微分積分II冬課題

平成27年1月2日

$\boxed{\textbf{1}}$

(1)

$$\frac{x+5}{(x+2)(x-1)} = \frac{a}{x+2} + \frac{b}{x-1}$$
$$x+5 = (x-1)a + (x+2)b$$
$x=-2$ とすると $a=-1$
$x=1$ とすると $b=2$

(2)

$$\frac{1}{(x-3)(x-1)} = \frac{a}{x-3} + \frac{b}{x-1}$$
$$1 = (x-1)a + (x-3)b$$
$x=3$ とすると $a=\dfrac{1}{2}$
$x=1$ とすると $b=-\dfrac{1}{2}$

(3)

$$\frac{2x}{(x-1)(x^2+1)} = \frac{a}{x-1} + \frac{bx+c}{x^2+1}$$
$$2x = (x^2+1)a + (x-1)(bx+c)$$
$x=1$ とすると $a=1$
$$2x = ax^2 + a + bx^2 + cx - bx - c$$
$$= (a+b)x^2 + (c-b)x + a - c$$
$a=1$ だから
$$2x = (1+b)x^2 + (c-b)x + 1 - c$$
係数を比較して
$a=1, b=-1, c=1$

$\boxed{\textbf{2}}$

(1)

$$\begin{aligned}(証明)(右辺) &= \frac{1}{2}(1-\cos 2x)\\ &= \frac{1}{2}(1-\cos^2 x+\sin^2 x)\\ &= \frac{1}{2}(2\sin^2 x)\\ &= \sin^2 x = (左辺)\end{aligned}$$

(2)

$$\begin{aligned}(証明)(左辺) &= \cos^2 \frac{x}{2}\\ &= \cos x + \sin^2 \frac{x}{2}\\ &= \cos x + \frac{1}{2} - \frac{\cos x}{2}\\ &= \frac{1}{2}(1+\cos x) = (右辺)\end{aligned}$$

(3)

$$y = \frac{1}{2x-3}$$
$$y' = -\frac{2}{(2x-3)^2}$$

$\boxed{\textbf{3}}$

(1)

$$\sin 2x \cos x = \frac{1}{2}(\sin 3x + \sin x)$$

(2)

$$\begin{aligned}\cos 2x \sin 3x &= \frac{1}{2}(\sin 5x - \sin -x)\\ &= \frac{1}{2}(\sin 5x + \sin x)\end{aligned}$$

(3)

$$\begin{aligned}\cos 3x \cos 5x &= \frac{1}{2}(\cos 8x + \cos -2x)\\ &= \frac{1}{2}(\cos 8x + \cos 2x)\end{aligned}$$

(4)

$$\sin 4x \sin 2x = -\frac{1}{2}(\cos 6x - \cos 2x)$$

$\boxed{\textbf{4}}$

(1)

$$\lim_{x\to 3} \frac{x^2-x-6}{x^2-4x+3}$$
$$= \lim_{x\to 3} \frac{(x-3)(x+2)}{(x-3)(x-1)}$$
$$= \frac{5}{2}$$

(2)

$$\begin{aligned}&\lim_{x\to1}\frac{x^3-1}{3x^2-x-2}\\&=\lim_{x\to1}\frac{(x-1)(x^2+x+1)}{(3x+2)(x-1)}\\&=\frac{3}{5}\end{aligned}$$

(3)

$$\begin{aligned}&\lim_{x\to\infty}\frac{2+5^x}{5^{x+1}-3^x}\\&=\lim_{x\to\infty}\frac{\frac{2}{5^x}+1}{5-\frac{3^x}{5^x}}\\&=\frac{1}{5}\end{aligned}$$

(4)

$$\begin{aligned}&\lim_{x\to0}\frac{1-\cos x}{x^2}\\&=\lim_{x\to0}\frac{\sin^2 x}{x^2}\lim_{x\to0}\frac{1}{1+\cos x}\\&=\frac{1}{2}\end{aligned}$$

