Three Dimensional Adaptive Surface Re-Meshing For Large Displacement Finite Element Analysis

A.I. Khan† and B.H.V. Topping ‡
†*Lecturer in Computing and Information Technology*
‡*Professor of Structural Engineering*
Heriot-Watt University, Riccarton,
Edinburgh, EH14 4AS, United Kingdom

Keywords: Unstructured meshes, mesh generation, adaptive finite element analysis.

Abstract This paper describes a local interpolation method and proposes its use for the generation of three dimensional surface meshes. Large displacements from non-linear finite element analysis may be represented realistically by mapping a two dimensional adaptive un-structured mesh onto the displaced geometry of the initial coarse finite element mesh. Hence an interpolated three dimensional surface may be generated having first order derivative continuity from a relatively coarser initial mesh. Through the use of adaptive meshes greater resolution is readily achieved in the areas of the surface where it is most needed i.e. the regions of high displacement variations.

1 Introduction

When considering the use of adaptive meshes for three dimensional large displacement finite element analysis the following aspects have to be taken into consideration. An efficient way of performing adaptive finite element distretization is by using unstructured meshes. Unstructured meshes when used for adaptive finite element descretizations require the use of an automatic finite element mesh generator. The adaptive un-structured finite element mesh generation for two dimensional domains is computationally efficient to implement in contrast with a three dimensional surface meshing implementation.

Zienkiewicz and Phillips [1] proposed an isoparametric mapping approach to mesh plane and curved surfaces where the arbitrarily curved surfaces were first divided into zones and each zone in the three dimensional space was represented by a two dimensional isoparametric quadrilateral. A curvi-linear co-ordinate system ξ and η with values ranging from 1 to -1 at the extremities was defined at the center of the quadrilateral. The data points $P(x, y, z)$ representing nodes on the curved surface were generated by using the shape function associated with the pre-defined boundary nodes of the quadrilateral. Lohner and Parikh [2] used a similar approach for mapping an unstructured mesh generated using a two dimensional unstructured mesh generator to three dimensional surfaces. Lo [4] divided the three dimensional surfaces into five elementary surfaces i.e. spatial plane, cylinder, cone, sphere and surface of revolution. Nodes were generated on theses surfaces and triangulation was performed considering the geometrical properties of each surface. Rao, Hinton and Ozakca [5] proposed the use of Coon's patches for the parametric representation of surfaces. The surface to be traingulated was divided into sufficient number of patches then an unstructured mesh created in two dimensional co-ordinate system was mapped to a three dimensional co-ordinate system.

The above techniques are based upon the division of a curved surface into sufficient number of patches or quadrilateral or into basic geometrical shapes for transforming a mesh created in two dimensional co-ordinate system to a three dimensional co-ordinate system. When dealing with large displacements of plates and membranes the deformations are generally in a plane perpendicular to the surface. Adaptive finite element analysis is used in these cases to accurately determine the final deformed shape of the plate or membrane. The techniques mentioned above have been formulated keeping in mind the importance of having an accurate representation of the initial curved geometry of plates and shells. With these methods it is generally assumed that the final shape of the surface remains unchanged with respect to the initial shape. Hence these methods do not provide an efficient route for determining the large displacement response of plates with transverse loading. A three dimensional surface meshing techinique is presented which fits a surface through scattered data points in a three dimensional space. For data interpolation through a scatter of data points; the method of inverse distance weighted least squares interpolation proposed by Franke and Neilson [6] has been used to transform two dimensional adaptive meshes to three dimensional co-ordinate system. This is done with respect to the node displacements obtained after the first finite element analysis for the initial coarse finite element mesh.

2 The large displacement problem for plates and membranes

Wall panels in a building experiencing an internal gas explosion or an external blast load may undergo large out of plane displacements before failure. Similarly the wings of a high performance aircraft may under go large displacements during its operation. Form-finding constitutes an important feature in the design of architectural membrane structures [7]. In these structures, it is important to determine the deformed shape of the plate or the membrane accurately. The displaced shape may be determined using highly non-linear finite element analysis schemes such as dynamic relaxation.

Finite element analysis may be conducted to determine the displacements in structures. The displacements which are uniquely determined within the finite element analysis are the ones associated with the nodes of the finite element mesh. Generally a greater number of nodes are required to determine the accurate response within the regions of high displacements in the structure.

