Faculty of Aerospace Engineering

AE4140 - Gas Dynamics

Task 1

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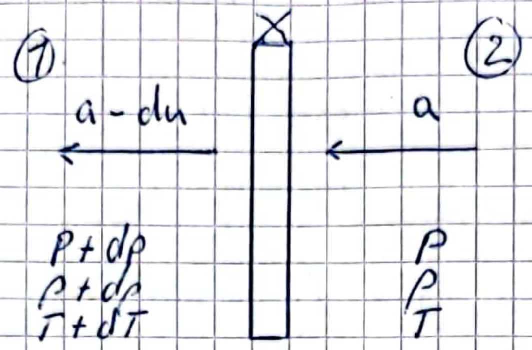
Problem 1.1

a



b

mass conservation in integral form:

$$\frac{d}{dt}\iiint_V \rho \, dV + \iint_S \rho \bar{V}\cdot\bar{n}\, dS = 0$$

Apply divergence theorem:

$$\iint_S \rho \bar{V}\cdot\bar{n}\, dS = \iiint_V \nabla\cdot\rho\bar{V}\, dV$$

$$\Longrightarrow \frac{d}{dt}\iiint_V \rho\, dV + \iiint_V \nabla\cdot\rho\bar{V}\, dV = 0$$

$$\Longrightarrow \frac{\partial \rho}{\partial t} + \nabla\cdot\rho\bar{V} = 0$$

The problem is one dimensional:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\rho\bar{V} = 0$$

And the problem is steady:

$$\frac{\partial \rho}{\partial t} = 0$$

$$\Longrightarrow \frac{\partial}{\partial x}\rho\bar{V} = 0 \Longrightarrow \rho\bar{V} = \text{constant}$$

$$(\rho + d\rho)(a - du) = \rho a$$

$$\rho a - \rho\, du + d\rho\, a - d\rho\, du = \rho a$$

We assume $d\rho\, du = 0$, because both terms are infinitesimally small

$$-\rho\, du + d\rho\, a = 0$$

$$\boxed{a = \rho \frac{du}{d\rho}}$$

Conservation of momentum in integral form

$$\frac{d}{dt}\iiint_V \rho\bar{v}\,dV + \iint_S \rho\bar{v}\bar{v}\cdot\bar{n}\,dS + \iint_S p\cdot\bar{n}\,dS \ldots$$

$$\ldots = \iiint_V \rho\bar{f}\,dV + \bar{F}_{viscous} + \bar{F}_{external}$$

Some terms are zero for this problem leaving:

$$\frac{d}{dt}\iiint_V \rho\bar{v}\,dV + \iint_S \rho\bar{v}\bar{v}\cdot\bar{n}\,dS + \iint_S p\cdot\bar{n}\,dS = 0$$

And after applying the divergence theorem:

$$\frac{d}{dt}\iiint_V \rho\bar{v}\,dV + \iiint_V \nabla\cdot\rho\bar{v}\bar{v}\,dV + \iiint_V \nabla\cdot p\,dV = 0$$

$$\frac{\partial}{\partial t}\rho\bar{v} + \bar{\nabla}\cdot\rho\bar{v}\bar{v} + \bar{\nabla}\cdot p\,dV = 0$$

Flow is steady and one dimensional:

$$\frac{\partial}{\partial x}\rho\bar{v}\bar{v} + \frac{\partial}{\partial x}p = 0$$

$$\Longrightarrow \quad \rho\bar{v}\bar{v} + p = \text{constant}$$

$$\underbrace{(\rho+d\rho)(a-du)^2}_{} + (p+dp) = \rho a^2 + p$$

$$= (\rho+d\rho)(a^2 - 2a\,du + \cancelto{=0}{du^2})$$

$$= \rho a^2 - 2\rho a\,du + d\rho a^2 - \cancelto{=0}{2d\rho a\,du}$$

$$= \rho a^2 - 2\rho a\,du + d\rho a^2$$

$$\longrightarrow \rho a^2 - 2\rho a\,du + d\rho a^2 + p + dp = \rho a^2 + p$$

$$-2\rho a\,du + d\rho a^2 + dp = 0$$

$$d\rho\, a^2 = 2\rho a\,du - dp$$

So now, from conservation of mass and momentum we have:

$$\underbrace{\boxed{a = \rho\frac{du}{d\rho}}}_{\text{conservation of mass}} \qquad \underbrace{\boxed{a^2 = 2\rho a\frac{du}{d\rho} - \frac{dp}{d\rho}}}_{\text{conservation of momentum}}$$

(1) (2)

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If we take equation ② and reorganize to:

$$a^2 = 2a\underbrace{\rho\frac{du}{d\rho}}_{=a \text{ , according to ①}} - \frac{dp}{d\rho}$$

$$\Longrightarrow \quad a^2 = 2a^2 - \frac{dp}{d\rho}$$

$$-a^2 = -\frac{dp}{d\rho}$$

$$\boxed{a^2 = dp/d\rho}$$

C <u>Energy equation in integral form:</u>

$$\frac{d}{dt}\iiint_V \rho E\, dV + \iint_S \rho E \bar{V}\cdot\bar{n}\, dS + \iint_S p\bar{V}\cdot\bar{n}\, dS \;\ldots$$

$$\ldots = \iiint_V \rho \bar{f}\cdot\bar{V}\, dV + \dot{Q} + \dot{W}_{viscous} + \dot{W}_{external}$$

Leaving the zero terms out and using divergence theorem leads to:

$$\frac{d}{dt}\iiint_V \rho E\, dV + \iiint_V \nabla\cdot\rho E\bar{V}\, dV + \iiint_V \nabla\cdot p\bar{V}\, dV = 0$$

$$\frac{\partial}{\partial t}\rho E + \nabla\cdot\rho E\bar{V} + \nabla\cdot p\bar{V} = 0$$

The problem is steady and one-dimensional:

$$\frac{\partial}{\partial x}\rho E\bar{V} + \frac{\partial}{\partial x}p\bar{V} = 0$$

$$\longrightarrow \rho E\bar{V} + p\bar{V} = \text{constant} \quad , \quad E = e + \tfrac{1}{2}q^2$$

$$(\rho+d\rho)\underbrace{(e + de + \tfrac{1}{2}(a-du)^2)(a-du)}_{} + (p+dp)(a-du) = \rho(e+\tfrac{1}{2}a^2)a + pa$$

$$= (e + de + \tfrac{1}{2}a^2 - a\,du)(a - du)$$

$$= ea + dea + \tfrac{1}{2}a^3 - a^2du - edu - \tfrac{1}{2}a^2du$$

$$= ea + dea + \tfrac{1}{2}a^3 - \tfrac{3}{2}a^2du - edu$$

$$\rightarrow \underbrace{(\rho+d\rho)(ea + dea + \tfrac{1}{2}a^3 - \tfrac{3}{2}a^2du - edu)}_{} + (p+dp)(a-du) = \rho(e+\tfrac{1}{2}a^2)a + pa$$

$$= ea\rho + dea\rho + \tfrac{1}{2}a^3\rho - \tfrac{3}{2}a^2du\rho - edu\rho + ead\rho + \tfrac{1}{2}a^3d\rho$$

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$$\rightarrow \; ea\rho + dea\rho + \tfrac{1}{2}a^3\rho - \tfrac{3}{2}a^2 du\rho - edu\rho + ead\rho + \tfrac{1}{2}a^3 d\rho \ldots$$

$$\ldots + \underbrace{(p+dp)(a-du)}_{= pa - pdu + dpa} = \underbrace{\rho(e+\tfrac{1}{2}a^2)a + pa}_{= \rho e a + \frac{1}{2}\rho a^3 + pa}$$

$$\rightarrow \; \cancel{ea\rho} + dea\rho + \cancel{\tfrac{1}{2}a^3\rho} - \tfrac{3}{2}a^2 du\rho - edu\rho + ead\rho + \tfrac{1}{2}a^3 d\rho \ldots$$

$$\ldots - pdu + dpa = \cancel{\rho e a} + \cancel{\tfrac{1}{2}\rho a^3}$$

$$\rightarrow \; dea\rho - \tfrac{3}{2}a^2 du\rho - edu\rho + ead\rho + \tfrac{1}{2}a^3 d\rho - pdu + dpa = 0$$

From conservation of mass, we know that:

$$du = a\frac{d\rho}{\rho}$$

$$\rightarrow \; dea\rho - \tfrac{3}{2}a^3 d\rho - ead\rho + ead\rho + \tfrac{1}{2}a^3 d\rho - pa\frac{d\rho}{\rho} + dpa = 0$$

$$de\rho - a^2 d\rho - \cancel{a}\,p\cancel{a}\frac{d\rho}{\rho} + dp = 0$$

And from b. we also know that:

$$a^2 = dp/d\rho$$

$$\rightarrow \; de\rho - dp - p\frac{d\rho}{\rho} + dp = 0$$

$$de\rho - p\frac{d\rho}{\rho} = 0$$

$$de - p\frac{d\rho}{\rho^2} = 0 = ds$$

$$\boxed{ds = de - p\frac{d\rho}{\rho^2} = 0}$$

Now, we can conclude that sound propagation is indeed an isentropic process

Problem 1.2

a. A steady shock, so:

$V_0 [U] = [F]$, where $V_0 = 0$

$\rightarrow [F] = 0$, $F_2 = F_1$

$$\rightarrow \begin{bmatrix} \rho_2 u_2 \\ p_2 + \rho_2 u_2^2 \\ \rho_2 u_2 H_2 \end{bmatrix} = \begin{bmatrix} \rho_1 u_1 \\ p_1 + \rho_1 u_1^2 \\ \rho_1 u_1 H_1 \end{bmatrix}$$

From mass conservation:

$$\rho_2 u_2 = \rho_1 u_1 \rightarrow \boxed{u_1 = \frac{\rho_2 u_2}{\rho_1}}$$

From momentum conservation:

$$p_2 + \rho_2 u_2^2 = p_1 + \rho_1 u_1^2$$

$$\rho_1 u_1^2 = p_2 - p_1 + \rho_2 u_2^2 \quad , \quad p_2 - p_1 = [p]$$

$$\rho_2 u_2^2 = \rho_1 u_1^2 - [p]$$

$$\rho_2 u_2^2 = \rho_1 \frac{\rho_2^2 u_2^2}{\rho_1^2} - [p]$$

$$\frac{\rho_2^2}{\rho_1} u_2^2 - \rho_2 u_2^2 = [p]$$

$$u_2^2 \left(\frac{\rho_2^2}{\rho_1} - \rho_2 \right) = [p]$$

$$u_2^2 = \frac{\rho_1}{\rho_2^2 - \rho_2 \rho_1} [p] \rightarrow \boxed{u_2^2 = \frac{\rho_1}{\rho_2} \frac{[p]}{\rho_2 - \rho_1}}$$

