Eigenvalues, Characteristic Polynomials, and the Trace of Matrix

Instructor: Vishnu Boddeti *Scribe: Bibin Varghese, Vaishnavi Sundaraganapathi*

1 Introduction

Linear transformations often do more than just move vectors around—they can stretch or compress them along specific directions. When a transformation T acts on a vector v such that

$$T(v) = \lambda v,$$

we say that v is an **eigenvector** of T and λ is its corresponding **eigenvalue**. In essence, the transformation simply scales v by λ without changing its direction.

To find these special values, we use the **characteristic polynomial** of a matrix A, defined as

$$P_A(t) = \det(A - tI).$$

The roots of this polynomial give us the eigenvalues of A.

Another key concept is the **trace** of a matrix, which is the sum of its diagonal entries. Interestingly, if a matrix is diagonalizable, its trace is equal to the sum of its eigenvalues, offering quick insight into the overall effect of the matrix.

In the sections that follow, you will delve into these ideas, exploring their properties and applications in a clear manner.

2 Eigenvalues and Eigenvectors

Definition 1. *Let $T : V \to V$ be a linear transformation on a vector space V over a field F. A scalar $\lambda \in F$ is called an **eigenvalue** if there exists a nonzero vector $v \in V$ such that*

$$Tv = \lambda v.$$

*The vector v ($v \neq 0$) is then called an **eigenvector** corresponding to λ. We also define the **eigenspace** associated with λ as*

$$E(\lambda, T) = \{v \in V \mid T(v) = \lambda v\} = \ker(T - \lambda I).$$

which is the set of all the eigenvectors corresponding to λ (including the zero vector).

Remark 2. *The following points highlight some important aspects of eigenvalues and eigenvectors:*

- ***Scaling Property:*** *When $T(v) = \lambda v$, the transformation scales v by λ without changing its direction.*

- ***Existence in Different Fields:*** *Many linear maps (like rotations) in $\mathbb{R}$ do not have eigenvectors. However, over an algebraically closed field such as $\mathbb{C}$ or a complex field, every linear operator has at least one eigenvalue. An **algebraically closed field** is one in which every non-constant polynomial has at least one root within the field. While the complex numbers $\mathbb{C}$ are algebraically closed, the real numbers $\mathbb{R}$ are not. For example, consider the polynomial*

$$x^2 + 1 = 0.$$

This equation has no solution in $\mathbb{R}$ because the square of any real number is non-negative. However, in $\mathbb{C}$, it has the two solutions $x = i$ and $x = -i$. Thus, $\mathbb{R}$ is not algebraically closed, which is one reason why some real linear transformations (like rotations) may not have real eigenvalues, even though they always have complex eigenvalues when viewed over $\mathbb{C}$.

- ***Multiple Eigenvectors:*** *If λ is an eigenvalue, then there are infinitely many eigenvectors corresponding to it. For instance, if v is an eigenvector, any nonzero scalar multiple $a \cdot v$ (with $a \in K$) is also an eigenvector, since*

$$T(a \cdot v) = a \cdot T(v) = a \cdot \lambda v = \lambda(a \cdot v).$$

- ***Linear Independence for Distinct Eigenvalues:*** *Eigenvectors corresponding to distinct eigenvalues are linearly independent. To illustrate, suppose v_1 and v_2 correspond to λ_1 and λ_2 with $\lambda_1 \neq \lambda_2$. If we assume v_1 and v_2 are linearly dependent, $v_2 = c \cdot v_1$ for some scalar c, then*

$$T(v_1) = \lambda_1 v_1 \quad \text{and} \quad T(v_2) = T(c \cdot v_1) = c \cdot T(v_1) = c\lambda_1 v_1,$$

but also $T(v_2) = \lambda_2 v_2 = \lambda_2(c \cdot v_1) = c\lambda_2 v_1$. Since $\lambda_1 \neq \lambda_2$, this yields a contradiction.

- ***Dependence within the Same Eigenspace:*** *Eigenvectors corresponding to the same eigenvalue need not be linearly independent. For example, both v and $c \cdot v$ (with $c \neq 0$) are eigenvectors for λ, but they are linearly dependent.*

- ***Choosing a Basis:*** *Even though any two scalar multiples of the same eigenvector are dependent, it is often possible to select a maximal set of linearly independent eigenvectors from the eigenspace. For example, if $A = I$ (the identity matrix), every vector is an eigenvector with eigenvalue 1, yet we can choose a basis for the space.*

- ***Eigenspace is a Subspace:*** *Finally, the eigenspace $E(\lambda, T) = \ker(T - \lambda I)$ is always a linear subspace of V.*

A Visual Insight

The following diagram helps visualize the idea: the blue arrow represents an eigenvector v and the red arrow shows $T(v) = \lambda v$, which is a scaled version of v along the same direction.