The accuracy of the finite element analysis depends upon the quality of the finite element mesh used to discretize the domain. Adaptive finite element analysis techniques provide means for automatically determining the optimal meshes for finite element problems. The adaptive h-refinement technique varies the mesh density according to the local error. The local error is estimated from the difference in the stress resultants computed from the finite element analysis and the smoothed stresses. A relatively coarse mesh is first analysed for adaptive finite element analysis and then the mesh parameters are computed for this coarse mesh. These mesh parameters are used to generate a finer mesh. Thus for an adaptive finite element analysis the number of elements in the mesh will increase in the regions of high stress gradients and remain the same or reduce (depending upon the type of mesh generation [3]) in the regions of low stress gradients. Hence from equation 1 the unknown nodal displacements are calculated then these displacements are translated into strains using equation 2. The stresses are recovered from the strained state of the structure using equation 3.

$$\mathbf{K}\mathbf{u} = \mathbf{P} \tag{1}$$

$$\epsilon = \mathbf{B}\mathbf{u} \tag{2}$$

$$\sigma = \mathbf{D}\epsilon \tag{3}$$

Where:

$\mathbf{K}$ = The global stiffness matrix for the structure;

$\mathbf{u}$ = The vector of unknown displacements at the nodes of the mesh;

$\mathbf{P}$ = The vector of applied loads at the nodes of the mesh;

ϵ = The strain vector;

$\mathbf{B}$ = A matrix which is based upon the geometrical properties of the mesh;

$\mathbf{D}$ = A matrix which is based upon the material properties of the mesh; and

σ = The stress vector.

From the finite element analysis procedure outlined above it may may inferred that the error norms, based upon stresses, also increases the accuracy of the finite element analysis with respect to displacements.

The geometrically non-linear analysis of a structure with a coarse mesh for a plate or a membrane results in an out of plane displacement of the non-restrained nodes of the mesh under the action of an out of plane applied load. Adaptive re-meshing of the domain in two dimensional co-ordinate system would result in a refined two dimensional finite element mesh. This mesh may then be mapped to form a refined displacement surface using the nodal co-ordinates from the coarse mesh. Un-structured meshes may be used with gradual adaptive refinement, each time using the relatively coarser mesh of the previous iteration, for calculating a more accurate displacement response of the structure.

3 Inverse distance weighted least squares interpolation

Franke and Nielson [6] formulated two interpolation techniques for interpolating a smooth three dimensional surface from a scatter of data points. The second technique referred to as Method-II in this reference has been used here since a triangulated domain is already available in the form of the coarse initial mesh.

Method-II is based upon McLains's [9] *Inverse Distance Weighted Least Squares Interpolation* for large sets of scattered data. A three dimensional surface is defined as $P = f(x, y, z)$, which may also be defined as

$$z_i = f(x_i, y_i), i = 1, \dots, N \tag{4}$$

Given the data $(x_i, y_i, z_i),\ i = 1, \dots, N$ a smooth bivariate interpolant S is defined which satisfies the condition $S(x_i, y_i) = z_i,\ i = 1, \dots, N$. The interpolant $S(x, y)$ derives information for interpolation from the data points lying within a certain radius of influence. The data points in our case are the nodes of the starting triangular mesh. If I is defined as an interpolation operator on the equation 4 then the interpolation function S may be written as

$$S(x, y) = I[z](x, y) \tag{5}$$

If ϕ_j represents a set of basis functions to be used for least square approximations ie. $\phi_j,\ j = 1, \dots m$. The Euclidean distance of the point of interpolation, (x, y), is defined as

$$\rho_i(x, y) = \sqrt{(x - x_i)^2 + (y - y_i)^2} \tag{6}$$

Then according to Mclain the general form of these interpolants is

$$I[z](x, y) = \sum_{j=1}^{m} \overline{a}_j(x, y)\phi_j(x, y) \tag{7}$$

where $\overline{a}_j(x, y),\ j = 1, \dots, m$ represent the solution of

$$min_{a_1,\dots,a_m} \sum_{i=1}^{N} \left(\frac{a_1\phi_1(x_i, y_i) + \dots + a_m\phi_m(x_i, y_i) - z_i}{\rho_i(x, y)} \right)^2 \tag{8}$$

