

1 Diagonalization

1.1 Definition

An operator $T \in L(V)$ is **diagonalizable** if there exists a basis of V such that the corresponding matrix is diagonal:

$$M(T) = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

Example: Any diagonal matrix D is diagonalizable because it is similar to itself. For instance,

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = I_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} I_3^{-1}.$$

1.2 Nice Property

The diagonal form is the best since we have the eigenvectors as the basis.

1.3 Proposition

Let V be a finite-dimensional vector space. $A \in L(V)$. Then the following statements are equivalent:

- (P1) A is diagonalizable.
- (P2) The characteristic polynomial P can be decomposed into linear factors, and the algebraic multiplicity of each root of P equals its geometric multiplicity.
- (P3) If $\lambda_1, \dots, \lambda_k$ are the pairwise distinct eigenvalues of A , then

$$V = E(A, \lambda_1) \oplus E(A, \lambda_2) \oplus \cdots \oplus E(A, \lambda_k)$$

Example of a diagonalizable matrix: SYMMETRIC MATRIX (property- all symmetrix matrices are diagonalizable).

*Not all matrices corresponding to linear maps are diagonalizable, in such cases, the next best thing we can hope for is a Triangular Matrix.

2 Triangular Matrices

A matrix is called **upper triangular** if it has the form:

$$\begin{pmatrix} \lambda_1 & * & * & \cdots & * \\ 0 & \lambda_2 & * & \cdots & * \\ 0 & 0 & \lambda_3 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are diagonal elements, all entries to the left of the diagonal are zeros, and the entries marked with * may be non-zero.

If all the entries both to the left and to the right of the diagonal are zeros, i.e., all the * entries are zero, the matrix reduces to a **diagonal matrix**:

$$\begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

*A diagonal matrix has non-zero entries only on its diagonal, with all off-diagonal elements equal to zero.

=> A square matrix whose all elements above the main diagonal are zero is called a lower triangular matrix.

=> A square matrix whose all elements below the main diagonal are zero is called an upper triangular matrix.

2.1 Proposition

Let $T \in L(V)$, and let $B = \{v_1, v_2, \dots, v_n\}$ be a basis. Then the following are equivalent:

(P1) The matrix representation $M(T, B)$ is **upper triangular**.

(P2) For any $j = 1, 2, \dots, n$,

$$T(v_j) \in \text{span}\{v_1, v_2, \dots, v_j\}$$

where v_j is a particular vector from the basis B .

If we apply this linear map to v_1 , we obtain:

$$T(v_1) = \begin{pmatrix} \lambda_1 & a_{12} & a_{13} \\ 0 & \lambda_2 & a_{23} \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ 0 \\ 0 \end{pmatrix} = \lambda_1 v_1$$

This result lies in the span of v_1 , i.e., $T(v_1) \in \text{span}(v_1)$.

Next, applying T to v_2 :

$$T(v_2) = \begin{pmatrix} \lambda_1 & a_{12} & a_{13} \\ 0 & \lambda_2 & a_{23} \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{12} \\ \lambda_2 \\ 0 \end{pmatrix} = a_{12} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = a_{12}v_1 + \lambda_2v_2$$

This result lies in the span of v_1 and v_2 , i.e.,

$$T(v_2) \in \text{span}(v_1, v_2)$$

This process allows us to build a linear operator in a step-by-step manner, where each transformed basis vector $T(v_j)$ is expressed as a linear combination of v_1, v_2, \dots, v_j .

=> Upper Triangular Matrix **always** has eigenvalues on its diagonal

2.2 Proposition

Suppose $T \in L(V)$, where V is a finite-dimensional vector space, and T has an upper triangular form. Then $M(T)$ has an **upper triangular form** for some **basis**.

*In a **COMPLEX FIELD**, every matrix can be expressed as an Upper Triangular matrix.

2.3 Proposition

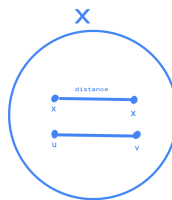
Suppose $T \in L(V)$, where V is a finite-dimensional vector space, and T has an upper triangular form, then the entries on the diagonal are precisely the eigenvalues of T .

