$= \Theta \cdot M[\ldots$

$D[\tilde{\Theta}_1] \geq \frac{1}{n\frac{1}{\Theta^2}} = \frac{\Theta^2}{n} = D[\tilde{\Theta}_1]$

$\tilde{r}$ – эффективн

№4

$\xi \sim R(\Theta; 2\Theta)$; $\rho(x,\Theta) = \frac{1}{\Theta} \ (\Theta, 2\Theta)$

$$F(x) = \begin{cases} 0, & x < \Theta \\ \frac{x-\Theta}{\Theta}, & \Theta \leq x \leq 2\Theta \\ 1, & x > 2\Theta \end{cases}$$

o) $M[\xi] = \int_{\Theta}^{2\Theta} x \cdot \frac{1}{\Theta} dx = \frac{1}{\Theta}\left(\frac{x^2}{2}\right)\Big|_{\Theta}^{2\Theta} = \frac{3\Theta}{2}$

$M[\xi^2] = \int_{\Theta}^{2\Theta} x^2 \frac{1}{\Theta} dx = \frac{7\Theta^2}{3}$

$D[\xi] = \frac{7\Theta^2}{3} - \frac{9\Theta^2}{4} = \frac{1\Theta^2}{12}$

a) ОММ: $\alpha_1 = \int_{\Theta}^{2\Theta} x \cdot \frac{1}{\Theta} dx = \frac{3\Theta}{2}; = \bar{X}$

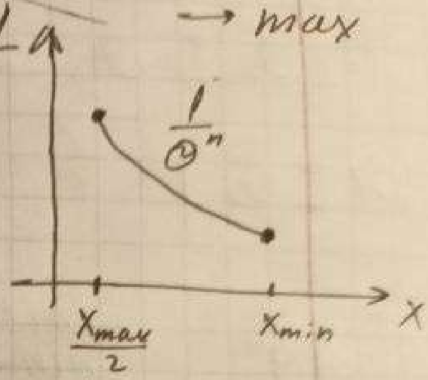
$\tilde{\Theta}_1 = \frac{2}{3}\bar{X}$

ОМП: $L(\theta) = \prod \rho(x_i,\theta) = \frac{1}{\theta^n}(\theta < x_i < 2\theta) =$ $\to$ max

$= \frac{1}{\theta^n}(\theta \le X_{min} \le X_{max} \le 2\theta)$

$(\frac{X_{max}}{2} \le \theta \le X_{min})$

L - max при $\tilde{\theta}_2 = \frac{X_{max}}{2}$



б) $\tilde{\theta}_1 = \frac{2}{3}\bar{X}$; $\tilde{\theta}_2 = \frac{X_{max}}{2}$; $\tilde{\theta}_3 = \frac{1}{5}(X_{min} + 2X_{max})$

$\tilde{\theta}_1$: $M[\tilde{\theta}_1] = \frac{2}{3}M[\bar{X}] = \frac{2}{3}M[\frac{1}{n}\sum x_i] = \frac{2}{3}\cdot\frac{1}{n}\cdot n M[x_i] = \frac{2}{3}\cdot\frac{3}{2}\theta = \theta$ — несмещ.

$D[\tilde{\theta}_1] = D[\frac{2}{3}\bar{X}] = \frac{4}{9}D[\bar{X}] = \frac{4}{9}\cdot\frac{1}{n^2}D[\sum x_i] = \frac{4}{9}\cdot\frac{1}{n^2}\cdot n\cdot\frac{\theta^2}{12} = \frac{\theta^2}{27n} \xrightarrow[n\to\infty]{} 0$ состоят.

$\tilde{\theta}_2$: $(\xi_1 \ldots \xi_n) \sim n[F(y)]^{n-1}$

$\varphi(y) = n[F(y)]^{n-1}\rho(y) = n\left[\frac{x-\theta}{\theta}\right]^{n-1}\cdot\frac{1}{\theta} = n\frac{(x-\theta)^{n-1}}{\theta^n}$

$M[\tilde{\theta}_2] = \frac{1}{2}M[X_{max}] = \frac{1}{2}\int_{\theta}^{2\theta} x n (x-\theta)^{n-1}\frac{1}{\theta^n}dx =$

$= \frac{n}{2\theta^n}\int_{\theta}^{2\theta} x(x-\theta)^{n-1}dx = \frac{n}{2\theta^n}\cdot\frac{(2n+1)\theta^{n+1}}{n(n+1)} = \frac{2n+1}{2(n+1)}\theta$ с смещ.