(5)

$$\begin{aligned}&\lim_{x\to2}\frac{x-2}{\sqrt{x^2+1}-\sqrt{5}}\\&=\lim_{x\to2}\frac{(x-2)(\sqrt{x^2+1}+\sqrt{5})}{x^2+1-5}\\&=\lim_{x\to2}\frac{(x-2)(\sqrt{x^2+1}+\sqrt{5})}{(x-2)(x+2)}\\&=\frac{\sqrt{5}}{2}\end{aligned}$$

(6)

$$\begin{aligned}&\lim_{x\to\infty}(\sqrt{x^2+1}-x)\\&=\lim_{x\to\infty}\frac{(\sqrt{x^2+1}-x)(\sqrt{x^2+1}+x)}{(\sqrt{x^2+1}+x)}\\&=\lim_{x\to\infty}\frac{1}{(\sqrt{x^2+1}+x)}\\&=0\end{aligned}$$

5

(1)

$$\begin{aligned}f(x)&=x^2\\f'(x)&=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}\\&=\lim_{h\to0}\frac{x^2+2xh+h^2-x^2}{h}\\&=\lim_{h\to0}2xh+h^2\\&=2x\end{aligned}$$

(2)

$$\begin{aligned}f(x)&=\sqrt{x}\\f'(x)&=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}\\&=\lim_{h\to0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\\&=\lim_{h\to0}\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}\\&=\frac{1}{2\sqrt{x}}\end{aligned}$$

(3)

$$\begin{aligned}f(x)&=\frac{1}{x}\\f'(x)&=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}\\&=\lim_{h\to0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}\\&=\lim_{h\to0}\frac{x-(x+h)}{hx(x+h)}\\&=\lim_{h\to0}\frac{-h}{hx(x+h)}\\&=-\frac{1}{x^2}\end{aligned}$$

6

(1)

$$\begin{aligned}y&=(5x+3)^7\\y'&=35(5x+3)^6\end{aligned}$$

(2)

$$\begin{aligned}y&=(x^2+3)^7\\y'&=14x(x^2+3)^6\end{aligned}$$

(3)

$$\begin{aligned}y&=\sqrt{4x+3}\\y'&=\frac{2}{\sqrt{4x+3}}\end{aligned}$$

(4)

$$\begin{aligned}y&=\sqrt[3]{6x+1}\\y'&=\frac{2}{\sqrt[3]{6x+1}^2}\end{aligned}$$

(5)

$$\begin{aligned}y&=\frac{4x-3}{x+1}\\y'&=\frac{4(x+1)-(4x-3)}{(x+1)^2}\\&=\frac{7}{(x+1)^2}\end{aligned}$$

(6)

$$x = \frac{1}{2t^2 - 1}$$
$$x' = -\frac{4t}{(2t^2-1)^2}$$

(7)

$$x = \frac{t}{(1-t)^2}$$
$$\begin{aligned} x' &= \frac{(1-t)^2 - t(-2(1-t))}{(1-t)^4} \\ &= \frac{(1-t)+2t}{(1-t)^3} \\ &= \frac{1+t}{(1-t)^3} \end{aligned}$$

(8)

$$y = \cos\frac{t}{2}$$
$$y' = -\frac{1}{2}\sin\frac{t}{2}$$

(9)

$$y = \sin^4 x$$
$$y' = 4\sin^3 x \cos x$$

(10)

$$y = \cos^3 5x$$
$$y' = -15\cos^2 5x \sin 5x$$

$\boxed{7}$

(1)

$$\begin{aligned} y &= e^{3x}\sin 2x \\ y' &= 3e^{3x}\sin 2x + e^{3x}2\cos 2x \\ &= (3\sin 2x + 2\cos 2x)e^{3x} \end{aligned}$$

(2)

$$\begin{aligned} y &= e^{-x}\cos 4x \\ y' &= -e^{-x}\cos 4x - 4e^{-x}\sin 4x \\ &= (-\cos 4x - 4\sin 4x)e^{-x} \end{aligned}$$

(3)

$$v = \frac{\log u}{u}$$
$$\begin{aligned} v' &= \frac{\frac{1}{u}\cdot u - \log u}{u^2} \\ &= \frac{1-\log u}{u^2} \end{aligned}$$

