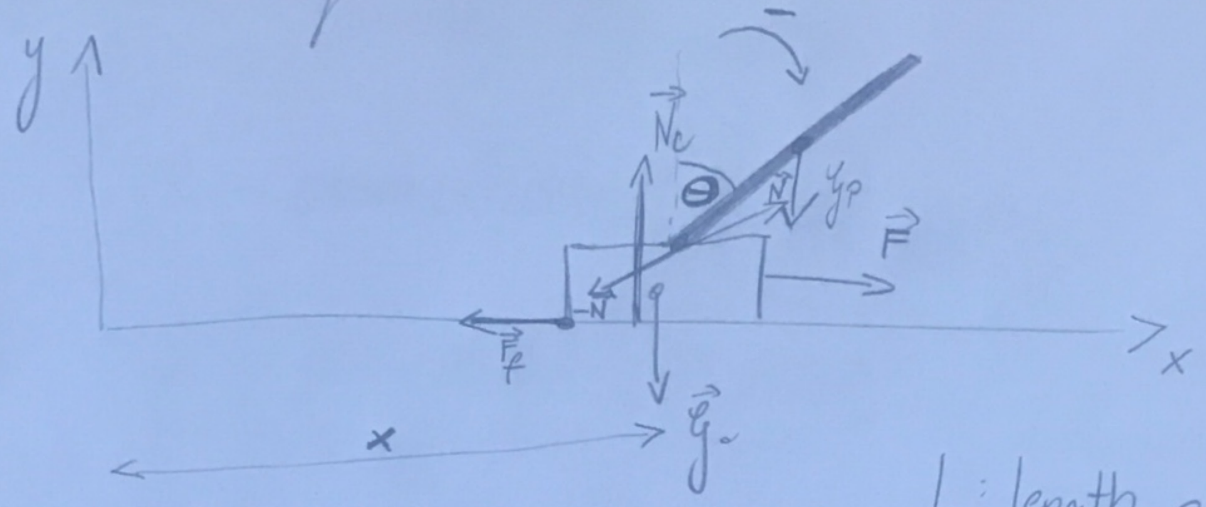
CartPole eq. derivation



L: length of the rod

$l = \frac{L}{2}$

Acceleration of the Cart

$$\vec{F} + \vec{F_f} + \vec{g_c} + \vec{N_c} - \vec{N} = m_c \vec{a_c}$$

$$\underline{F\vec{e_x}} \; \underline{-\mu_c |N_c| \operatorname{sign}(\dot{x}) \vec{e_x}} \; \overbrace{- g}^{g>0} m_c \vec{e_y} + N_c \vec{e_y} \; \underline{- N_x \vec{e_x}} - N_y \vec{e_y} = \underline{\ddot{x} \vec{e_x}} \, m_c$$

$$F - \mu_c |N_c| \operatorname{sign}(\dot{x}) - N_x = \ddot{x} m_c$$

$$-g m_c + N_c - N_y = 0$$

Acceleration of the ~~rod~~ pole

$$\vec{N} + \vec{g_p} = m_p \vec{a_p}$$

$$\underline{N_x \vec{e_x}} + N_y \vec{e_y} - g m_p \vec{e_y} = \underline{m_p a_{px} \vec{e_x}} + m_p a_{py} \vec{e_y}$$

$$N_x = m_p a_{px}$$

$$N_y - g m_p = m_p a_{py}$$

position of the centre of the pole

$$r_p = (x - l\sin\theta)\vec{e}_x + (l\cos\theta)\vec{e}_y$$

$$\dot{r}_p = (\dot{x} - \cancel{l\sin\theta}\, l\dot{\theta}\cos\theta)\vec{e}_x + (-l\dot{\theta}\sin\theta)\vec{e}_y$$