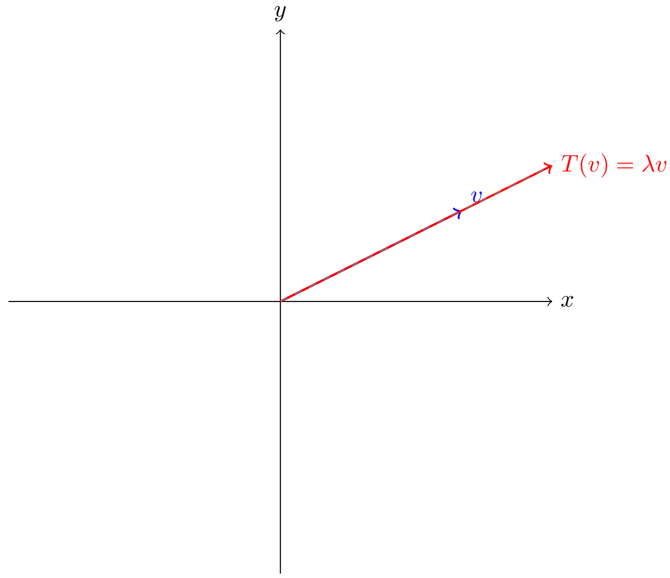

Figure 1: The eigenvector v (blue) is scaled by λ to produce $T(v)$ (red), which remains in the same direction.

Proposition 3. *For a finite-dimensional vector space V and a linear transformation $T \in L(V)$, the following statements are equivalent:*

1. *λ is an eigenvalue of T.*
2. *The transformation $T - \lambda I$ is not one-to-one (injective).*
3. *The transformation $T - \lambda I$ is not onto (surjective).*
4. *The transformation $T - \lambda I$ is not bijective.*

Proof. **Step 1.** Suppose (1) holds; that is, λ is an eigenvalue of T. By definition, there exists a nonzero vector $v \in V$ such that

$$T(v) = \lambda v.$$

Rearrange this equation to get:

$$(T - \lambda I)(v) = T(v) - \lambda v = 0.$$

Since v is nonzero, this shows that the kernel of $T - \lambda I$ is not just $\{0\}$ (i.e., it is nontrivial). A linear transformation with a nontrivial kernel cannot be one-to-one. Therefore, (1) implies (2). In other words, a linear transformation, $T - \lambda I$ is said to be one-to-one (injective) if its kernel is trivial, i.e.,

$$\ker(T - \lambda I) = \{0\}.$$

This occurs if and only if there is no nonzero vector v such that $(T - \lambda I)v = 0$. In the context of eigenvalues, this means that λ is not an eigenvalue of T. Conversely, if λ is an eigenvalue, then there exists a nonzero v satisfying $(T - \lambda I)v = 0$, and thus $T - \lambda I$ is not one-to-one.

Step 2. In a finite-dimensional vector space, a linear transformation is injective if and only if it is surjective. Thus, if $T - \lambda I$ is not injective (from Step 1), it must also be not surjective. This shows that (2) implies (3).

Step 3. If $T-\lambda I$ is not injective or not surjective, then it is not bijective. Hence, (2) (or equivalently (3)) implies (4).

Step 4. Finally, assume (4): that is, $T-\lambda I$ is not bijective. In particular, it is not injective, which means there exists a nonzero vector v such that

$$(T-\lambda I)(v)=0.$$

This rearranges to:

$$T(v)=\lambda v,$$

so v is an eigenvector corresponding to λ. Therefore, λ is an eigenvalue of T, showing that (4) implies (1).

