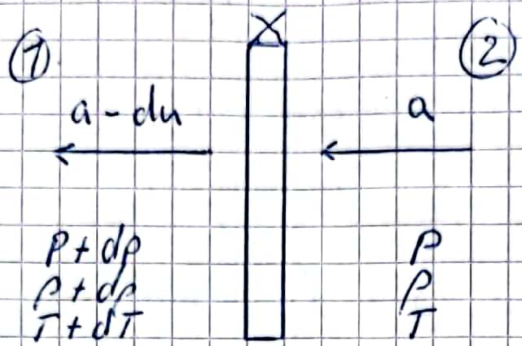
Problem 1.1

a



b mass conservation in integral form:

$$\frac{d}{dt}\iiint_V \rho \, dV + \iint_S \rho \bar{V}\cdot\bar{n}\, dS = 0$$

Apply divergence theorem:

$$\iint_S \rho\bar{V}\cdot\bar{n}\, dS = \iiint_V \nabla\cdot\rho\bar{V}\, dV$$

$$\Longrightarrow \frac{d}{dt}\iiint_V \rho\, dV + \iiint_V \nabla\cdot\rho\bar{V}\, dV = 0$$

$$\Longrightarrow \frac{\partial\rho}{\partial t} + \nabla\cdot\rho\bar{V} = 0$$

The problem is one dimensional:

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}\rho\bar{V} = 0$$

And the problem is steady:

$$\frac{\partial\rho}{\partial t} = 0$$

$$\Longrightarrow \frac{\partial}{\partial x}\rho\bar{V} = 0 \Longrightarrow \rho\bar{V} = \text{constant}$$

$$(\rho + d\rho)(a - du) = \rho a$$

$$\rho a - \rho\, du + d\rho\, a - d\rho\, du = \rho a$$

We assume $d\rho\, du = 0$, because both terms are infinitesimally small

$$-\rho\, du + d\rho\, a = 0$$

$$\boxed{a = \rho\frac{du}{d\rho}}$$

Conservation of momentum in integral form

$$\frac{d}{dt}\iiint_V \rho\bar{v}\,dV + \iint_S \rho\bar{v}\bar{v}\cdot\bar{n}\,dS + \iint_S p\cdot\bar{n}\,dS \ldots$$

$$\ldots = \iiint_V \rho\bar{f}\,dV + \bar{F}_{viscous} + \bar{F}_{external}$$

Some terms are zero for this problem leaving:

$$\frac{d}{dt}\iiint_V \rho\bar{v}\,dV + \iint_S \rho\bar{v}\bar{v}\cdot\bar{n}\,dS + \iint_S p\cdot\bar{n}\,dS = 0$$

And after applying the divergence theorem:

$$\frac{d}{dt}\iiint_V \rho\bar{v}\,dV + \iiint_V \nabla\cdot\rho\bar{v}\bar{v}\,dV + \iiint_V \nabla\cdot p\,dV = 0$$

$$\frac{\partial}{\partial t}\rho\bar{v} + \bar{\nabla}\cdot\rho\bar{v}\bar{v} + \bar{\nabla}\cdot p\,dV = 0$$

Flow is steady and one dimensional:

$$\frac{\partial}{\partial x}\rho\bar{v}\bar{v} + \frac{\partial}{\partial x}p = 0$$

$$\Longrightarrow \quad \rho\bar{v}\bar{v} + p = \text{constant}$$

$$\underbrace{(\rho+d\rho)(a-du)^2}_{} + (p+dp) = \rho a^2 + p$$

$$= (\rho+d\rho)(a^2 - 2a\,du + \cancel{du^2}^{=0})$$

$$= \rho a^2 - 2\rho a\,du + d\rho a^2 - \cancel{2d\rho a\,du}^{=0}$$

$$= \rho a^2 - 2\rho a\,du + d\rho a^2$$

$$\rightarrow \rho a^2 - 2\rho a\,du + d\rho a^2 + p + dp = \rho a^2 + p$$

$$-2\rho a\,du + d\rho a^2 + dp = 0$$

$$d\rho\, a^2 = 2\rho a\,du - dp$$

So now, from conservation of mass and momentum we have:

$$\underbrace{\boxed{a = \rho\frac{du}{d\rho}}}_{\text{conservation of mass}} \qquad \underbrace{\boxed{a^2 = 2\rho a\frac{du}{d\rho} - \frac{dp}{d\rho}}}_{\text{conservation of momentum}}$$

(1) (2)

