1. Find the particular solution of the following differential equation:

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0$$

where $y(0) = 6$, and $y'(0) = 3$.

Solution: The given differential equation has the form:

$$y'' + p^* y' + q^* y = 0,$$

Where

$$p = -8, \quad q = 16$$

It is called linear homogeneous second-order differential equation with constant coefficients.

The equation has an easy solution, we solve the corresponding homogeneous linear equation

$$y'' + p^* y' + q^* y = 0$$

First of all we should find the roots of the characteristic equation

$$q + (k^2 + kp) = 0$$

In this case, the characteristic equation, ~~and it~~ will be

$$k^2 - 8k + 16 = 0$$

- This is a simple quadratic equation

The root of this equation: $k_1 = 4$

As there is one root of the characteristic equation, and it is not complex, then solving the correspondent differential equation looks as follows:

$$y(x) = e^{k_1 x} C_1 + e^{k_1 x} C_2 x$$

Substituting

$$k_1 = 4$$

Hence the equation $y(x) = C_1 e^{4x} + C_2 x e^{4x}$

Further solving for C_1 & C_2.

At $x = 0$, the equation becomes

$$6 = C_1 + 0$$

$$\therefore C_1 = 6$$

$\because$ $y(x) = c_1 e^{4x} + c_2 x e^{4x}$

differentiating both sides w.r.t x

$y'(x) = 4c_1 e^{4x} + c_2 e^{4x} + 4c_2 x e^{4x}$

now at $x=0$, $y'(x) = 3$ (given in the question)

$$3 = 4c_1 + c_2$$

$\because$ $c_1 = 6$, the above inequality becomes

$\Rightarrow 3 = 24 + c_2$

$\therefore c_2 = -21$

Hence $c_1 = 6$, $c_2 = -21$

$\therefore$ The final solution is ~~$y(x) = 6e^{4x} + -21xe^{4x}$.~~

$y(x) = 6e^{4x} - 21xe^{4x}$.

2. Find the particular solution of the second-order differential equation:

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = e^{3x} - 10$$

where $y(0) = 2$, and $y'(0) = -\frac{1}{4}$.

Solution:

The differential equation has the form:

$$y'' + p^* y' + q^* y = s,$$

where $p = -7$

$q = 12$

$s = 10 - e^{3x}$

It is called linear inhomogeneous second-order differential equation with constant coefficients

The equation has an easy solution

we solve the corresponding homogeneous linear equation

$y'' + p^* y' + q^* y = 0$

First of all we should find the roots of the characteristics equation

$$q + (k^2 + kp) = 0$$

In this case, the characteristic equation will be:

$$k^2 - 7k + 12 = 0$$

- This is a simple quadratic equation

The roots of this equation: $k_1 = 3$, $k_2 = 4$

As there are two roots of the characteristic equation, and the roots are not complex, then solving the correspondent differential equation looks as follows:

$$y(x) = C_1 e^{k_1 x} + C_2 e^{k_2 x}$$

$$y(x) = C_1 e^{3x} + C_2 e^{4x}$$

We get a solution from the corresponding homogeneous equation. Now we should solve the inhomogeneous equation

$$y'' + p^* y' + q^* y = 5$$

use variation of parameters method.

Suppose that C_1 and C_2 - it is functions of x

The general solution is:

$$y(x) = C_1(x) e^{3x} + C_2(x) e^{4x}$$

where $C_1(x)$ and $C_2(x)$ by the method of variation of parameters, we find the solution from the system:

$$y_1(x) \frac{d}{dx} C_1(x) + y_2(x) \frac{d}{dx} C_2(x) = 0$$

$$\frac{d}{dx} C_1(x) \frac{d}{dx} y_1(x) + \frac{d}{dx} C_2(x) \frac{d}{dx} y_2(x) = f(x)$$

where

$y_1(x)$ and $y_2(x)$ - linearly independent particular solutions of Linear Ordinary Differential Equations,

$$y_1(x) = \exp(3^* x) \quad (C_1 = 1, C_2 = 0)$$

$$y_2(x) = \exp(4^* x) \quad (C_1 = 0, C_2 = 1)$$

The free term $f = -5$, or

$$f(x) = e^{3x} - 10$$

So, the system has the form:

$$e^{4x} \frac{d}{dx} C_2(x) + e^{3x} \frac{d}{dx} C_1(x) = 0$$

$$\frac{d}{dx} C_1(x) \frac{d}{dx} e^{3x} + \frac{d}{dx} C_2(x) \frac{d}{dx} e^{4x} = e^{3x} - 10$$

