Matcheva;Strobel (1999) — “Heating of Jupiter’s Thermosphere by Dissipation of Gravity Waves Due to Molecular Viscosity and Heat Conduction”

January 11, 2018; rev. January 11, 2018

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1 THEORETICAL CONSIDERATIONS

1.1 Dissipation of Gravity Waves by Molecular Thermal Conductivity and Viscosity

The linearized set of governing equations in Cartesian coordinates for a small amplitude wave in a dissipative, nonrotating ,deep, compressible, hydrostatic atmosphere with a constant zonal wind u_0 is given by: (from Matcheva and Strobel et al. 1999 eq(1)-eq(5))

$$\nabla \cdot \overrightarrow{V}^{'} - \frac{\omega^{'}}{H^{*}} = 0 \tag{1}$$

$$\left[\frac{\partial}{\partial t} + u_0\frac{\partial}{\partial x} - \frac{1}{\rho_0}\frac{\partial}{\partial z}\mu\frac{\partial}{\partial z}\right]u^{'} = -\frac{1}{\rho_0}\frac{\partial p^{'}}{\partial x} \tag{2}$$

$$\left[\frac{\partial}{\partial t} + u_0\frac{\partial}{\partial x} - \frac{1}{\rho_0}\frac{\partial}{\partial z}\mu\frac{\partial}{\partial z}\right]v^{'} = -\frac{1}{\rho_0}\frac{\partial p^{'}}{\partial y} \tag{3}$$

$$\left[\frac{\partial}{\partial t} + u_0\frac{\partial}{\partial x} - \frac{1}{\rho_0 p_r}\frac{\partial}{\partial z}\mu\frac{\partial}{\partial z}\right]T^{'} + \Gamma\omega^{'} = 0 \tag{4}$$

$$\frac{\partial}{\partial z}\left(\frac{p^{'}}{\rho_0}\right) = \frac{T^{'}R}{H} \tag{5}$$

Where $\omega_0^2 \ll N^2$, and $\frac{H-H^*}{H^*} \ll 1$, the parameters $H^* = \left(-\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial z}\right)^{-1}$, $H = \left(-\frac{1}{\rho_0}\frac{\partial p_0}{\partial z}\right)^{-1}$ are the density and the pressure scale height. $\Gamma = \frac{\partial T_0}{\partial z} + \frac{g}{c_p}$ is the static stability coefficient and P_r is Prandtl number.

Assume solutions of the form:

$$\left[u^{'}v^{'}w^{'}p^{'}T^{'}\right](x,y,z,t) = \left[\delta u(z)\,\delta v(z)\,\delta w(z)\,\delta p(z)\,\delta T(z)\right] exp\left[i\left(k_x x + k_y y - \omega_0 t\right)\right] \tag{6}$$

Substitute eq(6) in eq(1) to eq(5) and reduce the resulting system of equations to a single second-order differential equation for $\widetilde{w}(z)$:

$$\frac{d^2\widetilde{w}(z)}{dz^2} + k_z^2\widetilde{w}(z) = 0 \tag{7}$$

Where :

$$\widetilde{w}(z) = \delta w(z)\, exp\left(-\int\frac{dz}{2H^*}\right) \tag{8}$$

And:

$$k_z^2 = \frac{k_h^2 N^2}{\hat{w}(\hat{w}+i\beta)} - \frac{1}{4H^{*2}}\left[1 - 2\frac{dH^*}{dz}\right] \tag{9}$$

The parameters $\hat{w}$ and β are defined as:

$$\hat{\omega} = \omega_r + i\omega_i \tag{10}$$

$$\begin{aligned}\beta &= -Re\left[\frac{1}{T'}\left(\frac{1}{P_r}-1\right)\nu\frac{d^2}{dz^2}T^{'}\right]\\ &= \left(\frac{1}{P_r}-1\right)\nu\left[k_{zr}^2 - \left(\frac{1}{2H^*}-k_{zi}\right)^2\right]\end{aligned} \tag{11}$$