The bivariate quadratic proposed by Mclain for interpolation is given as below.

$$I[z](x_i, y_i) = \overline{a}_1 + \overline{a}_2 x + \overline{a}_3 y + \overline{a}_4 x^2 + \overline{a}_5 xy + \overline{a}_6 y^2 \tag{9}$$

The method proposed by Mclain is global as the interpolant depends on all the data points regardless how far away these points are from the point of evaluation of the interpolant. Franke and Nielson in their method have changed the global aspect of the interpolant to a local one by approximating $\overline{a}_j(x, y)$ with $A[\overline{a}_j](x, y)$ as long as:

$$A[\overline{a}_j](x_i, y_i) = \overline{a}_j(x_i, y_i)[5pt]i = 1, \ldots, N \tag{10}$$

A radius of influence has also been defined so that the least square interpolations are only based upon the data points lying within this radius only. This results in substantial reduction in the computational load for solving equation 8. It may be seen from equation 9 that the generated surface has first order continuity.

4 Mapping unstructured two dimensional meshes to three dimensional surfaces

A coarse initial mesh may be created and analysed for out of plane forces. The nodal points on this mesh act as the scattered data points in the three dimensional space. Using the method described in the previous section a fine mesh is interpolated using the node co-ordinates of the coarse initial mesh. The mapping of unstructured two dimensional meshes to three dimensional surfaces is described below is based upon Franke and Nielson Method-II [6].

Triangulation of a domain may be represented by a set of vertices $V_i = (x_i, y_i), i = 1, \ldots, N$. Sets of weight functions $W_i, i = 1, \ldots, N$ are defined for these vertices. Each W_i is a globally defined C^1 function with support $S_i = \bigcup_{jkl \in M_i} T_{jkl}$ where T_{jkl} denotes the triangles with vertices V_j, V_k, V_l and $M_i = \{jkl : T_{jkl} \text{ is a triangle with vertex } V_i\}$.

The weight function W_i and its first-order partial derivatives on $E = \bigcup_{kj \in N_e} e_{kj}$, where e_{kj} represent the edge with vertices V_k and V_j and $N_e = \{kj : \overline{V_k V_j} \text{ is an edge of the triangulation}\}$. Franke and Nielson incorporated a blending method for triangles for extending the definition to the interior of each triangles of the mesh. For an edge e_{ij} contained within S_i having V_i as an end point. A univariate function W_i along this edge must satisfy the following four conditions.

$$W_i(V_i) = 1 \tag{11}$$
$$W_i(V_j) = 0 \tag{12}$$

$$
\begin{aligned}
(x_j - x_i)\frac{\partial W_i}{\partial x}(V_i) + (y_j - y_i)\frac{\partial W_i}{\partial y}(V_i) &= 0 \\
(x_j - x_i)\frac{\partial W_i}{\partial x}(V_j) + (y_j - y_i)\frac{\partial W_i}{\partial y}(V_j) &= 0 \qquad (13)
\end{aligned}
$$

The above conditions may be satisfied by a cubic polynomial and so the following relationship is defined in reference [6].

$$
W_i((1-t)V_i + tV_j) \;=\; (1-t)^2(2t+1),\ 0 \le t \le 1 \qquad (14)
$$

W_i on all other edges is defined to be zero and the continuity of first order derivative across edges is maintained. For maintaining the continuity of first-order derivatives across the edges, derivatives which are taken normal to the edges may be defined as a linear function of the derivatives along the edges.

$$
\begin{aligned}
&(y_j - y_i)\frac{\partial W_i}{\partial x}((1-t)V_i + tV_j) - (x_j - x_i)\frac{\partial W_i}{\partial y}((1-t)V_i + tV_j) \\
&= (1-t)\left((y_j - y_i)\frac{\partial W_i}{\partial x}(V_i) - (x_j - x_i)\frac{\partial W_i}{\partial y}(V_i)\right) \\
&+ t\left((y_j - y_i)\frac{\partial W_i}{\partial x}(V_j) - (x_j - x_i)\frac{\partial W_i}{\partial y}(V_j)\right) \qquad (15)
\end{aligned}
$$

The relationships given above contain neccessary information for carrying out interpolation on the edges. The definition of W_i to the interior of each triangle is extended with the interpolation method presented Nielson [8] which proceeds by assuming a predescribed position and slope on the entire boundary of a triangular element. Subsituting the edge information the following expression for $(x, y) \in T_{ijk} \subset S_i$ has been determined by Franke and Nielson.

