Problem 2.3

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a. We need two basic relations to solve the problem

$$a^2 = \frac{\gamma p}{\rho} \quad ①$$

$$p = C\rho^{\gamma} \quad ② \quad \text{(Poisson)}$$

Both equations can be used under the assumption of an ideal and perfect gas

Combining ① and ② yields:

$$a^2 = C\gamma \frac{\rho^{\gamma}}{\rho} = C\gamma\rho^{(\gamma-1)}$$

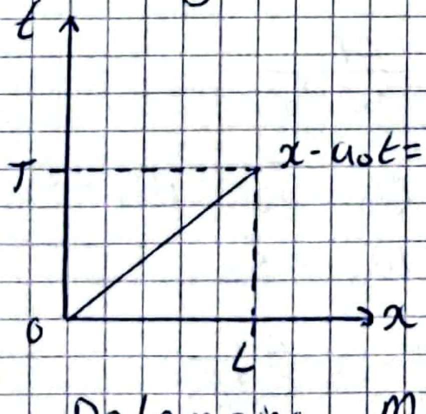
From this relation it follows that a will increase when density ρ increases. So:

when $\rho_4 > \rho_0$, $a_4 > a_0$

when $\rho_0 > \rho_4$, $a_0 > a_4$

b. The c.d. is not moving, because both region 0 and 4 are unperturbed $\rightarrow u_1 = u_4 = 0$
The particle paths go straight up and the c.d. does not move.

The time it takes before the c.d. starts moving can be obtained from the sketch below



$x - u_0 t = 0 \rightarrow L - u_0 T = 0 \rightarrow \boxed{T = \frac{L}{a_0}}$

Determining $m_1(x_1, t_1)$ and $\tilde{S}_1(x_1, t_1)$:

J^- : $m_1 - \tilde{S}_1 = m_0 - \tilde{S}_0 = 0$

$m_1 = \tilde{S}_1$

J^- : $m_p - \tilde{S}_p = m_0 - \tilde{S}_0 = 0$

$m_p = \tilde{S}_p$

$J^+:$ $m_1 + \tilde{s}_1 = m_p + \tilde{s}_p$

$2m_p = 2m_1$

$m_1 = m_p = \frac{u_p}{a_0} = \varepsilon$

$$\boxed{m_1(x_1, t_1) = \varepsilon \quad \text{and} \quad \tilde{s}_1(x_1, t_1) = \varepsilon}$$

$u_1 = m_1 a_0 = \varepsilon a_0$

$$\boxed{u_1 = u_p}$$

$\rho_1 = \rho_0 + \Delta\rho = \rho_0 + a_0^2 \Delta\rho = \rho_0 + a_0^2 \rho_0 \tilde{s}_1$

$$\boxed{\rho_1 = \rho_0 + a_0^2 \rho_0 \varepsilon}$$

C. $m_2 = \frac{u_2}{a_0}$, $m_3 = \frac{u_3}{a_4}$

$u_2 = u_3 = u_{c,D}$

$\rightarrow$ $m_2 = \frac{u_{c,D}}{a_0}$, $m_3 = \frac{u_{c,D}}{a_4}$

$\tilde{s}_2 = \frac{\Delta\rho_2}{\rho_0} = \frac{\Delta p_2}{a_0^2 \rho_0}$, $\tilde{s}_3 = \frac{\Delta\rho_3}{\rho_4} = \frac{\Delta p_3}{a_4^2 \rho_4}$

$\tilde{s}_2 = \frac{p_2 - p_0}{a_0^2 \rho_0}$, $\tilde{s}_3 = \frac{p_3 - p_4}{a_4^2 \rho_4}$

$p_2 = p_3$ and $p_0 = p_4$

$\rightarrow$ $\tilde{s}_2 = \frac{\Delta p}{a_0^2 \rho_0}$, $\tilde{s}_3 = \frac{\Delta p}{a_4^2 \rho_4}$

$\Gamma^+:$ $m_2 + \tilde{s}_2 = m_1 + \tilde{s}_1$, $\Gamma^-:$ $m_3 - \tilde{s}_3 = m_4 - \tilde{s}_4$

$m_2 + \tilde{s}_2 = 2\varepsilon$ $\qquad$ $m_3 = \tilde{s}_3$

$\frac{u_{c,D}}{a_0} + \frac{\Delta p}{a_0^2 \rho_0} = 2\varepsilon$ (1), $\quad \underbrace{\frac{u_{c,D}}{a_4} = \frac{\Delta p}{a_4^2 \rho_4}}$

$\Delta p = a_4 \rho_4 u_{c,D}$ (2)

Put (2) in (1)

$\frac{u_{c,D}}{a_0} + \frac{a_4 \rho_4 u_{c,D} \Delta p}{a_0^2 \rho_0} = 2\varepsilon$

$$u_{c,D}\left(\frac{1}{a_0}+\frac{a_4\rho_4}{a_0^2\rho_0}\right)=2\varepsilon$$

$$u_{c,D}\left(\frac{a_0\rho_0}{a_0^2\rho_0}+\frac{a_4\rho_4}{a_0^2\rho_0}\right)=2\varepsilon$$

$$u_{c,D}\left(\frac{a_0\rho_0+a_4\rho_4}{a_0^2\rho_0}\right)=2\varepsilon$$

$$u_{c,D}=\frac{a_0^2\rho_0}{a_4\rho_0+a_4\rho_4}\,2\varepsilon \qquad \text{where} \quad 2\varepsilon=\frac{u_p}{a_0}$$

$$\boxed{u_{c,D}=\frac{a_0\rho_0}{\frac{1}{2}(a_0\rho_0+a_4\rho_4)}\,u_p}$$

d. First we consider $\rho_4=\rho_0$

Region ③: $u_3=\frac{a_0\rho_0}{\frac{1}{2}(a_0\rho_0+a_4\rho_4)}u_p$

wave continues

$$M_3=\frac{a_0\rho_0}{\frac{1}{2}(a_0\rho_0+a_4\rho_4)}\frac{u_p}{a_4}$$

$$M_3=\frac{u_p}{a_4}=\frac{u_p}{a_0}$$

$$M_3=M_1$$

Region ⑥: $M_6=\frac{a_0\rho_0}{\frac{1}{2}(a_0\rho_0+a_4\rho_4)}-\varepsilon$

unperturbed region

$$M_6=\frac{u_p\rho_0}{\frac{1}{2}(a_0\rho_0+a_4\rho_4)}-\varepsilon$$

$$M_6=\frac{\varepsilon a_0\rho_0}{\frac{1}{2}(a_0\rho_0+a_4\rho_4)}-\varepsilon=0$$

Now we consider $\rho_4>>\rho_0$:

Region ③: $M_3=\frac{a_0\rho_0}{\frac{1}{2}(\rho_0a_0+a_4\rho_4)}\frac{u_p}{a_4}$

$\rho_4>>\rho_0 \rightarrow M_3=\frac{2a_0\rho_0}{a_4\rho_4}\frac{u_p}{a_4}=0$

unperturbed region

Region 6:

$$M_6 = \frac{u_p \rho_0}{\frac{1}{2}(a_0 \rho_0 + a_4 \rho_4)} - \varepsilon$$

$\rho_4 \gg \rho_0 \rightarrow$

$$M_6 = \frac{2 u_p \rho_0}{a_4 \rho_4} - \varepsilon$$

Simple re-flected wave

$$M_6 = -\varepsilon$$