims, so they look back through the past 200 occasions when the switches were replaced. The records show whether no switches, one switch or two switches had failed.

The data the owner collected are shown in the following table.

No switch fails	One switch fails	Two switches fail
118	72	10

(c) Perform a χ^2 goodness of fit test at the 5% significance level to test whether the manufacturer's claims are correct using the following hypotheses.

H_0: The manufacturer's claims are correct.
H_1: The manufacturer's claims are not both correct. [9]

5. [Maximum mark: 13]

Consider the differential equation $\frac{dy}{dx} = \frac{x}{e^{2y}}$.

(a) Identify which of the following diagrams, **A**, **B** or **C**, represents the slope field for the differential equation. Give a reason for your answer. [2]

A.

B.

C.

(This question continues on the following page)

(Question 5 continued)

It is given that, for a particular solution, $x = 0$ and $y = 0$.

(b) Find an expression for y, in terms of x, for this solution. [7]

(c) Find $\frac{dy}{dx}$, in terms of x, by differentiating your answer from part (b). [2]

(d) Hence verify that your answer to part (b) is a solution to $\frac{dy}{dx} = \frac{x}{e^{2y}}$. [2]

6. [Maximum mark: 13]

Taylor is playing a computer game in which they shoot at spaceships and battleships. The number of spaceships they hit per minute can be modelled by a Poisson distribution with mean 4.2. The number of battleships they hit per minute can be modelled by a Poisson distribution with a mean of 2.3. Any single hit occurs independently of all others.

(a) Find the probability Taylor hits

(i) at most 10 spaceships in 2 minutes.

(ii) a total of more than 10 spaceships and battleships in one minute. [5]

Every spaceship that is hit earns Taylor 3 points and every battleship 5 points. Let T be the total points earned in one minute.

(b) Find

(i) $\mathrm{E}(T)$.

(ii) $\mathrm{Var}(T)$. [3]

(c) State one reason why the distribution of T cannot be Poisson. [1]

Taylor intends to play the game for one hour.

(d) Use the central limit theorem to find the probability that Taylor's mean score per minute is greater than 25. [4]

7. [Maximum mark: 14]

The interior of a vase is modelled by rotating the region bounded by the curve $y = \frac{1}{2}x^2 - 1$, and the lines $x = 0$, $y = 0$ and $y = 15$, through 2π radians about the y-axis. The values of x and y are measured in centimetres.

The vase is filled with water to a height of $h\,\text{cm}$.

(a) Find an explicit expression for the volume of water in terms of h. [5]

The vase is filled at a rate of $20\,\text{cm}^3\,\text{s}^{-1}$.

(b) Find the time taken to completely fill the vase. [2]

(c) Find the rate at which the height is changing when $h = 10\,\text{cm}$. [7]

8. [Maximum mark: 18]

(a) Write down the adjacency matrix for the directed graph shown below. [2]

(b) Find the total number of walks of length 5 from A to B. [3]

A bird sits on one of three posts, labelled A, B and C, with B between A and C. When the bird moves, it will either fly to an adjacent post or return to the same post according to the following pattern.

- If it is on B, it will fly to A or C, each with a probability of 0.5.
- If it is on A or C, it will return to the same post with a probability of 0.5 or fly to B with a probability of 0.5.

The possible flights of the bird can be represented by the graph in part (a).

(c) (i) Every possible sequence of 5 flights by the bird has the same probability of occurring. State this probability.

(ii) Use your answer to part (b) to find the probability that if the bird was initially on post A, it will be on post B after 5 flights. [3]

A second bird often joins the first on the posts. Their flights now follow the pattern given below.

- The birds will never sit on the same post.
- They will always fly from the posts at the same time.
- If they are on adjacent posts, the bird on post B will **always** fly to the vacant end post. The other bird will fly to post B with a probability of 0.4 or return to the same post with a probability of 0.6.
- If they are each on one of the end posts, they will fly to post B with a probability of 0.5 or return to the same post with a probability of 0.5. However, if they both try to fly to post B at the same time, they will see the other one doing so and both will immediately return to the post they were previously on.

(This question continues on the following page)

(Question 8 continued)

The possible flights of the two birds can be represented by the following transition diagram, where the three vertices represent which posts are **occupied**.

diagram not to scale

(d) Write down the value of

(i) a.

(ii) b.

(iii) c. [3]

(e) Given that the birds are initially on posts A and B, find the probability they will be on posts B and C after 5 flights. [4]

The birds continue this pattern of flights for a long period.

(f) Given that the time between flights is always the same, find the post which is sat on least and the proportion of the time it is free. [3]

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Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 2

Friday 7 May 2021 (morning)

2 hours

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10 pages

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Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 19]

Give your answers in parts (a), (d)(i), (e) and (f) to the nearest dollar.