(4)

$$y = \log(x^2 + 3x + 1)$$
$$y' = \frac{2x+3}{x^2+3x-1}$$

(5)

$$y = \log\left|\frac{1+x}{1-x}\right|$$
$$\begin{aligned} y' &= \frac{\frac{(1-x)+(1+x)}{(1-x)^2}}{\frac{1+x}{1-x}} \\ &= \frac{2}{(1+x)(1-x)} \end{aligned}$$

(6)

$$y = e^{\sin 2x}$$
$$y' = 2\cos 2x e^{\sin 2x}$$

(7)

$$\begin{aligned} y &= (e^{2t} + e{-}2t)^4 \\ y' &= 4(e^{2t} + e{-}2t)^3(2e^{2t} - 2e^{-2t}) \\ &= 8(e^{2t} + e{-}2t)^3(e^{2t} - e^{-2t}) \end{aligned}$$

(8)

$$y = \tan^{-1} 2x$$
$$y' = \frac{2}{1+4x^2}$$

(9)

$$y = \sin^{-1}\frac{3}{x}$$
$$\begin{aligned} y' &= \frac{\frac{1}{3}}{\sqrt{1-\frac{x^2}{9}}} \\ &= \frac{1}{\sqrt{9-x^2}} \end{aligned}$$

(10)

$$x = \tan^{-1}\frac{2}{u}$$
$$\begin{aligned} x' &= \frac{-\frac{2}{u^2}}{1+\frac{4}{u^2}} \\ &= \frac{-2}{u^2+4} \end{aligned}$$

$\boxed{8}$

(1)

$y = x^2 + 2x - 1$

$y' = 2x + 2$

$x = 1$ の時 $y = 2$ で傾き $y' = 4$

$\therefore y - 2 = 4(x-1)$

$y = 4x - 2$

(2)

$$y=\frac{1}{x}$$
$$y'=-\frac{1}{x^2}$$
$$x=2 \text{ の時 } y=\frac{1}{2} \text{で傾き } y'=-\frac{1}{4}$$
$$\therefore y-\frac{1}{2}=-\frac{1}{4}(x-2)$$
$$y=-\frac{1}{4}x+1$$

(3)

$$y=x^2$$
$$y'=2x$$
$$x=-1 \text{ の時 } y=1, y'=-2 \text{ なので傾き } m=\frac{1}{2}$$
$$\therefore y-1=\frac{1}{2}(x+1)$$
$$y=\frac{1}{2}x+\frac{3}{2}$$

9

(1)

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=-\frac{\cos t}{\sin t}=-\cot t$$

(2)

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\sin t}{1-\cos t}$$

(3)

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{3\sin^2 t\cos t}{-3\cos^2 t\sin t}=-\tan t$$

10

(1)

$$y=x^3-5x^2+3x$$
$$y'=3x^2-10x+3$$
$$=(3x-1)(x-3)$$
$$y'=0 \text{ を解くと } x=\frac{1}{3},3$$

x	$\cdots$	$\frac{1}{3}$	$\cdots$	3	$\cdots$
y'	$+$	0	$-$	0	$+$
y	$\nearrow$	$\frac{13}{27}$	$\searrow$	-9	$\nearrow$

$$\therefore x=\frac{1}{3} \text{の時極大値 } y=\frac{13}{27}$$
$$x=3 \text{ の時極小値 } y=-9$$

(2)

$$y=x\log x$$
$$y'=\log x+1$$
$$y'=0 \text{ を解くと } x=\frac{1}{e}$$

x	0	$\cdots$	$\frac{1}{e}$	$\cdots$
y'	$/$	$-$	0	$+$
y	$/$	$\searrow$	$-\frac{1}{e}$	$\nearrow$

$$\therefore x=-\frac{1}{e}\text{の時極小値 } y=-\frac{1}{e}$$

11

(1)

$$y=x^3-3x$$
$$y'=3x^2-3$$
$$y'=0 \text{ を解くと } x=\pm 1$$

x	-2	$\cdots$	-1	$\cdots$	0
y'	9	$+$	0	$-$	-3
y	-2	$\nearrow$	2	$\searrow$	0

$$\therefore x=-2 \text{ の時最小値 } y=-2$$
$$x=-1 \text{ の時最大値 } y=2$$

(2)

