NEURAL NETWORKS FROM SCRATCH IN PYTHON

By Harrison Kinsley & Daniel Kukiela

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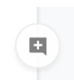

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Neural Networks from Scratch in Python

Harrison Kinsley & Daniel Kukieła

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Readme

The objective of this book is to break down an extremely complex topic, neural networks, into small pieces, consumable by anyone wishing to embark on this journey. Beyond breaking down this topic, the hope is to dramatically demystify neural networks. As you will soon see, this subject, when explored from scratch, can be an educational and engaging experience. This book is for anyone willing to put in the time to sit down and work through it. In return, you will gain a far deeper understanding than most when it comes to neural networks and deep learning.

This book will be easier to understand if you already have an understanding of Python or another programming language. Python is one of the most clear and understandable programming languages; we have no real interest in padding page counts and exhausting an entire first chapter with a basics of Python tutorial. If you need one, we suggest you start here: *https://pythonprogrammingnet/python-fundamental-tutorials/* To cite this material:

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Chapter 7

Derivatives

Randomly changing and searching for optimal weights and biases did not prove fruitful for one main reason: the number of possible combinations of weights and biases is infinite, and we need something smarter than pure luck to achieve any success. Each weight and bias may also have different degrees of influence on the loss — this influence depends on the parameters themselves as well as on the current sample, which is an input to the first layer. These input values are then multiplied by the weights, so the input data affects the neuron's output and affects the impact that the weights make on the loss. The same principle applies to the biases and parameters in the next layers, taking the previous layer's outputs as inputs. This means that the impact on the output values depends on the parameters as well as the samples — which is why we are calculating the loss value per each sample separately. Finally, the function of *how* a weight or bias impacts the overall loss is not necessarily linear. In order to know *how* to adjust weights and biases, we first need to understand their impact on the loss.

One concept to note is that we refer to weights and biases and their impact on the loss function. The loss function doesn't contain weights or biases, though. The input to this function is the output of the model, and the weights and biases of the neurons influence this output. Thus, even though we calculate loss from the model's output, not weights/biases, these weights and biases directly impact the loss.

In the coming chapters, we will describe exactly how this happens by explaining partial derivatives, gradients, gradient descent, and backpropagation. Basically, we'll calculate how much each singular weight and bias changes the loss value (how much of an impact it has on it) given a sample (as each sample produces a separate output, thus also a separate loss value), and how to change this weight or bias for the loss value to decrease. Remember — our goal here is to decrease loss, and we'll do this by using gradient descent. Gradient, on the other hand, is a result of the calculation of the partial derivatives, and we'll backpropagate it using the chain rule to update all of the weights and biases. Don't worry if that doesn't make much sense yet; we'll explain all of these terms and how to perform these actions in this and the coming chapters.

To understand partial derivatives, we need to start with derivatives, which are a special case of partial derivatives — they are calculated from functions taking single parameters.

The Impact of a Parameter on the Output

Let's start with a simple function and discover what is meant by "impact."
A very simple function $y=2x$, which takes x as an input:

```
def f(x):
    return 2*x
```

Now let's create some code around this to visualize the data — we'll import NumPy and Matplotlib, create an array of 5 input values from 0 to 4, calculate the function output for each of these input values, and plot the result as lines between consecutive points. These points' coordinates are inputs as x and function outputs as y:

```
import matplotlib.pyplot as plt
import numpy as np

def f(x):
    return 2*x
```

```
x = np.array(range(5))
y = f(x)


print(x)
print(y)


>>>
[0 1 2 3 4]
[0 2 4 6 8]


plt.plot(x, y)
plt.show()
```

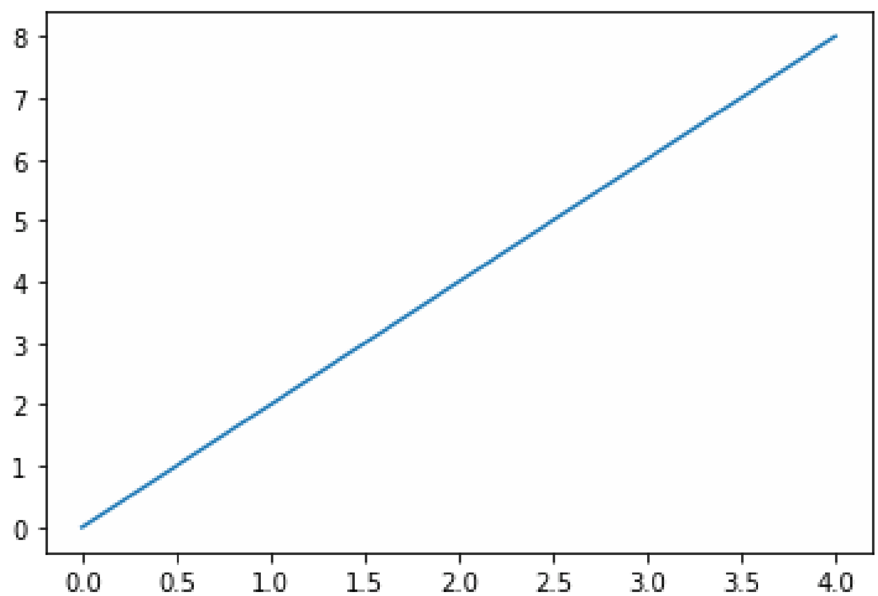

Fig 7.01: Linear function y=2x graphed

The Slope

This looks like an output of the *f(x) = 2x* function, which is a line. How might you define the *impact* that *x* will have on *y*? Some will say, *"y* is double *x*" Another way to describe the *impact* of a linear function such as this comes from algebra: the **slope**. "Rise over run" might be a phrase you recall from school. The slope of a line is:

$$\frac{Change\ in\ y}{Change\ in\ x} = \frac{\Delta y}{\Delta x}$$

It is change in *y* divided by change in *x,* or, in math — *delta y* divided by *delta x*. What's the slope of *f(x) = 2x* then?
To calculate the slope, first we have to take any two points lying on the function's graph and subtract them to calculate the change. Subtracting the points means to subtract their x and y dimensions respectively. Division of the change in y by the change in x returns the slope:

$$\begin{array}{l} \quad\quad\quad\; x\;\; y \\ \quad\quad\quad\; \downarrow\;\downarrow \\ \begin{cases} p_1 = [0, 0] \\ p_2 = [1, 2] \end{cases} \end{array}$$

$$\Delta x = p_{2x} - p_{1x} = 1 - 0 = 1$$
$$\Delta y = p_{2y} - p_{1y} = 2 - 0 = 2$$

$$slope = \frac{\Delta y}{\Delta x} = \frac{2}{1} = 2$$

Continuing the code, we keep all values of x in a single-dimensional NumPy array, `x`, and all results in a single-dimensional array, `y`. To perform the same operation, we'll take `x[0]` and `y[0]` for the first point, then `x[1]` and `y[1]` for the second one. Now we can calculate the slope between them:

```
print((y[1]-y[0]) / (x[1]-x[0]))
```

```
>>>
2.0
```

It is not surprising that the slope of this line is 2. We could say the measure of the impact that x has on y is 2. We can calculate the slope in the same way for any linear function, including linear functions that aren't as obvious.
What about a nonlinear function like $f(x)=2x^2$?

```
def f(x):
    return 2*x**2
```

This function creates a graph that does not form a straight line:

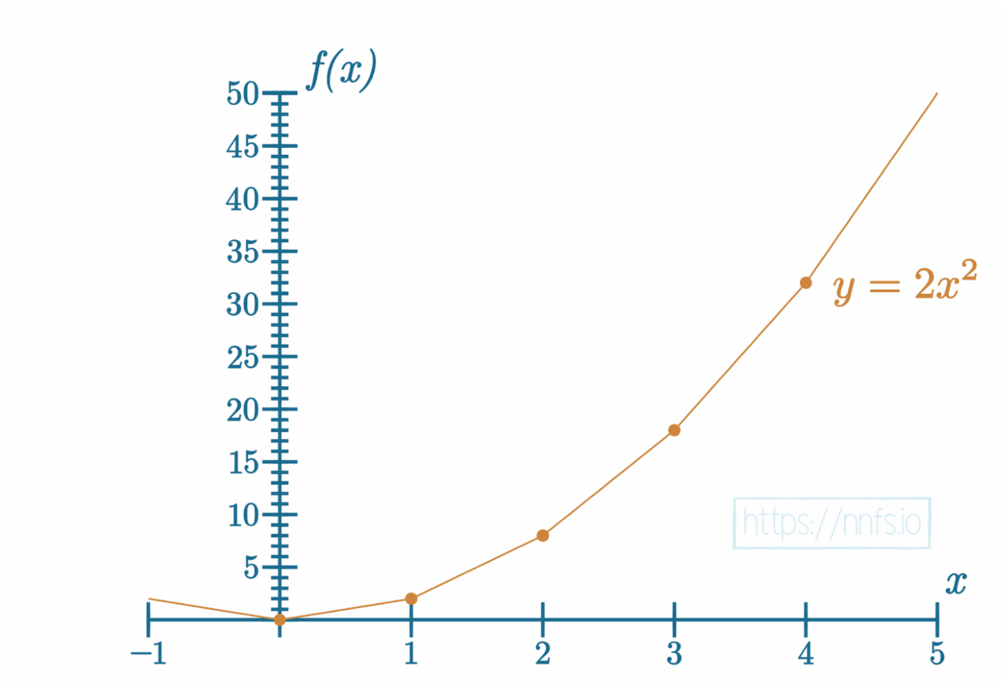

Fig 7.02: Approximation of the parabolic function $y=2x^2$ graphed

Can we measure the slope of this curve? Depending on which 2 points we choose to use, we will measure varying slopes:

```
y = f(x)  # Calculate function outputs for new function

print(x)
print(y)


>>>
[0 1 2 3 4]
[ 0  2  8 18 32]
```

Now for the first pair of points:

```
print((y[1]-y[0]) / (x[1]-x[0]))


>>>
2
```