From conservation of energy:

$$\rho_2 u_2 H_2 = \rho_1 u_1 H_1 \quad , \quad H = h + \tfrac{1}{2} u^2$$

$$\rho_2 u_2 \left(h_2 + \tfrac{1}{2} u_2^2 \right) = \rho_1 u_1 \left(h_1 + \tfrac{1}{2} u_{01}^2 \right)$$

From mass conservation we know: $\rho_2 u_2 = \rho_1 u_1$

$$\rightarrow h_2 + \tfrac{1}{2} u_2^2 = h_1 + \tfrac{1}{2} u_1^2 \quad , \quad h = e + \frac{p}{\rho}$$

$$e_2 + \frac{p_2}{\rho_2} + \tfrac{1}{2} u_2^2 = e_1 + \frac{p_1}{\rho_1} + \tfrac{1}{2} u_1^2$$

$$[e] + \frac{p_2}{\rho_2} - \frac{p_1}{\rho_1} = \tfrac{1}{2} \left(u_1^2 - u_2^2 \right)$$

$$= \tfrac{1}{2} \left(\frac{\rho_2}{\rho_1} \frac{[p]}{\rho_2 - \rho_1} - \frac{\rho_1}{\rho_2} \frac{[p]}{\rho_2 - \rho_1} \right)$$

$$= \tfrac{1}{2} \left(\frac{\rho_2^2 - \rho_1^2}{\rho_2 \rho_1} \frac{[p]}{\rho_2 - \rho_1} \right)$$

$$= \tfrac{1}{2} \left(\frac{(\rho_2 + \rho_1)(\rho_2 - \rho_1)}{\rho_2 \rho_1} \frac{[p]}{\rho_2 - \rho_1} \right)$$

$$[e] + \frac{p_2}{\rho_2} - \frac{p_1}{\rho_1} = \frac{[p]}{2}\left(\frac{(\rho_2+\rho_1)(\rho_2-\rho_1)}{\rho_2\rho_1(\rho_2-\rho_1)}\right)$$

$$[e] + \frac{p_2}{\rho_2} - \frac{p_1}{\rho_1} = \frac{[p]}{2}\,\frac{\rho_2+\rho_1}{\rho_2\rho_1}$$

$$[e] + \frac{p_2}{\rho_2} - \frac{p_1}{\rho_1} - \frac{[p]}{2}\,\frac{\rho_2+\rho_1}{\rho_2\rho_1} = 0$$

$$[e] + \frac{p_2}{\rho_2} - \frac{p_1}{\rho_1} - \frac{1}{2}p_2\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right) + \frac{1}{2}p_1\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right) = 0$$

$$[e] + \frac{p_2}{\rho_2} - \frac{p_1}{\rho_1} - \frac{1}{2}\frac{p_2}{\rho_1} - \frac{1}{2}\frac{p_2}{\rho_2} + \frac{1}{2}\frac{p_1}{\rho_1} + \frac{1}{2}\frac{p_1}{\rho_2} = 0$$

$$[e] + \frac{1}{2}\frac{p_2}{\rho_2} - \frac{1}{2}\frac{p_1}{\rho_1} - \frac{1}{2}\frac{p_2}{\rho_1} + \frac{1}{2}\frac{p_1}{\rho_2} = 0$$

$$[e] + \frac{1}{2}\left(\frac{p_2+p_1}{\rho_2} - \frac{p_2+p_1}{\rho_1}\right) = 0$$

$$[e] + \frac{1}{2}(p_2+p_1)\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right) = 0$$

$$\boxed{[e] + \langle p\rangle [v] = 0} \quad \text{I}$$

b. $p = (\gamma-1)\rho e \rightarrow e = \frac{p}{(\gamma-1)\rho}$

$$[e] = \frac{[p]}{(\gamma-1)[\rho]} = \frac{[p][v]}{(\gamma-1)}$$

We use the following property:

$$[p][v] = \langle p\rangle [v] + \langle v\rangle [p]$$

$$\rightarrow [e] = \frac{1}{(\gamma-1)}\left(\langle p\rangle [v] + \langle v\rangle [p]\right)$$

Now, we plut this in the equation from a.:

$$\frac{1}{(\gamma-1)}\left(\langle p\rangle [v] + \langle v\rangle [p]\right) + \langle p\rangle [v] = 0$$

$$\frac{1}{(\gamma-1)}\left(\langle p\rangle [v]\right) + \langle p\rangle [v] + \frac{1}{\gamma-1}\left(\langle v\rangle [p]\right) = 0$$

$$\left(\langle p\rangle [v]\right)\underbrace{\left(\frac{1}{\gamma-1}+1\right)}_{=\frac{\gamma}{\gamma-1}} + \left(\langle v\rangle [p]\right)\left(\frac{1}{\gamma-1}\right) = 0$$

$$\frac{\langle p \rangle}{[p]} \frac{\gamma}{\gamma - 1} + \frac{\langle V \rangle}{[V]} \frac{1}{\gamma - 1} = 0$$

$$\gamma \frac{\langle p \rangle}{[p]} + \frac{\langle V \rangle}{[V]} = 0$$

So finally we obtain :

$$\boxed{\frac{[p]}{\langle p \rangle} + \gamma \frac{[V]}{\langle V \rangle} = 0} \quad \text{(II)}$$

C. Equation (I) resembles the relation between the changes in internal energy and temperature (ideal gas law)

Equation (II) resembles the relation between the change in pressure and density of the fluid

Problem 1.3

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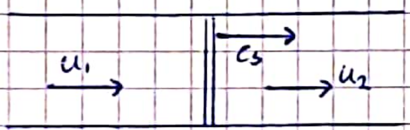
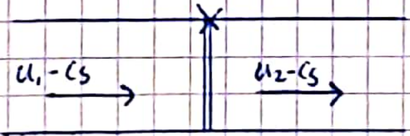
A few things should be noted that hold for problem 1.3.a – 1.3.cs. :

- We're considering a strong shock, so $\frac{\rho_2}{\rho_1} = 6$ and from conservation of mass we can therefor also say that the velocity in pre-shock state 1 and post-shock state 2 (V_i) has the following ratio in shock frame:
  $\frac{V_1}{V_2} = 6$
- The entropy condition must hold to have a valid shock

a. Laboratory frame:

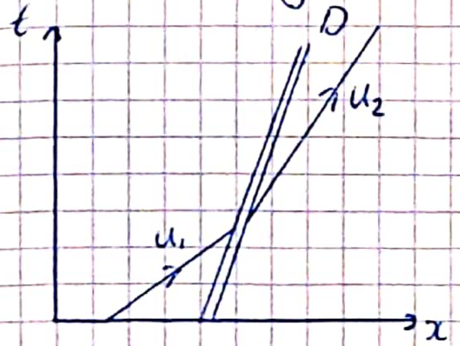


Shock frame:



velocity ratios:

t, x - diagram:



$c_s > 0, \quad u_1 > 0, \quad u_2 > 0$

$V_1 = u_1 - c_s$
$V_2 = u_2 - c_s$

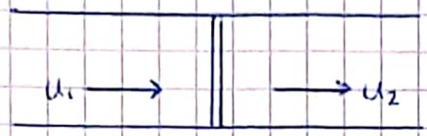
$\left.\frac{u_1 - c_s}{u_2 - c_s} = 6 \right\} \begin{array}{l} c_s = 1 \\ u_1 = 7 \\ u_2 = 2 \end{array}$

Entropy condition:

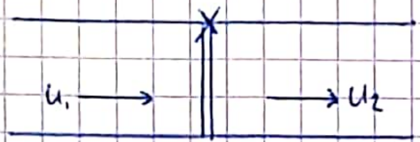
$(u_1 - a_1) > c_s > (u_2 - a_2)$

where, $- |u_1| > a_1$
$- |u_2| < a_2$

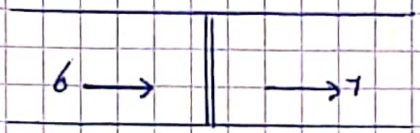
b. Laboratory frame:



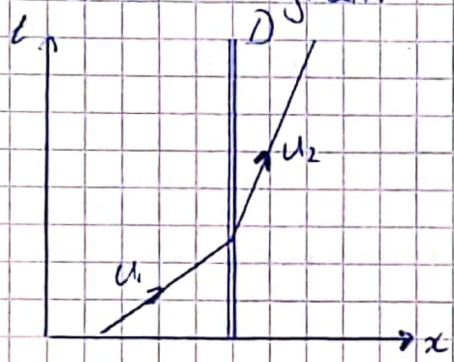
Shock frame:



Velocity ratios:



t,x - diagram:



$c_s = 0, \quad u_1 > 0, \quad u_2 > 0$

$V_1 = u_1$
$V_2 = u_2$

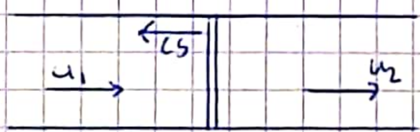
$\frac{u_1}{u_2} = 6 \Big\} \quad \begin{matrix} u_1 = 6 \\ u_2 = 1 \end{matrix}$

Entropy condition:

$|u_1| > a_1$
$|u_2| < a_1$

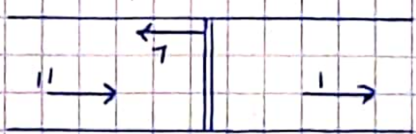
$(u_1 - a_1) > 0 > (u_2 - a_2)$

c1. Laboratory frame:



Shock frame:

Velocity ratios



$c_s \neq 0, \quad u_1 > 0, \quad u_2 > 0$

$V_1 = u_1 + c_s$
$V_2 = u_2 + c_s$

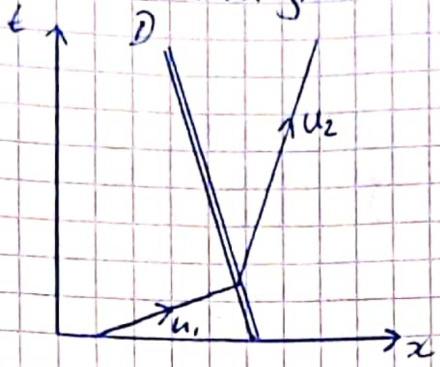
$\frac{u_1 + |c_s|}{u_2 + |c_s|} = 6 \Big\} \quad \begin{matrix} |c_s| = 1 \\ u_1 = 11 \\ u_2 = 1 \end{matrix}$