$$
\begin{aligned}
&W_i(x, y) = b_i^2(3 - 2b_i) + 3\frac{b_i^2 b_j b_k}{b_i b_j + b_i b_k + b_j b_k} \\
&\left\{ b_j \frac{(\|e_i\|^2 + \|e_k\|^2 - \|e_j\|^2)}{\|e_k\|^2} + b_k \frac{(\|e_i\|^2 + \|e_j\|^2 - \|e_k\|^2)}{\|e_j\|^2} \right\} (16)
\end{aligned}
$$

where b_i, b_j and b_k are the barycentric (area) co-ordinates of the point (x, y) with respect to the triangle T_{ijk} and $\|e_n\|, n = i, j,$ or k represents the length of the edge opposite $V_n, n = i, j,$ or k. The final interpolant is given by

$$G[f](x,y) = W_i(x,y)Q_i(x,y) + W_j(x,y)Q_j(x,y)$$
$$+W_k(x,y)Q_k(x,y),\ (x,y) \in T_{ijk} \quad (17)$$

The function Q is defined as

$$Q_k(x,y) = z_k + \overline{a}_{k2}(x - x_k) + \overline{a}_{k3}(y - y_k)$$
$$+\overline{a}_{k4}(x - x_k)^2 + \overline{a}_{k5}(x - x_k)(y - y_k) + \overline{a}_{k6}(y - y_k)^2 \quad (18)$$

and the solution of the following minimisation problem is required:

$$min_{a_{k2},\ldots,a_{k6}}$$
$$\sum_{i-1 i\neq k}^{N} \left(\frac{z_k + a_{k2}(x_i - x_k) + \ldots + a_{k6}(y_i - y_k)^2 - z_i}{\rho_i(x_k, y_k)} \right)^2 \quad (19)$$

where

$$\frac{1}{\rho_i} = \frac{(R_q - d_i)_+}{R_q d_i} \quad (20)$$

d_i is the distance from ith node of the coarse mesh to the point represented by (x_k, y_k). The radius of influence R_q determines the accuracy and efficiency of the method and is taken as

$$R_q = \frac{D}{2}\sqrt{\frac{N_q}{N}} \quad (21)$$

where $D = max_{i,j}\ d_i(x_i, y_i)$.

The initial value of N_q was taken as $0.55D$. However this may have to be increased two to three times depending on the density and the gradation of the mesh in order to obtain correct interpolation of the three dimensional surface.

Example 1

An initial mesh shown in Figure 1 comprising 21 nodes and 28 elements was assigned arbitrary displacements at the central nodes as shown in Figure 2.

A finer uniformly graded mesh was created conforming to the boundaries of the initial mesh. This mesh comprising 241 nodes and 416 elements, as shown in Figure 3, was mapped to the three dimensional coarse mesh geometery represented by Figure 2.

The three dimensional mapped mesh is shown in Figure 4. The side views of the three dimensional coarse and refined meshes are shown in Figure 5.

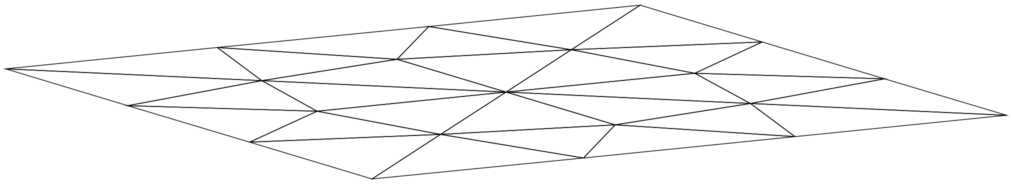

Figure 1. *Example 1: Initial un-disturbed mesh comprising 21 nodes and 28 elements*

