

Linear Mappings

Def: Let U, V be vector spaces over the same field F . A mapping $T: U \rightarrow V$ is called a linear map if $\forall u_1, u_2 \in U, \lambda \in F$

$$T(u_1 + u_2) = T(u_1) + T(u_2)$$

$$T(\lambda u_1) = \lambda T(u_1)$$

The set of all linear mappings

from $U \rightarrow V$ is denoted $L(U, V)$.

If $U = V$, then we denote $L(U)$.

Examples: • $T: C([a, b]) \rightarrow \mathbb{R}$, $t \mapsto \int_a^b t(x) dx$
(integration)

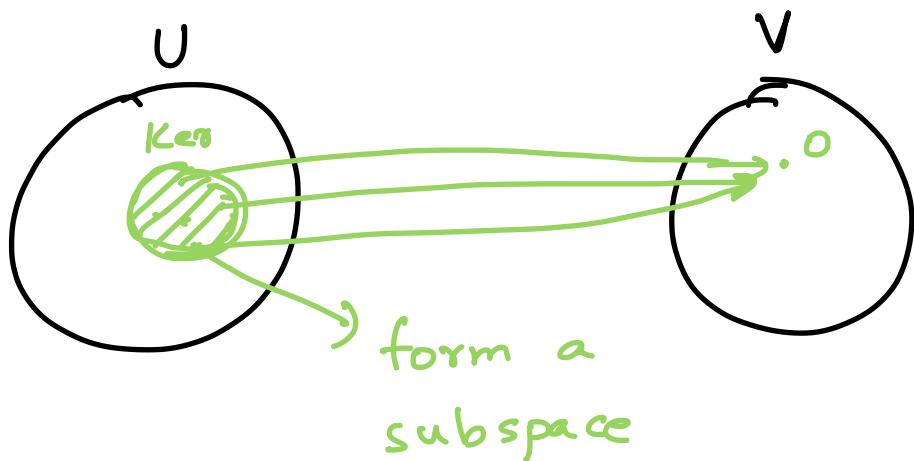
• $D: C^\infty([a, b]) \rightarrow C^\infty([a, b])$, $t \mapsto t'$ (differentiation)

Def: $T \in L(U, V)$. Then Kernel of T (null-space of T)

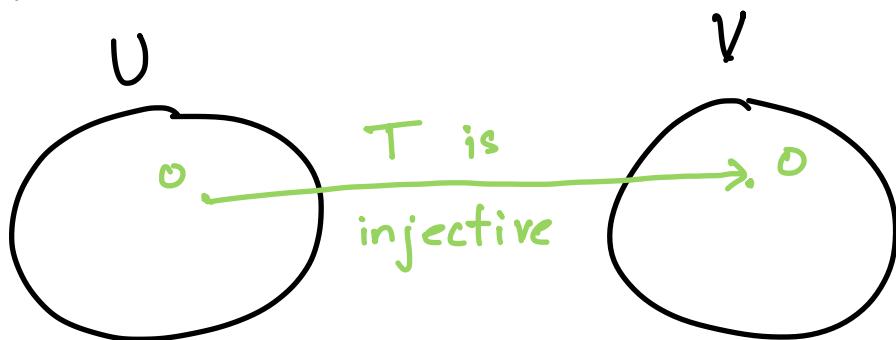
is defined as.

$$Ker(T) := \text{null}(T) := \{u \in U \mid Tu = 0\}$$

Proposition: • $\text{Ker}(T)$ is a subspace of U .



