CHAPTER 1

Introduction: Dependence Modeling

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1.1 Introduction

The vines described in this book are not *climbing* or *trailing plants*. Nor do they refer to the Australian rock band. Vines are graphical structures that represent joint probability distributions. They were named for their close visual resemblance to grapes (compare Figs. 1.1 and 1.5)

Vines first appeared in mathematical publications in the late 1990s. It took time before the community of researchers interested in this model grew sufficiently and before vines were recognized in applications. Vines are still young but now have sufficiently matured to deserve a comprehensive presentation. This book is a joint effort of the vine-community and contains established as well as the newest results concerning vines. Samuel Johnson once said: "What is written without effort is in general read without pleasure".

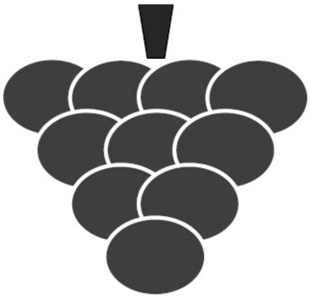

Figure 1.1. Grapes.

A lot of effort has gone into this book and we hope that it will be read with pleasure.

1.2 Investment Example

It is recognized that modeling dependence is of great importance for financial and engineering applications. We present the motivation for this book in a very simple financial example. When we invest \$1000 for five years, the five-year return is:

$$5yR = 1000(1 + r_1)(1 + r_2)(1 + r_3)(1 + r_4)(1 + r_5)$$

where r_1, r_2, r_3, r_4, r_5 are the interest rates in those five years. Interest rates are not known with certainty. We can find their distribution in principle from data but here they were assumed to be uniformly distributed on $[0.05, 0.15]$.

To find the distribution of our fortune after five years, we require the joint distribution of interest rates. If we assume that interest rates are independent, then their joint distribution is a product of marginal distributions and the five-year return can be easily calculated. If we recognize some sort of dependence between interest rates, we must build a joint distribution with given margins and given dependencies. A popular model recently used for this purpose is a copula. A copula is a distribution on the unit hypercube with uniform margins. It is also called a dependence function as it allows the separation of information coming from margins and dependence in the joint distribution. Different bivariate copulae, with the ability to model various features of a joint distribution (correlation, tail dependence), are available (see e.g., Refs. 12 and 20). In Fig. 1.2, scatter plots of the normal and

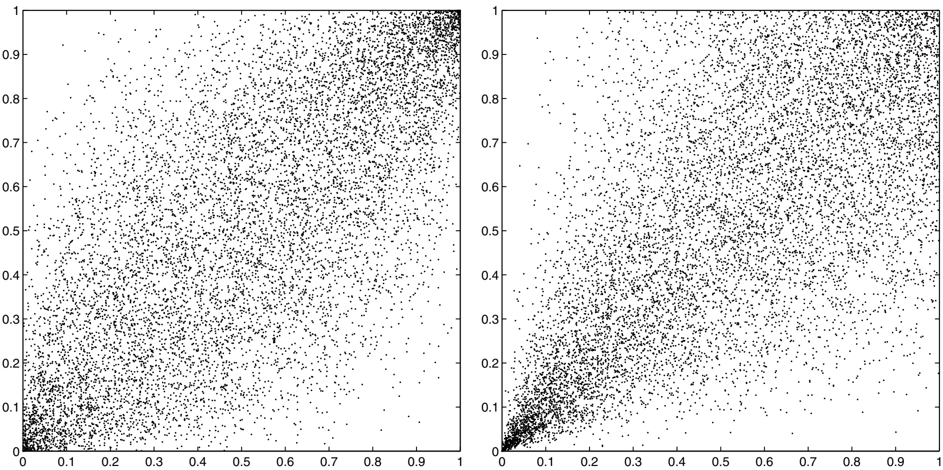

Figure 1.2. Scatter plots of normal (left) and Clayton (right) copulae with correlation 0.8.

Clayton copulae are shown. We can see how different these distributions are even for the same correlation value.

The choice of copula is an important question as this can affect the results significantly. In the bivariate case, this choice is based on statistical tests (see e.g., Ref. 9) when joint data are available. If only information about rank correlation is given, then the minimum information copula with given correlation is advocated.[16] Bivariate copulae are well studied, understood and applied (see e.g., Refs. 12 and 20).