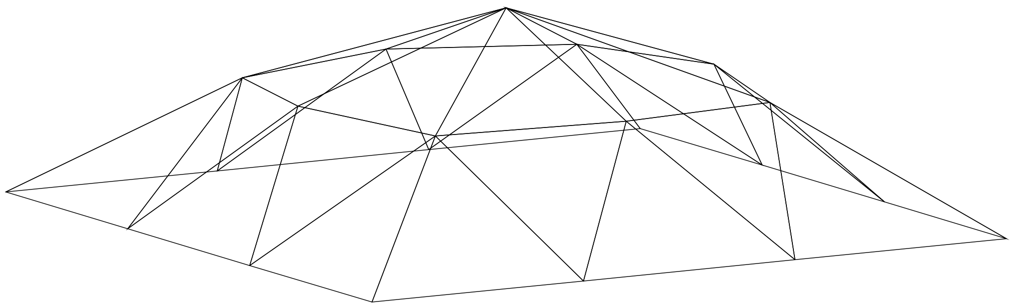

Figure 2. *Example 1: Central nodes of the initial mesh assigned arbitrary displacements*

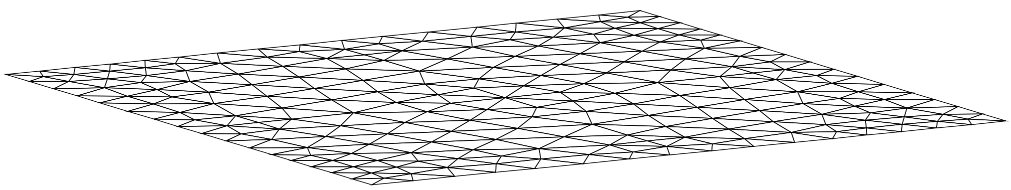

Figure 3. *Example 1: Fine mesh comprising 241 nodes and 416 elements*

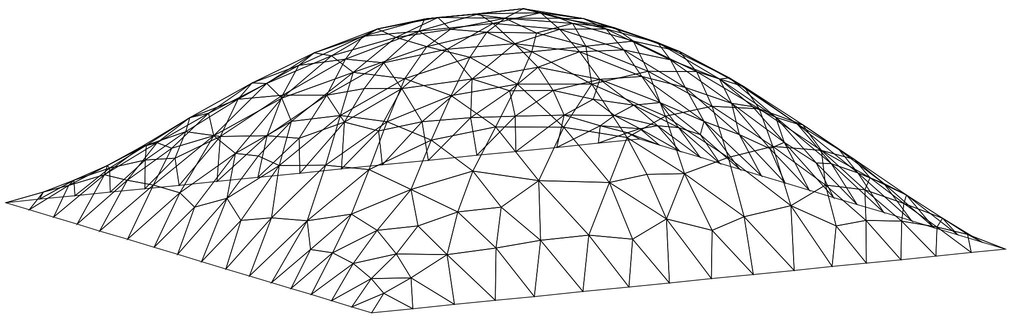

Figure 4. *Example 1: Fine mesh comprising 241 nodes and 416 elements mapped to the three dimensional geometry.*

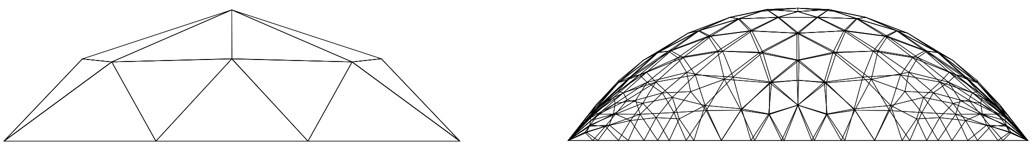

Figure 5. *Example 1: The side views of the coarse and refined three dimensional meshes*

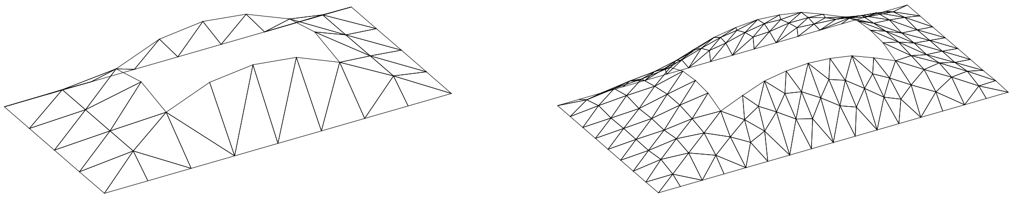

Figure 6. *Example 2: The coarse and refined three dimensional meshes*

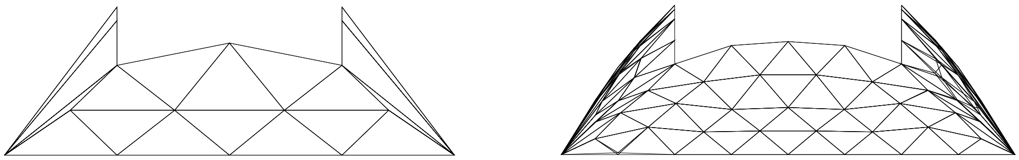

Figure 7. *Example 2: The side views of the coarse and refined three dimensional meshes*