$\tilde{\theta}_2^{)} = \frac{2(n+1)}{2n+1}\tilde{\theta}_2 = \frac{n+1}{2n+1}X_{max}$. — несмещ.

$$D[\tilde{\Theta}_2'] = \left(\frac{2n+2}{2n+1}\right)^2 D[X_{max}];$$

$$M[\tilde{\Theta}_2'] = \Theta; \quad \cancel{M[\tilde{\Theta}_2'^2] = \left(\frac{n+1}{2n+1}\right)^2 M[X_{max}^2]}$$

$$M[\tilde{\Theta}_2^2] = \frac{1}{4} M[X_{max}^2] = \frac{1}{4}\int_{\Theta}^{2\Theta} x^2 n (x-\Theta)^{n-1} \frac{1}{\Theta^n}\,dx =$$

$$= \frac{n}{4\Theta^n} \cdot \frac{2(2n(n+2)+1)\Theta^{n+2}}{n(n+1)(n+2)} = \frac{(2n(n+2)+1)\Theta^2}{2(n+1)(n+2)}$$

$$M[\tilde{\Theta}_2'^2] = \frac{(n+1)^2}{(2n+1)^2} \cdot \frac{(2n(n+2)+1)\Theta^2}{2(n+1)(n+2)}$$

$$D[\tilde{\Theta}_2'] = \frac{n\Theta^2}{(n+2)(2n+1)^2} \xrightarrow[n\to\infty]{} 0$$ — состоятельна

$\tilde{\Theta}_3$: $\tilde{\Theta}_3 = \frac{1}{5}[X_{min} + 2X_{max}]$

$$M[\tilde{\Theta}_3] = \frac{1}{5} M[X_{min}] + \frac{2}{5} M[X_{max}] =$$

$$= \frac{1}{5}\left[n\Theta\left(\frac{2}{n} - \frac{1}{n+1}\right)\right] + \frac{2}{5}\left[n\Theta\left(\frac{1}{n+1} + \frac{1}{n}\right)\right] = \Theta\,\frac{5n+4}{5n+5}$$

$$\tilde{\Theta}_3' = \frac{5(n+1)}{5n+4}\tilde{\Theta}_3 = \frac{(n+1)}{5n+4}(X_{min} + 2X_{max})$$ несмещ

$$D[\tilde{\Theta}_3] = \frac{1}{25} D[X_{min}] + \frac{4}{25} D[X_{max}] + 2\,\mathrm{cov}(X_{min}, X_{max})$$

$$D[X_{min}] = D[X_{max}] = \frac{n\Theta^2}{(n+2)(n+1)^2}$$

$$cov(X_{min}, X_{max}) = M[X_{min} X_{max}] - M[X_{min}] M[X_{max}]$$

$$M[X_{min} X_{max}] = \iint_D y x \cdot n(n-1)\, p(x)\, p(y) [F(y) - F(x)]^{n-2} =$$

$$= n(n-1)\frac{1}{\Theta^n} \int_{\Theta}^{2\Theta}\int_{x}^{2\Theta} x\, y (y-x)^{n-2} dx\, dy =$$

$$= \frac{n(n-1)}{\Theta^n} \int_{\Theta}^{2\Theta} x \left[\int_x^{2\Theta} y(y-x)^{n-2} dy \right] dx =$$

$$= \frac{n(n-1)}{\Theta^n} \int_{\Theta}^{2\Theta} x \left[\frac{(2\Theta - x)^n}{x} + \frac{x(2\Theta - x)^{n-1}}{n-1} \right] dx =$$