Daisy invested $37\,000$ Australian dollars (AUD) in a fixed deposit account with an annual interest rate of 6.4% compounded **quarterly**.

(a) Calculate the value of Daisy's investment after 2 years. [3]

After m months, the amount of money in the fixed deposit account has appreciated to more than $50\,000$ AUD.

(b) Find the minimum value of m, where $m \in \mathbb{N}$. [4]

Daisy is saving to purchase a new apartment. The price of the apartment is $200\,000$ AUD.

Daisy makes an initial payment of 25% and takes out a loan to pay the rest.

(c) Write down the amount of the loan. [1]

The loan is for 10 years, compounded monthly, with equal monthly payments of 1700 AUD made by Daisy at the end of each month.

(d) For this loan, find

(i) the amount of interest paid by Daisy.

(ii) the annual interest rate of the loan. [5]

After 5 years of paying off this loan, Daisy decides to pay the **remainder** in one final payment.

(e) Find the amount of Daisy's final payment. [3]

(f) Find how much money Daisy saved by making one final payment after 5 years. [3]

2. [Maximum mark: 16]

The cross-sectional view of a tunnel is shown on the axes below. The line $[AB]$ represents a vertical wall located at the left side of the tunnel. The height, in metres, of the tunnel above the horizontal ground is modelled by $y = -0.1x^3 + 0.8x^2$, $2 \le x \le 8$, relative to an origin O.

Point A has coordinates $(2, 0)$, point B has coordinates $(2, 2.4)$, and point C has coordinates $(8, 0)$.

(a) (i) Find $\frac{dy}{dx}$.

(ii) Hence find the maximum height of the tunnel. [6]

(b) Find the height of the tunnel when

(i) $x = 4$.

(ii) $x = 6$. [3]

(c) Use the trapezoidal rule, with three intervals, to estimate the cross-sectional area of the tunnel. [3]

(d) (i) Write down the integral which can be used to find the cross-sectional area of the tunnel.

(ii) Hence find the cross-sectional area of the tunnel. [4]

3. [Maximum mark: 13]

The stopping distances for bicycles travelling at $20\,\mathrm{km\,h^{-1}}$ are assumed to follow a normal distribution with mean $6.76\,\mathrm{m}$ and standard deviation $0.12\,\mathrm{m}$.

(a) Under this assumption, find, correct to four decimal places, the probability that a bicycle chosen at random travelling at $20\,\mathrm{km\,h^{-1}}$ manages to stop

(i) in less than $6.5\,\mathrm{m}$.

(ii) in more than $7\,\mathrm{m}$. [3]

1000 randomly selected bicycles are tested and their stopping distances when travelling at $20\,\mathrm{km\,h^{-1}}$ are measured.

(b) Find, correct to four significant figures, the expected number of bicycles tested that stop between

(i) $6.5\,\mathrm{m}$ and $6.75\,\mathrm{m}$.

(ii) $6.75\,\mathrm{m}$ and $7\,\mathrm{m}$. [3]

The measured stopping distances of the 1000 bicycles are given in the table.

Measured stopping distance	**Number of bicycles**
Less than $6.5\,\mathrm{m}$	12
Between $6.5\,\mathrm{m}$ and $6.75\,\mathrm{m}$	428
Between $6.75\,\mathrm{m}$ and $7\,\mathrm{m}$	527
More than $7\,\mathrm{m}$	33

It is decided to perform a χ^2 goodness of fit test at the $5\,\%$ level of significance to decide whether the stopping distances of bicycles travelling at $20\,\mathrm{km\,h^{-1}}$ can be modelled by a normal distribution with mean $6.76\,\mathrm{m}$ and standard deviation $0.12\,\mathrm{m}$.

(c) State the null and alternative hypotheses. [2]

(d) Find the p-value for the test. [3]

(e) State the conclusion of the test. Give a reason for your answer. [2]

4. [Maximum mark: 14]

Charlotte decides to model the shape of a cupcake to calculate its volume.

From rotating a photograph of her cupcake she estimates that its cross-section passes through the points $(0, 3.5)$, $(4, 6)$, $(6.5, 4)$, $(7, 3)$ and $(7.5, 0)$, where all units are in centimetres. The cross-section is symmetrical in the x-axis, as shown below:

She models the section from $(0, 3.5)$ to $(4, 6)$ as a straight line.

(a) Find the equation of the line passing through these two points. [2]

(This question continues on the following page)

(Question 4 continued)

Charlotte models the section of the cupcake that passes through the points $(4, 6)$, $(6.5, 4)$, $(7, 3)$ and $(7.5, 0)$ with a quadratic curve.