3 Metric Spaces

3.1 Definition

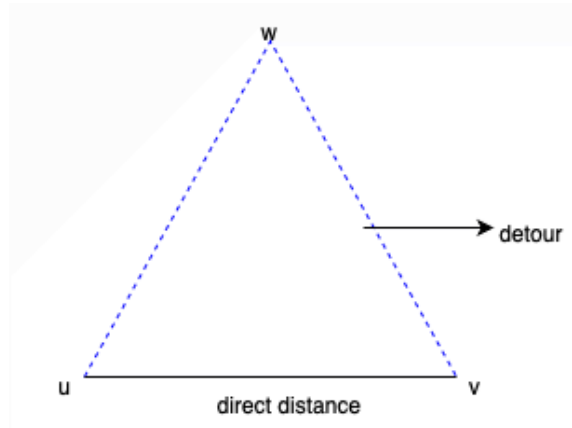
Let X be a set. A function $d : X \times X \rightarrow \mathbb{R}$ is called a **metric** if the following conditions hold:

(P1) $d(u, v) > 0$ if $u \neq v$, and $d(u, u) = 0$.



(P2) $d(u, v) = d(v, u)$ (symmetry).

(P3) $d(u, v) \leq d(u, w) + d(w, v)$ (triangle inequality).



$d(u, v)$ is the direct distance.

\Rightarrow Distance between the points must be positive.

3.2 Definition - Sequences

Sequence is basically an ordered set of elements.

A sequence (x_n) in a metric space (X, d) is called a **Cauchy sequence** if:

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n, m > N, d(x_n, x_m) < \epsilon$$

\Rightarrow Beyond a certain value of N , looking at the points, those points are going to be no further apart than ϵ , i.e., these points will get closer and closer to each other as we go further in the sequence or increase the index of the sequence.

\Rightarrow Every convergent sequence is a Cauchy Sequence.

3.3 Definition - Converge

A sequence $(x_n)_{n \in \mathbb{N}}$ **converges** to a point $x \in X$ if

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall n > N, d(x_n, x) < \epsilon$$

This means the points in the sequence get arbitrarily close to x as n increases.

*The point to which the sequence is converging also belongs to the set.

Notation:

$$x_n \rightarrow x \quad \text{or equivalently} \quad \lim_{n \rightarrow \infty} x_n = x$$

Consider the sequence $(x_n)_{n \in \mathbb{N}} = \frac{1}{n}$ on $X = (0, 1)$ (an open set).

Here, $(x_n)_{n \in \mathbb{N}}$ is a **Cauchy sequence**, but it does **not converge** in X .

(Note: All values are between 0 and 1, but 0 and 1 are not included in the set.)

*The sequence is approaching 0, which is not in X .)

Now consider the sequence $(x_n)_{n \in \mathbb{N}} = \frac{1}{n}$ on $\bar{X} = [0, 1]$ (a closed set).

Here, $(x_n)_{n \in \mathbb{N}}$ is a **Cauchy sequence** that **converges** in \bar{X} to 0.

3.4 Definition - Complete

A metric space is called **complete** if every Cauchy sequence converges.

Example: The set of real numbers \mathbb{R} with the standard metric is a **complete** set.

3.5 Definition - Epsilon ball

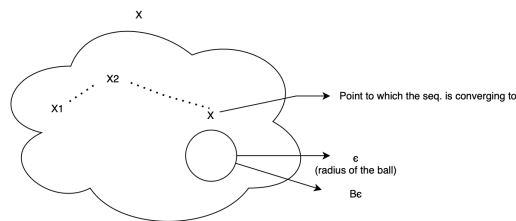
The Epsilon-Ball (ϵ -ball) around a point x is denoted by $B_\epsilon(x)$ and defined as:

$$B_\epsilon(u) := \{x \in X \mid d(x, u) < \epsilon\}$$

Where B_ϵ is the Epsilon-Ball

B_ϵ is defined as a set of points in a set X , such that the distance from x to u is less than (or equal to) ϵ .

This represents the set of all points in X whose distance from u is less than ϵ .



*If we can draw a B_ϵ around a point, then that set is open (does not include the boundary).

* $\epsilon > 0$

***Open Set:** $A \subseteq X$ (A is a subset of X .)

***Boundary:** ∂A (All the points whose distance is less than ϵ)

***Closed Set:** $A \cup \partial A$ (taking an open set and adding the boundary points to it)

3.6 Definition - Closed Set

A set $A \subseteq X$ is called **closed** if every Cauchy sequence in A converges to a limit in A .

