Первое задание:

№1

X_n - выборка:

$\xi \sim R(0,\theta)$

1) $\tilde{\theta}_1 = 2\bar{X} = \frac{2}{n}\sum_{i=1}^{n} x_i$

2) $\tilde{\theta}_2 = X_{min}$

3) $\tilde{\theta}_3 = X_{max}$

4) $\tilde{\theta}_4 = X_{min} + X_{max}$

5) $\tilde{\theta}_5 = X_1 + \frac{\sum_{k=2}^{n} X_k}{(n-1)}$

① $M\tilde{\theta}_1 = M\left[\frac{2}{n}\sum_{i=1}^{n} x_i\right] = \frac{2}{n}\sum M x_i =$

$x_i \sim R(0,\theta)$: $M\xi = \frac{\theta}{2}$ $= \frac{2}{n} n \frac{\theta}{2} =$

$= \theta$ - ~~несов~~ несмещенная

$D[\tilde{\theta}_1] = D\left[\frac{2}{n}\sum x_i\right] = \frac{4}{n^2}\sum D x_i = \frac{4}{n^2} n D\xi = \frac{\theta^2}{3n} \xrightarrow[n\to\infty]{} 0$

$D\xi = \frac{(b-a)^2}{12}$ - состоятельна

② $M\tilde{\theta}_2 = M[X_{min}]$: $\min(\xi_1 \dots \xi_n) \sim 1-(1-F(\xi))^n$

$\varphi(y) = n(1-F(y))^{n-1} p(y) =$

$= n\left(1-\frac{y}{\theta}\right)^{n-1}\frac{1}{\theta}$ $(0,\theta)$

$M[X_{min}] = \int_0^\theta y n \frac{1}{\theta}\left(1-\frac{y}{\theta}\right)^{n-1} dy =$

$= \left|t = \frac{y}{\theta}\right| = \int_0^1 t\theta n (1-t)^{n-1} dt = n\theta B(2,n) =$

$= n\theta \frac{\Gamma(2)\Gamma(n)}{\Gamma(n+2)} = \frac{\theta}{n+1}$ - смещена, не асс. несмещ.

Исправляем $\rightarrow \theta_2' = (n+1) X_{min}$ - несмещ.

$M[X_{min}^2] = \int_0^\theta y^2 n\left(1-\frac{y}{\theta}\right)^{n-1}\frac{1}{\theta} dy = \int_0^1 \theta^2 t^2 n(1-t)^{n-1} dt$

$= \theta^2 n B(3,n) = \frac{2\theta^2}{(n+2)(n+1)}$

$D[X_{min}] = \frac{2\theta^2}{(n+2)(n+1)} - \frac{\theta^2}{(n+1)^2} = \frac{\theta^2[(n+1)2 - n - 2]}{(n+1)^2(n+2)} =$

$$= \frac{\Theta^2 n}{(n+1)^2(n+2)}$$

~~состоятельн~~

$$D[\Theta_2'] = D[(n+1)X_{min}] = \frac{\Theta^2 n}{n+2} \underset{n\to\infty}{\not\to} 0$$

Проверка: Определено: $\tilde{\Theta}_2' \xrightarrow{P} \Theta$

$$\Theta_2' = (n+1)X_{min}; \quad \forall \varepsilon > 0 \quad P(|\Theta_2' - \Theta| \geq \varepsilon) \xrightarrow[n\to\infty]{} 0$$

$$P(|\Theta_2' - \Theta| \geq \varepsilon) \geq P(\Theta_2' \geq \Theta + \varepsilon) = P\left(X_{min} \geq \frac{\Theta+\varepsilon}{n+1}\right)$$

$$= P(X_1 \geq \ldots, X_2 \geq \ldots, X_n \geq \ldots) = \prod_{i=1}^{n} P\left(X_i \geq \frac{\Theta+\varepsilon}{n+1}\right) =$$

$$= \left(1 - F\left(\frac{\Theta+\varepsilon}{n+1}\right)\right)^n = \left(1 - \frac{\Theta+\varepsilon}{\Theta(n+1)}\right)^n \xrightarrow[n\to\infty]{} e^{-\frac{\Theta+\varepsilon}{\Theta}} > 0$$

↑ несостоятельна

③ $\tilde{\Theta}_3 = X_{max}$: $X_i \sim F(x)$; $X_{max} \sim (F(y))^n$

$$\rho_{X_{max}}(y) = n(F(y))^{n-1}\rho(y) \quad (0, \Theta)$$

$$M[\tilde{\Theta}_3] = \int_0^\Theta y\, n\left(\frac{y}{\Theta}\right)^{n-1}\frac{1}{\Theta}\,dy = \frac{n}{\Theta^n}\int_0^\Theta y^n\,dy = \frac{n\Theta}{n+1}$$

