Problem 1.2

a. A steady shock, so:

$V_0 [U] = [F]$, where $V_0 = 0$

$\rightarrow [F] = 0$, $F_2 = F_1$

$$\rightarrow \begin{bmatrix} \rho_2 u_2 \\ p_2 + \rho_2 u_2^2 \\ \rho_2 u_2 H_2 \end{bmatrix} = \begin{bmatrix} \rho_1 u_1 \\ p_1 + \rho_1 u_1^2 \\ \rho_1 u_1 H_1 \end{bmatrix}$$

From mass conservation:

$$\rho_2 u_2 = \rho_1 u_1 \rightarrow \boxed{u_1 = \frac{\rho_2 u_2}{\rho_1}}$$

From momentum conservation:

$$p_2 + \rho_2 u_2^2 = p_1 + \rho_1 u_1^2$$

$$\rho_1 u_1^2 = p_2 - p_1 + \rho_2 u_2^2 \quad , \quad p_2 - p_1 = [p]$$

$$\rho_2 u_2^2 = \rho_1 u_1^2 - [p]$$

$$\rho_2 u_2^2 = \rho_1 \frac{\rho_2^2 u_2^2}{\rho_1^2} - [p]$$

$$\frac{\rho_2^2}{\rho_1} u_2^2 - \rho_2 u_2^2 = [p]$$

$$u_2^2 \left(\frac{\rho_2^2}{\rho_1} - \rho_2 \right) = [p]$$

$$u_2^2 = \frac{\rho_1}{\rho_2^2 - \rho_2 \rho_1} [p] \rightarrow \boxed{u_2^2 = \frac{\rho_1}{\rho_2} \frac{[p]}{\rho_2 - \rho_1}}$$

From conservation of energy:

$$\rho_2 u_2 H_2 = \rho_1 u_1 H_1 \quad , \quad H = h + \tfrac{1}{2} u^2$$

$$\rho_2 u_2 (h_2 + \tfrac{1}{2} u_2^2) = \rho_1 u_1 (h_1 + \tfrac{1}{2} u_1^2)$$

From mass conservation we know: $\rho_2 u_2 = \rho_1 u_1$

$$\rightarrow h_2 + \tfrac{1}{2} u_2^2 = h_1 + \tfrac{1}{2} u_1^2 \quad , \quad h = e + \frac{p}{\rho}$$

$$e_2 + \frac{p_2}{\rho_2} + \tfrac{1}{2} u_2^2 = e_1 + \frac{p_1}{\rho_1} + \tfrac{1}{2} u_1^2$$

$$[e] + \frac{p_2}{\rho_2} - \frac{p_1}{\rho_1} = \tfrac{1}{2} (u_1^2 - u_2^2)$$

$$= \tfrac{1}{2} \left(\frac{\rho_2}{\rho_1} \frac{[p]}{\rho_2 - \rho_1} - \frac{\rho_1}{\rho_2} \frac{[p]}{\rho_2 - \rho_1} \right)$$

$$= \tfrac{1}{2} \left(\frac{\rho_2^2 - \rho_1^2}{\rho_2 \rho_1} \frac{[p]}{\rho_2 - \rho_1} \right)$$

$$= \tfrac{1}{2} \left(\frac{(\rho_2 + \rho_1)(\rho_2 - \rho_1)}{\rho_2 \rho_1} \frac{[p]}{\rho_2 - \rho_1} \right)$$

$$[e] + \frac{p_2}{\rho_2} - \frac{p_1}{\rho_1} = \frac{[p]}{2}\left(\frac{(\rho_2+\rho_1)(\rho_2-\rho_1)}{\rho_2\rho_1(\rho_2-\rho_1)}\right)$$

$$[e] + \frac{p_2}{\rho_2} - \frac{p_1}{\rho_1} = \frac{[p]}{2}\,\frac{\rho_2+\rho_1}{\rho_2\rho_1}$$

$$[e] + \frac{p_2}{\rho_2} - \frac{p_1}{\rho_1} - \frac{[p]}{2}\,\frac{\rho_2+\rho_1}{\rho_2\rho_1} = 0$$

$$[e] + \frac{p_2}{\rho_2} - \frac{p_1}{\rho_1} - \frac{1}{2}p_2\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right) + \frac{1}{2}p_1\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right) = 0$$

$$[e] + \frac{p_2}{\rho_2} - \frac{p_1}{\rho_1} - \frac{1}{2}\frac{p_2}{\rho_1} - \frac{1}{2}\frac{p_2}{\rho_2} + \frac{1}{2}\frac{p_1}{\rho_1} + \frac{1}{2}\frac{p_1}{\rho_2} = 0$$

$$[e] + \frac{1}{2}\frac{p_2}{\rho_2} - \frac{1}{2}\frac{p_1}{\rho_1} - \frac{1}{2}\frac{p_2}{\rho_1} + \frac{1}{2}\frac{p_1}{\rho_2} = 0$$

$$[e] + \frac{1}{2}\left(\frac{p_2+p_1}{\rho_2} - \frac{p_2+p_1}{\rho_1}\right) = 0$$

$$[e] + \frac{1}{2}(p_2+p_1)\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right) = 0$$

$$\boxed{[e] + \langle p\rangle [v] = 0} \quad \text{I}$$

b. $p = (\gamma-1)\rho e \rightarrow e = \frac{p}{(\gamma-1)\rho}$

$$[e] = \frac{[p]}{(\gamma-1)[\rho]} = \frac{[p][v]}{(\gamma-1)}$$

We use the following property:

$$[p][v] = \langle p\rangle [v] + \langle v\rangle [p]$$

$$\rightarrow [e] = \frac{1}{(\gamma-1)}(\langle p\rangle[v] + \langle v\rangle [p])$$

Now, we plut this in the equation from <u>a.</u>:

$$\frac{1}{(\gamma-1)}(\langle p\rangle[v] + \langle v\rangle[p]) + \langle p\rangle[v] = 0$$

$$\frac{1}{(\gamma-1)}(\langle p\rangle[v]) + \langle p\rangle[v] + \frac{1}{\gamma-1}(\langle v\rangle[p]) = 0$$

$$(\langle p\rangle[v])\underbrace{\left(\frac{1}{\gamma-1}+1\right)}_{=\frac{\gamma}{\gamma-1}} + (\langle v\rangle[p])\left(\frac{1}{\gamma-1}\right) = 0$$

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$$\frac{\langle p\rangle}{[p]}\frac{\gamma}{\gamma-1} + \frac{\langle V\rangle}{[V]}\frac{1}{\gamma-1} = 0$$

$$\gamma\frac{\langle p\rangle}{[p]} + \frac{\langle V\rangle}{[V]} = 0$$

So finally we obtain :

$$\boxed{\frac{[p]}{\langle p\rangle} + \gamma\frac{[V]}{\langle V\rangle} = 0} \quad \text{II}$$

C. Equation I resembles the relation between the changes in internal energy and temperature (ideal gas law)

Equation II resembles the relation between the change in pressure and density of the fluid