(b) (i) Find the equation of the least squares regression quadratic curve for these four points.

(ii) By considering the gradient of this curve when $x = 4$, explain why it may not be a good model. [3]

Charlotte thinks that a quadratic with a maximum point at $(4, 6)$ and that passes through the point $(7.5, 0)$ would be a better fit.

(c) Find the equation of the new model. [4]

Believing this to be a better model for her cupcake, Charlotte finds the volume of revolution about the x-axis to estimate the volume of the cupcake.

(d) (i) Write down an expression for her estimate of the volume as a sum of two integrals.

(ii) Find the value of Charlotte's estimate. [5]

5. [Maximum mark: 13]

Long term experience shows that if it is sunny on a particular day in Vokram, then the probability that it will be sunny the following day is 0.8. If it is not sunny, then the probability that it will be sunny the following day is 0.3.

The transition matrix $\boldsymbol{T}$ is used to model this information, where $\boldsymbol{T} = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$.

(a) It is sunny today. Find the probability that it will be sunny in three days' time. [2]

(b) Find the eigenvalues and eigenvectors of $\boldsymbol{T}$. [5]

The matrix $\boldsymbol{T}$ can be written as a product of three matrices, $\boldsymbol{PDP}^{-1}$, where $\boldsymbol{D}$ is a diagonal matrix.

(c) (i) Write down the matrix $\boldsymbol{P}$.

(ii) Write down the matrix $\boldsymbol{D}$. [2]

(d) Hence find the long-term percentage of sunny days in Vokram. [4]

6. [Maximum mark: 18]

An ice-skater is skating such that her position vector when viewed from above at time t seconds can be modelled by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\,\mathrm{e}^{bt}\cos t \\ a\,\mathrm{e}^{bt}\sin t \end{pmatrix}$$

with respect to a rectangular coordinate system from a point O, where the non-zero constants a and b can be determined. All distances are in metres.

(a) Find the velocity vector at time t. [3]

(b) Show that the magnitude of the velocity of the ice-skater at time t is given by

$$a\,\mathrm{e}^{bt}\sqrt{(1+b^2)}.$$ [4]

At time $t = 0$, the displacement of the ice-skater is given by $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$ and the velocity of the ice-skater is given by $\begin{pmatrix} -3.5 \\ 5 \end{pmatrix}$.

(c) Find the value of a and the value of b. [3]

(d) Find the magnitude of the velocity of the ice-skater when $t = 2$. [2]

At a point P, the ice-skater is skating parallel to the y-axis for the first time.

(e) Find OP. [6]

7. [Maximum mark: 17]

A biologist introduces 100 rabbits to an island and records the size of their population (x) over a period of time. The population growth of the rabbits can be approximately modelled by the following differential equation, where t is time measured in years.

$$\frac{dx}{dt} = 2x$$

(a) Find the population of rabbits 1 year after they were introduced. [5]

A population of 100 foxes is introduced to the island when the population of rabbits has reached 1000. The subsequent population growth of rabbits and foxes, where y is the population of foxes at time t, can be approximately modelled by the coupled equations:

$$\frac{dx}{dt} = x(2 - 0.01y)$$

$$\frac{dy}{dt} = y(0.0002x - 0.8)$$

(b) Use Euler's method with a step size of 0.25, to find

(i) the population of rabbits 1 year after the foxes were introduced.

(ii) the population of foxes 1 year after the foxes were introduced. [6]

(c) The graph of the population sizes, according to this model, for the first 4 years after the foxes were introduced is shown below.

Describe the changes in the populations of rabbits and foxes for these 4 years at

(i) point A.

(ii) point B. [3]

(d) Find the non-zero equilibrium point for the populations of rabbits and foxes. [3]

References:

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Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 2

Monday 9 May 2022 (morning)

2 hours

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

2222 – 7212

8 pages

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Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 13]

Scott purchases food for his dog in large bags and feeds the dog the same amount of dog food each day. The amount of dog food left in the bag at the end of each day can be modelled by an arithmetic sequence.

On a particular day, Scott opened a new bag of dog food and fed his dog. By the end of the third day there were 115.5 cups of dog food remaining in the bag and at the end of the eighth day there were 108 cups of dog food remaining in the bag.

(a) Find the number of cups of dog food

(i) fed to the dog per day;

(ii) remaining in the bag at the end of the first day. [4]

(b) Calculate the number of days that Scott can feed his dog with one bag of food. [2]

In 2021, Scott spent $\$625$ on dog food. Scott expects that the amount he spends on dog food will increase at an annual rate of 6.4%.

