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CSE 840: Computational Foundations in AI
Homework 1

Instructions:

- **Format:** Please submit a PDF file with typed answers (LaTeX or otherwise). Handwritten notes will not be graded.
- **Submission:** Only homeworks uploaded to Gradescope will be graded. Make sure to show all the steps of your derivations in order to receive full credit.
- **Integrity and Collaboration:** You are expected to work on the homeworks by yourself. You are not permitted to discuss them with anyone except the instructor. The homework that you hand in should be entirely your own work. You may be asked to demonstrate how you got any results that you report.
- **Clarifications:** If you have any question, please look at Piazza first. Other students may have encountered the same problem, and is solved already. If not, post your question there. We will respond as soon as possible.

1 **[8 points]** Which of the following subsets of R^2 are actually linear subspaces? Explain.

1. $\{(x, y) | xy = 0\}$
2. $\{(x, y) | x + y = 0\}$
3. $\{(x, y) | x + y = 2\}$
4. $\{(x, y) | x + y \geq 0\}$

2 **[8 points]** Which of the following sets are linear spaces? Explain why.

1. $x = (x_1, x_2, x_3) \in R^3$ with the property $x_1 - 2x_3 = 0$
2. The set of all 2×3 matrices with real coefficients?
3. The set of all 2×2 invertible real matrices?
4. The set of solutions x of $Ax = 0$ where A is an $m \times n$ matrix.

3 **[4 points]** Which of the following are valid linear maps? Explain why.

1. $T((x_1, x_2)) = (x_1 + x_2, x_1 - x_2 + 1)$
2. $T((x_1, x_2)) = x_1^2 + x_2^2$

4 [15 points] Let $\mathbf{p} \in \mathbb{R}^d$, $\mathbf{q} \in \mathbb{R}^d$, and $\mathbf{r} \in \mathbb{R}^d$ be d-dimensional real-valued vectors whereas $\mathbf{A} \in \mathbb{R}^{d\times d}$ is a square matrix. Use this information to answer each of the following True/False questions below. If true, you must provide formal proof in support of the statement to receive full credit. If false, you may provide a counter-example to disprove the statement.

1. If $\|\mathbf{p}\|_2 = \|\mathbf{q}\|_2$, then $\mathbf{p} + \mathbf{q}$ must be orthogonal to $\mathbf{p} - \mathbf{q}$.
2. If $\mathbf{p} \cdot \mathbf{q} = \mathbf{p} \cdot \mathbf{r}$, then $\mathbf{q} = \mathbf{r}$.
3. $\|\mathbf{p}\|_\infty \leq \|\mathbf{p}\|_1$.
4. If $\mathbf{Ax} = \mathbf{Ay}$, then $\mathbf{x} = \mathbf{y}$.
5. $\mathbf{x}^T\mathbf{Ax} = \mathrm{tr}[\mathbf{Axx}^T]$.

5 [10 points] Let $\mathcal{B} = \{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$ be the basis for a vector space V and $T : V \to \mathbb{R}^2$ be a linear transformation with the property that

$$T(x_1\boldsymbol{b}_1 + x_2\boldsymbol{b}_2 + x_3\boldsymbol{b}_3) = \begin{bmatrix} 2x_1 - 4x_2 + 5x_3 \\ -x_2 + 3x_5 \end{bmatrix} \tag{1}$$

1. Find a matrix $M(T, \mathcal{B}, \mathcal{S})$ where $\mathcal{S}$ is the standard basis for $\mathbb{R}^2$.
2. Find a matrix $M(T, \mathcal{B}, \mathcal{Q})$ where $\mathcal{Q}$ is the basis $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ for $\mathbb{R}^2$.

6 [5 points]

1. Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be the adjacency matrix of a directed graph that has n nodes, where $A_{ij} = 1$ if there is an outgoing edge from node i to node j in the graph. Suppose $\mathbf{1}_n$ denotes an n-dimensional vector of all 1s. What does each entry of the matrix-vector product $\mathbf{A}\mathbf{1}_n$ represent? What does each entry of the matrix-vector product $\mathbf{A}^T\mathbf{1}_n$ represent? You should explain the meaning of the entries from a graph perspective.

2. Suppose $\mathbf{A}$ denotes the adjacency matrix of a directed graph, where $A_{ij} = 1$ if there is an outgoing edge from node i to node j in the graph. Suppose you compute the matrix $\mathbf{A}^{10} = \mathbf{A} \times \mathbf{A} \times \cdots \times \mathbf{A}$ (10 times). What does each (i, j)-th entry of the matrix, $\mathbf{A}_{ij}^{10}$, represent? You should explain the meaning of the entries from a graph perspective.

3. A square matrix $\mathbf{P}$ is called idempotent if $\mathbf{P}^2 = \mathbf{P}$. Compute $\mathbf{A}^{500}$, where

$$\mathbf{A} = \begin{bmatrix} \mathbf{I} & \mathbf{P} \\ \mathbf{0} & \mathbf{P} \end{bmatrix},$$

where $\mathbf{I}$ is the identity matrix, $\mathbf{0}$ is the null matrix, and $\mathbf{P}$ is an idempotent matrix. Note that all 3 submatrices are assumed to have the same dimensions.

7 [10 points] Consider a square and symmetric 3×3 matrix $\mathbf{A}$, which can be decomposed into a product of the following 3 matrices:

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{U}^T,$$

where $\mathbf{U}$ is an orthogonal matrix and $\mathbf{\Sigma}$ is a diagonal matrix with elements 1, 2, and 4. Compute the following based on the given information.

1. $\mathrm{tr}[\mathbf{A}^2]$ and $\det[\mathbf{A}^2]$.
2. $\mathrm{tr}[\mathbf{A}^{-1}]$ and $\det[\mathbf{A}^{-1}]$.

8 [13 points] Answer the following, show your reasoning.

1. Show that if a matrix $\mathbf{A}$ satisfies $\mathbf{A}^2 = 4\mathbf{I}$, then the eigenvalues of $\mathbf{A}$ include the values 2 and -2.

2. Show that if a square matrix $\mathbf{P}$ satisfies $\mathbf{P}^2 = \mathbf{P}$, then all its eigenvalues must be 1 or 0. Note that such matrix is known as an idempotent matrix.

3. True or False? A matrix with all zero eigenvalues must be the zero matrix.

9 [7 points]

1. Consider a fully-connected graph of n nodes with self loops. The adjacency matrix of the graph is $A_{ij} = 1$ for all i and j. Find the largest eigenvalue of the adjacency matrix along with its corresponding eigenvector.

2. Consider a graph that has n nodes and k equal-sized cliques (a clique is a fully-connected subgraph). In other words, each clique has exactly n/k nodes. Assume all the edges/links have the same weight of 1 and each node has a self-loop, i.e., $A_{ii} = 1$ for all i. Find the non-zero eigenvalues and their corresponding eigenvectors of the adjacency matrix.