Where :

$$\begin{aligned}\omega_r &= -Im\left[\frac{1}{u'}\left(\frac{\partial}{\partial t} + u_0\frac{\partial}{\partial x} - \nu\frac{\partial^2}{\partial z^2}\right)u^{'}\right]\\ &= \widetilde{\omega}_0 + 2k_{zr}\nu\left(\frac{1}{2H^*}-k_{zi}\right)\end{aligned} \tag{12}$$

$$\begin{aligned}\omega_i &= +Re\left[\frac{1}{u'}\left(\frac{\partial}{\partial t} + u_0\frac{\partial}{\partial x} - \nu\frac{\partial^2}{\partial z^2}\right)u^{'}\right]\\ &= \nu\left[k_{zr}^2 - \left(\frac{1}{2H^*}-k_{zi}\right)^2\right]\end{aligned} \tag{13}$$

Where $\widetilde{\omega}_0 = \omega_0 - u_0k_x$ is the intrinsic frequency of the wave and $\nu = \mu/\rho_0$ is the kinetic viscosity.

The following is the derivations of these equations:

First, we assume that:

$$\frac{\nu}{\delta u}\frac{d^2\delta u}{dz^2}=\frac{\nu}{\delta v}\frac{d^2\delta v}{dz^2}=\frac{\nu}{\delta T}\frac{d^2\delta T}{dz^2}=\lambda=\lambda_r+i\lambda_i \tag{14}$$

$$\left(\frac{1}{Pr}-1\right)\frac{\nu}{\delta T}\frac{d^2\delta T}{dz^2}=\left(\frac{1}{Pr}-1\right)\lambda=-\beta+i\gamma \tag{15}$$

Then, we have:

$$\frac{1}{u'}\left(\frac{\partial}{\partial t}+u_0\frac{\partial}{\partial x}-\nu\frac{\partial^2}{\partial z^2}\right)u'=\frac{1}{v'}\left(\frac{\partial}{\partial t}+u_0\frac{\partial}{\partial x}-\nu\frac{\partial^2}{\partial z^2}\right)v'$$

$$=\frac{1}{T'}\left(\frac{\partial}{\partial t}+u_0\frac{\partial}{\partial x}-\nu\frac{\partial^2}{\partial z^2}\right)T'=-i\widetilde{\omega}_0-\lambda \tag{16}$$

Where $\widetilde{\omega}_0=\omega_0-u_0k_x$.

Then, we substitute eq(6) in eq(1) to eq(5):

$$ik_x\delta u(z)+ik_y\delta v(z)+\frac{d\delta w(z)}{dz}=\frac{\delta w(z)}{H^*} \tag{17}$$

$$(-i\widetilde{\omega}_0-\lambda)\,\delta u=-\frac{ik_x}{\rho_0}\delta p \tag{18}$$

$$(-i\widetilde{\omega}_0-\lambda)\,\delta v=-\frac{ik_y}{\rho_0}\delta p \tag{19}$$

$$(-i\widetilde{\omega}_0-\lambda)\,\delta T+\Gamma\delta w=(-\beta+i\gamma)\,\delta T \tag{20}$$

$$\frac{1}{\rho_0}\frac{d\delta p(z)}{dz}=\frac{R}{H}\delta T(z) \tag{21}$$

And, according to eq(18) and eq(19), we have:

$$\frac{\delta u}{k_x}=\frac{\delta v}{k_y} \tag{22}$$

According to eq(20) and eq(21):

$$(-i\widetilde{\omega}_0-\lambda+\beta-i\gamma)\,\frac{1}{\rho_0}\frac{d\delta p}{dz}+N^2\delta w=0 \tag{23}$$

Substitute eq(23) in eq(18), we have:

$$\frac{d\delta u}{dz}=\frac{ik_xN^2\delta w}{(-i\widetilde{\omega}_0-\lambda)\,(-i\widetilde{\omega}_0-\lambda+\beta-i\gamma)} \tag{24}$$

Substitute eq(22) in eq(17):

$$\frac{ik_h^2}{k_x}\frac{d\delta u}{dz}+\frac{d^2\delta w}{dz^2}=-\frac{1}{{H^*}^2}\frac{dH^*}{dz}\delta w+\frac{1}{H^*}\frac{d\delta w}{dz} \tag{25}$$

Where $N^2=\frac{R}{H}\Gamma=\frac{g}{T_0}\Gamma$, $k_h^2=k_x^2+k_y^2$.