(c) Determine the amount that Scott expects to spend on dog food in 2025. Round your answer to the nearest dollar. [3]

(d) (i) Calculate the value of $\sum_{n=1}^{10}(625 \times 1.064^{(n-1)})$.

(ii) Describe what the value in part (d)(i) represents in this context. [3]

(e) Comment on the appropriateness of modelling this scenario with a geometric sequence. [1]

2. [Maximum mark: 15]

A cafe makes x litres of coffee each morning. The cafe's profit each morning, C, measured in dollars, is modelled by the following equation

$$C = \frac{x}{10}\left(k^2 - \frac{3}{100}x^2\right)$$

where k is a positive constant.

(a) Find an expression for $\frac{\mathrm{d}C}{\mathrm{d}x}$ in terms of k and x. [3]

(b) Hence find the maximum value of C in terms of k. Give your answer in the form pk^3, where p is a constant. [4]

The cafe's manager knows that the cafe makes a profit of $\$426$ when 20 litres of coffee are made in a morning.

(c) (i) Find the value of k.

(ii) Use the model to find how much coffee the cafe should make each morning to maximize its profit. [3]

(d) Sketch the graph of C against x, labelling the maximum point and the x-intercepts with their coordinates. [3]

The manager of the cafe wishes to serve as many customers as possible.

(e) Determine the maximum amount of coffee the cafe can make that will not result in a loss of money for the morning. [2]

3. [Maximum mark: 18]

The Voronoi diagram below shows four supermarkets represented by points with coordinates $A(0, 0)$, $B(6, 0)$, $C(0, 6)$ and $D(2, 2)$. The vertices X, Y, Z are also shown. All distances are measured in kilometres.

diagram not to scale

(a) Find the midpoint of $[BD]$. [2]

(b) Find the equation of (XZ). [4]

The equation of (XY) is $y = 2 - x$ and the equation of (YZ) is $y = 0.5x + 3.5$.

(c) Find the coordinates of X. [3]

The coordinates of Y are $(-1, 3)$ and the coordinates of Z are $(7, 7)$.

(d) Determine the exact length of $[YZ]$. [2]

(e) Given that the exact length of $[XY]$ is $\sqrt{32}$, find the size of $X\hat{Y}Z$ in degrees. [4]

(f) Hence find the area of triangle XYZ. [2]

A town planner believes that the larger the area of the Voronoi cell XYZ, the more people will shop at supermarket D.

(g) State one criticism of this interpretation. [1]

4. [Maximum mark: 15]

A student investigating the relationship between chemical reactions and temperature finds the Arrhenius equation on the internet.

$$k = A\mathrm{e}^{-\frac{c}{T}}$$

This equation links a variable k with the temperature T, where A and c are positive constants and $T > 0$.

(a) Show that $\frac{\mathrm{d}k}{\mathrm{d}T}$ is always positive. [3]

(b) Given that $\lim_{T\to\infty} k = A$ and $\lim_{T\to 0} k = 0$, sketch the graph of k against T. [3]

The Arrhenius equation predicts that the graph of $\ln k$ against $\frac{1}{T}$ is a straight line.

(c) Write down

(i) the gradient of this line in terms of c;

(ii) the y-intercept of this line in terms of A. [4]

The following data are found for a particular reaction, where T is measured in Kelvin and k is measured in $\mathrm{cm}^3\ \mathrm{mol}^{-1}\ \mathrm{s}^{-1}$:

T	***k***
590	5×10^{-4}
600	6×10^{-4}
610	10×10^{-4}
620	14×10^{-4}
630	20×10^{-4}
640	29×10^{-4}
650	36×10^{-4}

(d) Find the equation of the regression line for $\ln k$ on $\frac{1}{T}$. [2]

(e) Find an estimate of

(i) c;

(ii) A.

It is not required to state units for these values. [3]

5. [Maximum mark: 12]

A geneticist uses a Markov chain model to investigate changes in a specific gene in a cell as it divides. Every time the cell divides, the gene may mutate between its normal state and other states.

The model is of the form

$$\begin{pmatrix} X_{n+1} \\ Z_{n+1} \end{pmatrix} = \boldsymbol{M} \begin{pmatrix} X_n \\ Z_n \end{pmatrix}$$

where X_n is the probability of the gene being in its normal state after dividing for the nth time, and Z_n is the probability of it being in another state after dividing for the nth time, where $n \in \mathbb{N}$.

Matrix $\boldsymbol{M}$ is found to be $\begin{pmatrix} 0.94 & b \\ 0.06 & 0.98 \end{pmatrix}$.

(a) (i) Write down the value of b.