And for another one:

```
print((y[3]-y[2]) /
(x[3]-x[2]))

>>>
10
```

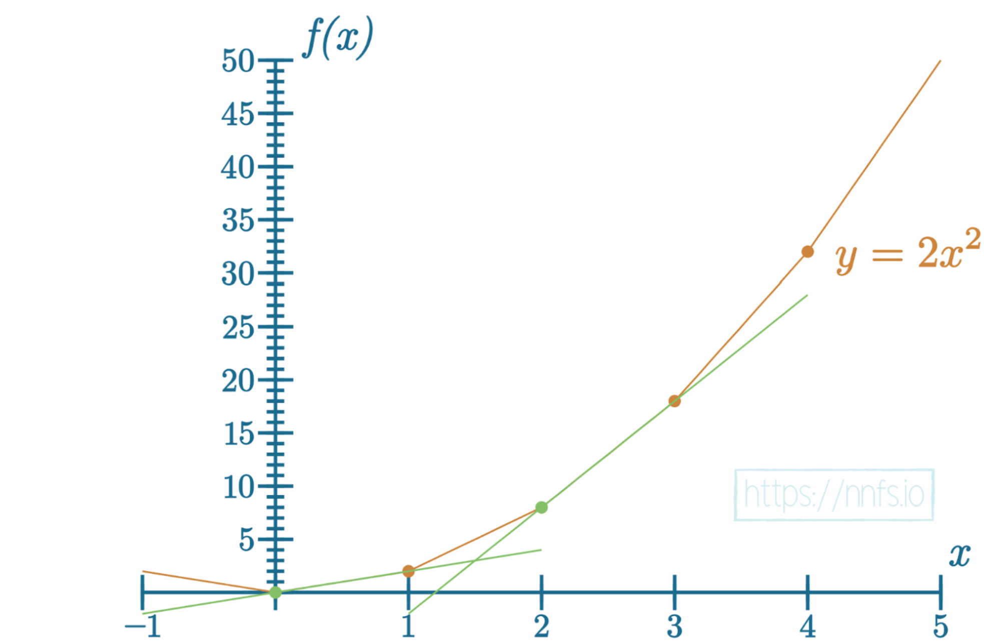

Fig 7.03: Approximation of the parabolic function's example tangents

Anim 7.03: https://nnfs.io/bro

How might we measure the impact that x has on y in this nonlinear function? Calculus proposes that we measure the slope of the **tangent line** at x (for a specific input value to the function), giving us the **instantaneous slope** (slope at this point)**,** which is the **derivative**. The **tangent line** is created by drawing a line between two points that are "infinitely close" on a curve, but this curve has to be differentiable at the derivation point. This means that it has to be continuous and smooth (we cannot calculate the slope at something that we could describe as a "sharp corner," since it contains an infinite number of slopes). Then, because this is a curve, there is no single slope. Slope depends on where we measure it. To give an immediate example, we can approximate a derivative of the function at x by using this point and another one also taken at x, but with a very small delta added to it, such as *0.0001*. This number is a common choice as it does not introduce too large an error (when estimating the derivative) or cause the whole expression to be numerically unstable (Δx might round to 0 due to floating-point number resolution). This lets us perform the same calculation for the slope as before, but on two points that are very close to each other, resulting in a good approximation of a slope at x:

```
p2_delta = 0.0001

x1 = 1
x2 = x1 + p2_delta  # add delta

y1 = f(x1)  # result at the derivation point
y2 = f(x2)  # result at the other, close point

approximate_derivative = (y2-y1)/(x2-x1)
print(approximate_derivative)
```

```
>>>
4.0001999999987845
```

As we will soon learn, the derivative of $2x^2$ at $x=1$ should be exactly *4*. The difference we see (~4.0002) comes from the method used to compute the tangent. We chose a delta small enough to approximate the derivative as accurately as possible but large enough to prevent a rounding error. To elaborate, an infinitely small delta value will approximate an accurate derivative; however, the delta value needs to be numerically stable, meaning, our delta can not surpass the limitations of Python's floating-point precision (can't be too small as it might be rounded to *0* and, as we know, dividing by *0* is "illegal"). Our solution is, therefore, restricted between estimating the derivative and remaining numerically stable, thus introducing this small but visible error.

The Numerical Derivative

This method of calculating the derivative is called **numerical differentiation** — calculating the slope of the tangent line using two *infinitely* close points, or as with the code solution — calculating the slope of a tangent line made from two points that were "sufficiently close." We can visualize why we perform this on two close points with the following:

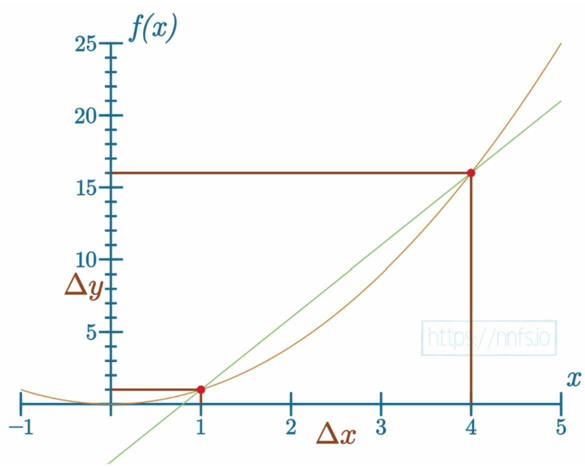

Fig 7.04: Why we want to use 2 points that are sufficiently close — large delta inaccuracy.

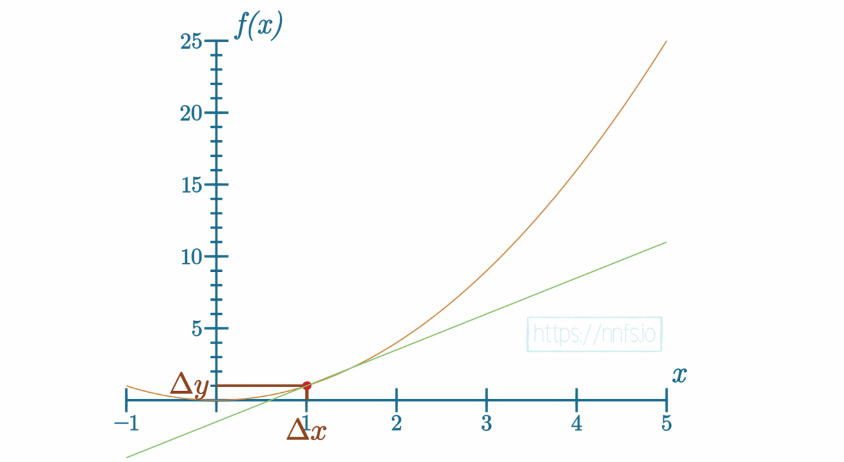

Fig 7.05: Why we want to use 2 points that are sufficiently close — very small delta inaccuracy.

Anim 7.04-7.05: https://nnfs.io/cat

We can see that the closer these two points are to each other, the more correct the tangent line appears to be.