Multivariate copulae are often limited in the range of dependence structures that they can handle. The most popular choice is the normal copula that can model the full range of correlation structures but does not allow for tail dependence. The student-t copula enjoys the flexibility of taking into account dependence in the tails of the distribution. However, it has only one parameter that controls tail dependence in all bivariate margins. Some models involving constructions with Archimedean copulae are also available (see e.g., Ref. 15 and Chapter 2). The choice of multivariate copula is usually based on its simplicity, popularity and possibility of modeling a given dependence structure.

Graphical models with bivariate copulae as building blocks of the joint distribution have recently become the tool of choice in dependence modeling. Coming back to our example, let us assume that successive yearly interest rates have a rank correlation of 0.7. Different dependence structures can be considered to satisfy the specified information. One possibility

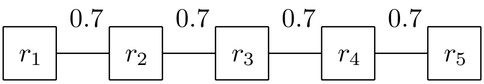

Figure 1.3. Tree for investment.

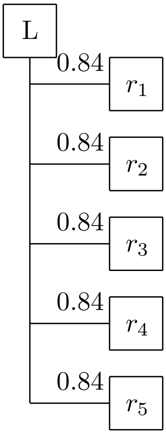

Figure 1.4. Tree for investment with latent variable.

is the dependence tree given in Fig. 1.3. With this structure, the correlation between r_1 and r_5 is much smaller than that between r_1 and r_2.

The second possibility is to correlate all yearly interests to a latent variable with a rank correlation of 0.84 (see Fig. 1.4). This dependence structure is symmetric and all interest rates are correlated at approximately 0.7.

Dependence structures that can be realized with trees are very limited. For joint distribution on n variables, we can specify only $n-1$ correlations and realize them with different bivariate copulae. A new graphical model introduced in 1997, called a regular vine, allows the specification of a joint distribution on n variables with given margins by specifying $\binom{n}{2}$ bivariate copulae and conditional copulae. Dependence trees are special cases of vines where conditional copulae are the independence copulae.

Returning to the investment example, we may reflect that if the interest is high in year 2, it is unlikely to be high in both years 1 and 3. We can capture this with a D-vine structure with a rank correlation of -0.7 between r_i and r_{i+2} conditional on r_{i+1}, $i = 1, 2, 3$, as shown in Fig. 1.5.

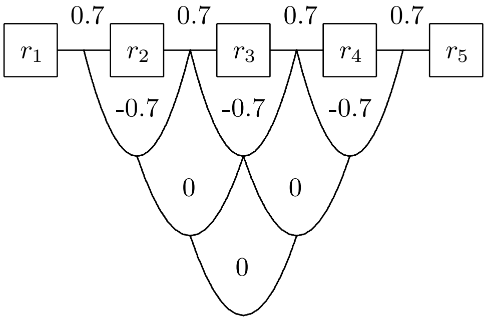

Figure 1.5. D-vine for investment.

Table 1.1. Quantiles, means and variances of distributions for five-year return in case of independence and for different dependence structures realized with the normal copula.

Model	5%-quant.	50%-quant.	95%-quant.	Mean	Variance
5yRind	1459.03	1609.42	1769.03	1611.13	9038.33
5yRtree	1367.43	1615.72	1889.93	1618.66	27384.70
5yRlatent	1348.15	1611.04	1913.49	1618.89	34352.21
5yRvine	1403.41	1607.61	1831.79	1610.58	16817.76

The results of our investment after five years in the case of independence between interest rates and for different dependence structures realized with the normal copula are shown in Table 1.1 and Fig. 1.6.

We see that the distributions of our fortune after five years in the case of different dependence structures are different even when realized by the same copula. Different choices of copula can be made and they will lead to different results (see results for Clayton copula in Table 1.2).

To decide if our investment is worth considering, imagine that there is another investment that in five years yields \$1900 with probability 95%. From Table 1.1 we see that only if the latent model with normal copula was appropriate for the joint distribution of interest rates would we prefer our original investment. However, Table 1.2 shows that with none of the dependence structures realized with the Clayton copula can we reach \$1900 with probability 95%. In this case, the competing investment would be preferred. From this simple example we can appreciate the importance of getting the dependence right. Flexible models that allow representation of a variety of