And according to eq(24) and eq(25), we have:

$$\frac{d^2\delta w}{dz^2}-\frac{1}{H^*}\frac{d\delta w}{dz}+\frac{1}{{H^*}^2}\frac{dH^*}{dz}\delta w-\frac{k_h^2N^2\delta w}{(-i\widetilde{\omega}_0-\lambda)\,(-i\widetilde{\omega}_0-\lambda+\beta-i\gamma)}=0 \tag{26}$$

However, according to eq(12), eq(13) and eq(16):

$$-i\widetilde{\omega}_0-\lambda=\omega_i-i\omega_r=-i\hat{\omega} \tag{27}$$

And substitute eq(27) in eq(26):

$$\frac{d^2\delta w}{dz^2}-\frac{1}{H^*}\frac{d\delta w}{dz}+\left(\frac{1}{{H^*}^2}\frac{dH^*}{dz}+\frac{k_h^2N^2}{\hat{\omega}\,(\hat{\omega}+i\beta+\gamma)}\right)\delta w=0 \tag{28}$$

If we let $\gamma=0$(why?), and according to eq(9), we have:

$$\frac{d^2\delta w}{dz^2}-\frac{1}{H^*}\frac{d\delta w}{dz}+\left(\frac{1}{2{H^*}^2}\frac{dH^*}{dz}+\frac{1}{4{H^*}^2}+k_z^2\right)\delta w=0 \tag{29}$$

And substitute eq(8) in eq(29), we have get eq(7).

In the limits of the WKB approximation we obtain a wave-like solution:

$$\widetilde{w}(z)=\Delta W(z_0)\left(\frac{k_{zr}(z_0)}{k_{zr}(z)}\right)^{1/2}exp\left[-\int_{z_0}^{z}k_{zi}dz\right]exp\left[i\int_{z_0}^{z}k_{zr}dz\right] \tag{30}$$

Where $k_{zr}=Rek_z$, $k_{zi}=Imk_z$.

Then, according to eq(8) and eq(30), we have:

$$\delta w=\Delta W(z_0)\left(\frac{k_{zr}(z_0)}{k_{zr}(z)}\right)^{1/2}exp\left[\int_{z_0}^{z}\left(ik_z+\frac{1}{2H^*}\right)dz\right] \tag{31}$$

If we assume that k_z is slowly change with z, then, we have:

$$\frac{d\delta w}{dz} = \left(ik_z + \frac{1}{2H^*}\right)\delta w \tag{32}$$

Now according to eq(17), eq(18) and eq(32), we have:

$$\frac{ik_h^2}{k_x}\delta u = \left(-ik_z + \frac{1}{2H^*}\right)\delta w \tag{33}$$

And:

$$\frac{ik_h^2}{k_x}\frac{d^2\delta u}{dz^2} = \left(-ik_z + \frac{1}{2H^*}\right)\frac{d^2\delta w}{dz^2} = \left(-ik_z + \frac{1}{2H^*}\right)\left(ik_z + \frac{1}{2H^*}\right)^2\delta w \tag{34}$$

Then we have:

$$\frac{1}{\delta u}\frac{d^2\delta u}{dz^2} = \left(ik_z + \frac{1}{2H^*}\right)^2 = \left(ik_{zr} + \frac{1}{2H^*} - k_{zi}\right)^2$$
$$= \left[\left(\frac{1}{2H^*} - k_{zi}\right)^2 - k_{zr}^2 + 2ik_{zr}\left(\frac{1}{2H^*} - k_{zi}\right)\right] \tag{35}$$

Now according to eq(14) and eq(27), we have:

$$\omega_r = \widetilde{\omega}_0 + \lambda_i = \widetilde{\omega}_0 + 2\nu k_{zr}\left(\frac{1}{2H^*} - k_{zi}\right) \tag{36}$$

$$\omega_i = -\lambda_r = \nu\left[{k_{zr}}^2 - \left(\frac{1}{2H^*} - k_{zi}\right)^2\right] \tag{37}$$

And according to eq(15), we have:

$$\beta = -\left(\frac{1}{Pr} - 1\right)\lambda_r = \left(\frac{1}{Pr} - 1\right)\omega_i$$
$$= \left(\frac{1}{Pr} - 1\right)\nu\left[{k_{zr}}^2 - \left(\frac{1}{2H^*} - k_{zi}\right)^2\right] \tag{38}$$