Continuing with **numerical differentiation**, let us visualize the tangent lines and how they change depending on where we calculate them. To begin, we'll make the graph of this function more granular using Numpy's `arange()`, allowing us to plot with smaller steps. The `np.arange` method takes in *start, stop,* and *step* parameters, allowing us to take fractions of a step, such as *0.001* at a time:

```
import matplotlib.pyplot as plt
import numpy as np


def f(x):
    return 2*x**2


# np.arange(start, stop, step) to give us smoother line
x = np.arange(0, 5, 0.001)
y = f(x)
```

```
plt.plot(x, y)

plt.show()
```

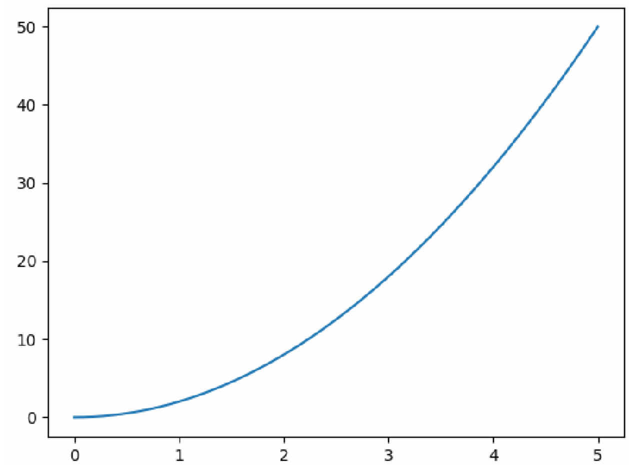

Fig 7.06: Matplotlib output that you should see from graphing y=2x^2.

To draw these tangent lines, we will derive the function for the tangent line at a point and plot the tangent on the graph at this point. The function for a straight line is $y = mx+b$. Where m is the slope or the *approximate_derivative* that we've already calculated. And x is the input which leaves b, or the y-intercept, for us to calculate. The slope remains unchanged, but currently, you can "move" the line up or down using the y-intercept. We already know x and m, but b is still unknown. Let's assume $m=1$ for the purpose of the figure and see what exactly it means:

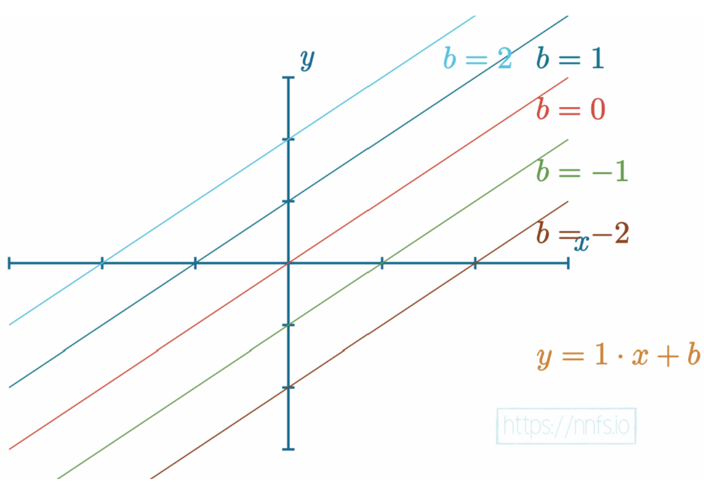

Fig 7.07: Various biases graphed where slope = 1.

Anim 7.07: https://nnfs.io/but

To calculate *b,* the formula is *b = y - mx*:

$$y = mx + b$$

$$y - mx = b$$

$$b = y - mx$$

So far we've used two points — the point that we want to calculate the derivative at and the "close enough" to it point to calculate the approximation of the derivative. Now, given the above equation for *b*, the approximation of the derivative and the same "close enough" point (its *x* and *y* coordinates to be specific), we can substitute them in the equation and get the y-intercept for the tangent line at the derivation point. Using code:

```
b = y2 - approximate_derivative*x2
```

Putting everything together:

```
import matplotlib.pyplot as plt
import numpy as np


def f(x):
    return 2*x**2


# np.arange(start, stop, step) to give us smoother line
x = np.arange(0, 5, 0.001)
y = f(x)

plt.plot(x, y)
```

```
# The point and the "close enough" point
p2_delta = 0.0001
x1 = 2
x2 = x1+p2_delta

y1 = f(x1)
y2 = f(x2)

print((x1, y1), (x2, y2))

# Derivative approximation and y-intercept for the tangent line
approximate_derivative = (y2-y1)/(x2-x1)
b = y2 - approximate_derivative*x2


# We put the tangent line calculation into a function so we can call
# it multiple times for different values of x
# approximate_derivative and b are constant for given function
# thus calculated once above this function
def approximate_tangent_line(x):
    return approximate_derivative*x + b


# plotting the tangent line
# +/- 0.9 to draw the tangent line on our graph
# then we calculate the y for given x using the tangent line function
# Matplotlib will draw a line for us through these points
to_plot = [x1-0.9, x1, x1+0.9]
plt.plot(to_plot, [approximate_tangent_line(point) for point in to_plot])

print('Approximate derivative for f(x)',
      f'where x = {x1} is {approximate_derivative}')

plt.show()
```

```
>>>
(2, 8) (2.0001, 8.000800020000002)
Approximate derivative for f(x) where x = 2 is 8.000199999998785
```