$$y=xe^{-x}$$
$$y'=e^{-x}-xe^{-x}$$
$$y'=0 \text{ を解くと } x=1$$

x	0	$\cdots$	1	$\cdots$	2
y'	1	$+$	0	$-$	$-\frac{1}{e^2}$
y	0	$\nearrow$	$\frac{1}{e}$	$\searrow$	$\frac{2}{e^2}$

$$\therefore x=0 \text{ の時最小値 } y=0$$
$$x=1 \text{ の時最大値 } y=\frac{1}{e}$$

(3)

$$y=x-2\sin x$$
$$y'=1-2\cos x$$
$$y'=0 \text{ を解くと } x=\frac{\pi}{3},\frac{5\pi}{3}$$

x	0	$\cdots$	$\frac{\pi}{3}$	$\cdots$	$\frac{5\pi}{3}$	$\cdots$	2π
y'	-1	$-$	0	$+$	0	$-$	-1
y	0	$\searrow$	$\frac{\pi}{3}-\sqrt{3}$	$\nearrow$	$\frac{5\pi}{3}-\sqrt{3}$	$\searrow$	2π

$$\therefore x=\frac{\pi}{3}\text{の時最小値 } y=\frac{\pi}{3}-\sqrt{3}$$
$$x=\frac{5\pi}{3}\text{の時最大値 } y=\frac{5\pi}{3}-\sqrt{3}$$

12

(1)

$$
\begin{aligned}
y &= x^4 - 2x^3 \\
y' &= 4x^3 - 6x^2 \\
&= 2x^2(2x-3)
\end{aligned}
$$

$y' = 0$ を解くと $x = \frac{3}{2}, 0$

$$
\begin{aligned}
y'' &= 12x^2 - 12x \\
&= 12x(x-1)
\end{aligned}
$$

$y'' = 0$ を解くと $x = 1, 0$

x	$\cdots$	0	$\cdots$	1	$\cdots$	$\frac{3}{2}$	$\cdots$
y'	$-$	0	$-$	$-$	$-$	0	$+$
y''	$+$	0	$-$	0	$+$	$+$	$+$
y	$\searrow$	0	$\searrow$	-1	$\searrow$	$\frac{27}{16}$	$\nearrow$

$\therefore$変曲点 $(0,0), (1,-1)$

(2)

$$
\begin{aligned}
y &= \frac{1}{x^2+1} \\
y' &= -\frac{2x}{(x^2+1)^2}
\end{aligned}
$$

$y' = 0$ を解くと $x = 0$

$$
\begin{aligned}
y'' &= \frac{-2(x^2+1)^2 + 8x^2(x^2+1)}{(x^2+1)^4} \\
&= \frac{6x^2-2}{(x^2+1)^3}
\end{aligned}
$$

$y'' = 0$ を解くと $x = \pm\frac{1}{\sqrt{3}}$

x	$\cdots$	$-\frac{1}{\sqrt{3}}$	$\cdots$	0	$\cdots$	$\frac{1}{\sqrt{3}}$	$\cdots$
y'	$+$	$+$	$+$	0	$-$	$-$	$-$
y''	$+$	0	$-$	$-$	$-$	0	$+$
y	$\nearrow$	$\frac{3}{4}$	$\nearrow$	1	$\searrow$	$\frac{3}{4}$	$\searrow$

$\therefore$変曲点 $(-\frac{1}{\sqrt{3}}, \frac{3}{4}), (\frac{1}{\sqrt{3}}, \frac{3}{4})$

(3)

$$
\begin{aligned}
y &= x^4 - 4x^3 + 6x^2 \\
y' &= 4x^3 - 12x^2 + 12x
\end{aligned}
$$

$y' = 0$ を解くと $x = 0$

$$
\begin{aligned}
y'' &= 12x^2 - 24x + 12 \\
&= 12(x-1)^2
\end{aligned}
$$

$y'' = 0$ を解くと $x = 1$

x	$\cdots$	0	$\cdots$	1	$\cdots$
y'	$-$	0	$+$	$+$	$+$
y''	$+$	$+$	$+$	0	$+$
y	$\searrow$	0	$\nearrow$	3	$\nearrow$