Using eq(6), eq(8) and eq(30) we obtain a final expression for the perturbed vertical velocity field:

$$w^{'}(x, y, z, t) = \Delta W(z)\, cos\varphi \tag{39}$$

Where:

$$\Delta W(z) = \Delta W(z_0)\left(\frac{k_{zr}(z_0)}{k_{zr}(z)}\right)^{1/2} exp\left[\int_{z_0}^{z}\left(\frac{1}{2H^*} - k_{zi}\right)dz\right] \tag{40}$$

$$\varphi = k_x x + k_y y + \int_{z_0}^{z} k_{zr}dz - \omega_0 t \tag{41}$$

The following is the derivations of eq(39) :

$$w^{'}(x, y, z, t) = \delta w(z)\, exp\left[i\left(k_x x + k_y y - \omega_0 t\right)\right]$$
$$= \widetilde{w}(z)\, exp\left(\int\frac{dz}{2H^*}\right) exp\left[i\left(k_x x + k_y y - \omega_0 t\right)\right]$$
$$= \Delta W(z_0)\left(\frac{k_{zr}(z_0)}{k_{zr}(z)}\right)^{1/2} exp\left[\int\frac{dz}{2H^*} - \int_{z_0}^{z}k_{zi}dz\right]e^{i\varphi}$$
$$= \Delta W(z)\, e^{i\varphi} \tag{42}$$

The corresponding expressions for the temperature and pressure perturbation fields are obtained as well:

$$T^{'}(x, y, z, t) = \frac{\Gamma}{\omega_r}\Delta W(z)\, cos\theta cos\left(\varphi - \frac{\pi}{2} - \theta\right) \tag{43}$$

$$p^{'}(x, y, z, t) = -\left(\frac{\omega_r k_{zr}}{k_h^2}\right)\Delta W\left[1 - \frac{\omega_i}{\omega_r}\left(\frac{k_{zi}}{k_{zr}} + \frac{1}{2H^* k_{zr}}\right)\right]\frac{cos\left(\varphi - \theta^{'}\right)}{cos\theta^{'}} \tag{44}$$

Where θ and $\theta^{'}$ are define as:

$$tan\theta = \frac{1}{P_r}\frac{\omega_i}{\omega_r} \tag{45}$$

$$tan\theta^{'} = \left(\frac{\omega_i}{\omega_r} + \frac{k_{zi}}{k_{zr}} + \frac{1}{2H^* k_{zr}}\right)\left[1 - \frac{\omega_i}{\omega_r}\left(\frac{k_{zi}}{k_{zr}} + \frac{1}{2H^* k_{zr}}\right)\right]^{-1} \tag{46}$$

The following is the derivations of eq(43) and eq(44) :

According to eq(20), we have:

$$\delta T = \frac{-\Gamma \delta w}{-i\tilde{\omega}_0 - \lambda + \beta} = \frac{-i\Gamma \delta w}{\hat{\omega} + i\beta} \tag{47}$$

Then:

$$T^{'}(x,y,z,t) = \frac{-i\Gamma}{\hat{\omega} + i\beta}\Delta W(z) e^{i\varphi} \tag{48}$$

And according to eq(15) and eq(27):

$$\frac{-i}{\hat{\omega}+i\beta}e^{i\varphi} = \frac{exp\left[i\left(\varphi - \frac{\pi}{2}\right)\right]}{\omega_r + i\frac{\omega_i}{Pr}} = \frac{1}{\omega_r\left(1 + i\frac{1}{Pr}\frac{\omega_i}{\omega_r}\right)}exp\left[i\left(\varphi - \frac{\pi}{2}\right)\right]$$

$$= \frac{1}{\omega_r\left(1 + itan\theta\right)}exp\left[i\left(\varphi - \frac{\pi}{2}\right)\right] = \frac{cos\theta}{\omega_r\left(cos\theta + isin\theta\right)}exp\left[i\left(\varphi - \frac{\pi}{2}\right)\right]$$

$$= \frac{cos\theta}{\omega_r}exp\left[i\left(\varphi - \frac{\pi}{2} - \theta\right)\right]$$

Then, eq(48) become:

$$T^{'}(x,y,z,t) = \frac{\Gamma}{\omega_r}\Delta W(z)\, cos\theta exp\left[i\left(\varphi - \frac{\pi}{2} - \theta\right)\right] \tag{49}$$

For eq(44):

Firstly, according to eq(33):

$$u^{'}(x,y,z,t) = -\frac{ik_x}{k_h^2}\left(-ik_z + \frac{1}{2H^*}\right)\Delta W(z) e^{i\varphi} \tag{50}$$