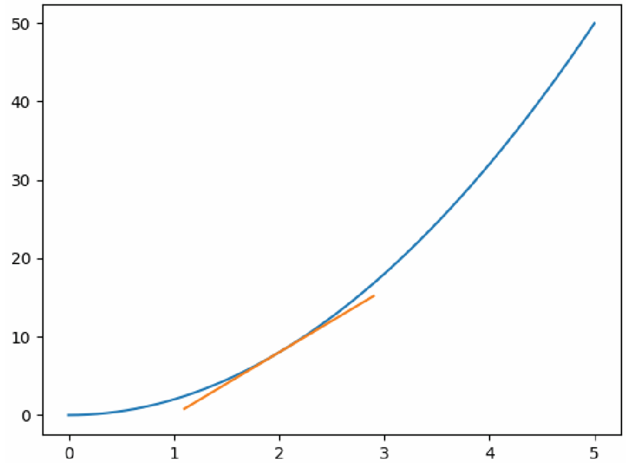

Fig 7.08: Graphed approximate derivative for f(x) where x=2

The orange line is the approximate tangent line at $x=2$ for the function $f(x) = 2x^2$. Why do we care about this? You will soon find that we care only about the *slope* of this tangent line but both visualizing and understanding the **tangent line** are very important. We care about the slope of the tangent line because it informs us about the *impact* that *x* has on this function at a particular point, referred to as the **instantaneous rate of change**. We will use this concept to determine the effect of a specific weight or bias on the overall loss function given a sample. For now, with different values for *x*, we can observe resulting impacts on the function. We can continue the previous code to see the tangent line for various inputs *(x)* - we put a part of the code in a loop over example *x* values and plot multiple tangent lines:

```
import matplotlib.pyplot as plt
import numpy as np


def f(x):
    return 2*x**2


# np.arange(start, stop, step) to give us a smoother curve
x = np.array(np.arange(0, 5, 0.001))
y = f(x)

plt.plot(x, y)

colors = ['k', 'g', 'r', 'b', 'c']


def approximate_tangent_line(x, approximate_derivative, b):
    return approximate_derivative*x + b
```

```
for i in range(5):
    p2_delta = 0.0001
    x1 = i
    x2 = x1+p2_delta

    y1 = f(x1)
    y2 = f(x2)

    print((x1, y1), (x2, y2))

    approximate_derivative = (y2-y1)/(x2-x1)
    b = y2 - approximate_derivative*x2

    to_plot = [x1-0.9, x1, x1+0.9]

    plt.scatter(x1, y1, c=colors[i])
    plt.plot(to_plot,
             [approximate_tangent_line(point, approximate_derivative, b)
                 for point in to_plot],
             c=colors[i])

    print('Approximate derivative for f(x)',
          f'where x = {x1} is {approximate_derivative}')

plt.show()
```

```
>>>
(0, 0) (0.0001, 2e-08)
Approximate derivative for f(x) where x = 0 is 0.00019999999999999998
(1, 2) (1.0001, 2.00040002)
Approximate derivative for f(x) where x = 1 is 4.0001999999987845
(2, 8) (2.0001, 8.000800020000002)
Approximate derivative for f(x) where x = 2 is 8.000199999998785
(3, 18) (3.0001, 18.001200020000002)
Approximate derivative for f(x) where x = 3 is 12.000199999998785
(4, 32) (4.0001, 32.00160002)
Approximate derivative for f(x) where x = 4 is 16.000200000016548
```

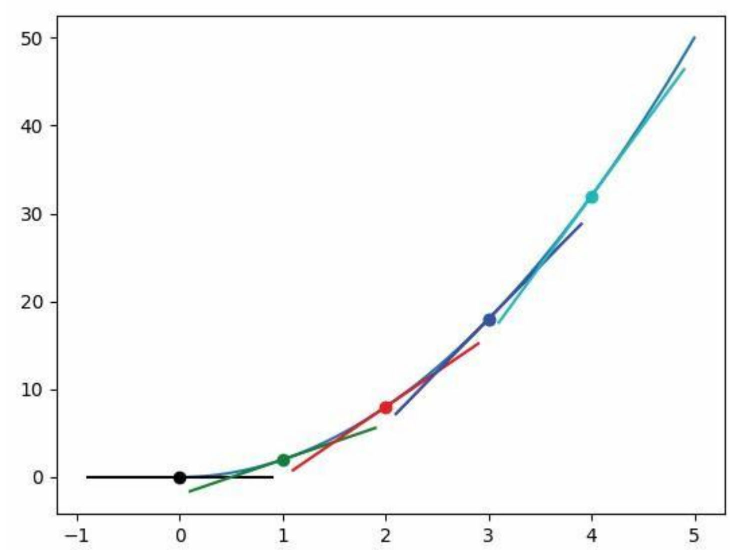

Fig 7.09: Derivative calculated at various points.

For this simple function, $f(x) = 2x^2$, we didn't pay a high penalty by approximating the derivative (i.e., the slope of the tangent line) like this, and received a value that was close enough for our needs.

The problem is that the *actual* function employed in our neural network is not so simple. The loss function contains all of the layers, weights, and biases — it's an absolutely massive function operating in multiple dimensions! Calculating derivatives using **numerical differentiation** requires multiple forward passes for a single parameter update (we'll talk about parameter updates in chapter 10). We need to perform the forward pass as a reference, then update a single parameter by the delta value and perform the forward pass through our model again to see the change of the loss value. Next, we need to calculate the **derivative** and revert the parameter change that we made for this calculation. We have to repeat this for every weight and bias and for every sample, which will be very time-consuming. We can also think of this method as brute-forcing the derivative calculations. To reiterate, as we quickly covered many terms, the **derivative** is the **slope** of the **tangent line** for a function that takes a single parameter as an input. We'll use this ability to calculate the slopes of the loss function at each of the weight and bias points — this brings us to the multivariate function, which is a function that takes multiple parameters and is a topic for the next chapter — the partial derivative.

The Analytical Derivative

Now that we have a better idea of what a derivative *is*, how to calculate the numerical (also called universal) derivative, and why it's not a good approach for us, we can move on to the **Analytical Derivative**, the actual solution to the derivative that we'll implement in our code.