or

$$e^{4x}\frac{d}{dx}C_2(x) + e^{3x}\frac{d}{dx}C_1(x) = 0$$

$$4e^{4x}\frac{d}{dx}C_2(x) + 3e^{3x}\frac{d}{dx}C_1(x) = e^{3x} - 10$$

Solve the system:

$$\frac{d}{dx}C_1(x) = -1 + 10e^{-3x}$$

$$\frac{d}{dx}C_2(x) = (e^{3x} - 10)e^{-4x}$$

- It is the simple differential equations, solve these equations

$$C_1(x) = C_3 + \int(-1 + 10e^{-3x})dx$$

$$C_2(x) = C_4 + \int(e^{3x} - 10)e^{-4x}dx$$

or

$$C_1(x) = C_3 - x - \frac{10e^{-3x}}{3}$$

$$C_2(x) = C_4 - e^{-x} + \frac{5e^{-4x}}{2}$$

Substituting $C_1(x)$ and $C_2(x)$ to

$$y(x) = C_1(x)e^{3x} + C_2(x)e^{4x}$$

$\therefore$ The final equation is

$$y(x) = C_3e^{3x} + C_4e^{4x} - xe^{3x} - e^{3x} - \frac{5}{6}$$

Further solving to get C_3 and C_4

at $x = 0$ $y(x) = 2$,

$$\therefore 2 = C_3 + C_4 - 1 - \frac{5}{6}$$

$$\Rightarrow \frac{23}{6} = C_3 + C_4 \quad \text{.... (i)}$$

Again differentiating $y(x) = C_3e^{3x} + C_4e^{4x} - xe^{3x} - e^{3x} - \frac{5}{6}$ w.r.t x

$$y'(x) = 3C_3e^{3x} + 4C_4e^{4x} - e^{3x} - 3xe^{3x} - 3e^{3x}$$

at $x = 0$ $y'(x) = -\frac{1}{4}$

$$\therefore -\frac{1}{4} = 3C_3 + 4C_4 - 1 - 3$$

$$\Rightarrow \frac{15}{4} = 3C_3 + 4C_4 \quad \text{- (ii)}$$

The two inequalities are as follows:

$$C_3 + C_4 = \frac{23}{6} \quad \cdots (i)$$

$$3C_3 + 4C_4 = \frac{15}{4} \quad \cdots (ii)$$

Multiplying equation (i) with 3, we get

$$3C_3 + 3C_4 = \frac{23}{2} \quad \cdots (i)$$

$$3C_3 + 4C_4 = \frac{15}{4} \quad \cdots (ii)$$

subtracting (ii) from (i) we get

$$-C_4 = \frac{31}{4}$$

$$\therefore C_4 = -\frac{31}{4}$$

Substituting C_4 in equation (ii)

$$3C_3 - 31 = \frac{15}{4}$$

$$\Rightarrow 3C_3 = \frac{15}{4} + 31$$

$$\Rightarrow C_3 = \frac{139}{3}$$

Hence the final solution is

$$y(x) = \frac{139}{3} e^{3x} + -\frac{31}{4} e^{4x} - x e^{3x} - e^{3x} - \frac{5}{6} \text{ (Ans)}$$

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3. Use Laplace transforms to solve to solve the differential equation:

$$\frac{d^2y}{dt^2} + y = 2, \text{ where } y(0) = 2,\ y'(0) = -\frac{1}{4}.$$

Given equation can be written as

$$y'' + y = 2$$

Taking laplace transformation of the side of equation we get.