If we take equation ② and reorganize to:

$$a^2 = 2a\underbrace{\rho\frac{du}{d\rho}}_{=a\text{ , according to ①}} - \frac{dp}{d\rho}$$

$$\Longrightarrow \quad a^2 = 2a^2 - \frac{dp}{d\rho}$$

$$-a^2 = -\frac{dp}{d\rho}$$

$$\boxed{a^2 = dp/d\rho}$$

<u>C</u> <u>Energy equation in integral form:</u>

$$\frac{d}{dt}\iiint_V \rho E\, dV + \iint_S \rho E\bar{V}\cdot\bar{n}\, dS + \iint_S p\bar{V}\cdot\bar{n}\, dS \ \ldots$$

$$\ldots = \iiint_V \rho\bar{f}\cdot\bar{V}\, dV + \dot{Q} + \dot{W}_{viscous} + \dot{W}_{external}$$

Leaving the zero terms out and using divergence theorem leads to:

$$\frac{d}{dt}\iiint_V \rho E\, dV + \iiint_V \nabla\cdot\rho E\bar{V}\, dV + \iiint_V \nabla\cdot p\bar{V}\, dV = 0$$

$$\frac{\partial}{\partial t}\rho E + \nabla\cdot\rho E\bar{V} + \nabla\cdot p\bar{V} = 0$$

The problem is steady and one-dimensional:

$$\frac{\partial}{\partial x}\rho E\bar{V} + \frac{\partial}{\partial x}p\bar{V} = 0$$

$$\longrightarrow \rho E\bar{V} + p\bar{V} = \text{constant} \quad , \quad E = e + \tfrac{1}{2}q^2$$

$$(\rho+d\rho)\underbrace{(e+de+\tfrac{1}{2}(a-du)^2)(a-du)}_{} + (p+dp)(a-du) = \rho(e+\tfrac{1}{2}a^2)a + pa$$

$$= (e + de + \tfrac{1}{2}a^2 - a\,du)(a-du)$$

$$= ea + dea + \tfrac{1}{2}a^3 - a^2du - e\,du - \tfrac{1}{2}a^2du$$

~~[illegible]~~

$$= ea + dea + \tfrac{1}{2}a^3 - \tfrac{3}{2}a^2du - e\,du$$

$$\rightarrow \underbrace{(\rho+d\rho)(ea+dea+\tfrac{1}{2}a^3-\tfrac{3}{2}a^2du-e\,du)}_{} + (p+dp)(a-du) = \rho(e+\tfrac{1}{2}a^2)a + pa$$

$$= ea\rho + dea\rho + \tfrac{1}{2}a^3\rho - \tfrac{3}{2}a^2du\rho - edu\rho + ead\rho + \tfrac{1}{2}a^3d\rho$$

$\rightarrow \quad ea\rho + dea\rho + \frac{1}{2}a^3\rho - \frac{3}{2}a^2 du\rho - edu\rho + ead\rho + \frac{1}{2}a^3 d\rho \ldots$

$\ldots + \underbrace{(p+dp)(a-du)}_{= pa - pdu + dpa} = \underbrace{\rho(e+\frac{1}{2}a^2)a + pa}_{= \rho ea + \frac{1}{2}\rho a^3 + pa}$

$\rightarrow \quad \cancel{ea\rho} + dea\rho + \cancel{\frac{1}{2}a^3\rho} - \frac{3}{2}a^2 du\rho - edu\rho + ead\rho + \frac{1}{2}a^3 d\rho \ldots$

$\ldots - pdu + dpa = \cancel{\rho ea} + \cancel{\frac{1}{2}\rho a^3}$

$\rightarrow \quad dea\rho - \frac{3}{2}a^2 du\rho - edu\rho + ead\rho + \frac{1}{2}a^3 d\rho - pdu + dpa = 0$

From conservation of mass, we know that:

$du = a\frac{d\rho}{\rho}$

$\rightarrow \quad dea\rho - \frac{3}{2}a^3 d\rho - ead\rho + ead\rho + \frac{1}{2}a^3 d\rho - pa\frac{d\rho}{\rho} + dpa = 0$

$de\rho - a^2 d\rho - p\frac{d\rho}{\rho} + dp = 0$

And from b. we also know that:

$a^2 = dp/d\rho$

$\rightarrow \quad de\rho - dp - p\frac{d\rho}{\rho} + dp = 0$

$de\rho - p\frac{d\rho}{\rho} = 0$

$de - p\frac{d\rho}{\rho^2} = 0 = ds$

$$\boxed{ds = de - p\frac{d\rho}{\rho^2} = 0}$$

Now, we can conclude that sound propagation is indeed an isentropic process