In mathematics, there are two general ways to solve problems: **numerical** and **analytical** methods. Numerical solution methods involve coming up with a number to find a solution, like the above approach with `approximate_derivative`. The numerical solution is also an approximation. On the other hand, the analytical method offers the exact and much quicker, in terms of calculation, solution. However, identifying the analytical solution for the derivative of a given function, as we'll quickly learn, will vary in complexity, whereas the numerical approach never gets more complicated — it's always calling the method twice with two inputs to calculate the approximate derivative at a point. Some analytical solutions are quite obvious, some can be calculated with simple rules, and some complex functions can be broken down into simpler parts and calculated using the so-called **chain rule**. We can leverage already-proven derivative solutions for certain functions, and others — like our loss function — can be solved with combinations of the above.

To compute the derivative of functions using the analytical method, we can split them into simple, elemental functions, finding the derivatives of those and then applying the **chain rule**, which we will explain soon, to get the full derivative. To start building an intuition, let's start with simple functions and their respective derivatives.

The derivative of a simple constant function:

$$f(x) = 1 \quad \rightarrow \quad \frac{d}{dx} f(x) = \frac{d}{dx} 1 = 0$$

$$f(x) = 1$$

$$f'(x) = \frac{d}{dx} 1$$

$$f'(x) = 0$$

Fig 7.10: Derivative of a constant function — calculation steps.

Anim 7.10: https://nnfs.io/cow

When calculating the derivative of a function, recall that the derivative can be interpreted as a slope. In this example, the result of this function is a horizontal line as the output value for any x is 1:

By looking at it, it becomes evident that the derivative equals 0 since there's no change from one value of x to any other value of x (i.e., there's no slope).

So far, we are calculating derivatives of the functions by taking a single parameter, *x* in our case, in each example. This changes with partial derivatives since they take functions with multiple parameters, and we'll be calculating the derivative with respect to only one of them at a time. For now, with derivatives, it's always with respect to a single parameter. To denote the derivative, we can use prime notation, where, for the function $f(x)$, we add a prime (') like $f'(x)$. For our example, $f(x) = 1$, the derivative $f'(x) = 0$. Another notation we can use is called the Leibniz's notation — the dependence on the prime notation and multiple ways of writing the derivative with the Leibniz's notation is as follows:

$$f'(x) = \frac{d}{dx}f(x) = \frac{df}{dx}(x) = \frac{df(x)}{dx}$$

Each of these notations has the same meaning — the derivative of a function (with respect to *x*).

In the following examples, we use both notations, since sometimes it's convenient to use one notation or another. We can also use both of them in a single equation.

In summary: the derivative of a constant function equals *0*:

$$f(x) = 1 \quad \rightarrow \quad f'(x) = 0$$

The derivative of a linear function:

$$f(x) = x \quad \rightarrow \quad \frac{d}{dx}f(x) = \frac{d}{dx}x = \frac{d}{dx}x^1 = 1 \cdot x^{1-1} = 1 \cdot x^0 = 1 \cdot 1 = 1$$

$$\begin{aligned}
f(x) &= x \\
f'(x) &= \frac{d}{dx}x \\
f'(x) &= \frac{d}{dx}x^1 \\
f'(x) &= \quad x^{1-1} \\
f'(x) &= 1 \cdot x^0 \\
f'(x) &= 1 \cdot 1 \\
f'(x) &= 1
\end{aligned}$$

https://nnfs.io

Fig 7.11: Derivative of a linear function — calculation steps.

Anim 7.11: https://nnfs.io/tob

In this case, the derivative is 1, and the intuition behind this is that for every change of x, y changes by the same amount, so y changes one times the x.

The derivative of the linear function equals *1* (but not in every case, which we'll explain next):

$$f(x) = x \quad \rightarrow \quad f'(x) = 1$$

What if we try 2x, which is also a linear function?

$$f(x) = 2x \quad \rightarrow \quad \frac{d}{dx}f(x) = \frac{d}{dx}2x = 2 \cdot \frac{d}{dx}x = 2 \cdot 1x^{1-1} = 2 \cdot 1x^0 = 2 \cdot 1 = 2$$

$$f(x) = 2x$$
$$f'(x) = \frac{d}{dx}2x$$
$$f'(x) = 2 \cdot \frac{d}{dx}x^1$$
$$f'(x) = 2 \cdot \; x^{1-1}$$
$$f'(x) = 2 \cdot 1x^0$$
$$f'(x) = 2 \cdot 1 \cdot 1$$
$$f'(x) = 2$$

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Fig 7.12: Derivative of another linear function — calculation steps.

Anim 7.12: https://nnfs.io/pop

When calculating the derivative, we can take any constant that function is multiplied by and move it outside of the derivative — in this case it's 2 multiplied by the derivative of x. Since we already determined that the derivative of *f(x) = x* was *1*, we now multiply it by *2* to give us the result.

The derivative of a linear function equals the slope, *m* In this case *m = 2*:

$$f(x) = 2x \quad \rightarrow \quad f'(x) = 2$$

If you associate this with numerical differentiation, you're absolutely right — we already concluded that the derivative of a linear function equals its slope:

$$f(x) = mx \quad \rightarrow \quad f'(x) = m$$

m, in this case, is a constant, no different than the value *2,* as it's not a parameter — every non-parameter to the function can't change its value; thus, we consider it to be a constant. We have just found a simpler way to calculate the derivative of a linear function and also generalized it for the equations of different slopes, *m*. It's also an exact derivative, not an approximation, as with the numerical differentiation.