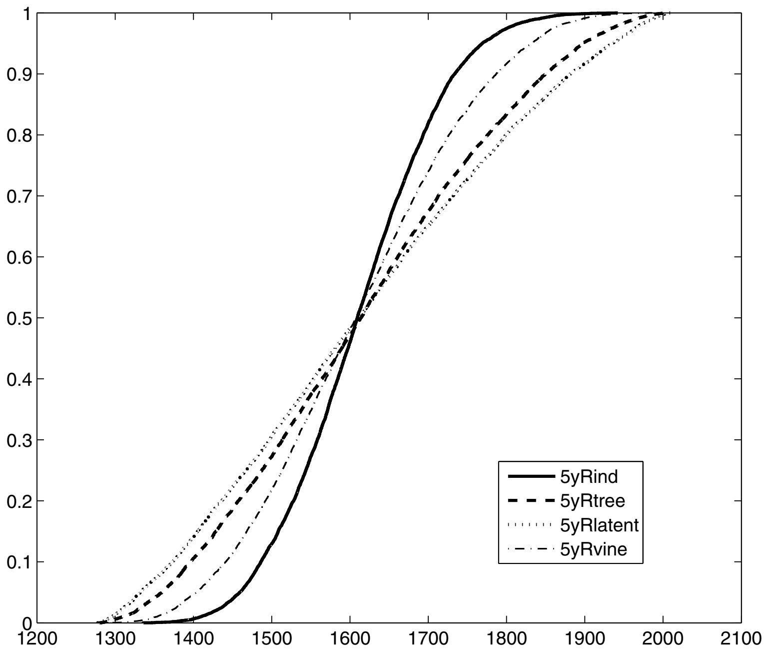

Figure 1.6. CDFs for five-year return in case of independence and for different dependence structures realized with the normal copula.

Table 1.2. Quantiles, means and variances of distributions for five-year return in case of independence and for different dependence structures realized with the Clayton copula.

Model	5%-quant.	50%-quant.	95%-quant.	Mean	Variance
5yRind	1459.33	1609.30	1768.85	1611.13	9039.08
5yRtree	1323.06	1638.60	1863.00	1618.64	27808.26
5yRlatent	1311.26	1639.51	1885.18	1619.20	33761.83
5yRvine	1337.70	1643.43	1792.84	1612.98	18665.36

dependence structures and different choices of copulae is essential for this kind of problem. Vines are very promising in this respect.

1.3 Vines

A vine on n elements $\mathcal{V} = (T_1, \dots, T_{n-1})$ is a nested set of trees where the edges of the tree j are nodes of the tree $j+1$ and each tree has the maximum number of edges. A regular vine on n elements is one in which two edges in tree j are joined by an edge in tree $j+1$ only if these edges share a common node.

For each edge of a vine, we define *constraint, conditioned* and *conditioning* sets of this edge as follows: the nodes of the first tree reachable from a given edge via the membership relation are called the constraint set of that edge. When two edges are joined by an edge in the next tree, the intersection of the respective constraint sets form the conditioning set, and the symmetric difference of the constraint sets is the conditioned set of this edge. Formal definitions can be found in Refs. 4, 3 and 13 are summarized in Chapter 3. Copulae can be assigned to the edges of the vine such that the conditioned variables correspond to the conditioned set, and the conditioning variables to the conditioning set of an edge.

1.3.1 *Graphical representation*

In Fig. 1.5, a special type of vine on five elements, the D-vine, is shown. Copulae and conditional copulae that can be assigned to the edges of this vine are (from left to right), in T_1, $c_{12}, c_{23}, c_{34}, c_{45}$; in T_2, $c_{13|2}, c_{24|3}, c_{35|4}$; $c_{14|23}, c_{25|34}$ in the third tree; and only one copula $c_{15|234}$ in the fourth. For the D-vine the graphical representation in Fig. 1.5 is quite clear; however, for other regular vines, this type of graphical representation can be a bit messy (see the C-vine in Fig. 1.7).

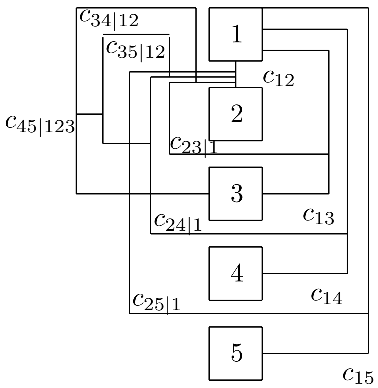

Figure 1.7. C-vine on five variables with copulae and conditional copulae assigned to the edges.

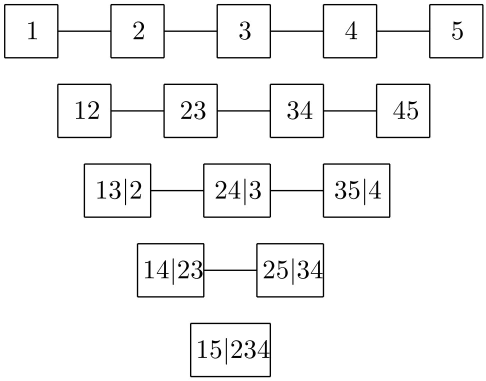

Figure 1.8. Trees for D-vine in Fig. 1.5 where nodes of each tree are enumerated by conditioned and conditioning sets of copula that can be assigned to it.

