

解析学A後期中間試験

平成 27 年 11 月 29 日

1 次の極限を求めよ

$$(1) \lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^2}$$

$x = r \cos \theta$ $y = r \sin \theta$ とおけば $(x, y) \rightarrow (0, 0)$ であることは $r \rightarrow 0$ と同義である

$$\begin{aligned} \frac{xy^2}{x^2 + y^2} &= \frac{r \cos \theta \cdot r^2 \sin^2 \theta}{r^2 \cos^2 \theta + r^2 \sin^2 \theta} \\ &= \frac{r \cos \theta \sin^2 \theta}{\cos^2 \theta + \sin^2 \theta} \\ &= r \cos \theta \sin^2 \theta \\ &= 0 \text{ as } r \rightarrow 0 \end{aligned}$$

$$(2) \lim_{(x,y) \rightarrow (0,0)} \frac{x+y}{\sqrt{x^2 + y^2}}$$

x 軸正の向きでは, $(x, y) = (x, 0), \sqrt{x^2 + 0^2} = |x| = x$ だから

$$\frac{x+y}{\sqrt{x^2 + y^2}} = \frac{x}{\sqrt{x^2}} = \frac{x}{x} \rightarrow 1 \text{ as } (x, y) \rightarrow (0, 0)$$

x 軸負の向きでは, $(x, y) = (0, y), \sqrt{x^2 + y^2} = |x| = -x$ だから

$$\frac{x+y}{\sqrt{x^2 + y^2}} = \frac{x}{\sqrt{x^2}} = \frac{x}{-x} \rightarrow -1 \text{ as } (x, y) \rightarrow (0, 0)$$

(x, y) の近づけ方により極限がことなる、つまり極限は存在しない

2 次のヤコビ行列を求めよ

$$(1) f(x, y) = \sqrt{2x + 3y}$$

$$\begin{aligned} z &= \sqrt{2x + 3y} \\ \frac{\partial z}{\partial(x, y)} &= \frac{1}{2\sqrt{2x + 3y}} \begin{pmatrix} 2 & 3 \end{pmatrix} \end{aligned}$$

$$(2) f(x, y) = \frac{x-y}{x+y}$$

$$\begin{aligned} z &= \frac{x-y}{x+y} \\ \frac{\partial z}{\partial(x, y)} &= \frac{1}{(x+y)^2} \begin{pmatrix} (x+y) - (x-y) & -(x+y) - (x-y) \end{pmatrix} \\ &= \frac{1}{(x+y)^2} \begin{pmatrix} 2y & -2x \end{pmatrix} \end{aligned}$$

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(1) $z = g(x, y) = x^2 y^2$

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} = f(t) &= \begin{pmatrix} t + e^t \\ t - e^t \end{pmatrix} \text{ のとき } \frac{\partial z}{\partial t} \text{ を求めよ} \\ \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial t} \\ &= (2xy^2 \quad 2x^2y) \begin{pmatrix} 1 + e^t \\ 1 - e^t \end{pmatrix} \\ &= 2\{xy^2(1 + e^t) + 2x^2y(1 - e^t)\} \end{aligned}$$

(2) $z = g(x, y) = \log(xy)$

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} = f(t) &= \begin{pmatrix} u^2 + v^2 \\ 2uv \end{pmatrix} \text{ のとき } (g \circ f)'(u, v) = \frac{\partial z}{\partial(u, v)} \text{ を求めよ} \\ \frac{\partial z}{\partial(u, v)} &= \frac{\partial z}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(u, v)} \\ &= \begin{pmatrix} \frac{1}{x} & \frac{1}{y} \end{pmatrix} \begin{pmatrix} 2u & 2v \\ 2v & 2u \end{pmatrix} \\ &= \frac{2}{xy} (y \quad x) \begin{pmatrix} u & v \\ v & u \end{pmatrix} \\ &= \frac{2}{xy} (uy + vx \quad vy + ux) \end{aligned}$$