Example 2

Figures 6 represents an arbitrary coarse three dimensional mesh with a cut out and a three dimensional mapping of a refined mesh over the coarse mesh geometry. The side elevetions of the coarse and refined meshed surfaces are shown in Figure 7

Example 3

The initial coarse mesh shown in Figure 8 comprising 21 nodes and 28 elements was restrained at the four corners and then a distributed out-of-plane load was applied to the central portion of the mesh. The membrane was analysed using a dynamic relaxation based non-linear finite element analysis scheme. The displaced shape after the analysis is shown in Figure 9.

Adaptive finite element calculation were performed using the stress resultants obtained from the finite element analysis. The domain was re-meshed in two dimensional

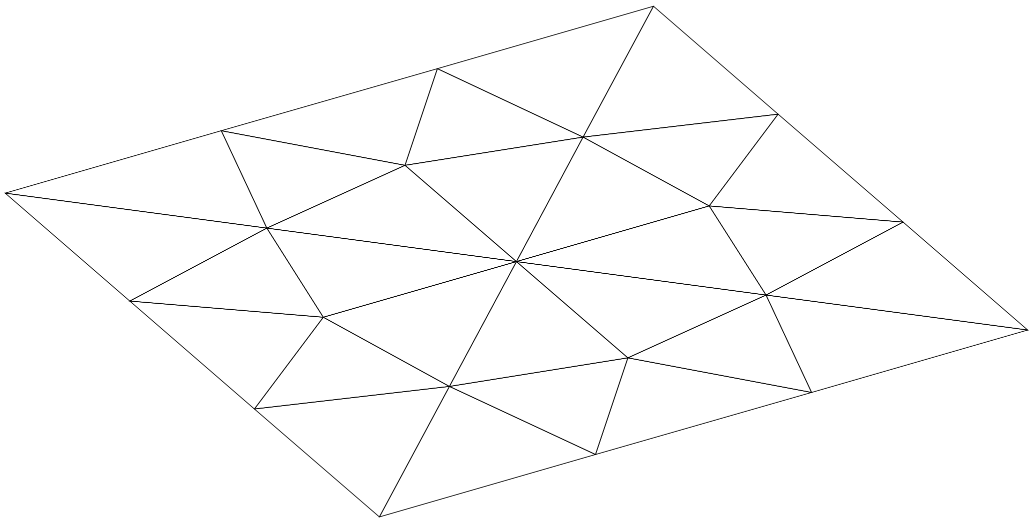

Figure 8. *Example 3: Initial coarse mesh comprising 21 nodes and 28 elements*

using the mesh parameters computed within the adaptivity module. The re-meshed two dimensional domain is shown in Figure 10.

Using the Franke and Nielson's inverse distance weighted least square interpolation the two dimensional re-meshed domain was mapped over the three dimensional geometry represented in Figure 9. The adptively generated three dimensional surface is shown in Figure 11.

5 Conclusions

For the analysis of wall panels subjected to blast loads, the analysis of the wings of a high performance aircraft and form-finding of membrane structures it is important to compute the displaced surface accurately.

The definition of the generated surface depends upon the the degree of the availabilty of the data within the regions of large out of plane deformations. The degree of availablity of data may be taken as the number of nodes available within the radius of influence of the surface point to be plotted.