Entropy condition:

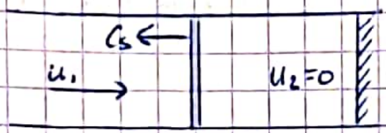
$|u_1| > a_1$
$|u_2| < a_2$

$(a_1 - u_1) < |c_s| < (a_2 - u_2)$

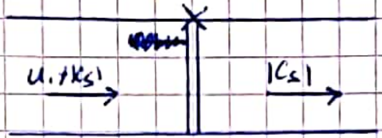
t, x - diagram:



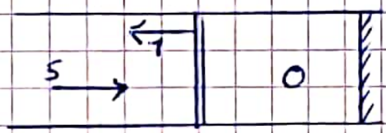
C_2. Laboratory frame:



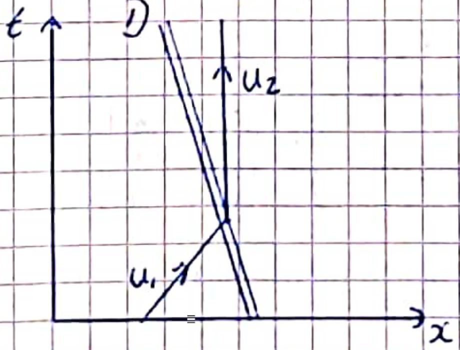
Shock frame:



velocity ratios:



t, x - diagram:



$c_s < 0, \quad u_1 > 0, \quad u_2 = 0$

$v_1 = u_1 + |c_s|$

$v_2 = |c_s|$

$$\left.\frac{u_1 + |c_s|}{|c_s|} = 6 \right\} \quad \begin{aligned} |c_s| &= 1 \\ u_1 &= 5 \\ u_2 &= 0 \end{aligned}$$

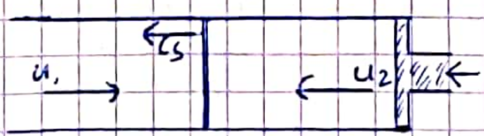
Entropy condition:

$|u_1| > a_1$

$|u_2| < a_2$

$(a_1 - u_1) < |c_s| < a_2$

C_3. Laboratory frame:



Shock frame:

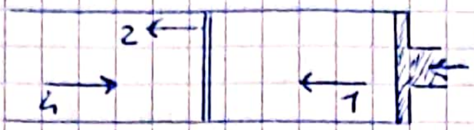
$c_s < 0, \quad u_1 > 0, \quad u_2 < 0$

$v_1 = c_s + u_1$

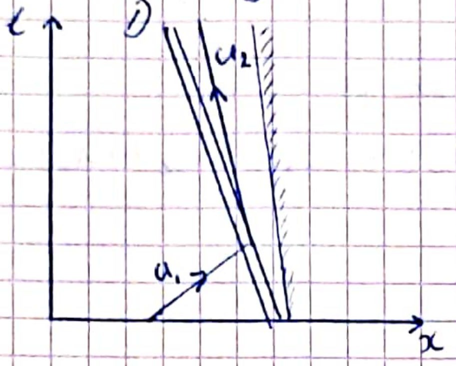
$v_2 = c_s - u_1$

$$\left.\frac{c_s + u_1}{c_s - u_1} = 6 \right\} \quad \begin{aligned} c_s &= 2 \\ u_1 &= 4 \\ u_2 &= 1 \end{aligned}$$

velocity ratios



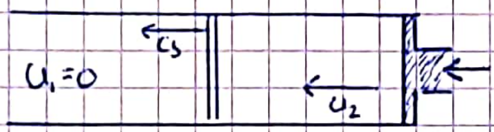
t,x - diagram:



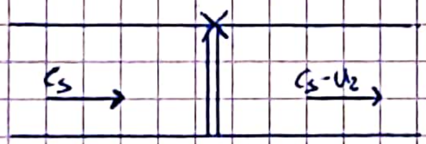
Entropy condition

$a_1 - u_1 < |c_s| < a_2 + |u_2|$

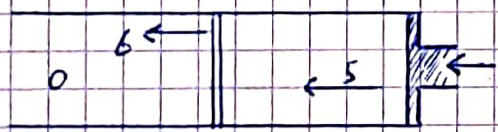
C_s. Laboratory frame:



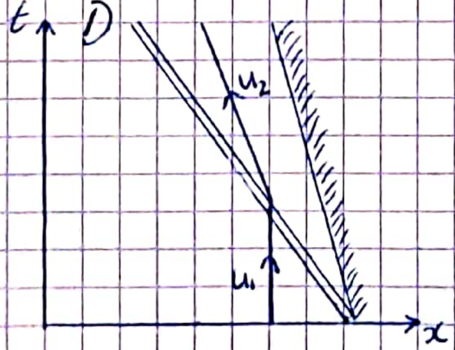
Shock frame:



velocity ratios:



t,x - diagram:



$c_s < 0$, $u_1 = 0$, $u_2 < 0$

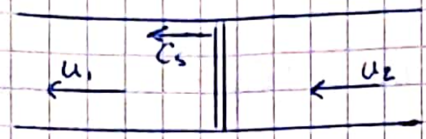
$v_1 = c_s$
$v_2 = c_s - u_2$

$$\left.\frac{c_s}{c_s - u_2} = 6 \right\} \begin{aligned} c_s &= 6 \\ u_1 &= 0 \\ u_2 &= 5 \end{aligned}$$

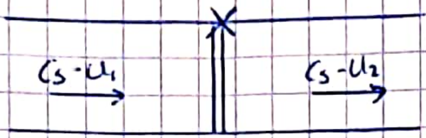
Entropy condition:

$a_1 < |c_s| < a_2 + |u_2|$

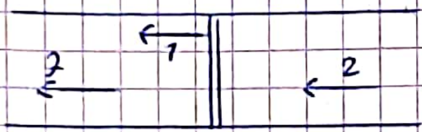
C5. Laboratory frame



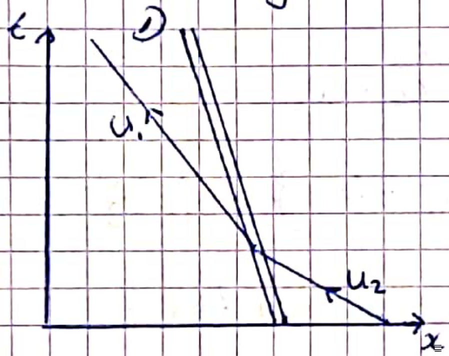
Shock frame:



velocity ratios



t,x - diagram



$c_s < 0, \quad u_1 < 0, \quad u_2 < 0$

$v_1 = c_s - u_1$

$v_2 = c_s - u_2$

$$\left.\frac{c_s - u_1}{c_s - u_2} = 6 \quad \right\} \quad \begin{array}{l} c_s = 1 \\ u_1 = 7 \\ u_2 = 2 \end{array}$$

Entropy condition:

$a_1 + |u_1| < |c_s| < a_2 + |u_2|$

Problem 1.4

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a. Mass conservation should hold

$AO = u_1 \Delta t$ and $BO = u_2 \Delta t$

$u_1 \Delta t \rho_1 = u_2 \Delta t \rho_2$

$u_1 \rho_1 = u_2 \rho_2$

Laboratory frame: Shock frame:

The strong shock limit implies the following:

$$\frac{\Delta \rho}{\rho_1} >> 1 \longrightarrow \frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} = \frac{\gamma+1}{\gamma-1}$$

In this case, $\gamma = 5/3$, while helium is a mono-atomic gas.

$$\rightarrow \frac{v_1}{v_2} = 4 \quad , \text{where} \begin{cases} v_1 = u_1 + |c_s| \\ v_2 = |c_s| \end{cases}$$

$$\frac{u_1 + |c_s|}{|c_s|} = 4 \quad , \quad u_1 + |c_s| = 4|c_s|$$

$|c_s| = \frac{1}{3} u_1$

We know that $c_s < 0$ (see laboratory frame)

$$\boxed{c_s = -\frac{1}{3} u_1}$$

b. $V_D [U] = [F]$

$$U = \begin{pmatrix} \rho \\ \rho u \\ \rho E \end{pmatrix} \text{ and } F = \begin{pmatrix} \rho u \\ p + \rho u^2 \\ \rho u H \end{pmatrix}$$

If we take the momentum jump equation:

$$V_D (\rho_2 u_2 - \rho_1 u_1) = (p_2 + \rho_2 u_2^2) - (p_1 + \rho_1 u_1^2)$$

In this case V_D is equal to the shock speed c_s

$$c_s (\rho_2 u_2 - \rho_1 u_1) = (p_2 + \rho_2 u_2^2) - (p_1 + \rho_1 u_1^2)$$

Scanned with CamScanner

The post shock velocity $u_2 = 0$

$c_s(-\rho_1 u_1) = p_2 - (p_1 + \rho_1 u_1^2)$

From a. we have: $c_s = -\frac{1}{3}u_1$

$\frac{1}{3}\rho_1 u_1^2 = p_2 - (p_1 + \rho_1 u_1^2)$

$p_2 = \frac{4}{3}\rho_1 u_1^2 + p_1$

The speed of sound for an ideal gas is defined as:

$a_1 = \sqrt{\gamma \frac{p_1}{\rho_1}} \rightarrow p_1 = \frac{a_1^2 \rho_1}{\gamma}$

$\rightarrow p_2 = \frac{4}{3}\rho_1 u_1^2 + \frac{1}{\gamma}\rho_1 a_1^2$

$p_2 = \rho_1(\frac{4}{3}u_1^2 + \frac{3}{5}a_1^2)$

$u_1 => a_1$, so we can discard the second term

$\boxed{p_2 = \frac{4}{3}\rho_1 u_1^2}$

c. Callorically perfect gas:

$p = (\gamma - 1)\rho e \rightarrow p = \frac{2}{3}\rho e \rightarrow e = \frac{3}{2}\frac{p}{\rho}$

$e_2 = \frac{3}{2}\frac{p_2}{\rho_2}$, from b. we know $p_2 = \frac{4}{3}\rho_1 u_1^2$