- T injective iff $\text{ker}(T) = \{0\}$



Def: The range of T (image of T) is defined as,

$$\text{range}(T) := \text{image}(T) := \{Tu \mid u \in U\}$$

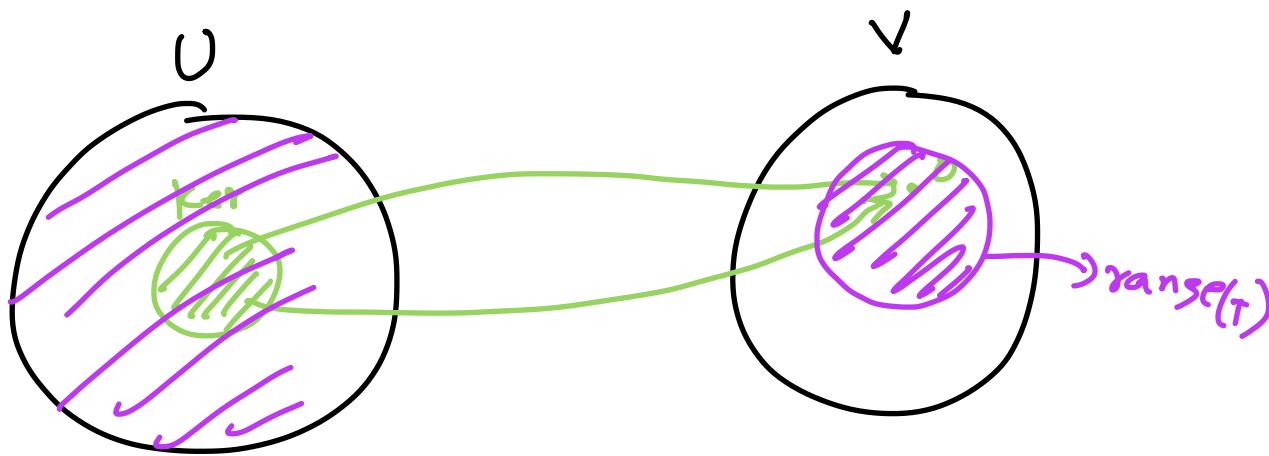
Proposition: • The range is always a subspace of V .
• T is surjective iff $\text{range}(T) = V$

Def: Let V' be any subset of V i.e. $V' \subset V$.

The pre-image of V' is defined as

$$T^{-1}(V') := \{u \in U \mid Tu \in V'\}$$

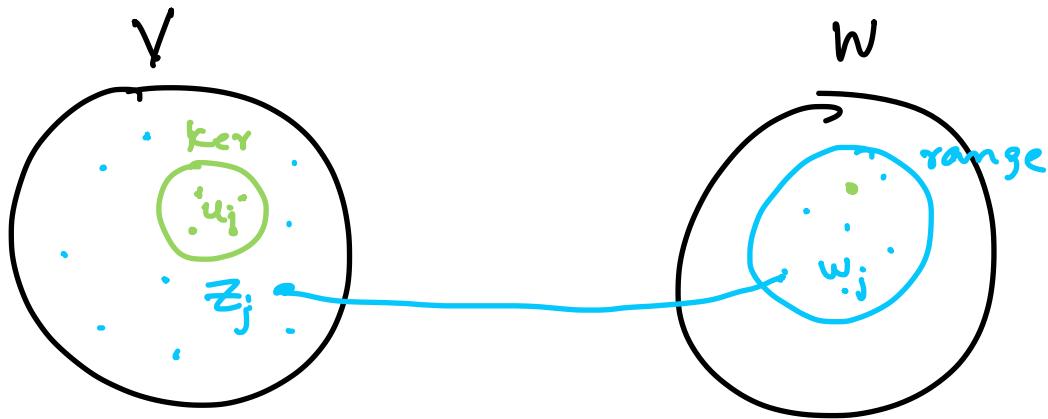
Proposition: If $V' \subset V$ is a subspace of V ,
then $T^{-1}(V')$ is a subspace of U .



Theorem: Let V be finite-dim, W any VS,
 $T \in \mathcal{L}(V, W)$. Let (u_1, \dots, u_n) be a basis of $\ker(T) \subset V$
Let w_1, \dots, w_m be a basis of $\text{range}(T) \subset W$. Then.
 $u_1, \dots, u_n, T^{-1}(w_1), \dots, T^{-1}(w_m) \subset V$ form a basis of V .

In particular, $\dim(V) = \dim(\ker(T)) + \dim(\text{range}(T))$.

Proof: Denote $T^{-1}(w_1) = z_1, \dots, T^{-1}(w_m) = z_m$



Step 1: $V \subset \text{span}\{u_1, \dots, u_n, z_1, \dots, z_m\}$

Let $v \in V$ consider $Tv \in \text{range}(T)$

$\Rightarrow \exists \lambda_1, \dots, \lambda_m$ s.t.

$$Tv = \lambda_1 w_1 + \lambda_2 w_2 + \dots + \lambda_m w_m$$

$$= \lambda_1 T(z_1) + \lambda_2 T(z_2) + \dots + \lambda_m T(z_m)$$

$$= T(\lambda_1 z_1 + \lambda_2 z_2 + \dots + \lambda_m z_m)$$

$$\Rightarrow Tv - T(\lambda_1 z_1 + \lambda_2 z_2 + \dots + \lambda_m z_m) = 0$$

$$\Rightarrow T(v - (\lambda_1 z_1 + \lambda_2 z_2 + \dots + \lambda_m z_m)) = 0$$

$\underbrace{v - (\lambda_1 z_1 + \lambda_2 z_2 + \dots + \lambda_m z_m)}$ $\in \ker(T)$

$\Rightarrow \exists \mu_1, \dots, \mu_n$ s.t.