$$= \frac{n(n-1)}{\Theta^n} \left[\frac{\Theta^{n+1}}{n+1} + \ldots \right] = \Theta^2 \left(2 + \frac{1}{n+2} \right)$$

$$cov(X_{min}, X_{max}) = \Theta^2 \left(2 + \frac{1}{n+2} \right) - \left[n\Theta \left(\frac{1}{n+1} + \frac{1}{n} \right) \cdot n\Theta \left(\frac{2}{n} - \frac{1}{n+1} \right) \right]$$

$$= \frac{\Theta^2 (2n-1)}{(n+2)(n+1)^2}$$

$$D[\tilde{\Theta}_3] = \frac{1}{25} \cdot \frac{n\Theta^2}{(n+2)(n+1)^2} + \frac{4}{25} \cdot \frac{n\Theta^2}{(n+2)(n+1)^2} +$$

$$+ 2 \cdot \frac{2}{25} \cdot \frac{(2n-1)\Theta^2}{(n+2)(n+1)^2} = \frac{(13n-4)\Theta^2}{25(n+2)(n+1)^2}$$

$$D[\tilde{\Theta}_3'] = \frac{25(n+1)^2}{(5n+4)^2} \cdot \frac{(13n-4)\Theta^2}{25(n+2)(n+1)^2} = \frac{(13n-4)\Theta^2}{(n+2)(5n+4)^2} \xrightarrow[n \to \infty]{} 0$$

— состоятельна

c) $\tilde{\Theta}_1 = \frac{2}{3}\bar{X}$; $\mathcal{D}[\tilde{\Theta}_1] = \frac{\Theta^2}{27n}$

$\tilde{\Theta}_2 = \frac{n+1}{2n+1} X_{max}$; $\mathcal{D}[\tilde{\Theta}_2'] = \frac{n\Theta^2}{(n+2)(2n+1)^2}$

$\tilde{\Theta}_3 = \frac{n+1}{5n+4}\left(X_{min} + 2X_{max}\right)$; $\mathcal{D}[\tilde{\Theta}_3'] = \frac{(13n-4)}{(n+2)(5n+4)}\Theta^2$

При $n \geq 4$ – эффективна $\tilde{\Theta}_2$
при $n \to \infty$ – аналогично.

d) $X_i \sim R(\Theta, 2\Theta)$

$y_i = \frac{X_i}{\Theta} - 1 \sim R(0,1)$ $\quad y_n = \frac{X_n}{\Theta} - 1$

$F_y(y) = y^n \; [0,1] \quad p(y) = ny^{n-1} \; [0,1]$

$$P\left(t_1 < \frac{X_n}{\Theta} - 1 < t_2\right) = P\left(q_{\frac{1-\beta}{2}} < \frac{X_n}{\Theta} - 1 < q_{\frac{1+\beta}{2}}\right) =$$

$$P\left(\frac{X_n}{t_2+1} < \Theta < \frac{X_n}{t_1+1}\right) \geq \beta$$

t_1: $P(0 < y_n < t_1) = F_y(t_1) - F_y(0) = t_1^n - 0 = \frac{1-\beta}{2}$

$t_1 = \sqrt[n]{\frac{1-\beta}{2}}$

t_2: $t_2 = \sqrt[n]{\frac{1+\beta}{2}}$;

$$P_\Theta\left(\frac{X_n}{\sqrt[n]{\frac{1+\beta}{2}}+1} < \Theta < \frac{X_n}{\sqrt[n]{\frac{1-\beta}{2}}+1}\right) \geq \beta$$

e) ОММ: $\tilde{\Theta} = \frac{2}{3}\overline{X}$

$$\frac{f(\tilde{\lambda}) - f(\lambda)}{\sigma(\lambda)}\sqrt{n} \rightsquigarrow N(0,1)$$

$$f(\tilde{\lambda}) = \frac{2}{3}\lambda_1 = \Theta \; ; \quad f'(\lambda) = \frac{2}{3} \; ; \; K_{11} = \lambda_2 - \lambda_1^2$$

