# Problem 2.1

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**a.** Unperturbed regions : 1, 4, 6 $\rightarrow$ $M = \tilde{S} = 0$

Simple wave regions : 2 $\Rightarrow$ only $J^+$ varies

5 $\rightarrow$ only $J^-$ varies

Non-simple wave regions : 3 $\Rightarrow$ both $J^-$ and $J^+$ vary

**b.** $J^-$ : $M_A - \tilde{S}_A = M_2 - \tilde{S}_2 = 0$

$\rightarrow M_2 = \tilde{S}_2$

$J^+$ : $M_B + \tilde{S}_B = M_2 + \tilde{S}_2 = 2M_2$

$J^-$ : $M_C - \tilde{S}_C = M_B - \tilde{S}_C = 0$

$M_B = \tilde{S}_C$

$\rightarrow 2M_B = 2M_2$

$\hookrightarrow M_B = M_p = \frac{u_p}{a_0}$

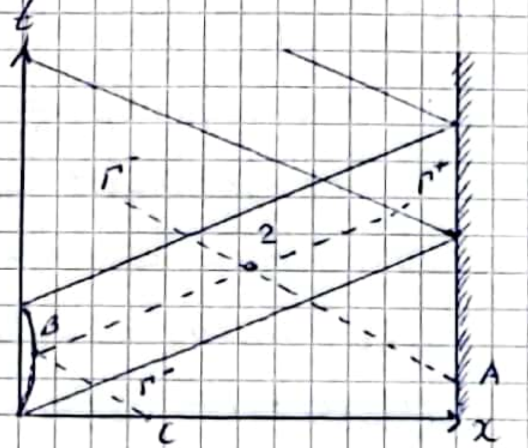
$M_2(x_2, t_2) = M_p(x_p, t_p)$

$\Gamma^+$ : $x_2 - a_0 t_2 = x_p - a_0 t_p$ , where $x_p = 0$

$\rightarrow$ ~~$x_p$~~ $t_p = t_2 - \frac{x_2}{a_0}$

$M_2(x_2, t_2) = M_p(0, t_2 - \frac{x_2}{a_0}) = \varepsilon \sin[\omega(t_2 - \frac{x_2}{a_0})]$

$$\boxed{M_2(x_2, t_2) = \tilde{S}(x_2, t_2) = \varepsilon \sin[\omega(t_2 - \frac{x_2}{a_0})]}$$




**c.** $J^-$ : $M_3 - \tilde{S}_3 = M_D - \tilde{S}_D = -\tilde{S}_D$

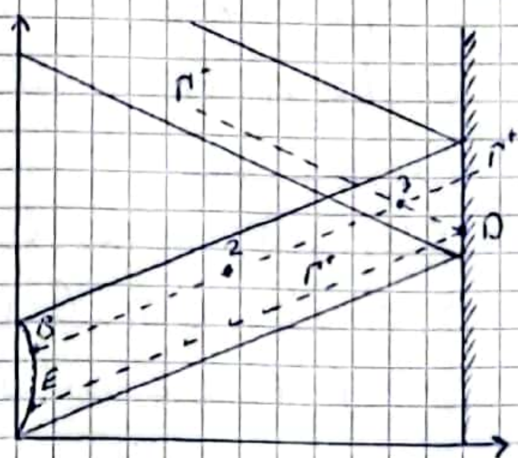
$M_3 = \tilde{S}_3 - \tilde{S}_D$ (1)

$J^+$ : $M_2 + \tilde{S}_2 = M_3 + \tilde{S}_3 = 2M_p$

$M_3 = 2M_p - \tilde{S}_3$ (2)

Combine eq. (1) and (2)

$\rightarrow M_3 = -\tilde{S}_D/2 + M_p$



Now, we need an expression for $\tilde{S}_D$

$J^+$ : $m_D + \tilde{S}_D = m_E + \tilde{S}_E$ , $m_D = 0$

$\tilde{S}_D = m_E + \tilde{S}_E = 2m_P$

$\tilde{S}_D(x_D, t_D) = 2m_P(x_P, t_P)$

$\Gamma^+$ : $x_D - a_0 t_D = x_P - a_0 t_P$ , $x_D = L$ and $x_P = 0$

$\rightarrow t_P = t_D - \frac{L}{a_0}$ and $t_D = t_P + \frac{L}{a_0}$

$\Gamma^-$ : $x_D + a_0 t_D = x_3 + a_0 t_3$

$L + a_0(t_P + \frac{L}{a_0}) = x_3 + a_0 t_3$

$\rightarrow t_P = (t_3 + \frac{x_3}{a_0} - \frac{2L}{a_0})$

So, know we have for $\tilde{S}_D$ :

$S_D(x_D, t_D) = 2m_P(0, t_3 + \frac{x_3}{a_0} - \frac{2L}{a_0})$

$m_3(x_3, t_3) = m_P(0, t_3 - \frac{x_3}{a_0}) - m_P(0, t_3 + \frac{x_3}{a_0} - \frac{2L}{a_0})$