What happens when we introduce exponents to the function?

$$f(x) = 3x^2 \quad \rightarrow \quad \frac{d}{dx}f(x) = \frac{d}{dx}3x^2 = 3 \cdot \frac{d}{dx}x^2 = 3 \cdot 2x^{2-1} = 3 \cdot 2x^1 = 6x$$

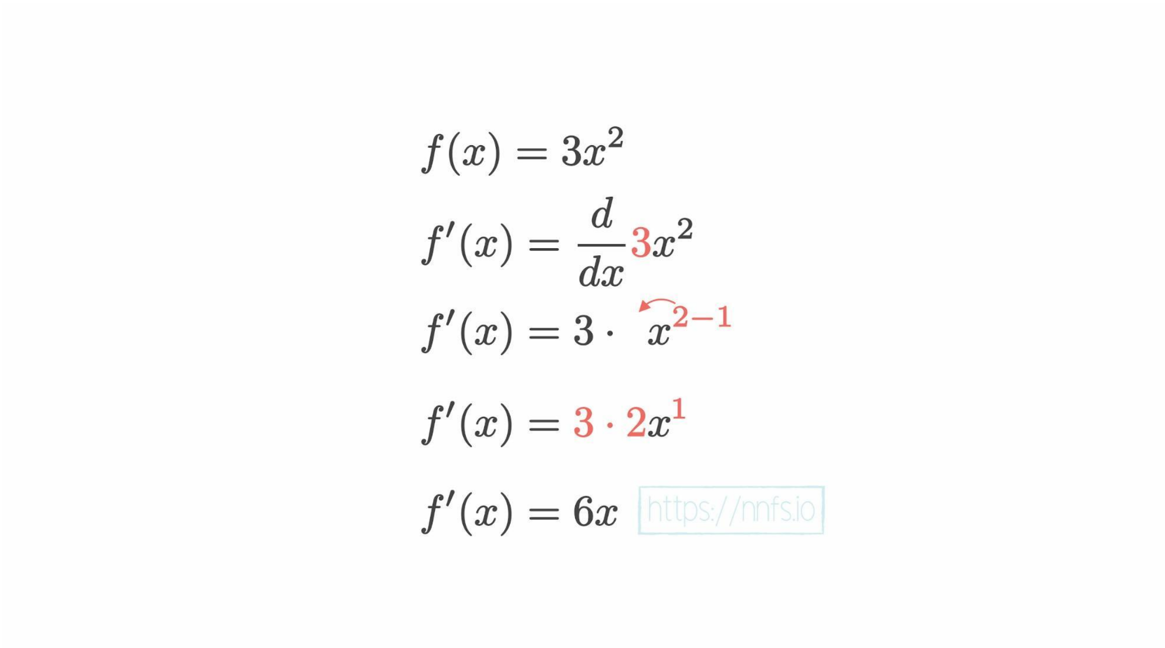

Fig 7.13: Derivative of quadratic function — calculation steps.

Anim 7.13: https://nnfs.io/rok

First, we are applying the rule of a constant — we can move the coefficient (the value that multiplies the other value) outside of the derivative. The rule for handling exponents is as follows: take the exponent, in this case a *2*, and use it as a coefficient for the derived value, then, subtract 1 from the exponent, as seen here: $2 - 1 = 1$.

If $f(x) = 3x^2$ then $f'(x) = 3 \cdot 2x^1$ or simply $6x$. This means the slope of the tangent line, at any point, x, for this quadratic function, will be $6x$. As discussed with the numerical solution of the quadratic function differentiation, the derivative of a quadratic function depends on the x and in this case it equals $6x$:

$$f(x) = 3x^2 \quad \rightarrow \quad f'(x) = 6x$$

A commonly used operator in functions is addition. How do we calculate the derivative in this case?

$$f(x) = 3x^2 + 5x \quad \rightarrow \quad \frac{d}{dx}f(x) = \frac{d}{dx}[3x^2 + 5x] =$$

$$= \frac{d}{dx}3x^2 + \frac{d}{dx}5x^1 =$$

$$= 3 \cdot \frac{d}{dx}x^2 + 5 \cdot \frac{d}{dx}x^1 =$$

$$= 3 \cdot 2x^{2-1} + 5 \cdot 1x^{1-1} =$$

$$= 3 \cdot 2x^1 + 5 \cdot x^0 =$$

$$= 6x + 5$$

$$f(x) = 3x^2 + 5x$$
$$f'(x) = \frac{d}{dx}[3x^2 + 5x]$$
$$f'(x) = \frac{d}{dx}3x^2 + \frac{d}{dx}5x$$
$$f'(x) = 3 \cdot x^{2-1} + 5 \cdot x^{1-1}$$
$$f'(x) = 3 \cdot 2x^1 + 5 \cdot 1x^0$$
$$f'(x) = 6x + 5$$

Fig 7.14: Derivative of quadratic function with addition — calculation steps.

Anim 7.14: https://nnfs.io/mob

The derivative of a sum operation is the sum of derivatives, so we can split the derivative of a more complex sum operation into a sum of the derivatives of each term of the equation and solve the rest of the derivative using methods we already know.

The derivative of a sum of functions equals their derivatives:

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x) = f'(x) + g'(x)$$

In this case, we've shown the rule using both notations.

Let’s try a couple more examples:

$$f(x) = 5x^5 + 4x^3 - 5 \quad \rightarrow \quad \frac{d}{dx}f(x) = \frac{d}{dx}[5x^5 + 4x^3 - 5] =$$