– смещена, асс. несмещ.

$$\Theta_3' = \frac{n+1}{n}X_{max} \text{ – несмещ.}$$

$$M[\tilde{\Theta}_3^2] = \int_0^\Theta y^2 n\left(\frac{y}{\Theta}\right)^{n-1}\frac{1}{\Theta}\,dy = \frac{n}{\Theta^n}\int_0^\Theta y^{n+1}\,dy = \frac{\Theta^2 n}{n+2}$$

$$D[\tilde{\Theta}_3] = \frac{\Theta^2 n}{n+2} - \frac{n^2\Theta^2}{(n+1)^2} = \Theta^2\frac{n}{(n+2)(n+1)^2}$$

$$D[\Theta_3'] = \frac{(n+1)^2}{n^2}\frac{n\Theta^2}{(n+2)(n+1)^2} = \frac{\Theta^2}{n(n+2)} \xrightarrow[n\to\infty]{} 0$$

$\Rightarrow$ состоятельна

④ $\tilde{\Theta}_4 = X_{min} + X_{max}$

$$M[\tilde{\Theta}_4] = M[X_{min}] + M[X_{max}] = \frac{\Theta}{n+1} + \frac{n\Theta}{n+1} = \Theta$$

несмещ.

$$D[\tilde{\Theta}_4] = D[X_{min}] + D[X_{max}] + 2\operatorname{cov}(X_{max}, X_{min})$$

$$\operatorname{cov}(X_{max}, X_{min}) = M[X_{min}X_{max}] - M[X_{min}]\cdot M[X_{max}]$$

$$p(y,z) = n(n-1)(F(z)-F(y))^{n-2}p(z)p(y)$$

$$M[X_{min}X_{max}] = \iint_D yz\cdot n(n-1)\left(\frac{z}{\Theta}-\frac{y}{\Theta}\right)^{n-2}\frac{1}{\Theta}\frac{1}{\Theta}\,dz\,dy$$

$$= \int_0^\Theta dz\int_0^z yz(n-1)n\left(\frac{z}{\Theta}-\frac{y}{\Theta}\right)^{n-2}\frac{1}{\Theta^2}\,dy = \left\{t=\frac{y}{z}\right\} =$$

$$= \frac{n(n-1)}{\Theta^n}\int_0^\Theta dz\int_0^1 t z^2 (z-tz)^{n-2}\,dt =$$

$$= \frac{n(n-1)}{\Theta^n}\int_0^\Theta z^{n+1}\,dz\int_0^1 t(1-t)^{n-2}\,dt =$$

$$= \frac{n(n-1)}{\Theta^n}\int_0^\Theta z^{n+1}\frac{1}{n(n-1)}\,dz = \frac{\Theta^{n+2}}{(n+2)\Theta^n} = \frac{\Theta^2}{n+2}$$

$$\operatorname{cov}(X_{min}, X_{max}) = \frac{\Theta^2}{n+2} - \frac{\Theta}{n+1}\cdot\frac{n\Theta}{n+1} = \frac{\Theta^2}{(n+2)(n+1)^2}$$

$$D[\tilde{\Theta}_4] = \frac{\Theta^2 n}{(n+1)^2(n+2)} + \frac{\Theta^2 n}{(n+1)^2(n+2)} + \frac{2\Theta^2}{(n+2)(n+1)^2} =$$

$$= \frac{2\Theta^3}{(n+1)(n+2)} \xrightarrow[n\to\infty]{} 0$$ состоятельная.

5. $M[\tilde{\Theta}_5] = M[X_1] + \frac{n}{n-1} \cdot \frac{n}{n-1} M[X_2] =$

$= \frac{\Theta}{2}\left(1 + \frac{n-1}{n-1}\right) = \Theta$; $\quad x_1 \sim R(0,\Theta)$: $M[X_i] = \frac{\Theta}{2}$

несмещ.

$D[\tilde{\Theta}_5] = D[X_1] + \frac{(n-1)}{(n-1)^2} D[X_2] = \frac{\Theta^2}{12}\left(1 + \frac{1}{n-1}\right) \underset{n\to\infty}{\not\to} 0$

$X_1 + \frac{1}{n-1}\sum_{m=2}^{n} X_m \xrightarrow{P} X_1 + \frac{\Theta}{2}$ — не явл. состоят.