Reminder: u_1, \dots, u_n are basis of $\ker(T)$

$$V - (\lambda_1 z_1 + \lambda_2 z_2 + \dots + \lambda_m z_m) = \mu_1 u_1 + \mu_2 u_2 + \dots + \mu_n u_n$$

$$\Rightarrow V = \lambda_1 z_1 + \lambda_2 z_2 + \dots + \lambda_m z_m + \mu_1 u_1 + \mu_2 u_2 + \dots + \mu_n u_n$$

Step 2: $u_1, u_2, \dots, u_n, z_1, z_2, \dots, z_m$ are linearly independent.

Assume that $\mu_1 u_1 + \dots + \mu_n u_n + \lambda_1 z_1 + \dots + \lambda_m z_m = 0$ (*)

Now consider: $\lambda_1 w_1 + \dots + \lambda_m w_m$

$$= \lambda_1 T(z_1) + \dots + \lambda_m T(z_m)$$

$$= \lambda_1 T(z_1) + \dots + \lambda_m T(z_m) + \mu_1 T(u_1) + \dots + \mu_n T(u_n)$$

$$= T(\lambda_1 z_1 + \dots + \lambda_m z_m + \mu_1 u_1 + \dots + \mu_n u_n) = 0$$

$= 0$ by (*)

$$\Rightarrow \lambda_1 w_1 + \dots + \lambda_m w_m = 0 \xrightarrow[w_1, \dots, w_m \text{ basis}]{} \lambda_1 = 0, \dots, \lambda_m = 0$$

$$\Rightarrow \mu_1 u_1 + \dots + \mu_n u_n = 0$$

$$\Rightarrow \mu_1 = \mu_2 = \dots = \mu_n = 0 \text{ since } u_1, \dots, u_n \text{ are basis}$$



Proposition: $T \in \mathcal{L}(V, V)$, V is finite-dim.

Then the following statements are equivalent.

- (i) T is injective
- (ii) T is surjective
- (iii) T is bijective

Proof: Direct consequence of theorem.

⚠ Does not hold in ∞ -dim spaces.

Matrices and Linear Maps

Notation:

$$A = \left\{ \begin{array}{c} \text{m-rows} \\ \left(\begin{array}{cccc} a_{11} & \dots & \dots & a_{1n} \\ \vdots & & & \vdots \\ a_{m1} & \dots & \dots & a_{mn} \end{array} \right) \end{array} \right. \overset{n-\text{cols}}{\leftarrow} = (a_{ij})_{\substack{i=1, \dots, m \\ j=1, \dots, n}}$$

Consider $T \in \mathcal{L}(V, W)$, V, W finite-dim,

let v_1, \dots, v_n be a basis of V

w_1, \dots, w_m be a basis of W

- $v = \lambda_1 v_1 + \dots + \lambda_n v_n$

$$T(v) = T(\lambda_1 v_1 + \dots + \lambda_n v_n)$$

$$= \lambda_1 T(v_1) + \dots + \lambda_n T(v_n)$$

- Each image vector $T(v_j)$ can be expressed in basis w_1, \dots, w_m :

there exist co-efficients a_{1j}, \dots, a_{mj} s.t.

$$T(v_j) = a_{1j} w_1 + \dots + a_{mj} w_m$$

- We now stack these co-efficients in a matrix:

m rows,
one for each
basis vector
of W .

$$\begin{pmatrix} a_{11} & \dots & \boxed{a_{1j}} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \dots & \boxed{a_{mj}} & \dots & a_{mn} \end{pmatrix}$$

matrix of
mapping T
w.r.t. the basis
 v_1, \dots, v_n of V
 w_1, \dots, w_m of W

n-cols, one for each basis vector of V .