Since we have shown that each statement leads to the next in a cycle, all four statements are equivalent. □

Proposition 4. *Suppose V is a finite-dimensional vector space, $T \in L(V)$, and $\lambda_1, \lambda_2, \ldots, \lambda_m$ are distinct eigenvalues of T. Then the sum of the corresponding eigenspaces*

$$E(\lambda_1,T)+E(\lambda_2,T)+\cdots+E(\lambda_m,T)$$

is a direct sum. In particular,

$$\dim E(\lambda_1,T)+\dim E(\lambda_2,T)+\cdots+\dim E(\lambda_m,T) \leq \dim V.$$

Proof. (Outline) The proof is based on the fact that eigenvectors corresponding to distinct eigenvalues are linearly independent. Assume a nontrivial linear combination of eigenvectors from different eigenspaces yields the zero vector. Using the fact that T acts by scaling each eigenvector by its corresponding eigenvalue, one deduces that the coefficients in the linear combination must be zero. This shows the set of eigenvectors is linearly independent, so the sum of the eigenspaces is direct. □

Theorem 5. *Every linear operator $T: V \to V$ on a finite-dimensional vector space V over the complex field $\mathbb{C}$ has at least one eigenvalue.*

Proof. Let $n=\dim V$ and choose a nonzero vector $v \in V$. Then the set

$$\{v,\, T(v),\, T^2(v),\, \ldots,\, T^n(v)\}$$

consists of $n+1$ vectors in an n-dimensional space, and hence must be linearly dependent. Therefore, there exist scalars $a_0, a_1, \ldots, a_n$, not all zero, such that

$$a_0 v + a_1 T(v) + a_2 T^2(v) + \cdots + a_n T^n(v) = 0.$$

Define the polynomial

$$P(z)=a_0+a_1 z+a_2 z^2+\cdots+a_n z^n.$$

Since $\mathbb{C}$ is algebraically closed, $P(z)$ can be factorized as

$$P(z)=c(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_m),$$

with $m \leq n$. Replacing z by T in the polynomial, we have

$$\left(a_0+a_1 T+\cdots+a_n T^n\right) v = c(T-\lambda_1 I)(T-\lambda_2 I)\cdots(T-\lambda_m I)v=0.$$

Since $v \neq 0$, at least one factor $T-\lambda_i I$ must have a nontrivial kernel, showing that λ_i is an eigenvalue of T. □

Problem 1: Verifying Proposition 3

Problem: Consider the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

Show that $\lambda = 2$ is an eigenvalue and verify that $A - 2I$ is not injective. Compute $\ker(A - 2I)$ and explain the result.

Solution: We compute

$$A - 2I = \begin{pmatrix} 2-2 & 1 \\ 0 & 2-2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The equation $(A - 2I)x = 0$ becomes:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus, $x_2 = 0$ and x_1 is free. Therefore,

$$\ker(A - 2I) = \left\{ \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \mid x_1 \in \mathbb{R} \right\}.$$

Since the kernel is nontrivial, $A - 2I$ is not injective, confirming that 2 is indeed an eigenvalue of A.

Problem 2: Direct Sum of Eigenspaces (Proposition 4)

Problem: Let

$$B = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}.$$

Identify the eigenvalues and the corresponding eigenspaces of B, and verify that the sum of these eigenspaces is a direct sum.

Solution: The eigenvalues of B are $\lambda_1 = 3$ and $\lambda_2 = 5$. The corresponding eigenspaces are:

$$E(3, B) = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \mid x \in \mathbb{R} \right\}, \quad E(5, B) = \left\{ \begin{pmatrix} 0 \\ y \end{pmatrix} \mid y \in \mathbb{R} \right\}.$$

Any vector in $\mathbb{R}^2$ can be uniquely expressed as a sum of a vector from $E(3, B)$ and a vector from $E(5, B)$. Therefore, the sum is direct, and we have:

$$\dim E(3, B) + \dim E(5, B) = 1 + 1 = 2 = \dim \mathbb{R}^2.$$

3 Characteristic Polynomial

Before we define the characteristic polynomial, let's build some intuition. Suppose A is an $n \times n$ matrix and there exists a nonzero vector v (i.e., $v \neq 0$) such that

$$Av = \lambda v.$$

This equation means that v is an eigenvector of A corresponding to the eigenvalue λ. Rearranging the equation gives

$$(A - \lambda I)v = 0.$$

Since v is nonzero, this tells us that v lies in the kernel of $A - \lambda I$, so

$$\ker(A - \lambda I) \neq \{0\}.$$

In a finite-dimensional vector space, the Rank-Nullity Theorem states that

$$\dim V = \dim \ker(A - \lambda I) + \dim \operatorname{im}(A - \lambda I).$$

Because $\dim \ker(A - \lambda I) \geq 1$ (since it contains v), it follows that the rank of $A - \lambda I$ is less than n. In other words, $A - \lambda I$ is not invertible, which means

$$\det(A - \lambda I) = 0.$$

This equation is key—it tells us that any eigenvalue λ must satisfy $\det(A - \lambda I) = 0$. This condition is precisely what we use to define the characteristic polynomial.