$$s^2 y' - s y(0) - y'(0) + y' = \frac{2}{s}$$

using initial condition

$$s^2 y' - s \times 2 - 1 + y' = \frac{2}{s}$$

$y'(s^2) - 2s - 1 > 2/s$

$y'(s^2) - 2s - 1$

$$y' = \frac{\frac{2}{s} + 2s + 1}{s^2} \quad -(i)$$

By partial fraction we write

$$y' = \frac{2 + 2s^2 + s}{s^3}$$

$$\frac{2s^2 + s + 2}{s(s^2+1)} = \frac{A}{s} + \frac{Bs + C}{s^2+1}$$

$$= \frac{As^2 + A + s^2B + sC}{s(s^2+1)}$$

$$\frac{2s^2 + s + 2}{s(s^2+1)} = \frac{s^2(A+B) + sC + A}{s(s^2+1)}$$

$A + B = 2$

$A = 2$

$C = 1$

So $B = 0$

substitution value of A, B, C we get

$$y' = \frac{A}{s} + \frac{Bs + C}{s^2+1}$$

$$y' = \frac{2}{s} + \frac{1}{s^2+1}$$

On inversion we get

$$L^{-1}(y^{-1}) = 2L^{-1}(1/s) + L^{-1}\left(\frac{1}{1+s^2}\right)$$

$y(t) \Rightarrow 2 \times 1 + \sin t$

$y(t) = \cancel{2t}\ 2 + \sin t$

$$\frac{d^2y}{dt^2} + y = 2, \quad y(0) = 2, \quad y'(0) = 1$$

given equation can be written as

$$y'' + y = 2$$

Taking laplace transformation of the equation we get

$$L\{y''\} + L\{y\} = L(2)$$

$$s^2 y' - s y(0) - y'(0) + y' = \frac{2}{s}$$

using initial condition

$$s^2 y' - s \times 2 - 1 + y' = \frac{2}{s}$$

$$y'(s^2+1) - 2s - 1 = 2/s$$

$$y' = \frac{\frac{2}{s} + 2s + 1}{s^2+1} \rightarrow (i)$$

By partial fraction we write

$$y' = \frac{2 + 2s^2 + s}{s(s^2+1)} \rightarrow (ii)$$

$$\frac{2s^2+s+2}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} \quad \ldots \quad (iii)$$

$$\frac{As^2 + A + s^2 B + sC}{s(s^2+1)}$$

$$\frac{2s^2+s+2}{s(s^2+1)} = \frac{s^2(A+B) + sC + A}{s(s^2+1)}$$

$$A+B = 2$$

$$A = 2$$

$$C = 1$$

so $B = 0$

substitution value of A, B, C we get

$$y' = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

$$y' = \frac{2}{s} + \frac{1}{s^2+1}$$

On inversion we get

$$L^{-1}(y^{-1}) \Rightarrow 2L^{-1}(1/s) + L^{-1}\left(\frac{1}{1+s^2}\right)$$

$$y(t) \Rightarrow 2 + \sin t$$

5) Determine the particular solution of the following differential equation:

$$\frac{dy}{dx} = \frac{x^2+y^2}{xy} \quad \text{given } y(1) = 4$$

=> We are given,

$$\frac{dy}{dx} = \frac{x^2+y^2}{xy}$$

Let $y = vx$.

Now $\frac{dy}{dx} = \frac{x^2+y^2}{xy} = \frac{x^2+v^2x^2}{x*vx}$ $(\because y = vx)$

$$= \frac{x^2(1+v^2)}{vx^2} = \frac{1+v^2}{v}.$$

Now, $y = vx$.

$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$.

$\Rightarrow v + x\frac{dv}{dx} = \frac{1+v^2}{v}$.

$\Rightarrow x\frac{dv}{dx} = \frac{1+v^2}{v} - v = \frac{1+v^2-v^2}{v} = \frac{1}{v}$.

Separate.

$v\,dv = \frac{dx}{x}$, Now integrate

$\Rightarrow \frac{v^2}{2} = \ln(x) + c$.

Let $c = \ln(C)$

$\therefore \frac{v^2}{v} = \ln(Cx)$

$$\frac{y^2}{x^2} = 2\ln(Cx)$$

$$\Rightarrow y^2 = 2x^2 \ln(Cx)$$

Now, putting $y(1) = 4$, we get

$$\Rightarrow (4)^2 = 2 * (1)^2 * \ln(C)$$

$$\Rightarrow \ln(C) = 8$$

$$\therefore y^2 = 2x^2 * [\ln(C) + \ln(x)]$$

$$\Rightarrow y^2 = 2[8 + \ln(x)]x^2$$

$$\Rightarrow y^2 = x^2 * [16 + \ln x^2]$$

$\therefore$ the partialar solution is

$$\boxed{y^2 = x^2 [16 + \ln x^2]}$$

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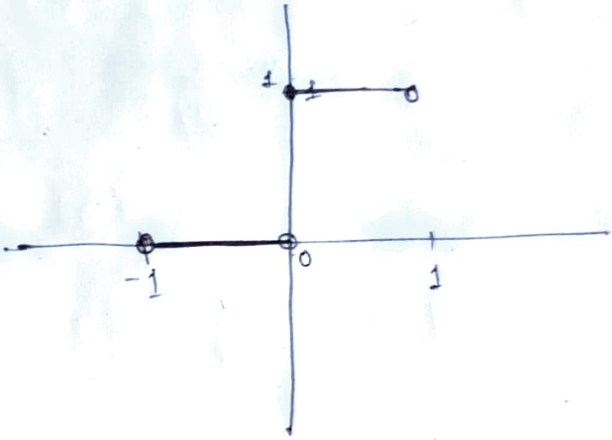
6) Determine the Fourier series for the function defined below:

$$f(t) = \begin{cases} 0 & , -1 < t < 0 \\ 1 & , 0 < t < 1 \end{cases} \qquad T = 2$$

Sketch the graph of $f(t)$ over 3 cycles, from $t = -3T/2$ to $t = 3T/2$.

Given function.

$$f(t) = \begin{cases} 0 & , -1 < t < 0 \\ 1 & , 0 < t < 1 \end{cases} \qquad T = 2$$



The Fourier coefficients are:

$$a_0 = \frac{1}{1} \int_{1}^{-1} f(t)\,dt = 1$$

For $n \geq 1$:

$$a_n = \int_{-1}^{1} f(t) \cos(nt)\,dt =$$

$$= \int_{0}^{1} f(t) \cos(nt)\,dt + \int_{-1}^{0} f(t) \cos(nt)\,dt$$

$$= \int_{0}^{1} f(t) \cos(nt)\,dt$$

$$= \int_{0}^{1} \cos(nt)\,dt$$

$$= \frac{1}{n} \sin(nt) \Big|_{0}^{1}$$

$$= \frac{1}{n} (\sin n - 0) = \frac{\sin n}{n}.$$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(t)\sin(nt)\,dt.$$

$$= \int_{0}^{1} f(t)\sin(nt)\,dt + \int_{-1}^{0} f(t)\sin(nt)\,dt$$

$$= \int_{0}^{1} \sin(nt)\,dt$$

$$= \left.-\frac{1}{n}\cos(nt)\right|_{0}^{1}$$

$$= -\frac{1}{n}\left[\cos n - 1\right]$$

$$= \left[\frac{1-\cos n}{n}\right]$$

∴ Series is.

$$\cancel{\frac{1}{2} + 1 + \left(\sin t + \sin 3t\right.}$$

$$\frac{1}{2} + \left(\cancel{\cos}\ \frac{1-\cos t}{1} + \frac{1-\cos 2t}{2} + \frac{1-\cos 3t}{3} - \cdots\right)$$

$$+ \left(\sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \cdots\right)$$

7) Determine the Fourier Series for the function defined below.

$f(x) = 2x$, in the range $x = 0$ to $x = 2\pi$,

$$f(x + 2\pi) = f(x)$$

$\Rightarrow$ Given function,

$$f(x) = 2x \quad [0 \leq x \leq 2\pi]$$

Now, since $f(x)$ is an odd function, so all the cosine terms will be 0,

$\therefore$ the sines terms are:

$$\cancel{b_n = \frac{1}{2\pi} \int_0^{2\pi} 2x \, dx}$$

$$\cancel{= \frac{1}{2\pi} x^2 \Big|_0^{2\pi}}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} 2x \sin\left(\frac{n\pi x}{2\pi}\right) dx$$

$$= \frac{2}{\pi} \int_0^{2\pi} x \sin\left(\frac{nx}{2}\right) dx.$$

$$= \frac{2}{\pi} \left[\frac{2}{n} \left(-x \cos\left(\frac{nx}{2}\right) + \sin\left(\frac{nx}{2}\right) \right) \right] \Bigg|_0^{2\pi}$$

$$= \frac{4}{n\pi} \left(-2\pi \cos(n\pi) + \sin n\pi + 0 - 0 \right)$$

$$= \frac{4}{n\pi} \left(\sin n\pi - 2\pi \cos(n\pi) \right)$$

$$= \frac{4}{n\pi} \times -2\pi \cos(n\pi)$$

$$= -\frac{8}{n} \cos(n\pi)$$

$\therefore b_n = \begin{cases} \frac{-8}{n} & n \text{ even.} \\ \frac{8}{n} & n \text{ odd.} \end{cases}$

$\therefore$ the fourier series is:

$$8\left(\sin\frac{\pi t}{2} - \sin\frac{2\pi t}{2} + \sin\frac{3\pi t}{2} - \cdots\right)$$