If the adaptive analysis are based upon h - refinement

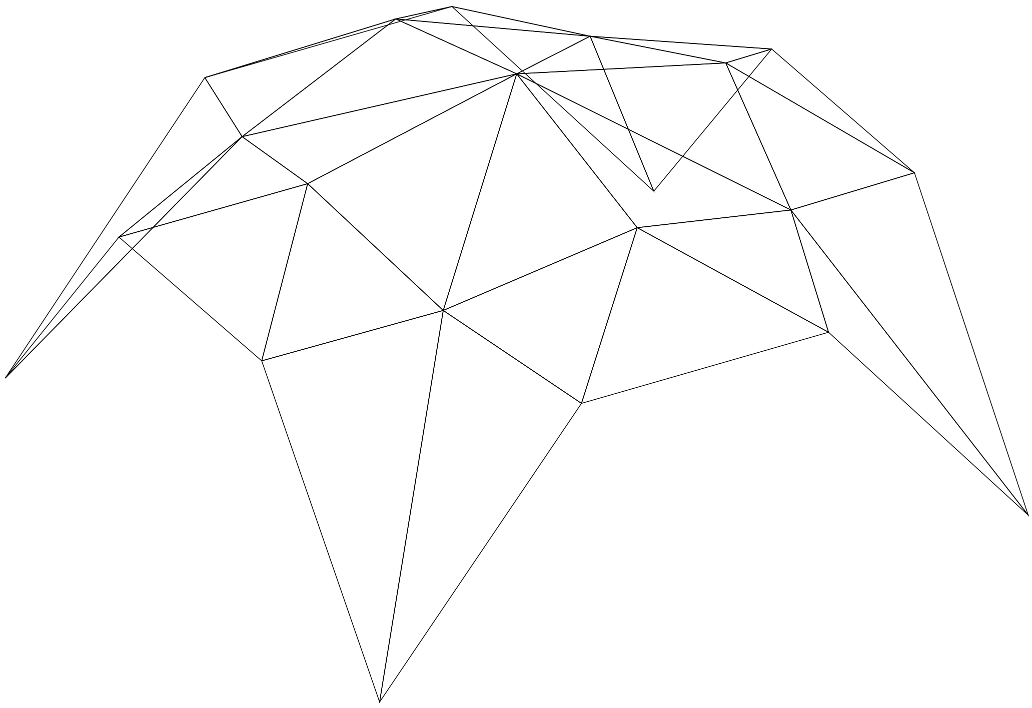

Figure 9. *Example 3: The displaced geometry of the initail mesh after nonlinear finite element analysis*