Similarly, we have:

$$v^{'}(x,y,z,t) = -\frac{ik_y}{k_h^2}\left(-ik_z + \frac{1}{2H^*}\right)\Delta W(z) e^{i\varphi} \tag{51}$$

And according to eq(18) and eq(50), we have:

$$p^{'}(x,y,z,t) = -\frac{\rho_0}{ik_x}\left(-i\tilde{\omega}_0 - \lambda\right)u^{'}(x,y,z,t)$$

$$= \frac{i\rho_0\hat{\omega}}{k_h^2}\left(ik_z - \frac{1}{2H^*}\right)\Delta W(z) e^{i\varphi}$$

Now we consider:

$$\hat{\omega}\left(ik_z - \frac{1}{2H^*}\right) = \left(\omega_r + i\omega_i\right)\left(ik_{zr} - \frac{1}{2H^*} - k_{zi}\right)$$

$$= \omega_r k_{zr}\left(1 + i\frac{\omega_i}{\omega_r}\right)\left[i - \left(\frac{1}{2H^*k_{zr}} + \frac{k_{zi}}{k_{zr}}\right)\right]$$

$$= \omega_r k_{zr}\left[i - \left(\frac{1}{2H^*k_{zr}} + \frac{k_{zi}}{k_{zr}}\right) - \frac{\omega_i}{\omega_r} - i\frac{\omega_i}{\omega_r}\left(\frac{1}{2H^*k_{zr}} + \frac{k_{zi}}{k_{zr}}\right)\right]$$

$$= \omega_r k_{zr}\left\{-\left(\frac{\omega_i}{\omega_r} + \frac{1}{2H^*k_{zr}} + \frac{k_{zi}}{k_{zr}}\right) + i\left[1 - \frac{\omega_i}{\omega_r}\left(\frac{1}{2H^*k_{zr}} + \frac{k_{zi}}{k_{zr}}\right)\right]\right\}$$

$$= \omega_r k_{zr}\left[1 - \frac{\omega_i}{\omega_r}\left(\frac{1}{2H^*k_{zr}} + \frac{k_{zi}}{k_{zr}}\right)\right]\left\{i - \frac{\frac{\omega_i}{\omega_r} + \frac{1}{2H^*k_{zr}} + \frac{k_{zi}}{k_{zr}}}{1 - \frac{\omega_i}{\omega_r}\left(\frac{1}{2H^*k_{zr}} + \frac{k_{zi}}{k_{zr}}\right)}\right\}$$

$$= \omega_r k_{zr}\left[1 - \frac{\omega_i}{\omega_r}\left(\frac{1}{2H^*k_{zr}} + \frac{k_{zi}}{k_{zr}}\right)\right]\left(i - tan\theta^{'}\right)$$

$$= \omega_r k_{zr}\left[1 - \frac{\omega_i}{\omega_r}\left(\frac{1}{2H^*k_{zr}} + \frac{k_{zi}}{k_{zr}}\right)\right]\frac{icos\theta^{'} - sin\theta^{'}}{cos\theta^{'}}$$

$$= \omega_r k_{zr}\left[1 - \frac{\omega_i}{\omega_r}\left(\frac{1}{2H^*k_{zr}} + \frac{k_{zi}}{k_{zr}}\right)\right]\frac{ie^{i\theta^{'}}}{cos\theta^{'}}$$

Where $tan\theta^{'} = \left(\frac{\omega_i}{\omega_r} + \frac{1}{2H^*k_{zr}} + \frac{k_{zi}}{k_{zr}}\right)\left[1 - \frac{\omega_i}{\omega_r}\left(\frac{1}{2H^*k_{zr}} + \frac{k_{zi}}{k_{zr}}\right)\right]^{-1}$.

Then, we have:

$$p^{'}(x,y,z,t) = -\frac{\rho_0\omega_r k_{zr}}{k_h^2}\Delta W(z)\left[1 - \frac{\omega_i}{\omega_r}\left(\frac{1}{2H^*k_{zr}} + \frac{k_{zi}}{k_{zr}}\right)\right]\frac{e^{i\left(\varphi+\theta^{'}\right)}}{cos\theta^{'}} \tag{52}$$

Reference

Matcheva;Strobel (1999). Heating of jupiter's thermosphere by dissipation of gravity waves due to molecular viscosity and heat conduction. Icarus, 140.