$\rightarrow e_2 = \frac{3}{2}\cdot\frac{4}{3}\cdot\frac{\rho_1}{\rho_2}\rho u_1^2 = \frac{3}{2}\cdot\frac{4}{3}\cdot\frac{1}{4}u_1^2$

$\boxed{e_2 = \frac{1}{2}u_1^2}$

d. The gas is brought to a complete rest, so all kinetic energy is converted into internal energy and the kinetic energy equation is the following:

$e_k = \frac{1}{2}u^2$

Problem 2.1

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a. Unperturbed regions : 1, 4, 6 $\rightarrow$ $M = \tilde{S} = 0$

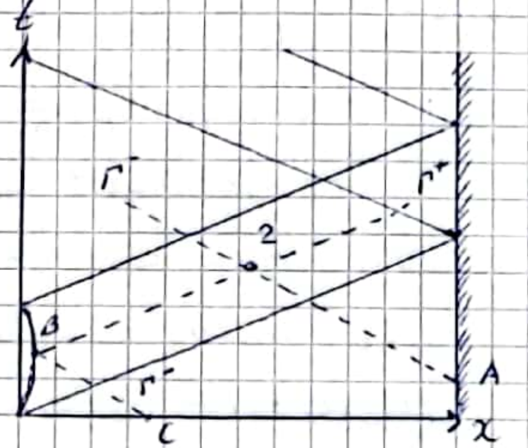
Simple wave regions : 2 $\Rightarrow$ only J^+ varies

5 $\rightarrow$ only J^- varies

Non-simple wave regions : 3 $\Rightarrow$ both J^- and J^+ vary

b. J^- : $M_A - \tilde{S}_A = M_2 - \tilde{S}_2 = 0$

$\rightarrow M_2 = \tilde{S}_2$



J^+ : $M_B + \tilde{S}_B = M_2 + \tilde{S}_2 = 2M_2$

J^- : $M_C - \tilde{S}_C = M_B - \tilde{S}_C = 0$

$M_B = \tilde{S}_C$

$\rightarrow 2M_B = 2M_2$

$\hookrightarrow M_B = M_p = \frac{u_p}{a_0}$

$M_2(x_2, t_2) = M_p(x_p, t_p)$

Γ^+ : $x_2 - a_0 t_2 = x_p - a_0 t_p$, where $x_p = 0$

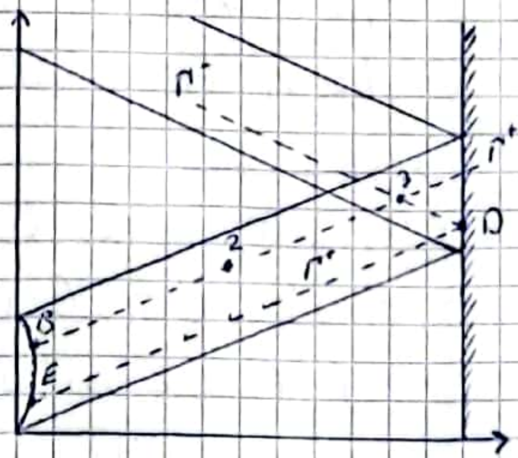
$\rightarrow$ ~~x_p~~ $t_p = t_2 - \frac{x_2}{a_0}$

$M_2(x_2, t_2) = M_p(0, t_2 - \frac{x_2}{a_0}) = \varepsilon \sin[\omega(t_2 - \frac{x_2}{a_0})]$

$$\boxed{M_2(x_2, t_2) = \tilde{S}(x_2, t_2) = \varepsilon \sin[\omega(t_2 - \frac{x_2}{a_0})]}$$

c. J^- : $M_3 - \tilde{S}_3 = M_D - \tilde{S}_D = -\tilde{S}_D$

$M_3 = \tilde{S}_3 - \tilde{S}_D$ (1)



J^+ : $M_2 + \tilde{S}_2 = M_3 + \tilde{S}_3 = 2M_p$

$M_3 = 2M_p - \tilde{S}_3$ (2)

Combine eq. (1) and (2)

$\rightarrow M_3 = -\tilde{S}_D/2 + M_p$

Now, we need an expression for $\tilde{S}_0$

J^+ : $M_0 + \tilde{S}_0 = M_E + \tilde{S}_E$, $M_0 = 0$

$\tilde{S}_0 = M_E + \tilde{S}_E = 2M_p$

$\tilde{S}_0(x_0, t_0) = 2M_p(x_p, t_p)$

Γ^+ : $x_0 - a_0 t_0 = x_p - a_0 t_p$, $x_0 = L$ and $x_p = 0$

$\rightarrow t_p = t_0 - \frac{L}{a_0}$ and $t_0 = t_p + \frac{L}{a_0}$

Γ^- : $x_0 + a_0 t_0 = x_3 + a_0 t_3$

$L + a_0(t_p + \frac{L}{a_0}) = x_3 + a_0 t_3$

$\rightarrow t_p = (t_3 + \frac{x_3}{a_0} - \frac{2L}{a_0})$

So, know we have for $\tilde{S}_0$:

$S_0(x_0, t_0) = 2M_p(0, t_3 + \frac{x_3}{a_0} - \frac{2L}{a_0})$

$M_3(x_3, t_3) = M_p(0, t_3 - \frac{x_3}{a_0}) - M_p(0, t_3 + \frac{x_3}{a_0} - \frac{2L}{a_0})$

Remember that $M_p = \frac{u_p}{a_0}$, so our solution is:

$$\boxed{M_3(x_3, t_3) = \mathcal{E}\sin[\omega(t_3 - \frac{x_3}{a_0})] - \mathcal{E}\sin[\omega(t_3 + \frac{x_3}{a_0} - \frac{2L}{a_0})]}$$

d. We need the following relation to compute the pressure:

$\Delta p = a_0^2 \Delta\rho$, where $\Delta\rho = \tilde{S}\rho_0$

$\rightarrow \Delta p = a_0^2 \rho_0 \tilde{S}$

$p = p_0 + \Delta p = p_0 + a_0^2 \rho_0 \tilde{S} = p_0 + p_0 \tilde{S}$

$p = p_0(1 + \tilde{S})$

- Problem 1 : At the piston $(x = 0)$

Region 2:

$\tilde{S}(x_2, t_2) = M(x_2, t_2) = \mathcal{E}\sin(\omega(t_2 - \frac{x_2}{a_0}))$

$S(0, t_2) = \mathcal{E}\sin(\omega t_2)$, $0 \leq t_2 \leq \frac{2\pi}{\omega}$

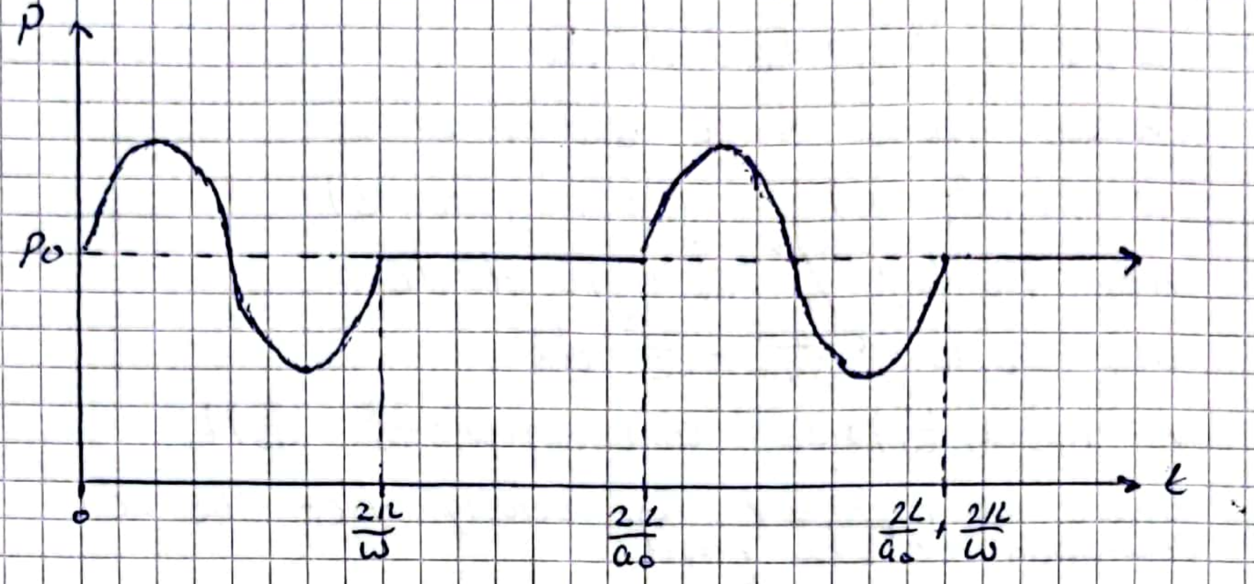
$p = p_0(1 + \mathcal{E}\sin(\omega t))$, $0 \leq t \leq \frac{2\pi}{\omega}$

Region 4: Unperturbed region

$\tilde{S}(0, t) = 0$, $\frac{2\pi}{\omega} \leq t \leq \frac{2L}{a_0}$

$p = p_0$, $\frac{2\pi}{\omega} \leq t \leq \frac{2L}{a_0}$

This will be repeated as the wave bounces back and forth in the tube, so the pressure signal will look like this:



- Problem ②: At the wall $(x=L)$

<u>Region 1:</u> Unperturbed

$\tilde{S}(x_1,t_1)=0$, $\tilde{S}(L,t)=0$, $0 \le t \le \frac{L}{a_0}$

$p=p_0$, $0 \le t \le \frac{L}{a_0}$

<u>Region 3:</u>

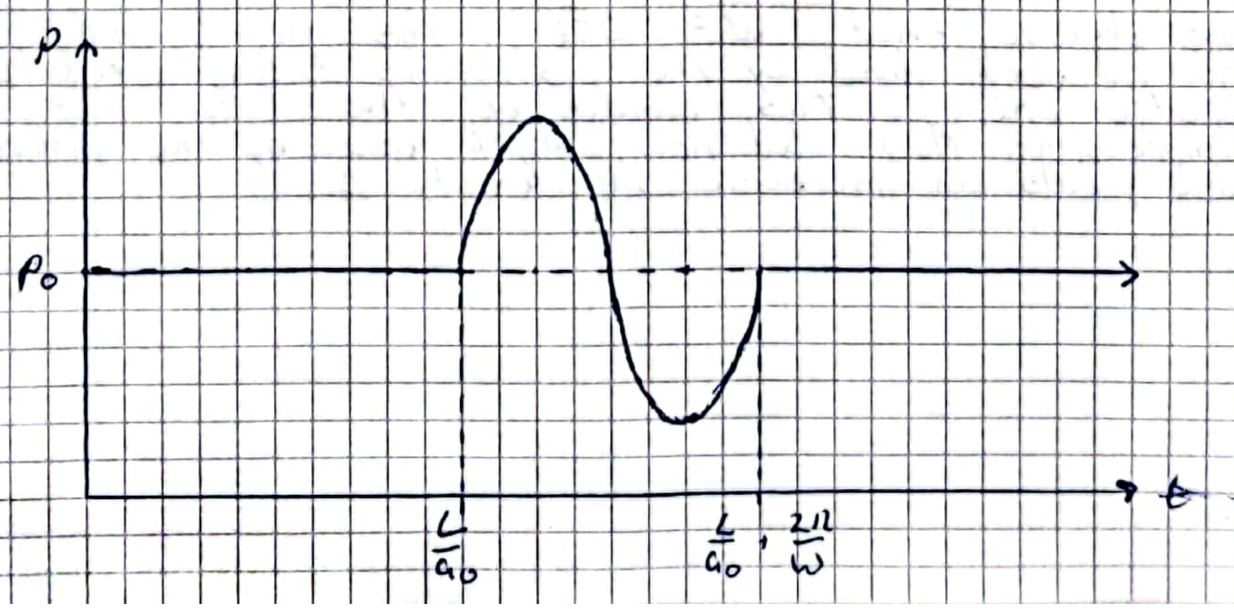
$\tilde{S}_0 = 2m_p$ (from <u>C.</u>)

$\tilde{S}_0 = 2\varepsilon \sin(\omega(t_3+\frac{x_3}{a_0}-\frac{2L}{a_0}))$

$\tilde{S}_3(L,t) = 2\varepsilon \sin(\omega(t_3-\frac{L}{a_0}))$, $\frac{L}{a_0} \le t \le \frac{L}{a_0}+\frac{2\pi}{\omega}$

<u>Region 6:</u> Unperturbed

$\tilde{S}(x_6,t_6)=0$, $\tilde{S}(L,t)=0$, $\frac{L}{a_0}+\frac{2\pi}{\omega} \le t \le \frac{3L}{a_0}$



e. First nothing, then compression and expansion. This repeats.

f. • Kinetic Energy

$$\Delta E_k = \int_0^x \int_0^v \rho A_t \bar{v} \, d\bar{v} \, d\bar{x}$$

where mass is a function of v

$$\rho = \rho_0 \left(\frac{v}{a_0} + 1\right) \quad \text{(in region 2)}$$

$$\Delta E_k = \rho_0 A_t \int_0^x \int_0^v \left(\frac{\bar{v}}{a_0} + 1\right) \bar{v} \, d\bar{v} \, d\bar{x}$$

$$\Delta E_k = \rho_0 A_t \int_0^x \frac{v^3}{3a_0} + \frac{v^2}{2} \, d\bar{x}$$

$$v = M_2\left(x, \frac{2\pi}{\omega}\right) \cdot a_0 = a_0 \varepsilon \sin\left(\omega\left(\frac{2\pi}{\omega} - \frac{x}{a_0}\right)\right)$$

where v represent the velocity after the acceleration of the fluid.

$$\Delta E_k = \rho_0 A_t a_0^2 \varepsilon^2 \int_0^x \frac{1}{3} \varepsilon \sin^3\left(\omega\left(\frac{2\pi}{\omega} - \frac{\bar{x}}{a_0}\right)\right) + \frac{1}{2} \sin^2\left(\omega\left(\frac{2\pi}{\omega} - \frac{\bar{x}}{a_0}\right)\right) d\bar{x}$$

I evaluated this integral using an online tool and put in $x = 2\pi a_0/\omega$ (distance until $t = 2\pi/\omega$)

This yields:

$$\boxed{\Delta E_k = \frac{A a_0^3 \varepsilon^2 \rho_0}{2\omega}}$$

• Work Done

$$W = \int F \, dx = \int A_t p \, dx = \int A p = A \int_0^{2\pi/\omega} p(t) u(t) \, dt$$

$$= A_t \int_0^{2\pi/\omega} (p_0 + a_0^2 \rho_0 \varepsilon \sin(\omega t)) \varepsilon a_0 \, dt$$

$$\boxed{W = \frac{A_t a_0^3 \varepsilon^2 \rho_0}{\omega}}$$

We observe that $W = 2 \Delta E_k$, this means that not all work done by the piston is transferred to kinetic energy. This would be the case for incompressible flow. But now, work done by the piston also yields an increase in density ρ.

Problem 2.2

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a. Diagram 1:

$m_0 = 0$, $\tilde{S}_0 = 0$

$m_1 = \varepsilon$, $\tilde{S}_1 = \varepsilon$

$m_2 = \varepsilon$, $\tilde{S}_2 = -\varepsilon$

$m_4 = 2\varepsilon$, $\tilde{S}_4 = 0$

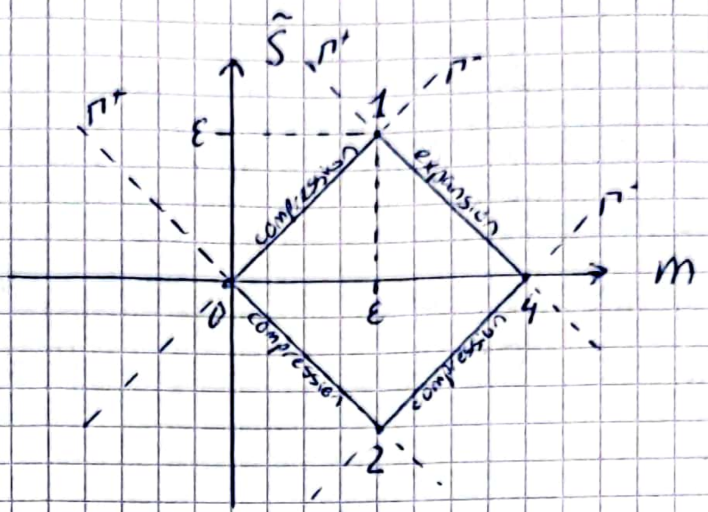


Diagram 2:

$m_1 = \varepsilon$, $\tilde{S}_1 = \varepsilon$

$m_3 = 0$, $\tilde{S}_3 = 0$

$m_4 = 2\varepsilon$, $\tilde{S}_4 = 0$

$m_6 = \varepsilon$, $\tilde{S}_6 = -\varepsilon$

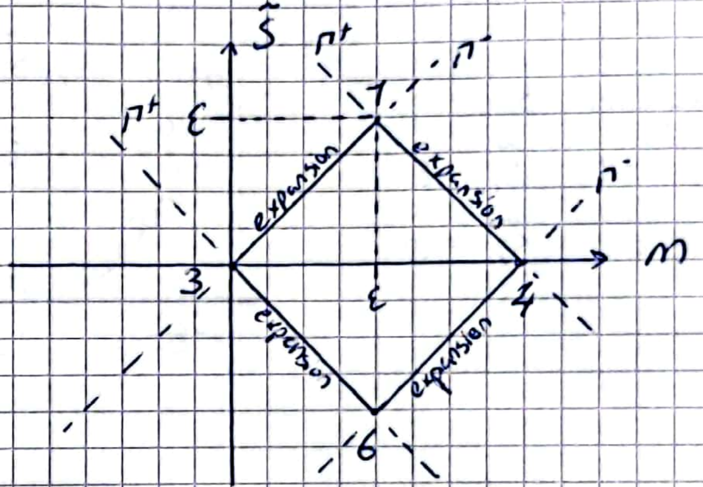


Diagram 3:

$m_2 = \varepsilon$, $\tilde{S}_2 = -\varepsilon$

$m_4 = 2\varepsilon$, $\tilde{S}_4 = 0$

$m_5 = 0$, $\tilde{S}_5 = 0$

$m_7 = \varepsilon$, $\tilde{S}_7 = \varepsilon$

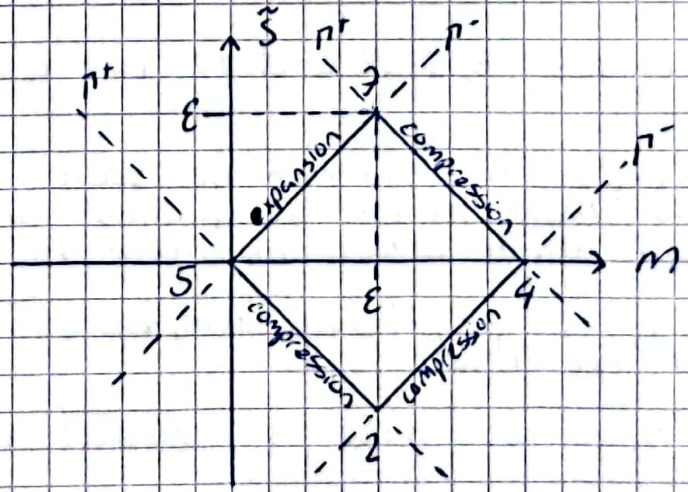


Diagram 4:

$M_6 = \varepsilon, \quad \tilde{S}_6 = -\varepsilon$

$M_7 = \varepsilon, \quad \tilde{S}_7 = \varepsilon$

$M_8 = 0, \quad \tilde{S}_8 = 0$

$M_9 = 2\varepsilon, \quad \tilde{S}_9 = 0$

$M_{10} = -\varepsilon, \quad \tilde{S}_{10} = \varepsilon$

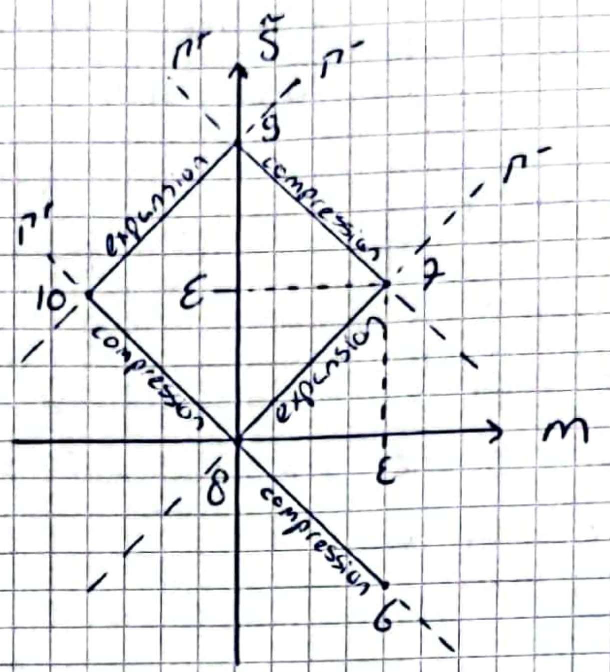
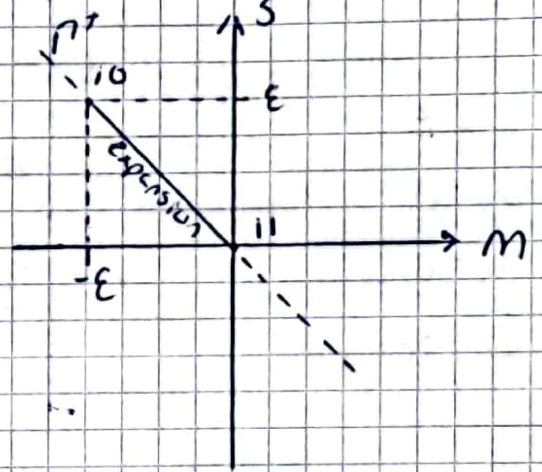


Diagram 5:

$M_{10} = -\varepsilon, \quad \tilde{S}_{10} = \varepsilon$

$M_{11} = 0, \quad \tilde{S}_{11} = 0$



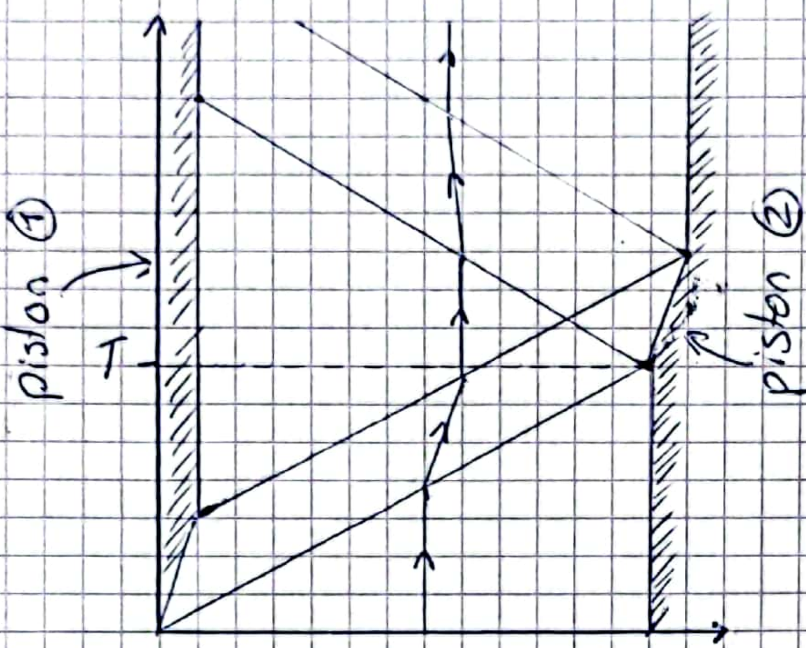
b. Regions 3, 5, 8 and 11 are also unpurturbed

c. Region 4. As can be seen in the diagrams above the Mach number is highest here. Both pistons comple-ment each others movement in region 4.

d. Region 9. Highest compression, because of reflection of the wall

e. Region 10. Wave is reflected of wall

f.



Activating piston ② when the wave from piston ① reaches it. The number of waves are minimized

Problem 2.3

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a. We need two basic relations to solve the problem

$$a^2 = \frac{\gamma p}{\rho} \quad ①$$

$$p = C\rho^{\gamma} \quad ② \quad \text{(Poisson)}$$

Both equations can be used under the assumption of an ideal and perfect gas

Combining ① and ② yields:

$$a^2 = C\gamma \frac{\rho^\gamma}{\rho} = C\gamma\rho^{(\gamma-1)}$$

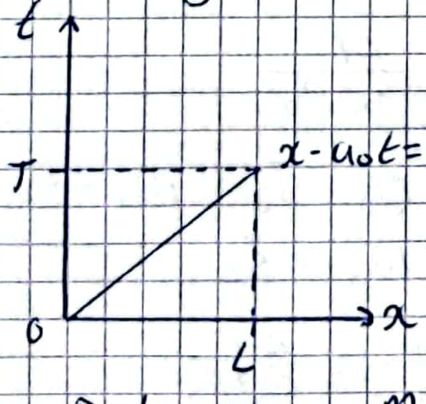
From this relation it follows that a will increase when density ρ increases. So:

when $\rho_4 > \rho_0$, $a_4 > a_0$

when $\rho_0 > \rho_4$, $a_0 > a_4$

b. The c.d. is not moving, because both region 0 and 4 are unperturbed $\rightarrow u_1 = u_4 = 0$
The particle paths go straight up and the c.d. does not move.

The time it takes before the c.d. starts moving can be obtained from the sketch below



$x - a_0 t = 0 \rightarrow L - a_0 T = 0 \rightarrow \boxed{T = \frac{L}{a_0}}$

Determining $m_1(x_1, t_1)$ and $\tilde{S}_1(x_1, t_1)$:

J^-: $m_1 - \tilde{S}_1 = m_0 - \tilde{S}_0 = 0$

$m_1 = \tilde{S}_1$

J^-: $m_p - \tilde{S}_p = m_0 - \tilde{S}_0 = 0$

$m_p = \tilde{S}_p$

$J^+ : \quad m_1 + \tilde{S}_1 = m_p + \tilde{S}_p$

$2m_p = 2m_1$

$m_1 = m_p = \frac{u_p}{a_0} = \varepsilon$

$$\boxed{m_1(x_1, t_1) = \varepsilon \quad \text{and} \quad \tilde{S}_1(x_1, t_1) = \varepsilon}$$

$u_1 = m_1 a_0 = \varepsilon a_0$

$$\boxed{u_1 = u_p}$$

$p_1 = p_0 + \Delta p = p_0 + a_0^2 \Delta \rho = p_0 + a_0^2 \rho_0 \tilde{S}_1$

$$\boxed{p_1 = p_0 + a_0^2 \rho_0 \varepsilon}$$

C. $\quad m_2 = \frac{u_2}{a_0} \quad , \quad m_3 = \frac{u_3}{a_4}$

$u_2 = u_3 = u_{c,D}$

$\rightarrow \quad m_2 = \frac{u_{c,D}}{a_0} \quad , \quad m_3 = \frac{u_{c,D}}{a_4}$

$\tilde{S}_2 = \frac{\Delta \rho_2}{\rho_0} = \frac{\Delta p_2}{a_0^2 \rho_0} \quad , \quad \tilde{S}_3 = \frac{\Delta \rho_3}{\rho_4} = \frac{\Delta p_3}{a_4^2 \rho_4}$

$\tilde{S}_2 = \frac{p_2 - p_0}{a_0^2 \rho_0} \quad , \quad \tilde{S}_3 = \frac{p_3 - p_4}{a_4^2 \rho_4}$

$p_2 = p_3$ and $p_0 = p_4$

$\rightarrow \quad \tilde{S}_2 = \frac{\Delta p}{a_0^2 \rho_0} \quad , \quad \tilde{S}_3 = \frac{\Delta p}{a_4^2 \rho_4}$

$\Gamma^+ : \quad m_2 + \tilde{S}_2 = m_1 + \tilde{S}_1 \quad , \quad \Gamma^- : \quad m_3 - \tilde{S}_3 = m_4 - \tilde{S}_4$

$m_2 + \tilde{S}_2 = 2\varepsilon \qquad m_3 = \tilde{S}_3$

$\frac{u_{c,D}}{a_0} + \frac{\Delta p}{a_0^2 \rho_0} = 2\varepsilon \quad (1), \qquad \underbrace{\frac{u_{c,D}}{a_4} = \frac{\Delta p}{a_4^2 \rho_4}}$

$\Delta p = a_4 \rho_4 u_{c,D} \quad (2)$

Put (2) in (1)

$\frac{u_{c,D}}{a_0} + \frac{a_4 \rho_4 u_{c,D} \Delta p}{a_0^2 \rho_0} = 2\varepsilon$

$$u_{c,D}\left(\frac{1}{a_0}+\frac{a_4\rho_4}{a_0^2\rho_0}\right)=2\varepsilon$$

$$u_{c,D}\left(\frac{a_0\rho_0}{a_0^2\rho_0}+\frac{a_4\rho_4}{a_0^2\rho_0}\right)=2\varepsilon$$

$$u_{c,D}\left(\frac{a_0\rho_0+a_4\rho_4}{a_0^2\rho_0}\right)=2\varepsilon$$

$$u_{c,D}=\frac{a_0^2\rho_0}{a_0\rho_0+a_4\rho_4}\,2\varepsilon \quad \text{where} \quad 2\varepsilon=\frac{u_p}{a_0}$$

$$\boxed{u_{c,D}=\frac{a_0\rho_0}{\frac{1}{2}(a_0\rho_0+a_4\rho_4)}\,u_p}$$

d. First we consider $\rho_4=\rho_0$

Region 3: $u_3=\dfrac{a_0\rho_0}{\frac{1}{2}(a_0\rho_0+a_4\rho_4)}\,u_p$

wave continues

$$M_3=\frac{a_0\rho_0}{\frac{1}{2}(a_0\rho_0+a_4\rho_4)}\,\frac{u_p}{a_4}$$

$$M_3=\frac{u_p}{a_4}=\frac{u_p}{a_0}$$

$$M_3=M_1$$

Region 6: $M_6=\dfrac{a_0\rho_0}{\frac{1}{2}(a_0\rho_0+a_4\rho_4)}-\varepsilon$

unperturbed region

$$M_6=\frac{u_p\rho_0}{\frac{1}{2}(a_0\rho_0+a_4\rho_4)}-\varepsilon$$

$$M_6=\frac{\varepsilon\, a_0\rho_0}{\frac{1}{2}(a_0\rho_0+a_4\rho_4)}-\varepsilon=0$$

Now we consider $\rho_4>>\rho_0$:

Region 3: $M_3=\dfrac{a_0\rho_0}{\frac{1}{2}(\rho_0a_0+a_4\rho_4)}\,\dfrac{u_p}{a_4}$

$\rho_4>>\rho_0 \rightarrow M_3=\dfrac{2a_0\rho_0}{a_4\rho_4}\,\dfrac{u_p}{a_4}=0$

unperturbed region

Region ⑥: $M_6 = \dfrac{u_p \rho_0}{\frac{1}{2}(a_0\rho_0 + a_4\rho_4)} - \varepsilon$

$\rho_4 >> \rho_0 \rightarrow$ $M_6 = \dfrac{2 u_p \rho_0}{a_4 \rho_4} - \varepsilon$

Single reflected wave $M_6 = -\varepsilon$

Problem 2.4

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a. $J^+: \quad m_2 + \tilde{S}_2 = m_1 + \tilde{S}_1$

$J^-: \quad m_1 - \tilde{S}_1 = m_0 - \tilde{S}_0 = 0 \rightarrow m_1 = \tilde{S}_1$

$\rightarrow \quad m_2 + \tilde{S}_2 = 2m_1$

$J^-: \quad m_2 - \tilde{S}_2 = m_0 - \tilde{S}_0 = 0 \rightarrow m_2 = \tilde{S}_2$

$\rightarrow \quad m_1 = \tilde{S}_1 = m_2 = \tilde{S}_2 = \varepsilon$

$m_2 + \tilde{S}_2 = 2\varepsilon$

$J^-: \quad m_2 - \tilde{S}_2 = m_R - \tilde{S}_{R,2}$

$\tilde{S}_2 = \tilde{S}_{R,2} \rightarrow m_2 = m_R = \frac{u_R(t)}{a_0}$

$\tilde{S}_2 = 2\varepsilon - m_2 = 2\varepsilon - m_R$

$$\boxed{\tilde{S}_2 = 2\varepsilon - \frac{u_R(t)}{a_0}}$$

$J^-: \quad m_3 - \tilde{S}_3 = m_0 - \tilde{S}_0 = 0 \rightarrow m_3 = \tilde{S}_3$

$J^-: \quad m_3 - \tilde{S}_3 = m_R - \tilde{S}_{R,3}$

$\tilde{S}_3 = \tilde{S}_{R,3} \rightarrow m_3 = m_R$

$\tilde{S}_3 = m_3 = m_R = \frac{u_R(t)}{a_0}$

$$\boxed{\tilde{S}_3 = \frac{u_R(t)}{a_0}}$$

b. $\Sigma F = m \cdot a$

$\Sigma F = F_R^- - F_R^+ = (p_R^- - p_R^+)A$

acoustic wave: $a^2 = \Delta p / \Delta \rho \rightarrow \Delta p = a_0^2 \Delta \rho$

$\Sigma F = (p_R^- - p_R^+)A = (\Delta p_R^- - \Delta p_R^+)A = (\Delta \rho_R^- - \Delta \rho_R^+) A a_0^2$

And we know: $\tilde{S} = \Delta\rho / \rho_0 \rightarrow \Delta \rho = \tilde{S}\rho_0$

$\Sigma F = (\tilde{S}_{R,2} - \tilde{S}_{R,3})\, a_0^2 A \rho_0$

$\Sigma F = (\tilde{S}_2 - \tilde{S}_3)\, a_0^2 A \rho_0$

Now, we have:

$(\tilde{S}_2 - \tilde{S}_3)\, a_0^2 A \rho_0 = m \cdot a = m \frac{d u_R(t)}{dt}$

From c_1, we know $\tilde{S}_2$ and $\tilde{S}_3$:

$$\left(2\varepsilon - \frac{u_R(t)}{a_0} - \frac{u_R(t)}{a_0}\right) a_0^2 A \rho_0 = m \frac{du_R(t)}{dt}$$

$$2\varepsilon a_0^2 A \rho_0 - 2\frac{u_R(t)}{a_0} a_0^2 A \rho_0 = m \frac{du_R(t)}{dt}$$

$$\frac{du_R(t)}{dt} + \frac{2 A a_0 \rho_0}{m} u_R(t) = \frac{2 A a_0 \rho_0}{m} \varepsilon a_0$$

$$\boxed{\frac{du_R(t)}{dt} + \lambda u_R(t) = \lambda \varepsilon a_0 \quad \text{with} \quad \lambda = \frac{2 A a_0 \rho_0}{m}}$$

c. $u_R(0) = 0$ for $0 \le t < \frac{L}{a_0}$

$u_R(0) = Ce^0 + a_0 \varepsilon = 0$

$\rightarrow \quad C = -a_0 \varepsilon$ for $0 \le t < \frac{L}{a_0}$

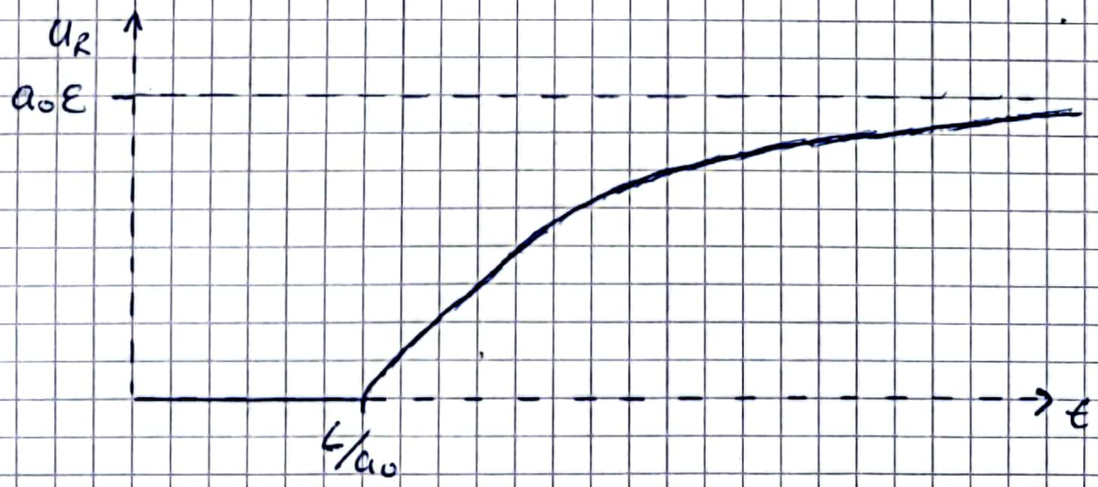
$u_R(\frac{L}{a_0}) = 0$

$u_R(\frac{L}{a_0}) = Ce^{-\lambda \frac{L}{a_0}} + a_0 \varepsilon = 0$

$\rightarrow \quad C = -\frac{a_0 \varepsilon}{e^{-\lambda (L/a_0)}}$ for $t \ge \frac{L}{a_0}$

d. $\lim_{t \to \infty} u_R(t) = a_0 \varepsilon$

This means that $u_R(t)$ approaches $a_0 \varepsilon$ as time goes to infinity



Problem 3.1

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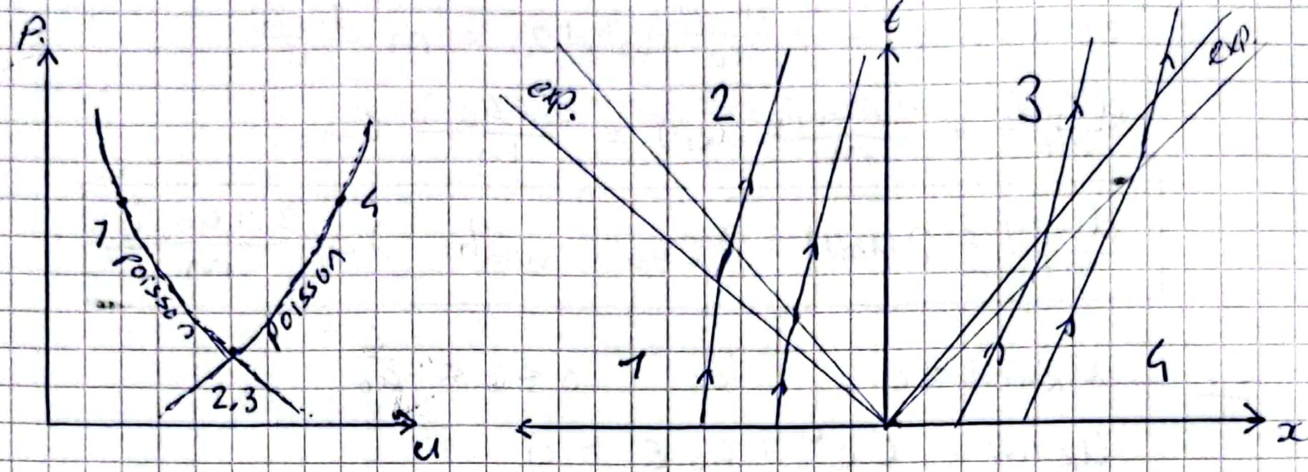
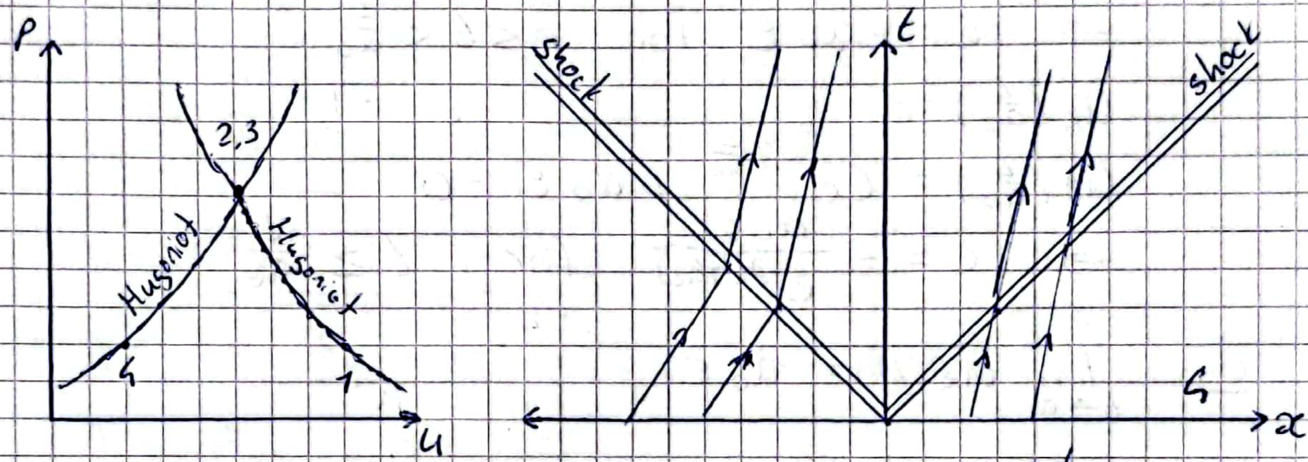
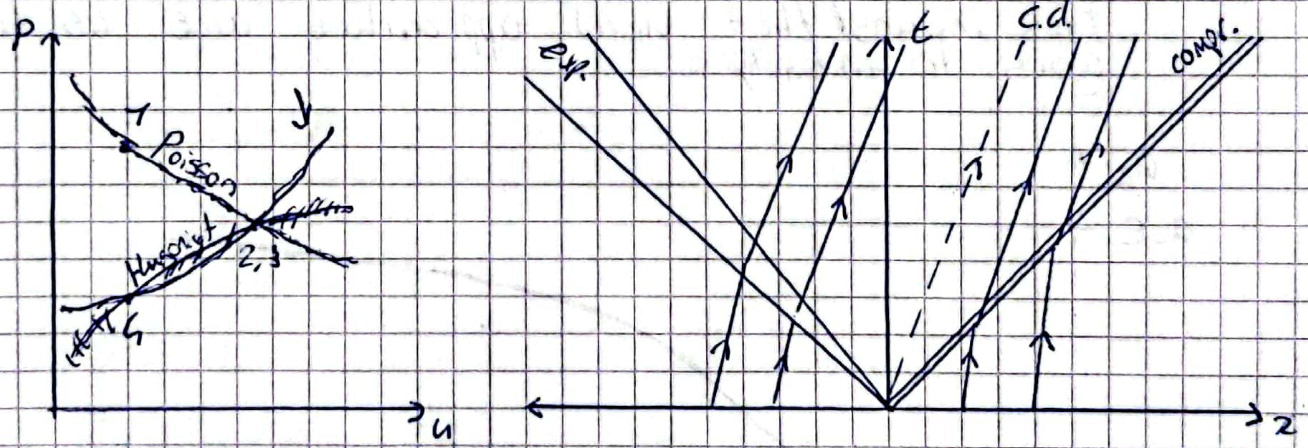
a.