4 ヤコビ行列とヘッセ行列を求めよ

(1) $z = f(x, y) = (x + y)^2 - 2(x^4 + y^4)$

$$\begin{aligned} f'(x, y) &= 2(x + y)[1 \quad 1] - 2 \cdot 4[x^3 \quad y^3] \\ &= 2[x + y - 4x^3 \quad x + y - 4y^3] \\ f''(x, y) &= 2 \begin{bmatrix} 1 - 12x^2 & 1 \\ 1 & 1 - 12y^2 \end{bmatrix} \end{aligned}$$

(2) $z = f(x, y) = e^{-x^2 - y^2}$

$$\begin{aligned} f'(x, y) &= e^{x^2 - y^2} [-2x \quad -2y] \\ &= -2e^{x^2 - y^2} [x \quad y] \\ f''(x, y) &= \begin{bmatrix} x \\ y \end{bmatrix} - 2e^{x^2 - y^2} [-2x \quad -2y] - 2e^{x^2 - y^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= -2e^{x^2 - y^2} \left\{ \begin{bmatrix} -2x^2 & -2xy \\ -2xy & -2y^2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \\ &= -2e^{x^2 - y^2} \begin{bmatrix} 1 - 2x^2 & -2xy \\ -2xy & 1 - 2y^2 \end{bmatrix} \end{aligned}$$

5 $f(x, y) = (x - y)^3 - 8xy = 0$ によって定まる曲線 C について

(1) 点 $A(2\sqrt{3} + 2, 2\sqrt{3} - 2)$ が C 上にあることを確かめよ

$$\begin{aligned} x - y &= 2\sqrt{3} + 2 + 2\sqrt{3} - 2 = 4 \\ xy &= (2\sqrt{3} + 2) \cdot (2\sqrt{3} - 2) = 12 - 4 = 8 \\ (x - y)^3 - 8xy &= 4^3 - 8 \cdot 8 = 64 - 64 = 0 \\ \therefore \text{点 } A &\text{ は } C \text{ 上にある} \end{aligned}$$

(2) 点 A における C の接線の方程式を求めよ

$$F = (x - y)^3 - 8xy \text{ とすれば}$$

$$F_x = 3(x - y)^2 - 8y \quad F_y = -3(x - y)^2 - 8x$$

$$F_y \neq 0 \text{ だから}$$

$$\frac{dy}{dx} = \frac{3(x - y)^2 - 8y}{3(x - y)^2 + 8x}$$

点 A の値を代入して

$$\begin{aligned} \frac{F_x(2\sqrt{3} + 2, 2\sqrt{3} - 2)}{-F_y(2\sqrt{3} + 2, 2\sqrt{3} - 2)} &= \frac{3 \cdot 4^2 - 8(2\sqrt{3} - 2)}{3 \cdot 4^2 + 8(2\sqrt{3} + 2)} \\ &= \frac{48 + 16 - 16\sqrt{3}}{48 + 16 + 16\sqrt{3}} \\ &= \frac{64 - 16\sqrt{3}}{64 + 16\sqrt{3}} \\ &= \frac{4 - \sqrt{3}}{4 + \sqrt{3}} \end{aligned}$$

よって直線の方程式は

$$\begin{aligned} y - (2\sqrt{3} - 2) &= \frac{4 - \sqrt{3}}{4 + \sqrt{3}}(x - (2\sqrt{3} + 2)) \\ &= \frac{(4 - \sqrt{3})^2}{(4 + \sqrt{3})(4 - \sqrt{3})}x - \frac{(4 - \sqrt{3})(2\sqrt{3} + 2)}{4 + \sqrt{3}} + (2\sqrt{3} - 2) \\ &= \frac{16 + 3 - 8\sqrt{3}}{16 - 3}x - \frac{2 + 6\sqrt{3}}{4 + \sqrt{3}} + \frac{-2 + 6\sqrt{3}}{4 + \sqrt{3}} \\ &= \frac{19 - 8\sqrt{3}}{13}x - \frac{2 + 6\sqrt{3} + 2 - 6\sqrt{3}}{4 + \sqrt{3}} \\ &= \frac{19 - 8\sqrt{3}}{13}x - \frac{4}{4 + \sqrt{3}} \end{aligned}$$