

In the new model, equations 1-3 are the same with the older model. Equations 1 and 2 are the basic Wilson-Cowan column.

$$\dot{e}_j = -e_j + (k_e - r_e e_j) \mathcal{F}[c_1 e_j - c_2 i_j + k_s \sum_{k \in \mathbb{N}_j} e_k + P + \phi(t)], \quad (1)$$

$$\dot{i}_j = -i_j + (k_i - r_i e_j) \mathcal{F}[c_3 e_j - c_4 i_j + k_s \sum_{k \in \mathbb{N}_j} e_k + Q]; \quad (2)$$

where

$$\mathcal{F}(x) = \frac{1}{1 + \exp[-a(x - \theta)]} - \frac{1}{1 + \exp(a\theta)}. \quad (3)$$

$\phi(t)$ is the slow gating variable:

$$\dot{\phi} = -\mu_1 \phi + \left(\frac{\mu_2 \exp(\kappa(M - \eta))}{1 + \exp(\kappa(M - \eta))} \right). \quad (4)$$

M is the metabolic substrate, which is regeneration term minus consumption:

$$\dot{M} = g_r(e) - g_c(e). \quad (5)$$

$$g_r = k_r \beta; g_c = k_c \left(\frac{e^2}{1 + e^2} \right); \quad (6)$$

where β is the recovery variable with homeostatic coupling:

$$\dot{\beta} = -\nu_1 \beta + \left(\frac{\nu_2 \exp(\phi - \zeta)}{1 + \exp(\phi - \zeta)} \right). \quad (7)$$

Using equation 6 as the energy substrate equation, the system behavior is similar as the old system in the previous EMBC paper. There exists a stable periodic solution in the slow gating ϕ so that the fast system manifests burst-like activity.

Synchronization Analysis

Neural Coupling

Consider two coupled burst suppression models with excitatory synaptic coupling in the fast dynamics. Denote them as columns $j, k, j, k = 1, 2, j \neq k$. Then the synaptic input to macrocolumn j, k is $k_f e_k, k_f e_j$ for $j, k = 1, 2, j \neq k$. k_f represents the coupling strength of fast neuronal coupling. The system reads:

$$\dot{e}_j = -e_j + (k_e - r_e e_j) \mathcal{F}[c_1 e_j - c_2 i_j + k_j e_k + P + \phi_j(t)] \quad (8)$$

$$\dot{i}_j = -i_j + (k_i - r_i e_j) \mathcal{F}[c_3 e_j - c_4 i_j + Q], \quad (9)$$

$$\dot{\phi}_j = -\mu_1 \phi_j + \left(\frac{\mu_2 e^{(\kappa M_j)}}{1 + e^{(\kappa M_j)}} \right) \quad (10)$$

$$\dot{M}_j = k_r \beta_j - k_c \left(\frac{e_j^2}{1 + e_j^2} \right) \quad (11)$$

$$\dot{\beta} = -\nu_1 \beta + \left(\frac{\nu_2 e^\phi}{1 + e^\phi} \right) \quad (12)$$

Since $e_k \geq 0, k_j > 0, p_j(t) = k_j e_k + P_j + \phi_j(t) \geq P_j + \phi_j(t)$, meaning the coupling adds more excitation in each column. Thus two columns tend to synchronize under this coupling.

Definition (Burst Synchronization): For columns j, k , burst synchronization occurs when $\|\phi_k(t) - \phi(t)_j\| < \epsilon$, where positive ϵ is arbitrarily small. (Note: Here we use norm $\|x(t)\| = \sup_t |x(t)|$ for continuous functions $x(t)$.)

Statement: There exists a coupling strength value k_f^* so that for $k_f > k_f^*$, burst synchronization occurs.

Proof: Suppose $\mu_k = \mu_j = \mu, \kappa = 1$. It suffices to show for $\phi_\perp = \phi_j - \phi_k, \dot{\phi}_\perp \rightarrow 0$.

$$\dot{\phi}_\perp = \mu(-\phi_\perp + \frac{e^{M_j} - e^{M_k}}{(1 + e^{M_j})(1 + e^{M_k})}) \quad (13)$$

Notice $(1 + e^{M_j})(1 + e^{M_k}) \geq 1 > 0$, if $\|e^{M_j} - e^{M_k}\| \rightarrow 0$ then the equation above is stable and there is synchronization. Since exponential is monotonic, we want to show $\|M_j - M_k\| \rightarrow 0$. But $0 \leq \|M_j\| - \|M_k\| \leq \|M_j - M_k\|$, so we seek a coupling value k_f^* s.t. $\|M_j\| = \|M_k\|$.

Consider two isolated macrocolumns j, k , and they have bursts from 0 to time τ_j, τ_k respectively. For example, in column j , at time τ_j $\phi_j(\tau_j)$ starts to drop below the threshold for bursting, i.e. $\phi_j(t) + P < P^*$ for $t > \tau_j$ and consequently $e_j(t) = 0$. By continuity in e_j , there is a small neighborhood of $\phi_j(\tau_j)$ over which $e_j(t)$ and $\phi_j(t)$ have one-to-one correspondence.

