Problem 1.4

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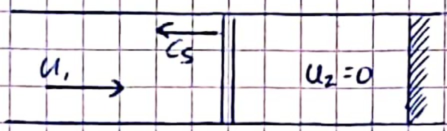
a. Mass conservation should hold

$AO = u_1 \Delta t$ and $BO = u_2 \Delta t$

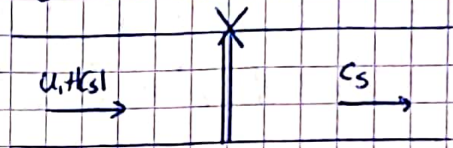
$u_1 \Delta t \rho_1 = u_2 \Delta t \rho_2$

$u_1 \rho_1 = u_2 \rho_2$

Laboratory frame:



Shock frame:



The strong shock limit implies the following:

$$\frac{\Delta p}{p_1} >> 1 \longrightarrow \frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} = \frac{\gamma+1}{\gamma-1}$$

In this case, $\gamma = 5/3$, while helium is a mono-atomic gas

$$\longrightarrow \frac{v_1}{v_2} = 4 \quad , \text{where} \begin{cases} v_1 = u_1 + |c_s| \\ v_2 = |c_s| \end{cases}$$

$$\frac{u_1 + |c_s|}{|c_s|} = 4 \quad , \quad u_1 + |c_s| = 4|c_s|$$

$|c_s| = \frac{1}{3} u_1$

We know that $c_s < 0$ (see laboratory frame)

$$\boxed{c_s = -\frac{1}{3} u_1}$$

b. $V_D [U] = [F]$

$$U = \begin{pmatrix} \rho \\ \rho u \\ \rho E \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} \rho u \\ p + \rho u^2 \\ \rho u H \end{pmatrix}$$

If we take the momentum jump equation:

$V_D (\rho_2 u_2 - \rho_1 u_1) = (p_2 + \rho_2 u_2^2) - (p_1 + \rho_1 u_1^2)$

In this case V_D is equal to the shock speed c_s

$c_s (\rho_2 u_2 - \rho_1 u_1) = (p_2 + \rho_2 u_2^2) - (p_1 + \rho_1 u_1^2)$

The post shock velocity $u_2 = 0$

$C_S(-\rho_1 u_1) = p_2 - (p_1 + \rho_1 u_1^2)$

From a. we have: $C_S = -\frac{1}{3} u_1$

$\frac{1}{3} \rho_1 u_1^2 = p_2 - (p_1 + \rho_1 u_1^2)$

$p_2 = \frac{4}{3} \rho_1 u_1^2 + p_1$

The speed of sound for an ideal gas is defined as:

$a_1 = \sqrt{\gamma \frac{p_1}{\rho_1}} \rightarrow p_1 = \frac{a_1^2 \rho_1}{\gamma}$

$\rightarrow p_2 = \frac{4}{3} \rho_1 u_1^2 + \frac{1}{\gamma} \rho_1 a_1^2$

$p_2 = \rho_1 (\frac{4}{3} u_1^2 + \frac{3}{5} a_1^2)$

$u_1 => a_1$, so we can discard the second term

$\boxed{p_2 = \frac{4}{3} \rho_1 u_1^2}$

c. Callorically perfect gas:

$p = (\gamma - 1) \rho e \rightarrow p = \frac{2}{3} \rho e \rightarrow e = \frac{3}{2} \frac{p}{\rho}$

$e_2 = \frac{3}{2} \frac{p_2}{\rho_2}$, from b. we know $p_2 = \frac{4}{3} \rho_1 u_1^2$

$\rightarrow e_2 = \frac{3}{2} \cdot \frac{4}{3} \cdot \frac{\rho_1}{\rho_2} \rho u_1^2 = \frac{3}{2} \cdot \frac{4}{3} \cdot \frac{1}{4} u_1^2$

$\boxed{e_2 = \frac{1}{2} u_1^2}$

d. The gas is brought to a complete rest, so all kinetic energy is converted into internal energy and the kinetic energy equation is the following:

$e_k = \frac{1}{2} u^2$