Remember that $m_P = \frac{u_P}{a_0}$ , so our solution is:

$$\boxed{m_3(x_3, t_3) = \varepsilon \sin[\omega(t_3 - \frac{x_3}{a_0})] - \varepsilon \sin[\omega(t_3 + \frac{x_3}{a_0} - \frac{2L}{a_0})]}$$

d. We need the following relation to compute the pressure:

$\Delta p = a_0^2 \Delta\rho$ , where $\Delta\rho = \tilde{S}\rho_0$

$\rightarrow \Delta p = a_0^2 \rho_0 \tilde{S}$

$p = p_0 + \Delta p = p_0 + a_0^2 \rho_0 \tilde{S} = p_0 + p_0 \tilde{S}$

$p = p_0(1 + \tilde{S})$

- Problem ① : At the piston $(x = 0)$

Region 2 :

$\tilde{S}(x_2, t_2) = m(x_2, t_2) = \varepsilon \sin(\omega(t_2 - \frac{x_2}{a_0}))$

$S(0, t_2) = \varepsilon \sin(\omega t_2)$ , $0 \leq t_2 \leq \frac{2\pi}{\omega}$

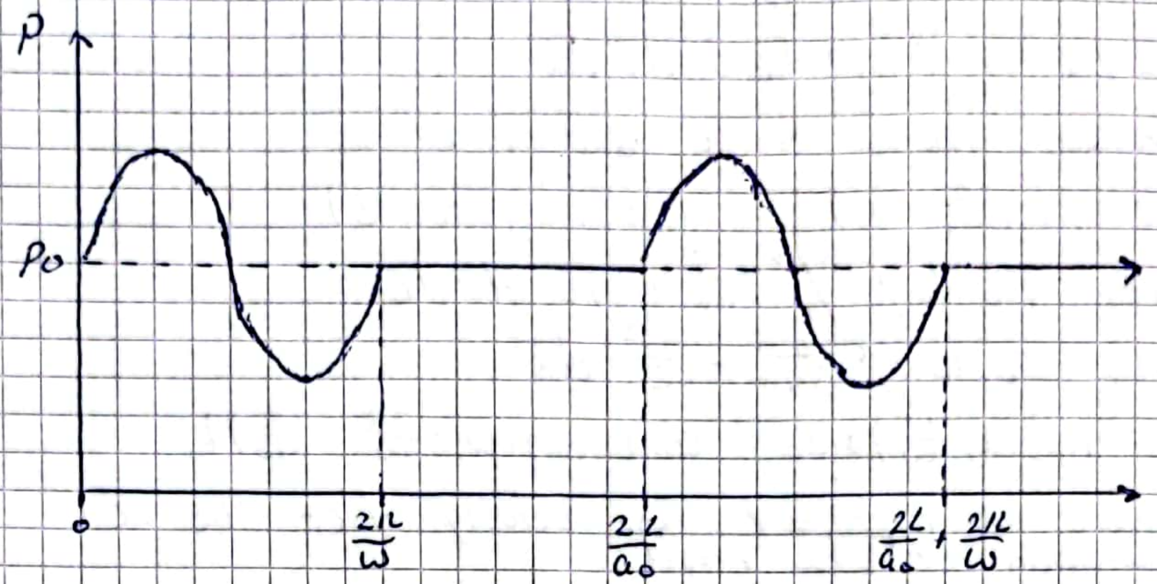
$p = p_0(1 + \varepsilon \sin(\omega t))$ , $0 \leq t \leq \frac{2\pi}{\omega}$

Region 4 : Unperturbed region

$\tilde{S}(0, t) = 0$ , $\frac{2\pi}{\omega} \leq t \leq \frac{2L}{a_0}$

$p = p_0$ , $\frac{2\pi}{\omega} \leq t \leq \frac{2L}{a_0}$

This will be repeated as the wave bounces back and forth in the tube, so the pressure signal will look like this:




- Problem ② : At the wall $(x = L)$

Region 1: Unperturbed

$\tilde{S}(x_1,t_1) = 0$ , $\tilde{S}(L,t) = 0$ , $0 \le t \le \frac{L}{a_0}$

$p = p_0$ , $0 \le t \le \frac{L}{a_0}$

Region 3:

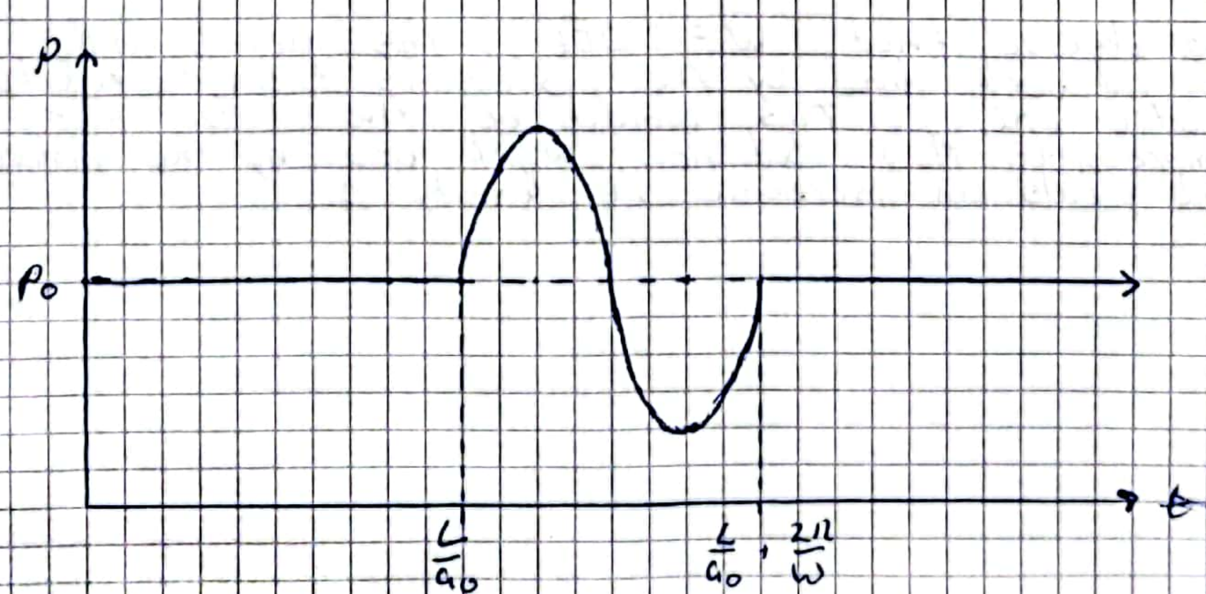
$\tilde{S}_0 = 2M_p$ (from c.)

$\tilde{S}_0 = 2\varepsilon \sin(\omega(t_3 + \frac{x_3}{a_0} - \frac{2L}{a_0}))$

$\tilde{S}_3(L,t) = 2\varepsilon \sin(\omega(t_3 - \frac{L}{a_0}))$ , $\frac{L}{a_0} \le t \le \frac{L}{a_0} + \frac{2\pi}{\omega}$

Region 6: Unperturbed

$\tilde{S}(x_6,t_6) = 0$, $\tilde{S}(L,t) = 0$ , $\frac{L}{a_0} + \frac{2\pi}{\omega} \le t \le \frac{3L}{a_0}$



**e.** First nothing, then compression and expansion. This repeats.

**f.**
- Kinetic Energy

$$\Delta E_k = \int_0^x \int_0^v \rho A_t \bar{v} \, d\bar{v} \, d\bar{x}$$

where mass is a function of $v$

$$\rho = \rho_0 \left(\frac{v}{a_0} + 1\right) \qquad \text{(in region 2)}$$

$$\Delta E_k = \rho_0 A_t \int_0^x \int_0^v \left(\frac{\bar{v}}{a_0} + 1\right) \bar{v} \, d\bar{v} \, d\bar{x}$$

$$\Delta E_k = \rho_0 A_t \int_0^x \frac{v^3}{3a_0} + \frac{v^2}{2} \, d\bar{x}$$

$$v = M_2\left(x, \frac{2\pi}{\omega}\right) \cdot a_0 = a_0 \varepsilon \sin\left(\omega\left(\frac{2\pi}{\omega} - \frac{x}{a_0}\right)\right)$$

where $v$ represent the velocity after the acceleration of the fluid.

$$\Delta E_k = \rho_0 A_t a_0^2 \varepsilon^2 \int_0^x \frac{1}{3} \varepsilon \sin^3\left(\omega\left(\frac{2\pi}{\omega} - \frac{\bar{x}}{a_0}\right)\right) + \frac{1}{2} \sin^2\left(\omega\left(\frac{2\pi}{\omega} - \frac{\bar{x}}{a_0}\right)\right) d\bar{x}$$

I evaluated this integral using an online tool and put in $x = 2\pi a_0/\omega$ (distance until $t = 2\pi/\omega$)

This yields:

$$\boxed{\Delta E_k = \frac{A a_0^3 \varepsilon^2 \rho_0}{2\omega}}$$

- Work Done

$$W = \int F \, dx = \int A_t p \, dx = \int A_p = A \int_0^{2\pi/\omega} p(t) u(t) \, dt$$

$$= A_t \int_0^{2\pi/\omega} \left(p_0 + a_0^2 \rho_0 \varepsilon \sin(\omega t)\right) \varepsilon a_0 \, dt$$

$$\boxed{W = \frac{A_t a_0^3 \varepsilon^2 \rho_0}{\omega}}$$

We observe that $W = 2\Delta E_k$, this means that not all work done by the piston is transferred to kinetic energy. This would be the case for incompressible flow. But now, work done by the piston also yields an increase in density $\rho$.