(ii) What does b represent in this context? [2]

(b) Find the eigenvalues of $\boldsymbol{M}$. [3]

(c) Find the eigenvectors of $\boldsymbol{M}$. [3]

(d) The gene is in its normal state when $n = 0$. Calculate the probability of it being in its normal state

(i) when $n = 5$;

(ii) in the long term. [4]

6. [Maximum mark: 21]

At an archery tournament, a particular competition sees a ball launched into the air while an archer attempts to hit it with an arrow.

The path of the ball is modelled by the equation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} + t\begin{pmatrix} u_x \\ u_y - 5t \end{pmatrix}$$

where x is the horizontal displacement from the archer and y is the vertical displacement from the ground, both measured in metres, and t is the time, in seconds, since the ball was launched.

- u_x is the horizontal component of the initial velocity
- u_y is the vertical component of the initial velocity.

In this question both the ball and the arrow are modelled as single points. The ball is launched with an initial velocity such that $u_x = 8$ and $u_y = 10$.

(a) (i) Find the initial speed of the ball.

(ii) Find the angle of elevation of the ball as it is launched. [4]

(b) Find the maximum height reached by the ball. [3]

(c) Assuming that the ground is horizontal and the ball is not hit by the arrow, find the x coordinate of the point where the ball lands. [3]

(d) For the path of the ball, find an expression for y in terms of x. [3]

An archer releases an arrow from the point $(0, 2)$. The arrow is modelled as travelling in a straight line, in the same plane as the ball, with speed $60\,\mathrm{m\,s^{-1}}$ and an angle of elevation of 10°.

(e) Determine the two positions where the path of the arrow intersects the path of the ball. [4]

(f) Determine the time when the arrow should be released to hit the ball before the ball reaches its maximum height. [4]

7. [Maximum mark: 16]

An environmental scientist is asked by a river authority to model the effect of a leak from a power plant on the mercury levels in a local river. The variable x measures the concentration of mercury in micrograms per litre.

The situation is modelled using the second order differential equation

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 0$$

where $t \geq 0$ is the time measured in days since the leak started. It is known that when $t = 0$, $x = 0$ and $\frac{dx}{dt} = 1$.

(a) Show that the system of coupled first order equations:

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -2x - 3y$$

can be written as the given second order differential equation. [2]

(b) Find the eigenvalues of the system of coupled first order equations given in part (a). [3]

(c) Hence find the exact solution of the second order differential equation. [5]

(d) Sketch the graph of x against t, labelling the maximum point of the graph with its coordinates. [2]

If the mercury levels are greater than 0.1 micrograms per litre, fishing in the river is considered unsafe and is stopped.

(e) Use the model to calculate the total amount of time when fishing should be stopped. [3]

The river authority decides to stop people from fishing in the river for 10% longer than the time found from the model.

(f) Write down one reason, with reference to the context, to support this decision. [1]

References:

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Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 2

9 May 2023

Zone A afternoon | **Zone B** morning | **Zone C** afternoon

2 hours

Instructions to candidates

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- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

10 pages

2223 – 7212
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1. [Maximum mark: 13]

The diagram shows points in a park viewed from above, at a specific moment in time.

The distance between two trees, at points A and B, is $6.36\,\text{m}$.

Odette is playing football in the park and is standing at point O, such that $\text{OA} = 25.9\,\text{m}$ and $\text{O}\hat{\text{A}}\text{B} = 125^\circ$.

diagram not to scale

(a) Calculate the area of triangle AOB. [3]

(This question continues on the following page)

(Question 1 continued)

Odette's friend, Khemil, is standing at point K such that he is $12\,\mathrm{m}$ from A and $\mathrm{K\hat{A}B} = 45^\circ$.

diagram not to scale

(b) Calculate Khemil's distance from B. [3]

XY is a semicircular path in the park with centre A, such that $\mathrm{K\hat{A}Y} = 45^\circ$. Khemil is standing on the path and Odette's football is at point X. This is shown in the diagram below.

diagram not to scale

The length $\mathrm{KX} = 22.2\,\mathrm{m}$, $\mathrm{K\hat{O}X} = 53.8^\circ$ and $\mathrm{O\hat{K}X} = 51.1^\circ$.

(c) Find whether Odette or Khemil is closer to the football. [4]

Khemil runs along the semicircular path to pick up the football.

(d) Calculate the distance that Khemil runs. [3]

2. [Maximum mark: 12]

A scientist is conducting an experiment on the growth of a certain species of bacteria.

The population of the bacteria, P, can be modelled by the function

$$P(t) = 1200 \times k^t,\ t \geq 0,$$

where t is the number of hours since the experiment began, and k is a positive constant.

(a) (i) Write down the value of $P(0)$.