**Limit point: the point to which the Cauchy Sequence is converging to

3.7 Definition - Open set

A set $A \subset X$ is called **open** if:

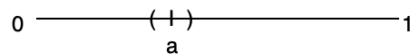
$$\forall a \in A, \exists \epsilon > 0 : B_\epsilon(a) \subset A$$

* Open sets can be bounded too.

3.8 Examples of Sets

=> Closed: $[0,1]$; 0,1, set of all real numbers

=> Open: $(0,1)$



Here, $B = (\epsilon, a + \epsilon)$

=> A set can be neither open nor closed, eg: $[0,1)$

=> A set can be both open and closed

3.9 Definition - Interior Point

Def: A point $a \in A$ is an **interior point** of A if

$$\exists \epsilon > 0, \text{ such that } B_\epsilon(a) \subset A$$

e.g. $A = [0, 1]$, then $x \in (0, 1)$ are **interior points**.

3.10 Definition - Closure

Definition: The (topological) **closure** of a set A is defined as the set of points that can be approximated by Cauchy sequences in A :

$$w \in \overline{A} \iff \forall \varepsilon > 0, \exists z \in A : d(w, z) < \varepsilon$$

Here, we basically take the open set A and its union with all of the boundary points.

Notation: \overline{A} is the closure of A .

* $A \cup \partial A$, is always closed.

3.11 Definition - (Topological) Interior

The **interior** of a set A is defined as the set of interior points of A :

Notation: A°

3.12 Definition - (Topological) Boundary

The **boundary** of a set A is defined as the set:

$$\overline{A} \setminus A^\circ$$

* We simply take the closure and remove the interior.

Examples:

$$X = \{0, 1\}$$

$$\overline{X} = [0, 1] \quad (\text{closure})$$

$$X^\circ = (0, 1) \quad (\text{interior})$$

$$\Rightarrow \text{boundary} \quad \partial X = \overline{X} \setminus X^\circ = \{0, 1\}$$

$$\text{sometimes} \quad \partial X = X \setminus X^\circ = \{0\}$$

3.13 Definition - Dense

A set A is **dense** in X if we can approximate **every** $x \in X$ by a sequence in A .

$$\text{Formally, } \forall x \in X, \forall \varepsilon > 0, B_\varepsilon(x) \cap A \neq \emptyset$$

Example: $\mathbb{Q} \subset \mathbb{R}$ is **dense** in \mathbb{R} .

*Taking any real number, we can have a sequence of rational numbers that can get arbitrarily close to the real number, however the sequence will never reach the real number.

3.14 Definition - Bounded

A set $A \subset X$ is **bounded** if there exists $D > 0$ such that

$$\forall u, v \in A, \quad d(u, v) \leq D$$

* Where D is some scalar

* Taking any 2 pairs of points from set A , then the largest possible distance is smaller than D

* $D > 0$

4 Norms

In general, there are no algebraic operations defined on a metric space, only a distance function. Most of the spaces that arise in analysis are vector, or linear, spaces, and the metrics on them are usually derived from a norm, which gives the "length" of a vector

4.1 Definition

Let V be a vector space. A **norm** on V is a **function** $\| \cdot \| : V \rightarrow \mathbb{R}$ such that for all $x, y \in V$ and $\lambda \in \mathbb{F}$, the following conditions hold:

$$(P1) \quad \|\lambda x\| = |\lambda| \|x\| \quad (\text{homogeneous})$$

$$(P2) \quad \|x + y\| \leq \|x\| + \|y\| \quad (\text{triangle inequality})$$

$$(P3) \quad x = 0 \implies \|x\| = 0 \quad (\text{norm property})$$

$$(P4) \quad \|x\| = 0 \implies x = 0 \quad (\text{vector property})$$

$\| \cdot \|$ is a **semi-norm** if (P1)–(P3) are satisfied.

Intuition: The norm of x represents the "**length of x** " or the **distance** between $(x, 0)$, (where 0 is the origin).