Эффективность: Θ_2 и Θ_5 — выкинули, Θ_3 тоже

Исследуем $\tilde{\Theta}_1, \tilde{\Theta}_3', \tilde{\Theta}_4$: $D\tilde{\Theta}_1 = \frac{\Theta^2}{3n}$; $D\tilde{\Theta}_3' = \frac{\Theta^2}{n(n+2)}$;

$D\tilde{\Theta}_4 = \frac{2\Theta^2}{(n+1)(n+2)}$

1–3: $\frac{\Theta^2}{3n} > \frac{\Theta^2}{n(n+2)}$ для $n > 1$

3–4: $\frac{\Theta^2}{n(n+2)} < \frac{2\Theta^2}{(n+1)(n+2)}$ для $n > 1$ $\Rightarrow \tilde{\Theta}_3'$ лучше

№2

a) $p(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$; $\quad \gamma = \frac{\tilde{M}_3}{(\tilde{m}_2)^{\frac{3}{2}}}$

$\alpha_1 = \int_0^\infty x e^{-x} dx = -xe^{-x}\Big|_0^\infty + \int_0^\infty e^{-x} dx = -e^{-x}\Big|_0^\infty = 1$

~~$m_2 = \frac{1}{n}\sum(x_i - \bar{x})^2$~~

$m_2 = \int_0^\infty (x-1)^2 e^{-x} dx = \int_0^\infty x^2 e^{-x} dx - 2\int_0^\infty x e^{-x} dx + \int_0^\infty e^{-x} dx = 2 - 2 + 1 = 1$

$$\mu_3 = \int_0^\infty (x-1)^3 e^{-x}dx = \int_0^\infty x^3 e^{-x}dx - 3\int_0^\infty x^2 e^{-x}dx$$

$$+ 3\int_0^\infty x e^{-x}dx - \int_0^\infty e^{-x}dx = 6 - 3\cdot 2 + 3\cdot 1 - 1 = 2$$

$$\gamma = \frac{\mu_3}{(\mu_2)^{\frac{3}{2}}} = 2$$

c) Ядерная оценка плотности:

$$h = 2{,}344 \frac{S}{\sqrt[5]{n}}; \quad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

$$\mu_2 = \frac{1}{n}\sum (x_i - \bar{x})^n, \quad \mu_2 n = \sum_{i=1}^{n}(x_i - \bar{x})^2$$

$$S^2 = \frac{\mu_2 n}{n-1}; \quad S = \sqrt[2]{\frac{\mu_2 n}{n-1}}$$

$$\tilde{p}(z) = \frac{1}{nh}\sum_{i=1}^{n} q\left(\frac{z - x_i}{h}\right) = \frac{1}{nh}\sum_{i=1}^{n}\left[\frac{3}{4}\left(1 - \left(\frac{z-x_i}{h}\right)^2\right)\right]$$

$$\tilde{p}(z) = \frac{3}{4nh}\sum_{i=1}^{n}\left(1 - \left(\frac{z-x_i}{h}\right)^2\right) = \cancel{\frac{3}{4nh}\sum_{i=1}^{n}\left(\frac{h-z+x_i}{h}\right)^2}$$

d) ЦПТ: $\sqrt{n}\dfrac{\tilde{\alpha}_n - \alpha_n}{\sqrt{\tilde{\alpha}_{2n} - \tilde{\alpha}_n^2}} \rightsquigarrow N(0,1)$

$$\tilde{\alpha}_1 = \frac{1}{n}\sum_{i=1}^{n} x_i; \quad \tilde{\alpha}_2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2;$$

$$\alpha_1 = \int_0^{+\infty} x e^{-x}dx; \quad (\tilde{\alpha}_1 - \alpha_1) \rightsquigarrow \frac{\sqrt{\tilde{\alpha}_2 - \tilde{\alpha}_1^2}}{\sqrt{n}} N(0,1)$$

$$\tilde{\alpha}_1 - \alpha_1 \to N\left(0, \frac{\sqrt{\alpha_2 - \alpha_1^2}}{n}\right); \quad \alpha_1 \to N\left(\tilde{\alpha}_1, \frac{\sqrt{\alpha_2 - \alpha_1^2}}{n}\right)$$

№3

$$\xi \sim p(x) = \begin{cases} \dfrac{e^{-\frac{x}{\theta}}}{\theta}, & x \geq 0 \\ 0, & x < 0 \end{cases}; \quad \theta > 0$$

$$F(x) = 1 - e^{-\frac{x}{\theta}}$$

$n = 3$: $\overline{X}_n = (X_1, \dots, X_n)$ – выборка.

