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Exercise 1

Let X_i be a random variable that denotes the result of the i-th roll, for $i = 1, 2, 3...10$. Since X_i is distributed uniformly over $1, 2, 3, 4, 5, 6$ it follows that

$$E[X_i] = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + ... \frac{1}{6} \cdot 6 = \frac{1}{6} \cdot \frac{6 \cdot 7}{2} = \frac{7}{2} \qquad \text{and}$$

$$Var[X_i] = E[X_i^2] - E[X_i]^2 = \frac{1}{6}(1^2 + 2^2 + ... + 6^2) - \left(\frac{7}{2}\right)^2 = \frac{1}{6} \cdot \frac{6 \cdot 7 \cdot 13}{6} - \frac{49}{4}$$
$$= \frac{91}{6} - \frac{49}{4} = \frac{182 - 147}{12} = \frac{35}{12}$$

Now let X be a random variable that denotes sum of these 10 independent dice rolls, therefore $X = \sum X_i$. Since the X_i are i.i.d it follows

$$Var(X) = Var(\sum_{i=1}^{10} X_i) = 10 \cdot Var(X_i) = 10 \cdot \frac{35}{12} = \frac{175}{6}$$

Now we use Chebychev's inequality for $k = 10$ which yields

$$Pr[|X - \mu| \geq 10] \leq \frac{Var(X)}{100} = \frac{175}{600} = \frac{7}{24} \approx 0.292$$

Exercise 2

Consider the function $f(x_1, x_2, x_3, x_4)$ with $f : \{0, 1\}^4 \rightarrow \{0, 1\}$.
Therefore we have 2^4 possible inputs (x_1, x_2, x_3, x_4) that each are mapped to either 0 or 1.
Changing these image values between 0 and 1 results into creating different functions. As we have again 2 possible image values we get

$$2^{2^4} = 2^{16} = 65,536$$

possible functions. Generalizing this result to functions $f : \{0, 1\}^n \rightarrow \{0, 1\}$ yields

$$2^{2^n}$$

The problem is that the search space for the real functions is very big. Simply brute-forcing all or even most possible functions for finding an optimal one is too time consuming.

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Exercise 3

Let $S, h, f, S|_x, \mathcal{D}, m$ be defined as in the specification. Then

$$\begin{aligned}
E_{S|_x \sim \mathcal{D}^m}[L_S(h)] &= E_{S|_x \sim \mathcal{D}^m}\left[\frac{1}{m}\sum_{i=1}^{m} 1_{h(x_i)\neq f(x_i)}\right] \\
&\overset{\text{lin.}}{=} \frac{1}{m}\sum_{i=1}^{m} E_{S|_x \sim \mathcal{D}^m}\left[1_{h(x_i)\neq f(x_i)}\right] \\
&\overset{\text{i.i.d}}{=} \frac{1}{m}\sum_{i=1}^{m} E_{x \sim \mathcal{D}}\left[1_{h(x)\neq f(x)}\right] \\
&= \frac{1}{m} \cdot m \cdot E_{x \sim \mathcal{D}}\left[1_{h(x)\neq f(x)}\right] = E_{x \sim \mathcal{D}}\left[1_{h(x)\neq f(x)}\right] \overset{\text{Def. 1.1}}{=} L_{\mathcal{D},f}(h)
\end{aligned}$$