Example:

- **Euclidean norm** on \mathbb{R}^d :

$$\|x\| = \left(\sum_{i=1}^d x_i^2 \right)^{1/2}$$

Each x_i is one of the coordinates of x

- **Manhattan distance:**

$$\|x\| = \sum_{i=1}^d |x_i|$$

Absolute value of the coordinates of x

4.2 P-Norm

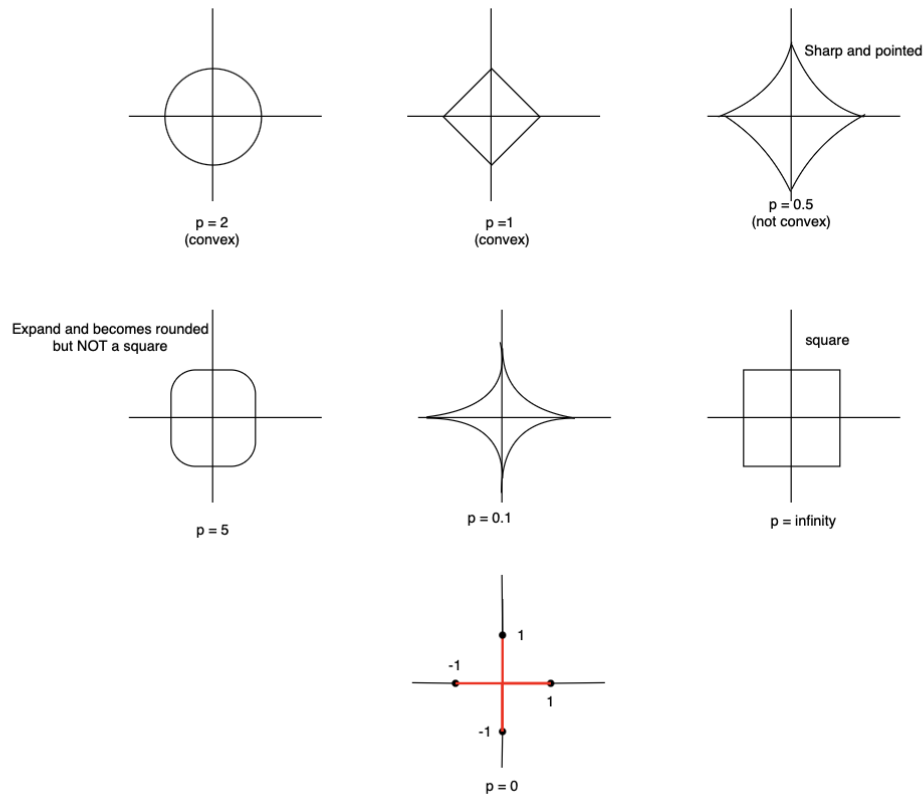
Consider $V = \mathbb{R}^d$. Define:

$$\|x\|_p := \left(\sum_{i=1}^d |x_i|^p \right)^{1/p} \quad \text{for } 0 < p < \infty$$

- $\|\cdot\|_p$ is a **norm** if $p \geq 1$.
- **Unit balls:** The unit ball of a norm is the set of points such that:

$$B_p := \{x \in \mathbb{R}^d \mid \|x\|_p \leq 1\}$$

Examples:



Def:

$$\|x\|_\infty := \max |x_i| \quad (\text{is a norm})$$

This represents the maximum value over the coordinates of the vector.

$$\|x\|_0 := \text{number of non-zero coordinates}$$

This can be written as:

$$\|x\|_0 = \sum_{i=1}^d \mathbf{1}_{\{x_i \neq 0\}}$$

where $\mathbf{1}$ is the indicator function, which equals 1 when the condition $x_i \neq 0$ is satisfied and 0 otherwise.

Note: $\|x\|_0$ is not a norm.

Example:

$$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \|x\|_0 = 1$$

$$\lambda x = \begin{bmatrix} 5 \\ 0 \end{bmatrix}, \quad \|\lambda x\|_0 = 1 \quad (\lambda = 5)$$

$$\|\lambda x\|_0 \neq \lambda \|x\|_0$$

5 References

Diagonalization. (n.d.). <https://textbooks.math.gatech.edu/ila/diagonalization.html>

Triangular Matrix - Lower and upper triangular matrix, examples. (n.d.). Cuemath. <https://www.cuemath.com/algebra/matrix/>

Bert, M. (n.d.). Metric spaces. In Metric Spaces (pp. 93–96). https://www.math.ucdavis.edu/hunter/m125a/intro_analysis