$\overline{X}_3 = (X_1, X_2, X_n)$

$$\tilde{\theta}_1 = \overline{X}; \quad \tilde{\theta}_2 = \frac{X_{min} + X_{max}}{2}; \quad \tilde{\theta}_3 = X_{(2)}$$

0) $\xi \sim p(x)$ $\quad M[\xi] = \int_0^\infty x \cdot \frac{e^{-\frac{x}{\theta}}}{\theta} dx = \frac{1}{\theta} \int_0^\infty x e^{-\frac{x}{\theta}} dx =$

$= \frac{1}{\theta} \cdot \theta^2 = \theta$

$$M[\xi^2] = \frac{1}{\theta} \int_0^\infty x^2 e^{-\frac{x}{\theta}} dx = \frac{1}{\theta} \cdot 2 \cdot \theta^3 = 2\theta^2$$

$$D[\xi] = M[\xi^2] - M[\xi]^2 = \theta^2$$

a) $M[\tilde{\theta}_1] = M\left[\frac{1}{n} \sum X_i\right] = \frac{1}{n} M\left[\sum X_i\right] = \frac{1}{n} \cdot n M[\xi] =$

$= \theta$ несмещ.

$\cancel{D[\tilde{\theta}_1] = D\left[\frac{1}{n} \sum X_i\right] = \frac{1}{n^2} n \sum D[X_i] = \frac{\theta^2}{n}}$

, 50 деталей с точны... ...В первой партии оказалось ... деталей в двух партиях по 10...

$$M[\tilde{\Theta}_2] = M\left[\frac{X_{min}+X_{max}}{2}\right] = \frac{1}{2}\left[M[X_{min}] + M[X_{max}]\right] =$$

$$= \frac{1}{2}\left[\frac{\Theta}{n+1} + \frac{n\Theta}{n+1}\right] = \frac{\Theta}{2} \neq \Theta$$ - смещено

Исправлено $\Rightarrow$ $\tilde{\Theta}_2' = (X_{min} + X_{max})$

$= \frac{1}{2}$

min: $\Phi(y) = 1 - (1 - F(y))^n$

$$\varphi(y) = n\,(1 - F(y))^{n-1} p(y)$$

$$\varphi(y) = n\left(1 - (1 - e^{-\frac{x}{\theta}})\right)^{n-1} \frac{e^{-\frac{x}{\theta}}}{\theta} = \frac{n}{\theta} e^{-\frac{xn}{\theta}}$$

$$M[\tilde{\Theta}_2] = \int_0^\infty x \frac{n}{\theta} e^{-\frac{xn}{\theta}} dx = \frac{\theta}{n}$$

$\cancel{\tilde{\Theta}_2' = \tilde{\Theta}_2 \cdot n = \frac{n}{2}(X_{min} + X_{max})}$

max: $\varphi(y) = n(1 - e^{-\frac{x}{\theta}})^{n-1} \frac{1}{\theta} e^{-\frac{x}{\theta}}$

$M[$

$M[\tilde{\Theta}_3]$: $\tilde{\Theta}_3 = X_{(2)}$: $\tilde{X}_k \sim n C_{n-1}^{k-1} \frac{e^{-\frac{x}{\theta}}}{\theta} (1 - e^{-\frac{x}{\theta}})^{k-1} (e^{-\frac{x}{\theta}})^{n-k}$

$n = 2$: $\tilde{X}_{(2)} \sim n C_{n-1}^{1} \frac{e^{-\frac{x}{\theta}}}{\theta} (1 - e^{-\frac{x}{\theta}})^1 (e^{-\frac{x}{\theta}})^{n-2} =$

$$= \frac{n(n-1)}{\theta} (1 - e^{-\frac{x}{\theta}}) (e^{-\frac{x}{\theta}})^{n-1},$$

$$M[\tilde{\Theta}_{(2)}] = \frac{n(n-1)}{\Theta}\int_0^\infty x\left(1-e^{-\frac{x}{\Theta}}\right)e^{-\frac{x}{\Theta}(n-1)}dx =$$

$$= \frac{n(n-1)}{\Theta}\left(\int_0^\infty x e^{-\frac{x}{\Theta}(n-1)}dx + \int_0^\infty x e^{-\frac{x}{\Theta}n}dx\right) =$$

$$= \frac{n(n-1)}{\Theta}\left(\frac{\Theta^2}{(n+1)^2} - \frac{\Theta^2}{n^2}\right) = \frac{n\Theta}{n-1} - \frac{(n-1)\Theta}{n} = \frac{2n-1}{n(n-1)}\Theta$$

// смещена

$$\tilde{\Theta}_3^{1} = \frac{n(n-1)}{2n-1}\bar{X}_{(2)}$$ - несмещ.

б) Нер-во Крамера-Рао: $D[\tilde{\Theta}] \geq \frac{1}{nJ(\Theta)}$