(ii) Interpret what this value means in this context. [2]

3 hours after the experiment began, the population of the bacteria is $18\,750$.

(b) Find the value of k. [2]

(c) Find the population of the bacteria 1 hour and 30 minutes after the experiment began. [2]

The scientist conducts a second experiment with a different species of bacteria.
The population of this bacteria, S, can be modelled by the function

$$S(t) = 5000 \times 1.65^t,\ t \geq 0,$$

where t is the number of hours since both experiments began.

(d) Find the value of t when the two populations of bacteria are equal. [2]

It takes 2 hours and m minutes for the number of bacteria in the second experiment to reach $19\,000$.

(e) Find the value of m, giving your answer as an integer value. [4]

3. [Maximum mark: 16]

A particular park consists of a rectangular garden, of area $A\,\mathrm{m}^2$, and a concrete path surrounding it. The park has a total area of $1200\,\mathrm{m}^2$.

The width of the path at the north and south side of the park is $2\,\mathrm{m}$.

The width of the path at the west and east side of the park is $1.5\,\mathrm{m}$.

The length of the park (along the north and south sides) is x metres, $3 < x < 300$.

diagram not to scale

(a) Show that $A = 1212 - 4x - \dfrac{3600}{x}$. [5]

(b) Find the possible dimensions of the park if the area of the garden is $800\,\mathrm{m}^2$. [4]

(c) Find an expression for $\dfrac{\mathrm{d}A}{\mathrm{d}x}$. [3]

(d) Use your answer from part (c) to find the value of x that will maximize the area of the garden. [2]

(e) Find the maximum possible area of the garden. [2]

4. [Maximum mark: 19]

The following graph shows five cities of the USA connected by weighted edges representing the cheapest direct flights in dollars ($) between cities.

(a) Explain why the graph can be described as “connected”, but not “complete”. [2]

(b) Find a minimum spanning tree for the graph using Kruskal’s algorithm.

State clearly the order in which your edges are added, and draw the tree obtained. [3]

(c) Using only the edges obtained in your answer to part (b), find an upper bound for the travelling salesman problem. [2]

Ronald lives in New York City and wishes to fly to each of the other cities, before finally returning to New York City. After some research, he finds that there exists a direct flight between Los Angeles and Dallas costing $26. He updates the graph to show this.

(d) By using the nearest neighbour algorithm and starting at Los Angeles, determine a better upper bound than that found in part (c).

State clearly the order in which you are adding the vertices. [3]

(This question continues on the following page)

(Question 4 continued)

(e) (i) By deleting the vertex which represents Chicago, use the deleted vertex algorithm to determine a lower bound for the travelling salesman problem.

(ii) Similarly, by instead deleting the vertex which represents Seattle, determine another lower bound. [5]

(f) Hence, using your previous answers, write down your best inequality for the **least** expensive tour Ronald could take. Let the variable C represent the total cost, in dollars, for the tour. [2]

(g) Write down a tour that is strictly greater than your lower bound and strictly less than your upper bound. [2]

5. [Maximum mark: 14]

The three countries of Belgium, Germany and The Netherlands meet at a single point called Vaalserberg.

To support future transport planning, a $10\,\text{km}$ circle was drawn around Vaalserberg on a map. A study was carried out over five years to determine what percentage of people living in each of these countries (within the $10\,\text{km}$ circular region) either stayed in their own country or moved to another country within the circle.

From this study, the following movements were observed during the five years.

- From Belgium, $5\,\%$ moved to Germany, and $0.5\,\%$ moved to The Netherlands.
- From Germany, $2\,\%$ moved to The Netherlands, and $1.5\,\%$ moved to Belgium.
- From The Netherlands, $3\,\%$ moved to Germany, and $2\,\%$ moved to Belgium.

All additional population changes within the circular region may be ignored.

(a) Represent the above information in a transition matrix $\boldsymbol{T}$. [3]

At the end of the study, the population of the Belgian side was $26\,000$, the population of the German side was $240\,000$, and the population of The Netherlands side was $50\,000$.

(b) By using $\boldsymbol{T}$, find the expected population of the German side of Vaalserberg 30 years after the end of the study. [4]

For matrix $\boldsymbol{T}$ there exists a steady state vector

$$\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix},$$

where u_1, u_2 and u_3 are the proportions of the total population on the Belgian side, the German side and The Netherlands side respectively.

The steady state vector $\boldsymbol{u}$ may be found by solving a system of equations.

(c) (i) Determine these equations that are to be solved.