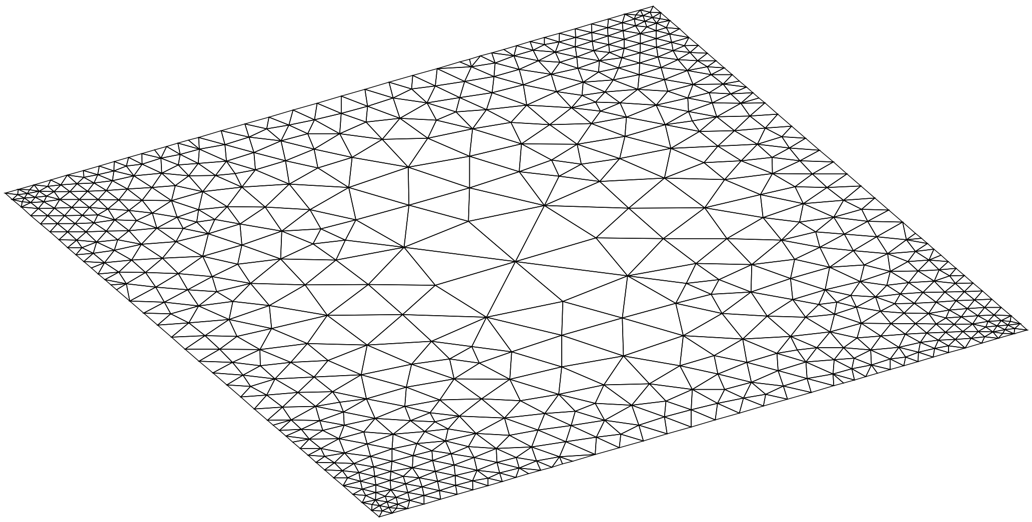

Figure 10. *Example 3: two dimensional domain adaptively re-meshed*

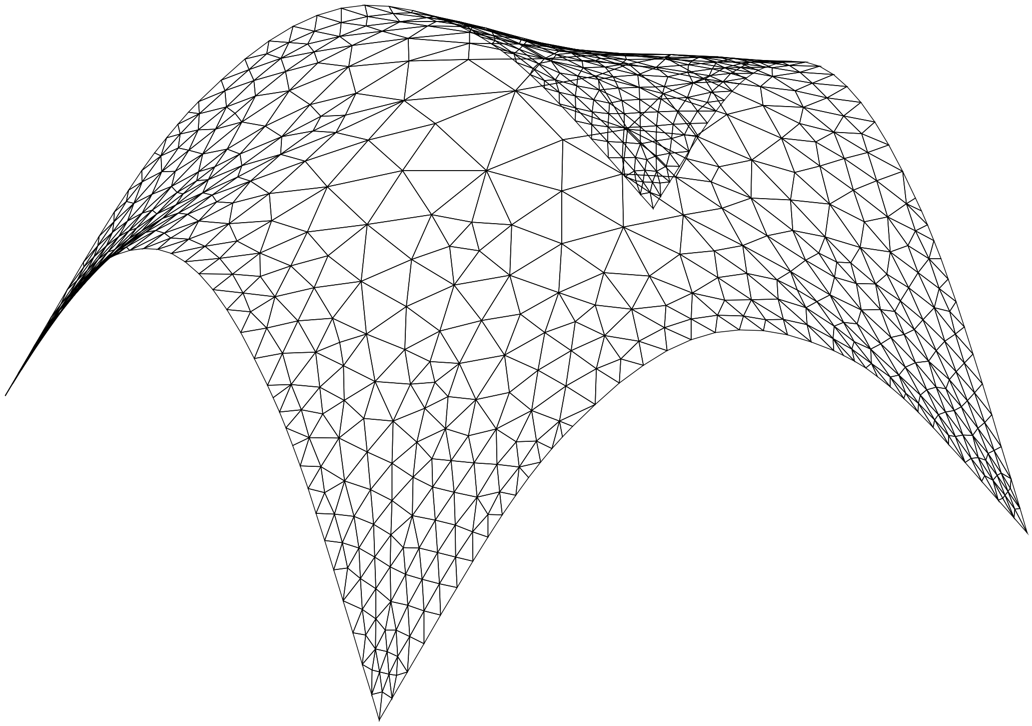

Figure 11. *Example 3: Adaptively generated three dimensional surface*

then the number of triangular elements would increase automatically in the vicinity of large out of plane displacements and consequently the definition of the generated surface will receive greater resolution in the regions of large deformations.

It is therefore anticipated that the three dimensional mesh mapping strategy discussed above would lead to efficient modelling of the surfaces generated through large displacement finite element analysis.

Acknowledgment The research described in this paper was supported by Marine Technology Directorate Limited Research Grant No SERC/GR/J22191.

References

[1] Zienkiewicz O. C., Phillips D. V., *"An Automatic Mesh Generation Scheme for Plane and Curved Surfaces by Isoparametric Co-ordinates"*, International Journal for Numerical Methods in Engineering, vol. 3, 519-528, 1971.

[2] Lohner R., Parikh P., *"Generation of Three-Dimensional Unstructured Grids by the Advancing-Front Method"*, International Journal for Numerical Methods in Fluids, vol. 8, 1135-1149, 1988.

[3] A.I. Khan, B.H.V. Topping, *"Parallel Adaptive mesh generation"*, Computing Systems in Engineering, vol. 2, no. 1, 75-102, 1991.

[4] Lo S. H., *" Finite Element Mesh Generation over Curved Surfaces"*, Computers & Structures, vol. 29, 731-742, 1988.

[5] Rao N. V. R., Hinton E., Ozakca M., *"Adaptive Finite Element Mesh Generation on Shell Surfaces"*, Department of Civil Engineering, University college of Swansea, Internal Report No. CR/952/90, Oct. 1990.

[6] Franke R., Nielson G., *"Smooth Interpolation of Large Sets of Scattered Data"*, International Journal for Numerical Methods in Engineering, vol 15, 1691-1704, 1980.

[7] Barnes, M.R. *"Computer Aided Design of the shade membrane roofs for EXPO 88"*, Structural Engineering Review, v.1, 3-13, 1988.

[8] Nielson G., *"Minimum norm interpolation in triangles"*, SIAM Numerical Analysis, vol 17, no 1, Febuary 1980.

[9] McLain D. H., *"Two dimensional interpolation from random data"*, Comp. J., vol 19, 178-181, 1976.