$$= \frac{d}{dx}5x^5 + \frac{d}{dx}4x^3 - \frac{d}{dx}5 =$$

$$= 5 \cdot \frac{d}{dx}x^5 + 4 \cdot \frac{d}{dx}x^3 - \frac{d}{dx}5 =$$

$$= 5 \cdot 5x^{5-1} + 4 \cdot 3x^{3-1} - 0 =$$

$$= 5 \cdot 5x^4 + 4 \cdot 3x^2 =$$

$$= 25x^4 + 12x^2$$

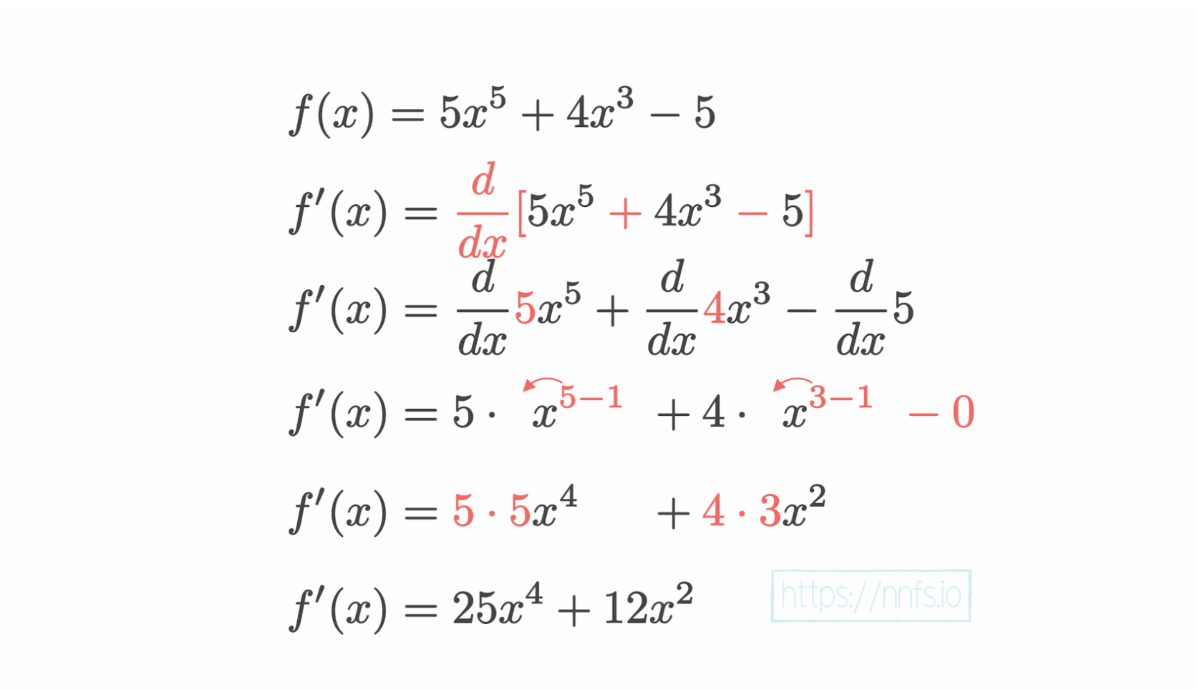

Fig 7.15: Analytical derivative of multi-dimensional function example — calculation steps.

Anim 7.15: https://nnfs.io/tom

The derivative of a constant 5 equals 0, as we already discussed at the beginning of this chapter. We also have to apply the other rules that we've learned so far to perform this calculation.

$$f(x) = x^3 + 2x^2 - 5x + 7 \quad \rightarrow \quad \frac{d}{dx}f(x) = \frac{d}{dx}[x^3 + 2x^2 - 5x + 7] =$$

$$= \frac{d}{dx}x^3 + \frac{d}{dx}2x^2 - \frac{d}{dx}5x + \frac{d}{dx}7 =$$

$$= \frac{d}{dx}x^3 + 2 \cdot \frac{d}{dx}x^2 - 5 \cdot \frac{d}{dx}x + \frac{d}{dx}7 =$$

$$= 3x^{3-1} + 2 \cdot 2x^{2-1} - 5 \cdot 1x^{1-1} + 0 =$$

$$= 3x^2 + 2 \cdot 2x^1 - 5 \cdot 1 + 0 =$$

$$= 3x^2 + 4x - 5$$

$$f(x) = x^3 + 2x^2 - 5x + 7$$

$$f'(x) = \frac{d}{dx}[x^3 + 2x^2 - 5x + 7]$$

$$f'(x) = \frac{d}{dx}x^3 + \frac{d}{dx}2x^2 - \frac{d}{dx}5x + \frac{d}{dx}7$$

$$f'(x) = x^{3-1} + 2 \cdot x^{2-1} - 5 \cdot x^{1-1} + 0$$

$$f'(x) = 3x^2 + 2 \cdot 2x^1 - 5 \cdot 1x^0$$

$$f'(x) = 3x^2 + 4x - 5$$

Fig 7.16: Analytical derivative of another multi-dimensional function example — calculation steps.

Anim 7.16: https://nnfs.io/sun

This looks relatively straight-forward so far, but, with neural networks, we'll work with functions that take multiple parameters as inputs, so we're going to calculate the partial derivatives as well.

Summary

Let's summarize some of the solutions and rules that we have learned in this chapter.

Solutions:

The derivative of a constant equals 0 (m is a constant in this case, as it's not a parameter that we are deriving with respect to, which is *x* in this example):

$$\frac{d}{dx}1 = 0$$

$$\frac{d}{dx}m = 0$$

The derivative of *x* equals 1:

$$\frac{d}{dx}x = 1$$

The derivative of a linear function equals its slope:

$$\frac{d}{dx}mx + b = m$$

Rules:

The derivative of a constant multiple of the function equals the constant multiple of the function's derivative:

$$\frac{d}{dx}[k \cdot f(x)] = k \cdot \frac{d}{dx}f(x)$$

The derivative of a sum of functions equals the sum of their derivatives:

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x) = f'(x) + g'(x)$$

The same concept applies to subtraction:

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x) = f'(x) - g'(x)$$

The derivative of an exponentiation:

$$\frac{d}{dx}x^n = n \cdot x^{n-1}$$

We used the value x instead of the whole function f(x) here since the derivative of an entire function is calculated a bit differently. We'll explain this concept along with the chain rule in the next chapter.

Since we've already learned what derivatives are and how to calculate them analytically, which we'll later implement in code, we can go a step further and cover partial derivatives in the next chapter.

Supplementary Material: https://nnfs.io/ch7
Chapter code, further resources, and errata for this chapter.

Chapter 8 - Gradients, Partial Derivatives, and the Chain Rule