(ii) By solving your system of equations, find $\boldsymbol{u}$. [3]

(d) Use your answer to part (c)(ii) to determine the long-term expected population of the German side. [2]

(e) Suggest two reasons why your answer to part (d) is not likely to be accurate. You may comment on both the model and the situation in context. [2]

6. [Maximum mark: 18]

The gardener in a local park suggested that the number of snails found in the park can be modelled by a Poisson distribution.

(a) Suggest two observations that the gardener may have made that led him to suggest this model. [2]

Now assume that the model is valid and that the mean number of snails per m^2 is 0.2. The gardener inspects, at random, a $12\,\mathrm{m}^2$ area of the park.

(b) Find the probability that the gardener finds exactly four snails. [3]

(c) Find the probability that the gardener finds fewer than three snails. [2]

(d) Find the probability that, in three consecutive inspections, the gardener finds at least one snail per inspection. [3]

Following heavy rain overnight, the gardener wished to determine whether the number of snails found in a random $12\,\mathrm{m}^2$ area of the park had increased.

(e) State the hypotheses for the test. [2]

(f) Find the critical region for the test at the $1\,\%$ significance level. [3]

(g) Given that the mean number of snails per m^2 has actually risen to 0.75, find the probability that the gardener makes a Type II error. [3]

7. [Maximum mark: 18]

A biologist suggests that the rates of change of the population of fruit flies (after time $t \geq 0$) in a particular ecosystem are given by the following equations, where x is the population of male fruit flies and y is the population of female fruit flies.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -4x + 6y$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 9x - y$$

(a) Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{pmatrix} -4 & 6 \\ 9 & -1 \end{pmatrix}$. [6]

(b) Hence write down the general solution of the system, giving your answer in the form $\begin{pmatrix} x \\ y \end{pmatrix} = A\boldsymbol{p}_1 \mathrm{e}^{\lambda_1 t} + B\boldsymbol{p}_2 \mathrm{e}^{\lambda_2 t}$, where A, B, λ_1, λ_2 $(\lambda_2 > \lambda_1)$ are scalar constants and $\boldsymbol{p}_1$, $\boldsymbol{p}_2$ are vector constants. [2]

Initially $x = 500$ and $y = 125$.

(c) Determine the value of A and the value of B. [2]

(d) State the long-term ratio of male fruit flies to female fruit flies. [1]

(e) Find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at time $t = 0$. [3]

(f) Sketch the trajectory, on the phase portrait, for the population growth of the fruit flies. [4]

References:

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Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 2

Friday 7 May 2021 (morning)

2 hours

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

2221 – 7212

8 pages

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Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 15]

A farmer owns a field in the shape of a triangle ABC such that $\mathrm{AB} = 650\,\mathrm{m}$, $\mathrm{AC} = 1005\,\mathrm{m}$ and $\mathrm{BC} = 1225\,\mathrm{m}$.

diagram not to scale

(a) Find the size of $\mathrm{A\hat{C}B}$. [3]

The local town is planning to build a highway that will intersect the borders of the field at points D and E, where $\mathrm{DC} = 210\,\mathrm{m}$ and $\mathrm{C\hat{E}D} = 100^\circ$, as shown in the diagram below.

diagram not to scale

(b) Find DE. [3]

The town wishes to build a carpark here. They ask the farmer to exchange the part of the field represented by triangle DCE. In return the farmer will get a triangle of equal area ADF, where F lies on the same line as D and E, as shown in the diagram above.

(c) Find the area of triangle DCE. [5]

(d) Estimate DF. You may assume the highway has a width of zero. [4]

2. [Maximum mark: 16]

It is known that the weights of male Persian cats are normally distributed with mean $6.1\,\text{kg}$ and variance $0.5^2\,\text{kg}^2$.

(a) Sketch a diagram showing the above information. [2]

(b) Find the proportion of male Persian cats weighing between $5.5\,\text{kg}$ and $6.5\,\text{kg}$. [2]

A group of 80 male Persian cats are drawn from this population.

(c) Determine the expected number of cats in this group that have a weight of less than $5.3\,\text{kg}$. [3]

The male cats are now joined by 80 female Persian cats. The female cats are drawn from a population whose weights are normally distributed with mean $4.5\,\text{kg}$ and standard deviation $0.45\,\text{kg}$.

(d) Ten female cats are chosen at random.

(i) Find the probability that exactly one of them weighs over $4.62\,\text{kg}$.

(ii) Let N be the number of cats weighing over $4.62\,\text{kg}$.

Find the variance of N. [5]

A cat is selected at random from all 160 cats.