Definition 6. *For an $n \times n$ matrix A, the **characteristic polynomial** is defined by*

$$P_A(t) := \det(A - tI).$$

The eigenvalues of A are the roots of this polynomial.

Consider the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Its characteristic polynomial is computed as:

$$\begin{aligned} P_A(t) &= \det \begin{pmatrix} a_{11} - t & a_{12} \\ a_{21} & a_{22} - t \end{pmatrix} \\ &= (a_{11} - t)(a_{22} - t) - a_{12}a_{21} \\ &= t^2 - t(a_{11} + a_{22}) + (a_{11}a_{22} - a_{12}a_{21}). \end{aligned}$$

Remark 7 (Observation 8)**.** *The following observations hold for the characteristic polynomial $P_A(t)$:*

- *$P_A(t)$ is a polynomial of degree n if A is an $n \times n$ matrix.*
- *Characteristic polynomials are invariant under a change of basis. That is, if U is an invertible matrix, then the matrices A and UAU^{-1} have the same characteristic polynomial.*
  Proof: *Observe that*

  $$P_{UAU^{-1}}(t) = \det(UAU^{-1} - tI) = \det(U(A - tI)U^{-1}) = \det(U)\,\det(A - tI)\,\det(U^{-1}) = \det(A - tI).$$

- *The roots of $P_A(t)$ are exactly the eigenvalues of A.*
- *Over $\mathbb{C}$, the polynomial always has n roots (counting multiplicity), so A has n eigenvalues (not necessarily distinct).*
- *A matrix A is invertible if and only if 0 is not an eigenvalue (since if 0 is an eigenvalue, there exists a nonzero v such that $Av = 0$, making A non-injective).*
- *If $A \in L(V)$ and λ is an eigenvalue of A, then λ^2 is an eigenvalue of A^2 (since $A^2 v = \lambda^2 v$).*
- *If A is invertible and λ is an eigenvalue of A, then λ^{-1} is an eigenvalue of A^{-1}.*

Definition 8 (Geometric Multiplicity)**.** *For an operator A with eigenvalue λ, the **geometric multiplicity** is defined as the dimension of the corresponding eigenspace:*

$$\dim E(\lambda, A).$$

Definition 9 (Algebraic Multiplicity)**.** *The **algebraic multiplicity** of an eigenvalue λ is the number of times λ appears as a root of the characteristic polynomial $P_A(t)$.*

Remark 10. *In general, the geometric multiplicity is less than or equal to the algebraic multiplicity. That is, even if an eigenvalue appears multiple times in the characteristic polynomial, the number of linearly independent eigenvectors corresponding to that eigenvalue may be smaller. In other words, the two notions are not the samee.*

Remark 11 (Computing Eigenvalues and Eigenvectors)**.** *To compute the eigenvalues of a matrix A, one typically:*

1. *Writes down the characteristic polynomial $P_A(t) = \det(A - tI)$.*
2. *Finds the roots of $P_A(t)$, which are the eigenvalues.*

Once the eigenvalues are known, the corresponding eigenvectors can be computed by solving the system:

$$(A - \lambda I)x = 0.$$

A Visual Insight of the Characteristic Polynomial

The diagram below illustrates a schematic view of a characteristic polynomial. The polynomial $P_A(t)$ is plotted along the t-axis, and its roots (where the curve crosses the t-axis) correspond to the eigenvalues of A.

Characteristic Polynomial: $P_A(t) = t^2 - 7t + 10$

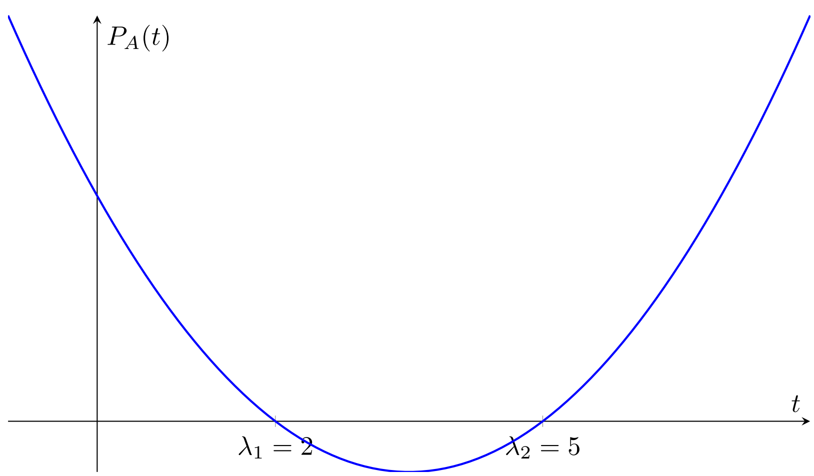

Figure 2: Plot of the characteristic polynomial $P_A(t) = t^2 - 7t + 10$ with eigenvalues at $t = 2$ and $t = 5$.