$$\sigma(\lambda) = \sqrt{\frac{2}{3}(\lambda_2 - \lambda_1^2)\frac{2}{3}} = \sqrt{\frac{4}{9}(\lambda_2 - \lambda_1^2)}$$

$$\frac{\tilde{\Theta} - \Theta}{\sqrt{\frac{4}{9}(\tilde{\lambda}_2 - \tilde{\lambda}_1^2)}}\sqrt{n} \rightsquigarrow N(0,1)$$

$$-1{,}96 < \frac{\tilde{\Theta} - \Theta}{\sqrt{\frac{4}{9}(\tilde{\lambda}_2 - \tilde{\lambda}_1^2)}} < 1{,}96$$

$$-1{,}96 \cdot \frac{2}{3}\sqrt{\tilde{\lambda}_2 - \tilde{\lambda}_1^2} + \tilde{\Theta} < \Theta < 1{,}96 \cdot \frac{2}{3}\sqrt{\tilde{\lambda}_2 - \tilde{\lambda}_1^2} + \tilde{\Theta}$$

с ОМП такие подходы

№5

$$p(x) = \begin{cases} \frac{\Theta - 1}{x^{\Theta}} & ; x \geq 1 \\ 0 & ; x < 1 \end{cases} \quad \Theta > 1 \; ; F(x) = 1 - x^{1-\Theta}$$

a) $L(\vec{x}, \Theta) = \prod p(x_i, \Theta)$

$$L(\Theta) = \prod_{i=1}^{n}\left(\frac{\Theta - 1}{x_i^{\Theta}}\right) = (\Theta - 1)^n \prod_{i=1}^{n}(x_i)^{-\Theta} \to \max$$

$$\ln L(\Theta) = n\ln(\Theta - 1) - \Theta\sum_{i=1}^{n}\ln x_i \to \max$$

$$\frac{\partial \ln L}{\partial \Theta} = \frac{n}{\Theta-1} - \sum_{i=1}^{n} \ln x_i = 0; \quad \tilde{\Theta} = 1 + \frac{n}{\sum_{i=1}^{n} \ln x_i}$$

Проверка: $\frac{\partial^2 \ln L}{\partial \Theta^2} = -\frac{n}{(\Theta-1)^2}\Big|_{\Theta=\tilde{\Theta}} = -\left(\sum_{i=1}^{n} \ln x_i\right)^2 < 0$

точка max

б) ОМП Асимптотика:

$~~\frac{f(\tilde{\lambda}) - f(\lambda)}{\sigma(\lambda)}~~$ $$\frac{f(\tilde{\Theta}) - f(\Theta)}{\sigma(\Theta)}\sqrt{n} \rightsquigarrow N(0,1)$$

$$\sigma(\Theta) = \sqrt{\nabla^T f(\Theta) \; J^{-1}(\Theta) \; \nabla f(\Theta)}$$

$f(\Theta) = X_{med}$

$$F(X_{med}) = \int_{-\infty}^{X_{med}} \frac{\Theta-1}{t^{\Theta}}\, dt = X_{med}^{1-\Theta} = \frac{1}{2}; \quad X_{med} = 2^{\frac{1}{\Theta-1}}$$

$$f'(\Theta) = 2^{\frac{1}{\Theta-1}} \ln 2 \cdot \frac{-1}{(1-\Theta)^2} = -\ln 2 \cdot 2^{\frac{1}{\Theta-1}} \cdot \frac{1}{(\Theta-1)^2}$$

$$J(\Theta) = M\left[\left(\frac{\partial \ln p}{\partial \Theta}\right)^2\right]; \quad \ln p = \ln(\Theta-1) - \Theta \ln x$$

$$\frac{\partial \ln p}{\partial \Theta} = \frac{1}{\Theta-1} - \ln x$$

$$J(\Theta) = \int_{1}^{+\infty} \frac{\Theta-1}{x^{\Theta}} \left(\frac{1}{\Theta-1} - \ln x\right)^2 dx = \frac{1}{(\Theta-1)^2}$$