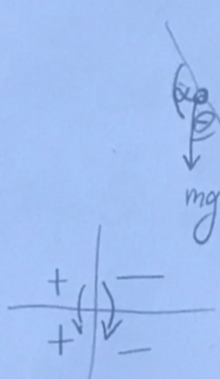
$$\ddot{r}_p = \underbrace{(\ddot{x} - l\ddot{\theta}\cos\theta + l\dot{\theta}^2\sin\theta)}_{a_x}\vec{e}_x + \underbrace{(-l\ddot{\theta}\sin\theta - l\dot{\theta}^2\cos\theta)}_{a_y}\vec{e}_y$$

$$N_x = m_p(\ddot{x} - l\ddot{\theta}\cos\theta + l\dot{\theta}^2\sin\theta)$$

$$N_y - gm_p = -m_p(l\ddot{\theta}\sin\theta + l\dot{\theta}^2\cos\theta)$$

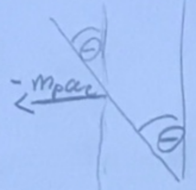
Torques acting on the pole
(It is easier to think of pole leaning to the left and so $\theta > 0$)

1) Gravity $\tau_g = mgl\sin\alpha$
$= mgl\sin(180-\theta)$
$= mgl\sin(\theta)$

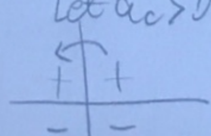


2) Inertia: consider rotation of the pole in a frame in which cart is not moving. Pole is than in accelareted ref. frame and there is a force $-m_p a_c$ acting on it

$\tau_i = +m_p a_c l\sin(90-\theta) =$
$= +m_p a_c l\cos\theta$

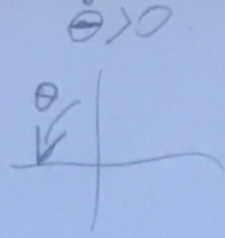


Let $a_c > 0$



3) Friction

$\tau_f = -\mu_p \dot{\theta}$



$$\tau_g + \tau_i + \tau_f = I\ddot{\theta}$$

$$mgl\sin\theta + m_p \ddot{x}\, l\cos\theta - \mu_p\dot{\theta} = I\ddot{\theta}$$

All eq.

1) $F - \mu_c |N_c|\,\mathrm{sign}(\dot{x}) - N_x = \ddot{x}\, m_c$

2) $-g m_c + N_c - N_y = 0$

3) $N_x = m_p(\ddot{x} - l\ddot{\theta}\cos\theta + l\dot{\theta}^2\sin\theta)$

4) $N_y - g m_p = -m_p l(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)$

5) $m_p g l\sin\theta + m_p\ddot{x}\, l\cos\theta - \mu_p\dot{\theta} = I\ddot{\theta}$

Eliminate: $N_x, N_y, N_c, \ddot{\theta}$

6)
1) + 3) $\ddot{x} = \dfrac{F - \mu_c|N_c|\,\mathrm{sign}(\dot{x}) + m_p l\ddot{\theta}\cos\theta - m_p l\dot{\theta}^2\sin\theta}{m_c + m_p}$

7)
6) + 5)

$$I\ddot{\theta} = mgl\sin\theta - \mu_p\dot{\theta} + \frac{(m_p l\cos\theta)^2}{m_c + m_p}\ddot{\theta} + m_p l\cos\theta\left[\frac{F - \mu_c|N_c|\,\mathrm{sign}(\dot{x}) - m_p l\dot{\theta}^2\sin\theta}{m_c + m_p}\right]$$

$$\frac{\ddot{\theta}\left[1-\frac{(m_p l\cos\theta)^2}{m_c+m_p}\right]}{m_p l}=g\sin\theta-\frac{\mu_p\dot{\theta}}{m_p l}+\cos\theta\left[\frac{F-m_p l\dot{\theta}^2\sin\theta-\mu_c|N_c|\,\mathrm{sign}\,\dot{x}}{m_c+m_p}\right]$$

2)+4)

$$N_c=g(m_c+m_p)-m_p l\left(\ddot{\theta}\sin\theta+\dot{\theta}^2\cos\theta\right)$$

No friction:

$$\ddot{x} = \frac{F + m_p l \ddot{\theta} \cos\theta - m_p l \dot{\theta}^2 \sin\theta}{m_c + m_p}$$

$$\ddot{\theta} = \alpha \left[g\sin\theta + \cos\theta \, \frac{F - m_p l \dot{\theta}^2 \sin\theta}{m_c + m_p} \right]$$

$$\alpha = \frac{m_p l}{1 - \frac{(m_p l \cos\theta)^2}{m_c + m_p}}$$