Problem 3: Understanding the Characteristic Polynomial

Problem: Let

$$A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}.$$

1. Compute the characteristic polynomial $P_A(t)$.
2. Find the eigenvalues of A.

Solution:

1. The characteristic polynomial is given by:

$$P_A(t) = \det(A - tI) = \det \begin{pmatrix} 4-t & 2 \\ 1 & 3-t \end{pmatrix}.$$

   Compute the determinant:

$$P_A(t) = (4-t)(3-t) - (2)(1) = (12 - 4t - 3t + t^2) - 2 = t^2 - 7t + 10.$$

2. Factorize the polynomial:

$$t^2 - 7t + 10 = (t-2)(t-5).$$

   Therefore, the eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = 5$.

Problem 4: Real vs. Complex Eigenvalues (Theorem 5)

Problem: Explain why every linear operator $T : V \to V$ on a finite-dimensional vector space over $\mathbb{C}$ has an eigenvalue, but a real matrix might not have a real eigenvalue. Use the rotation matrix

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

as an example, and find its eigenvalues when $\theta \neq k\pi$ (with k an integer).

Solution: The characteristic polynomial of $R(\theta)$ is

$$P(t) = \det\left(R(\theta) - tI\right) = \det \begin{pmatrix} \cos\theta - t & -\sin\theta \\ \sin\theta & \cos\theta - t \end{pmatrix}.$$

This expands to:

$$P(t) = (\cos\theta - t)^2 + \sin^2\theta = t^2 - 2\cos\theta\, t + (\cos^2\theta + \sin^2\theta) = t^2 - 2\cos\theta\, t + 1.$$

Using the quadratic formula, we obtain:

$$t = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \cos\theta \pm i\sin\theta.$$

Thus, for $\theta \neq k\pi$, the eigenvalues are complex. This shows that while every operator on a complex vector space has an eigenvalue (by Theorem 5), a real matrix like $R(\theta)$ may not have real eigenvalues because $\mathbb{R}$ is not algebraically closed.

4 Trace of a Matrix

Definition 12. *The **trace** of a square matrix $A \in \mathbb{F}^{n\times n}$ is the sum of its diagonal elements:*

$$\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

Remark 13 (Properties of the Trace)**.** *The trace function* tr *has several important properties:*

- $\mathrm{tr} : \mathbb{R}^{n\times n} \to \mathbb{R}$ *is a linear operator; in particular,*

  $$\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$$

  and for any scalar c, $\mathrm{tr}(cA) = c\,\mathrm{tr}(A)$.

- *It satisfies the cyclic property:*

  $$\mathrm{tr}(A \cdot B) = \mathrm{tr}(B \cdot A).$$

  (Note: In general, $\mathrm{tr}(A \cdot B) \neq \mathrm{tr}(A) \cdot \mathrm{tr}(B)$.)

- *The trace is invariant under a change of basis. That is, if $T \in L(V)$ and U and W are two bases of V, then the matrix representations $M(T, U)$ and $M(T, W)$ satisfy:*

  $$\mathrm{tr}(M(T, U)) = \mathrm{tr}(M(T, W)).$$

- *If A is diagonalizable (or more generally, when considered over $\mathbb{C}$), the trace of A equals the sum of its eigenvalues (counted with their algebraic multiplicities). For example, if*

  $$\widetilde{A} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

  then

  $$\mathrm{tr}(\widetilde{A}) = \lambda_1 + \lambda_2 + \cdots + \lambda_n.$$

- *The trace of A is also equal to the negative of the coefficient of t^{n-1} in the characteristic polynomial*

  $$P_A(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0.$$

  That is, $\mathrm{tr}(A) = -a_{n-1}$.