In each column, the energy substrate $\dot{M} = k_r \beta - k_c \left(\frac{e^2}{1 + e^2} \right)$ where $\beta \geq 0$ is close to a constant by choosing $\nu_1 \approx 0$ due to slow homeostasis. Thus:

$$M(t) = k_r \beta t - k_c \mathcal{C}(\min(t, \tau)) \quad (14)$$

where $\mathcal{C}(t)$ is the consumption term:

$$\mathcal{C}(t) = \int_0^t \left(\frac{e^2}{1 + e^2} \right) dt \quad (15)$$

Notice that $M(t)$ has an extreme value at time τ . If there exists a coupling strength k_f^* s.t. $\|M_j\| = \|M_k\|$, then $|M_j(\tau_j)| = |M_k(\tau_k)|$. Moreover, burst synchronization occurs when two columns have the same burst duration $\tau_j = \tau_k = \tau$. Therefore we seek a k_f^* so that after the fast neuronal coupling, two columns j, k have

$$|k_r \beta_j \tau - k_c \mathcal{C}(e_j, \tau)| = |k_r \beta_k \tau - k_c \mathcal{C}(e_k, \tau)| \quad (16)$$

If $\beta_j = \beta_k$, the problem simplifies to showing there exists a k_f^* so that

$$\int_0^\tau \frac{e_j^2}{1 + e_j^2} dt - \int_0^\tau \frac{e_k^2}{1 + e_k^2} dt = 0 \quad (17)$$

Integral has a solution over finite interval $[0, \tau]$. If the integral has a solution $\mathcal{G}(e)$, then $\mathcal{G}(e_j(t))|_0^\tau - \mathcal{G}(e_k(t))|_0^\tau = 0$, by one-to-one correspondence of $e(t)$ and $\phi(t)$ around time τ , we can write:

$$\mathcal{G}(\phi_j(\tau) + k_f^* \phi_k(\tau)) = \mathcal{G}(\phi_k(\tau) + k_f^* \phi_j(\tau)) \quad (18)$$

Observe that $k_f^* = 1$ is the threshold value. For $k_f > k_f^*$, $\phi_j(t) = \phi_k(t)$ and burst synchronization between two columns can be observed.

Substrate Coupling

Consider two columns coupled through diffusive coupling in the slow metabolic process. For columns $j, k = 1, 2, j \neq k$, the metabolic substrate variable equations are:

$$\dot{M}_j = k_r \beta_j - k_c \left(\frac{e_j^2}{1 + e_j^2} \right) + k_s (M_k - M_j) \quad (19)$$

where k_s represents the slow metabolic substrate coupling strength.

Statement: There exists a coupling strength (threshold) value k_s^* so that burst synchronization occurs for $k_s > k_s^*$.

Proof: From previous analysis, burst synchronization occurs when $\|M_j - M_k\| \rightarrow 0$, or $M_\perp(t) = M_j(t) - M_k(t) \rightarrow 0$ as $t \rightarrow \infty$. So it suffices to show M_\perp is stable at $M_\perp = 0$ under the slow coupling. Compute:

$$\dot{M}_\perp = -2k_s M_\perp + k_r(\beta_j - \beta_k) + k_c(g_c(e_k) - g_c(e_j)) \quad (20)$$

where $g_c(e_{k,j}) = \frac{e_{k,j}^2}{1 + e_{k,j}^2}$, and β_j, β_k can be viewed as constants. We want to show that exists a k_s^* , so that for $k_s > k_s^*$,

$$-2k_s + k_c \frac{d(g_c(e_k) - g_c(e_j))}{dM_\perp} < 0 \quad (21)$$

Or

$$\inf \{k_s\} > \sup \left| \frac{1}{2} k_c \frac{d(g_c(e_k) - g_c(e_j))}{dM_\perp} \right| \quad (22)$$

Then it suffices to show that $\frac{d(g_c(e_k) - g_c(e_j))}{dM_\perp}$ has a bounded derivative. To calculate $\frac{1}{2} k_c \frac{d(g_c(e_k) - g_c(e_j))}{dM_\perp}$:

$$\frac{1}{2} k_c \frac{d(g_c(e_k) - g_c(e_j))}{dM_\perp} = \frac{1}{2} k_c \frac{d(g_c(e_k) - g_c(e_j))}{dt} \frac{dt}{dM_\perp} = \frac{1}{2} k_c \left(\frac{2e_k \dot{e}_k}{(1 + e_k^2)^2} - \frac{2e_j \dot{e}_j}{(1 + e_j^2)^2} \right) \frac{dt}{dM_\perp} \quad (23)$$

and since $|e| \leq 1, |\dot{e}| \leq 1$:

$$\frac{1}{2} k_c \left(\frac{2e_k \dot{e}_k}{(1 + e_k^2)^2} - \frac{2e_j \dot{e}_j}{(1 + e_j^2)^2} \right) \leq k_c \left(\left| \frac{e_k \dot{e}_k}{(1 + e_k^2)^2} \right| + \left| \frac{e_j \dot{e}_j}{(1 + e_j^2)^2} \right| \right) \leq 2k_c \quad (24)$$