(e) Find the probability that the cat was female, given that its weight was over $4.7\,\text{kg}$. [4]

3. [Maximum mark: 15]

A hollow chocolate box is manufactured in the form of a right prism with a regular hexagonal base. The height of the prism is $h\,\text{cm}$, and the top and base of the prism have sides of length $x\,\text{cm}$.

diagram not to scale

(a) Given that $\sin 60^\circ = \frac{\sqrt{3}}{2}$, show that the area of the base of the box is equal to $\frac{3\sqrt{3}x^2}{2}$. [2]

(b) Given that the total external surface area of the box is $1200\,\text{cm}^2$, show that the volume of the box may be expressed as $V = 300\sqrt{3}\,x - \frac{9}{4}x^3$. [5]

(c) Sketch the graph of $V = 300\sqrt{3}\,x - \frac{9}{4}x^3$, for $0 \le x \le 16$. [2]

(d) Find an expression for $\frac{\mathrm{d}V}{\mathrm{d}x}$. [2]

(e) Find the value of x which maximizes the volume of the box. [2]

(f) Hence, or otherwise, find the maximum possible volume of the box. [2]

4. [Maximum mark: 18]

In a small village there are two doctors' clinics, one owned by Doctor Black and the other owned by Doctor Green. It was noted after each year that 3.5% of Doctor Black's patients moved to Doctor Green's clinic and 5% of Doctor Green's patients moved to Doctor Black's clinic. All additional losses and gains of patients by the clinics may be ignored.

At the start of a particular year, it was noted that Doctor Black had 2100 patients on their register, compared to Doctor Green's 3500 patients.

(a) Write down a transition matrix $\boldsymbol{T}$ indicating the annual population movement between clinics. [2]

(b) Find a prediction for the ratio of the number of patients Doctor Black will have, compared to Doctor Green, after two years. [2]

(c) Find a matrix $\boldsymbol{P}$, with integer elements, such that $\boldsymbol{T} = \boldsymbol{PDP}^{-1}$, where $\boldsymbol{D}$ is a diagonal matrix. [6]

(d) Hence, show that the long-term transition matrix $\boldsymbol{T}^{\infty}$ is given by $\boldsymbol{T}^{\infty} = \begin{pmatrix} \frac{10}{17} & \frac{10}{17} \\ \frac{7}{17} & \frac{7}{17} \end{pmatrix}$. [6]

(e) Hence, or otherwise, determine the expected ratio of the number of patients Doctor Black would have compared to Doctor Green in the long term. [2]

5. [Maximum mark: 14]

Hank sets up a bird table in his garden to provide the local birds with some food. Hank notices that a specific bird, a large magpie, visits several times per month and he names him Bill. Hank models the number of times per month that Bill visits his garden as a Poisson distribution with mean 3.1.

(a) Using Hank's model, find the probability that Bill visits the garden on exactly four occasions during one particular month. [1]

(b) Over the course of 3 consecutive months, find the probability that Bill visits the garden:

(i) on exactly 12 occasions.

(ii) during the first and third month only. [5]

(c) Find the probability that over a 12-month period, there will be exactly 3 months when Bill does not visit the garden. [4]

After the first year, a number of baby magpies start to visit Hank's garden. It may be assumed that each of these baby magpies visits the garden randomly and independently, and that the number of times each baby magpie visits the garden per month is modelled by a Poisson distribution with mean 2.1.

(d) Determine the least number of magpies required, including Bill, in order that the probability of Hank's garden having at least 30 magpie visits per month is greater than 0.2. [4]

6. [Maximum mark: 15]

A particle P moves along the x-axis. The velocity of P is $v\,\mathrm{m\,s^{-1}}$ at time t seconds, where $v = -2t^2 + 16t - 24$ for $t \geq 0$.

(a) Find the times when P is at instantaneous rest. [2]

(b) Find the magnitude of the particle's acceleration at 6 seconds. [4]

(c) Find the greatest speed of P in the interval $0 \leq t \leq 6$. [2]

(d) The particle starts from the origin O. Find an expression for the displacement of P from O at time t seconds. [4]

(e) Find the total distance travelled by P in the interval $0 \leq t \leq 4$. [3]

7. [Maximum mark: 17]

Consider the following system of coupled differential equations.

$$\frac{dx}{dt} = -4x$$
$$\frac{dy}{dt} = 3x - 2y$$

(a) Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{pmatrix} -4 & 0 \\ 3 & -2 \end{pmatrix}$. [6]

(b) Hence, write down the general solution of the system. [2]

(c) Determine, with justification, whether the equilibrium point $(0, 0)$ is stable or unstable. [2]

(d) Find the value of $\frac{dy}{dx}$

(i) at $(4, 0)$.

(ii) at $(-4, 0)$. [3]

(e) Sketch a phase portrait for the general solution to the system of coupled differential equations for $-6 \le x \le 6$, $-6 \le y \le 6$. [4]

References:

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