- *The trace of A equals the sum of its eigenvalues (when they exist)*

- *Finally, the determinant of A equals the product of its eigenvalues (when they exist):*

  $$\det(A) = \lambda_1\lambda_2 \cdots \lambda_n.$$

Consider the 2×2 rotation matrix

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

We compute several properties of $R(\theta)$:

- **Trace:** The trace is the sum of the diagonal elements:

$$\operatorname{tr}(R(\theta)) = \cos\theta + \cos\theta = 2\cos\theta.$$

- **Characteristic Polynomial:** The characteristic polynomial is given by

$$P_{R(\theta)}(t) = \det(R(\theta) - tI).$$

  First, compute

$$R(\theta) - tI = \begin{pmatrix} \cos\theta - t & -\sin\theta \\ \sin\theta & \cos\theta - t \end{pmatrix}.$$

  Then, the determinant is

$$\begin{aligned} P_{R(\theta)}(t) &= (\cos\theta - t)(\cos\theta - t) - (-\sin\theta)(\sin\theta) \\ &= (\cos\theta - t)^2 + \sin^2\theta \\ &= t^2 - 2\cos\theta\, t + (\cos^2\theta + \sin^2\theta) \\ &= t^2 - 2\cos\theta\, t + 1. \end{aligned}$$

- **Eigenvalues:** The eigenvalues of $R(\theta)$ are the roots of the characteristic polynomial

$$t^2 - 2\cos\theta\, t + 1 = 0.$$

  Using the quadratic formula,

$$t = \frac{2\cos\theta \pm \sqrt{(2\cos\theta)^2 - 4\cdot 1\cdot 1}}{2} = \cos\theta \pm i\sin\theta.$$

  Thus, the eigenvalues are $\cos\theta \pm i\sin\theta$; they are complex unless θ is an integer multiple of π.

Problem: Exploring a Rotation Matrix

Problem: Let

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

1. Compute the trace of B.
2. Determine the characteristic polynomial $P_B(t) = \det(B - tI)$.
3. Find the eigenvalues of B.

Solution:

1. **Trace:** The trace of B is the sum of its diagonal entries:

$$\operatorname{tr}(B) = 0 + 0 = 0.$$

2. **Characteristic Polynomial:** First, we form $B - tI$:

$$B - tI = \begin{pmatrix} 0 - t & -1 \\ 1 & 0 - t \end{pmatrix} = \begin{pmatrix} -t & -1 \\ 1 & -t \end{pmatrix}.$$

   Then, compute the determinant:

$$P_B(t) = \det(B - tI) = (-t)(-t) - (-1)(1) = t^2 + 1.$$

3. **Eigenvalues:** The eigenvalues are the roots of the characteristic polynomial:

$$t^2 + 1 = 0.$$

   Solving for t:

$$t^2 = -1 \quad \Rightarrow \quad t = \pm i.$$

   Thus, the eigenvalues of B are i and $-i$.

5 Applications in AI

Eigenvalues and eigenvectors are fundamental in many AI techniques. In this section, we present two simple applications: Principal Component Analysis (PCA) and Spectral Clustering.

5.1 Principal Component Analysis (PCA) in AI

Principal Component Analysis (PCA) is a widely used method for dimensionality reduction in machine learning. It transforms a high-dimensional dataset into a set of linearly uncorrelated variables, called principal components, that capture the most variance in the data.

- **Step 1:** Compute the covariance matrix Σ of the dataset X. For a dataset $X \in \mathbb{R}^{m \times n}$ (with m samples and n features), the covariance matrix is given by:

$$\Sigma = \frac{1}{m-1} X^T X.$$

- **Step 2:** Calculate the eigenvalues and eigenvectors of Σ.
- **Step 3:** The eigenvectors corresponding to the largest eigenvalues capture the directions of maximum variance. Projecting the data onto these eigenvectors reduces the dimensionality while preserving key information.

Example: In an image recognition task, each image is often represented as a high-dimensional vector. By applying PCA, you can reduce the dimensionality of the image data, thereby speeding up the classification process and often reducing noise.

For more details, see: Wikipedia: Principal Component Analysis.

5.2 Spectral Clustering in AI

Spectral clustering groups similar data points using the eigen-decomposition of a graph Laplacian. Briefly:

1. Build a similarity graph (nodes represent data points).
2. Form the Laplacian $L = D - W$, where W is the similarity matrix and D is the degree matrix.
3. Compute the eigenvectors corresponding to the smallest nonzero eigenvalues of L.
4. Cluster the data (e.g., using k-means) on the rows of the eigenvector matrix.

Example: Consider a social network where nodes represent individuals and edges represent friendships. Spectral clustering can identify communities by grouping nodes that are more densely interconnected.

For more details, see: Wikipedia: Spectral Clustering.

6 Bibliography

- Wikipedia: Eigenvalues and Eigenvectors
- Wikipedia: Principal Component Analysis
- Wikipedia: Spectral Clustering