$$\frac{2^{\frac{1}{\tilde{\theta}-1}} - 2^{\frac{1}{\theta-1}}}{\ln 2 \cdot 2^{\frac{1}{\tilde{\theta}-1}} \cdot \frac{1}{(\tilde{\theta}-1)^2} \cdot (\tilde{\theta}-1)} \sqrt{n} \rightsquigarrow N(0,1)$$

$$u_{\frac{1-\beta}{2}} < \frac{(\tilde{\theta}-1)}{\ln 2}\sqrt{n}\left(1 - 2^{\frac{1}{\theta-1}} \cdot 2^{-\frac{1}{\tilde{\theta}-1}}\right) < u_{\frac{1+\beta}{2}}$$

$$\frac{\ln 2}{(\tilde{\theta}-1)\sqrt{n}} u_{\frac{1-\beta}{2}} < 1 - X_{med} \cdot 2^{-\frac{1}{\tilde{\theta}-1}} < \frac{\ln 2}{(\tilde{\theta}-1)\sqrt{n}} u_{\frac{1+\beta}{2}}$$

$$2^{\frac{1}{\tilde{\theta}-1}}\left[1 - \frac{\ln 2}{(\tilde{\theta}-1)\sqrt{n}} u_{\frac{1+\beta}{2}}\right] < X_{med} < 2^{\frac{1}{\tilde{\theta}-1}}\left[1 - \frac{\ln 2}{(\tilde{\theta}-1)\sqrt{n}} u_{\frac{1-\beta}{2}}\right]$$

в)

$$\rho(y) = \begin{cases} e^{1-y}, & y \geq 1 \\ 0, & y < 1 \end{cases} \quad P(\theta|\vec{x_n}) = \text{const} \cdot \rho(\theta) \cdot L(\vec{x_n}, \theta)$$

$$\ln \rho(\theta|\vec{x_n}) = \underbrace{\ln C} + \underbrace{\ln L} + \underbrace{\ln \rho(\theta)} \to \max$$

$$L = \frac{(\theta-1)^n}{\prod_{i=1}^{n} x_i^{\theta}}; \quad \ln \rho = \underbrace{\ln C_1} + \underbrace{n\ln(\theta-1) - \theta \sum \ln x_i} + \underbrace{1-\theta} \to \max$$

$$\frac{\partial \ln \rho(\theta|\vec{x_n})}{\partial \theta} = \frac{n}{\theta-1} - 1 - \sum \ln x_i = 0; \quad \frac{n}{\theta-1} = 1 + \sum \ln x_i$$

$$\Rightarrow \tilde{\theta} = 1 + \frac{n}{1 + \sum \ln x_i} \text{ — оценка.}$$

$$\rho(\theta|\vec{x_n}) = A \cdot e^{1-\theta} \cdot \frac{(\theta-1)^n}{(\prod x_i)^{\theta}}$$

$$\int_1^{+\infty} A \cdot e^{1-\theta} \cdot \frac{(\theta-1)^n}{(\prod x_i)^\theta} d\theta = 1 \Rightarrow \begin{cases} \int_1^{q_1} p(\theta|\vec{X_n}) d\theta = 0{,}025 \\ \int_{q_2}^{+\infty} p(\theta|\vec{X_n}) d\theta = 0{,}025 \end{cases}$$

$q_1 = 5{,}75$; $q_2 = 8{,}05$

d) Асимптотика:

$$\frac{\tilde{\theta}-\theta}{\sigma(\tilde{\theta})}\sqrt{n} \rightsquigarrow N(0,1), \quad \sigma(\tilde{\theta}) = \sqrt{J^{-1}(\theta)}$$

$$u_{\frac{1-\beta}{2}} < (\tilde{\theta}-\theta)\frac{\sqrt{n}}{\tilde{\theta}-1} < u_{\frac{1+\beta}{2}} \qquad \tilde{\theta} - \text{ОМП}$$

$$\tilde{\theta} - \frac{\tilde{\theta}-1}{\sqrt{n}} u_{\frac{1+\beta}{2}} < \theta < \tilde{\theta} - \frac{\tilde{\theta}-1}{\sqrt{n}} u_{\frac{1-\beta}{2}};$$