$\therefore$変曲点なし

13

(1)

$$
\begin{aligned}
&\int (3x+2)^4 dx \\
&= \frac{(3x+2)^5}{15}
\end{aligned}
$$

(2)

$$
\begin{aligned}
&\int \frac{1}{\sqrt[3]{x}} dx \\
&= \frac{3}{2}\sqrt[3]{x}^2
\end{aligned}
$$

(3)

$$
\begin{aligned}
&\int \frac{1}{1-2x} dx \\
&= -\frac{1}{2}\log|1-2x|
\end{aligned}
$$

(4)

$$
\begin{aligned}
&\int (\cos 2x + \sin 3x) dx \\
&= \frac{1}{2} sin2x - \frac{1}{3}\cos 3x
\end{aligned}
$$

(5)

$$
\begin{aligned}
&\int \frac{e^{4x}+e^x}{e^{2x}} dx \\
&= \int (e^{2x} + e^{-x}) dx \\
&= \frac{1}{2}e^{2x} - e^{-x}
\end{aligned}
$$

(6)

$$
\begin{aligned}
&\int \frac{1}{4+x^2} \\
&= \frac{1}{2}\tan^{-1}\frac{x}{2}
\end{aligned}
$$

(7)

$$
\begin{aligned}
&\int \frac{dt}{\sqrt{9-t^2}} \\
&= \sin^{-1}\frac{t}{3}
\end{aligned}
$$

(8)

$$
\begin{aligned}
t &= \sqrt{x+1} \\
dt &= \frac{dx}{2\sqrt{x+1}} = \frac{dx}{2t} \\
&\int x\sqrt{x+1} dx \\
&= \int (t^2-1)t2tdt \\
&= \int (2t^4 - 2t^2) dt \\
&= \frac{2}{5}t^5 - \frac{2}{3}t^3 \\
&= \frac{2}{15}t^3(3t^2-5) \\
&= \frac{2}{15}\sqrt{x+1}^3(3x-2)
\end{aligned}
$$

(9)

$$\begin{aligned} t &= x^2+3 \\ dt &= 2xdx \end{aligned}$$

$$\begin{aligned} &\int x\sqrt{x^2+3}dx \\ &= \int \sqrt{t}x\frac{dt}{2x} \\ &= \frac{1}{2}\int \sqrt{t}dt \\ &= \frac{1}{3}\sqrt{(x^2+3)}^3 \end{aligned}$$

(10)

$$\begin{aligned} &\int \frac{x}{x^2+1}dx \\ &= \frac{1}{2}\int \frac{(x^2+1)'}{x^2+1}dx \\ &= \frac{1}{2}\log|x^2+1| \end{aligned}$$

(11)

$$\begin{aligned} t &= \log x \\ dt &= \frac{1}{x}dx \end{aligned}$$

$$\begin{aligned} &\int \frac{1}{x\log x}dx \\ &= \int \frac{1}{xt}xdt \\ &= \log t \\ &= \log|\log x| \end{aligned}$$

(12)

$$\begin{aligned} &\int x\cos xdx \\ &= x\sin x - \int \sin xdx \\ &= x\sin x + \cos x \end{aligned}$$

(13)

$$\begin{aligned} &\int x\sin xdx \\ &= -x\cos x - \int -\cos x \\ &= -x\cos x + \sin x \end{aligned}$$

(14)

$$\begin{aligned} \int xe^xdx &= xe^x - \int e^xdx \\ &= xe^x - e^x \\ &= (x-1)e^x \end{aligned}$$

(15)

$$\begin{aligned} &\int \log xdx \\ &= x\log x - \int x\frac{1}{x}dx \\ &= x\log x - x \end{aligned}$$

(16)

$$\begin{aligned} I &= \int e^x\cos xdx \\ I &= e^x\sin x - \int e^x\sin xdx \\ &= e^x\sin x - (-e^x\cos x - \int e^x\cos xdx) \\ &= e^x\sin x + e^x\cos x - \int e^x\cos xdx \\ 2I &= e^x\sin x + e^x\cos x \\ I &= \frac{1}{2}e^x(\sin x + \cos x) \end{aligned}$$