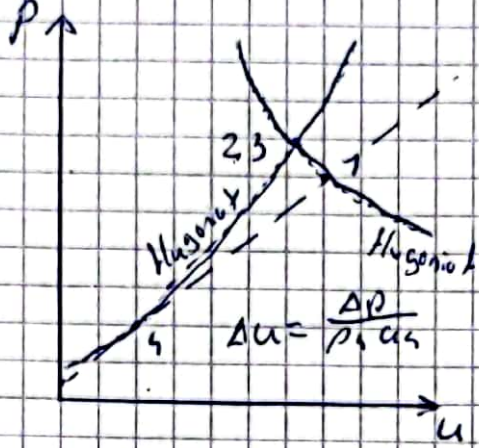
b.

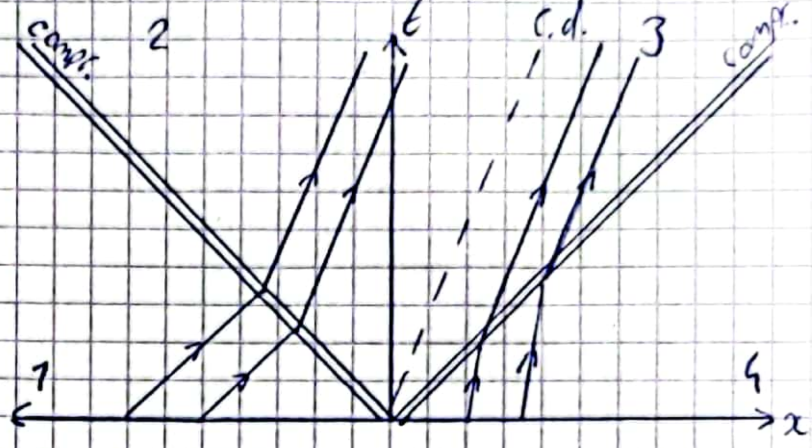


c.

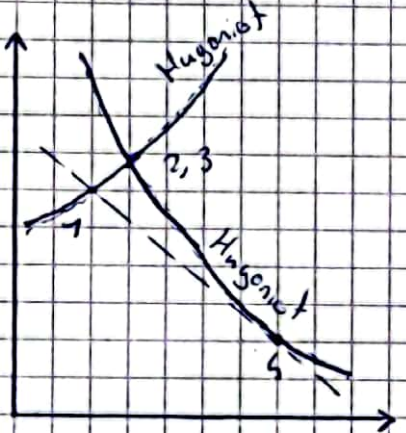


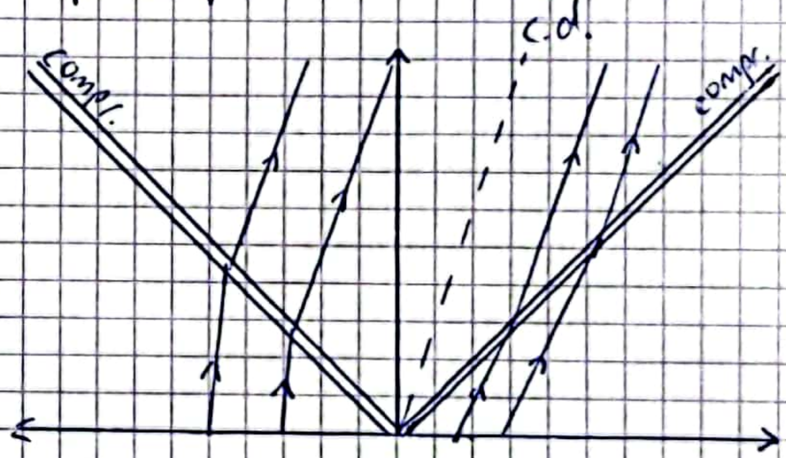
d. $u_1 > u_4$ and $p_1 > p_4$





e. $u_4 \geq u_1$ and $p_1 \geq p_4$





Above is for $u_4 > u_1$ and $p_1 > p_4$

Problem 3.2

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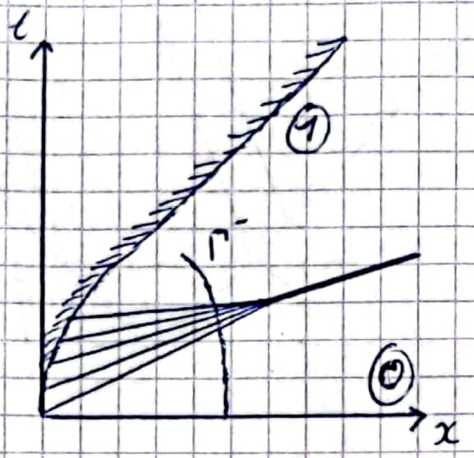
a.



$$J^-: \quad u_0 - \frac{2}{\gamma-1} a_0 = u_1 - \frac{2}{\gamma-1} a_1$$

$u_0 = 0$ and $u_1 = 3a_0$

$$-\frac{2}{\gamma-1} a_0 = 3a_0 - \frac{2}{\gamma-1} a_1$$

$$\rightarrow \quad \frac{a_1}{a_0} = \left(3 + \frac{2}{\gamma-1}\right)\frac{\gamma-1}{2}$$

$$\frac{\rho_1}{\rho_0} = \left(\frac{a_1}{a_0}\right)^{\frac{2}{\gamma-1}} \quad \text{where } \gamma = \frac{5}{3} \quad (He)$$

$$\boxed{\frac{\rho_1}{\rho_0} = 8}$$

b. The flow will no longer be homentropic, so the density ratio would be lower.

According to strong shock limit:

$$\frac{\rho_2}{\rho_1} = \frac{\gamma+1}{\gamma-1} = 4$$

Problem 3.3

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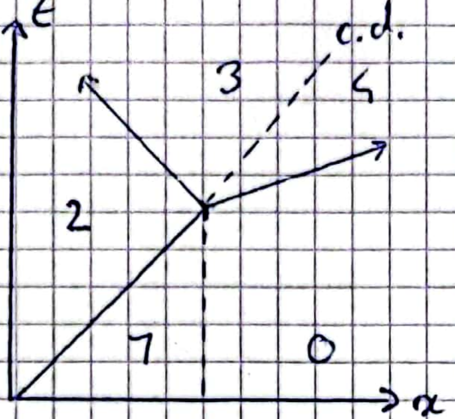
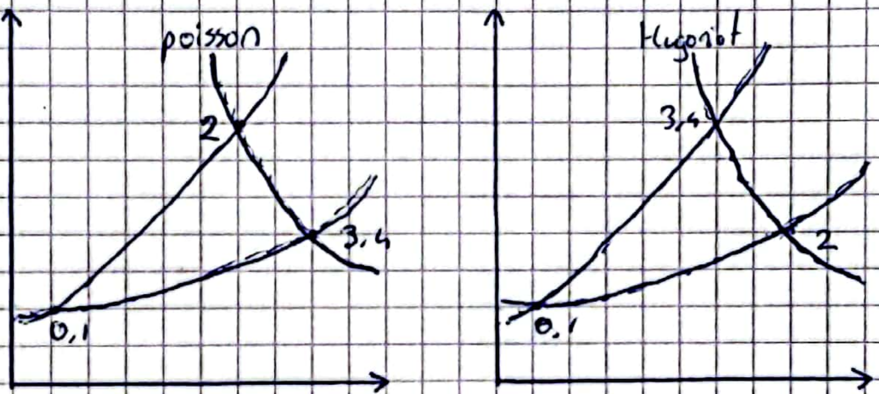


$p_1 > p_0$ $\quad$ $p_0 > p_1$



$p_1 > p_0$: Between $2 \to 3$ poisson expansion wave, because the pressure in 2 is higher

$p_0 > p_1$: Hugoniot compression from $2 \to 3$

Problem 3.4

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a. I: The unperturbed regions + simple compression
II: Behind the shock wave
III: After the shock wave

I: A, C, F
II: D_2, G_3, G_1
III: D_1, G_2, E, H, B

b. A, B, F, G_1, G_3

c. C : J^+ constant
D_2 : J^+ constant
H : J^- constant

d. D_1 and G_2