

Lecture 7

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1 Introduction

Metric spaces measure distances and lengths. The inner product extends this idea to vector spaces by defining angles and magnitudes.

2 Inner Product and Hilbert Spaces

Definition 1 Consider a vector space V . A mapping $\langle \cdot, \cdot \rangle: V \times V \rightarrow F$ is called an inner product if it satisfies the following properties:

- (P1): $\langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$
- (P1): $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$ ($\lambda \in F$)
- (P3): $\langle x, y \rangle = \overline{\langle y, x \rangle}$
- (P4): $\langle x, x \rangle \geq 0$
- (P5): $\langle x, x \rangle = 0 \Leftrightarrow x = 0$

Examples:

- Euclidean inner product on \mathbb{R}^n : $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$.
- On \mathbb{C}^n , $\langle x, y \rangle = \sum x_i \overline{y_i}$
- $\mathcal{C}([a, b])$: $\langle f, g \rangle = \int_a^b f(t)g(t)dt$ is an inner product (but space would not be complete)

Definition 2 A vector space with a norm is called a **normed space**. If every Cauchy sequence in the space converges, then it is called a **Banach space**. A vector space with an inner product is called a **pre-Hilbert space**. If it is additionally complete, then V is called a **Hilbert Space**.

Consider a vector space with an inner product $\langle \cdot, \cdot \rangle$. Define $\|\cdot\|: V \rightarrow \mathbb{R}$ as $\|x\| := \sqrt{\langle x, x \rangle}$. Then $\|\cdot\|$ is a norm on V , the norm is induced by $\langle \cdot, \cdot \rangle$. In general, the other way does not work.

Consider a vector space V with norm $\|\cdot\|$. Then $d: V \times V \rightarrow \mathbb{R}$, $d(x, y) := \|x - y\|$ is a metric on V , the metric is induced by the norm. In general, the other direction does not work.

3 Orthogonal Basis and Projection

Definition 3 Consider a pre-Hilbert space V . Two vectors $v_1, v_2 \in V$ are called **orthogonal** if $\langle v_1, v_2 \rangle = 0$

Notation: $v_1 \perp v_2$ Two sets $v_1, v_2 \subset V$ are called orthogonal if $\forall v_1 \in V_1, v_2 \in V_2 : \langle v_1, v_2 \rangle = 0$

Vectors are called orthogonal if additionally the two vectors have norm of 1.

A set of vectors v_1, v_2, \dots, v_n is called orthonormal if any two vectors are orthonormal. For a set $S \subseteq V$ we define its orthogonal complement S^\perp as follows:

$$S^\perp := \{v \in V | v \perp s, \forall s \in S\}$$

4 Orthogonal Projection

Definition 4 $A \in \mathcal{L}(V)$ is called a **projection** if $A^2 = A$.

Theorem 5 Let U be a finite-dim subspace of a pre-Hilbert-space H . Then there exists a linear projection $P_U : H \rightarrow U$, and $\text{Ker}(P_U) = U^\perp$. P_U is then called the **orthogonal projection** of H on U .

Construction: Let v_1, \dots, v_n be an orthogonal basis of U . Define $P_U : V \rightarrow U$ by $P_U(w) = \sum_{i=1}^n \frac{\langle w, v_i \rangle}{\|v_i\|^2} v_i$

Remark 6 In an orthonormal basis v_1, \dots, v_n , the representation of a vector is given by
它这里面写错了, 标准正交基应该是 u_1, u_2, \dots, u_n 才对

$$v = \sum_{i=1}^n \langle v, u_i \rangle u_i$$

5 Gram-Schmidt Orthogonalization

It is a procedure that takes any basis v_1, \dots, v_n of a finite-dim vector space and transforms it into another basis u_1, \dots, u_n that is orthonormal.

Intuition: iterative procedure

Step 1: $u_1 = \frac{v_1}{\|v_1\|}$, $U_1 = \text{span}\{u_1\}$

Step k: Assume that we already have u_1, u_2, \dots, u_{k-1} .

- Project v_k on U_{k-1} and keep "the rest":

$$\tilde{u}_k = v_k - P_{U_{k-1}}(v_k)$$

- Normalize:

$$u_k = \frac{\tilde{u}_k}{\|\tilde{u}_k\|}$$

In practice, use Householder reflection for a numerically stable orthogonalization.

6 Orthogonal Matrices

Definition 7 Let $Q \in \mathbb{R}^{n \times n}$ be a matrix with orthonormal column vectors (w.r.t Euclidean inner product). Then Q is called an orthogonal matrix.

If $Q \in \mathbb{C}^n$ and the columns are orthonormal (w.r.t the standard inner product on \mathbb{C}), then it is called unitary.

Examples:

- Identity:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Reflection:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- Permutation:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- Rotation:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

- Rotation in \mathbb{R}^3 :

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

- General rotation can be written as a product of "elementary" rotation.

Properties of orthogonal matrix Q :

- columns are orthogonal \Leftrightarrow rows are orthogonal.
- Q is always invertible, and $Q^{-1} = Q^T$
- Q realizes an isometry: $\forall v \in V : \|Qv\| = \|v\|$.
- Q preserves angles: $\langle Qu, Qv \rangle = \langle u, v \rangle, \forall u, v \in V$
- $|\det(Q)| = 1$

The respective properties also holds for unitary matrices U . ($U^{-1} = \bar{U}^T$)

Theorem 8 Let $S \in \mathcal{L}(V)$ for a real vector space V . Then the following are equivalent:

- S is an isometry: $\|Sv\| = \|v\|, v \in V$.

- There exists an orthonormal basis of V such that the matrix of S has the following form:

$$A = \begin{bmatrix} B_1 & 0 & 0 & \cdots & 0 \\ 0 & B_2 & 0 & \cdots & 0 \\ 0 & 0 & B_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & B_n \end{bmatrix}$$

where each B is a block, they are either a real number with value 1 or -1, or a 2×2 rotation matrix.