Therefore

$$\frac{1}{2} k_c \frac{d(g_c(e_k) - g_c(e_j))}{dM_\perp} \leq 2k_c \frac{dt}{dM_\perp} \quad (25)$$

To evaluate $\frac{dt}{dM_\perp}$ we consider the metabolic substrate difference M_\perp of two isolated columns before the coupling is introduced.

For each column, recall the consumption equation over one burst interval ($0 \leq t \leq \tau$), $\mathcal{C}(t) = \int_0^t g_c(e) dt$. Due to periodicity in the fast activity e , $\mathcal{C}(t)$ is monotonically increasing with an increment $\Delta C = \mathcal{C}(t + \Delta T) - \mathcal{C}(t) = \int_0^{\Delta T} g_c(e) dt$, where ΔT is the period of fast oscillatory activity. Define consumption rate $R := \Delta C / \Delta T$. Claim: $\mathcal{C}(t) \approx Rt, R \leq 1/2$ with a bounded error that vanishes at multiple times of ΔT .

(proof:) Observe that for $0 \leq t \leq \tau$, $\mathcal{C}(t) \leq \frac{1}{2} t$ is bounded since $|e| \leq 1$. Also for t being multiple times of ΔT , $\mathcal{C}(t) = Rt, R \leq \frac{1}{2}$ is a bounded linear operator. The approximation error is $v(t) = \mathcal{C}(t) - Rt$, then $\|v\| = \max |v(t)| = \max \left| \int_0^t g_c(e) dt - Rt \right|$. The maximal value occurs within the interval $[0, \Delta T]$, and $\|v\| \leq (\frac{1}{2} - R)\Delta T$ is bounded is small. Thus the approximation is reasonable. (end)

Then the energy substrate variable can be simplified as $M(t) = k_r \beta t - k_c R \min(t, \tau)$. For two columns $j, k = 1, 2, j \neq k$, we assume column k has longer burst ($\tau_k > \tau_j$), then:

$$\begin{cases} \dot{M}_\perp = k_r(\beta_j - \beta_k) + k_c(R_k - R_j), t \leq \min(\tau_j, \tau_k) \\ \dot{M}_\perp = k_r(\beta_j - \beta_k) + k_c R_k, \min(\tau_j, \tau_k) \leq t \leq \max(\tau_j, \tau_k) \\ \dot{M}_\perp = k_r(\beta_j - \beta_k), t > \max(\tau_j, \tau_k) \end{cases} \quad (26)$$

Notice that $\left| \dot{M}_\perp \right| \geq k_r |\beta_j - \beta_k| > 0$.

$$\left| \frac{dt}{dM_\perp} \right| \leq \left| \frac{1}{k_r(\beta_j - \beta_k)} \right| = \frac{1}{k_r |\beta_j - \beta_k|} \quad (27)$$

Hence

$$\sup \left| \frac{1}{2} k_c \frac{d(g_c(e_k) - g_c(e_j))}{dM_\perp} \right| \leq \frac{2k_c}{k_r |\beta_j - \beta_k|} < \infty \quad (28)$$

If we let $k_s^* = \frac{2k_c}{k_r |\beta_j - \beta_k|}$, then for $k_s > k_s^*$, the diffusive coupling strength is sufficient for burst synchronization.

Table 1: Parameterization

Parameter	Definition	Typical Value
k_e, k_i	The maximal values of the excitatory, inhibitory response functions	1,1
r_e, r_i	The absolute refractory period of the excitatory, inhibitory subpopulation	1,1
k_s	Synaptic coupling strength between columns	1.5
P	Level of background excitation in the excitatory subpopulation	-4
Q	Level of background excitation in the inhibitory subpopulation	0
c_1, c_3	Average number of excitatory synapses per cell	16,15
c_2, c_4	Average number of inhibitory synapses per cell	12,3
$\theta_e, \theta_i, a_e, a_i$	Maximal slope parameters of the logistic curve for the excitatory, inhibitory subpopulation	4,3.7,1.3,2
μ_1, μ_2	Modulation time scale constant	0.008, 0.008
κ	Sensitivity to the variations of the metabolic substrate	50
η, ζ	Threshold values to changes in the metabolic substrate, slow gating process	0,0
k_r, k_c	Metabolic recovery and consumption rate	0.24, 0.36
ν_1, ν_2	Homeostatic autoregulation time scale constant	0.09,0.09