Notation: Let $T: V \rightarrow W$ be linear, let B a basis of V , C basis of W . We denote by $M(T, B, C)$ the matrix corresponding to T w.r.t. bases B and C .

Convenient properties of matrices: Let V, W be vector spaces, consider the bases fixed. Let $S, T \in I(V, W)$,

- $M(S+T) = M(S) + M(T)$
- $M(\lambda S) = \lambda M(S)$

linear properties
of mapping extend
to matrices.

- For $v = \lambda_1 v_1 + \dots + \lambda_n v_n$ we have that

$$T(v) = M(T) \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

image of v
under T matrix-vector
product

where (v_1, \dots, v_n) is basis of V

- $T: U \rightarrow V, S: V \rightarrow W$ linear, then

$$M(S \circ T) = M(S) \cdot M(T)$$

↳ composition of the maps S and T .

Invertible maps and Matrices

Def: $T \in L(V, W)$ is called invertible if there exists a linear map $S \in L(W, V)$ such that

$$S \circ T = \text{Id}_V \quad \text{and} \quad T \circ S = \text{Id}_W$$

The map S is called the inverse of T , denoted by T^{-1}

Remark: Inverse maps exist and are unique.

Prop: A linear map is invertible iff it is injective and surjective i.e. bijective.

Proof: " \Rightarrow " invertible \Rightarrow injective:

Suppose $\underline{T(u)} = \underline{T(v)}$. Then $u = T^{-1}(T(u))$

$= T^{-1}(T(v)) = v \Rightarrow u = v \Rightarrow$ injective

invertible \Rightarrow surjective: $w \in W$. Then

$w = T(T^{-1}(w)) \Rightarrow w \in \text{range of } T$

\Rightarrow surjective

" \Leftarrow " injective & surjective \Rightarrow invertible

Let $w \in W$. There exists unique $v \in V$ s.t. $T(v) = w$

Define the mapping: $S(w) = v$. Clearly have $T \circ S = \text{Id}$

Let $v \in V$. Then $T((S \circ T)v)$

$$= (T \circ S)(Tv) = \text{Id} \circ Tv = Tv$$

↳ composition of mappings is associative

$$\Rightarrow (S \circ T)v = v \Rightarrow S \circ T = \text{Id} \Rightarrow S \text{ is inverse of } T$$

Still need to show S is a linear mapping

$$\begin{aligned} \text{Let } y_1, y_2 \in W, \alpha \in F : S(y_1 + y_2) &= S(y_1) + S(y_2) \\ S(\alpha y_1) &= \alpha S(y_1) \end{aligned}$$

Let $x_1, x_2 \in V$. s.t. $T(x_i) = y_i$. Then $S(y_i) = x_i$

$$\begin{aligned} S(y_1 + y_2) &= S(T(x_1) + T(x_2)) \\ &= S(T(x_1 + x_2)) \\ &= x_1 + x_2 \\ &= S(y_1) + S(y_2) \end{aligned} \quad \left| \begin{aligned} S(\alpha y_1) &= S(\alpha T(x_1)) \\ &= S(T(\alpha x_1)) \\ &= \alpha x_1 \\ &= \alpha S(y_1) \end{aligned} \right.$$

$\Rightarrow S$ is a linear transform.

Inverse Matrix

Def: A square matrix $A \in F^{n \times n}$ is invertible if there exists a square matrix $B \in F^{n \times n}$ such that : $A \cdot B = B \cdot A = \text{Id} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

The matrix B is called the inverse matrix, and is denoted by A^{-1} .

Proposition: The inverse matrix represents the inverse of the corresponding linear map, that is : $T: V \rightarrow V$

$$M(\underbrace{T^{-1}}_{\text{matrix of inverse map}}) = (\underbrace{M(T)}_{\text{inverse matrix of the original map}})^{-1}$$

In particular, a matrix is invertible iff the corresponding map is invertible.

Remarks:

- The inverse matrix does not always exist
- $(A^{-1})^+ = A$, $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$
- A^+ invertible $\Leftrightarrow A$ invertible
$$(A^+)^{-1} = (A^{-1})^+$$
- $A \in F^{n \times n}$ invertible $\Leftrightarrow \text{rank}(A) = n$
- The set of all invertible matrices is called general linear group:

$$GL(n, F) = \{ A \in F^{n \times n} \mid A \text{ invertible} \}$$