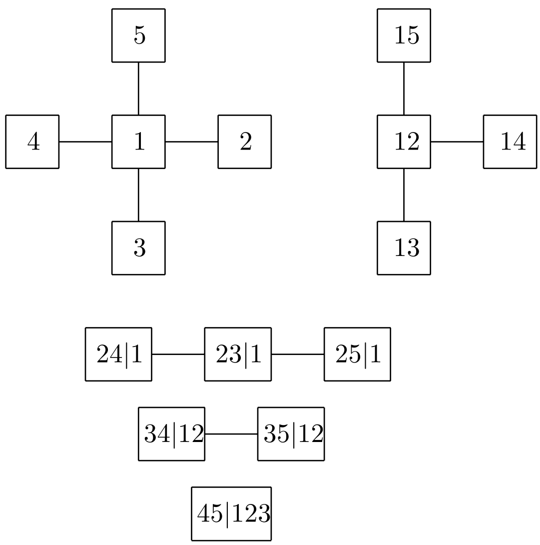

Figure 1.9. Trees for C-vine in Fig. 1.7 where nodes of each tree are enumerated by conditioned and conditioning sets of copula that can be assigned to it.

Often it is clearer to show all trees in a vine separately (see Figs. 1.8 and 1.9 where all trees are shown separately for the D-vine in Fig. 1.5 and the C-vine in Fig. 1.7 respectively).

A representation where all trees are kept separately is not very compact in terms of storing information necessary to represent a given vine. For storing information necessary for any regular vine, an n by n array can be used instead (for more information, see Ref. 18 or Chapters 9 and 10. We follow

Table 1.3. Array for the D-vine in Fig. 1.5 (left) and the C-vine in Fig. 1.7 (right).

3	3	3	2	4
	2	2	3	3
		4	4	2
			1	1
				5

1	1	1	1	1
	2	2	2	2
		3	3	3
			4	4
				5

here the notation used in Chapter 10). In Table 1.3, arrays containing all information for the D-vine in Fig. 1.5 and the C-vine in Fig. 1.7 are shown.

The diagonal elements in arrays printed in bold denote an ordering of variables in the vine, called the natural order. From the rightmost column of the left array in Table 1.3, we can read that variable 5 is in the conditioned set of the top node of the vine together with variable 1, and the conditioning set consists of variables $\{2, 3, 4\}$. In the third tree, the variable 5 forms with 2 the conditioned set of an edge, and the conditioning set of this edge is $\{3, 4\}$. In the second tree, 5 is paired with 3 and conditioned on 4, and in the first tree 5 is connected with 4. Notice the very simple structure of the array for the C-vine.

The bottom part of both arrays is empty and it can store, for instance, information about the parameters of copulae that are assigned to the edges of the vine.

1.3.2 *Vine density*

Bedford and Cooke[3] show that the joint density of a regular vine copula with margins $f_1, \dots, f_n$ is a product of (conditional) copula densities assigned to the edges of the vine and a product of marginal densities. For the C-vine in Fig. 1.7, the density is of the form:

$$f_{1\dots5} = f_1 \dots f_5 \prod_{i=1}^{4} \prod_{j=i+1}^{5} c_{ij|i+1\dots j-1}(F_{i|i+1\dots j-1}, F_{j|i+1\dots j-1}). \qquad (1.1)$$

On the other hand, given a positive joint density $f_{1\dots5}$ with standard factorization $\prod_{i=1}^{5} f_{i|1\dots i-1}$, we can see that the i^{th} term of this factorization $(i > 1)$ can be expressed as:

$$f_{i|1\dots i-1} = c_{i,i-1|1\dots i-2}(F_{i|1\dots i-2}, F_{i-1|1\dots i-2}) f_{i-1|1\dots i-2}.$$

This recursive representation leads to the conclusion that every positive multivariate density function can be expressed as a product of bivariate copula acting on several different conditional probability distributions. Hence, every positive joint density can be represented as a density of any regular vine copula.

For most densities, the conditional copulae will depend on conditioning variables as shown for the trivariate Frank's copula density in Example 1.1. Distributions belonging to the elliptical family factorize on a vine such that conditional copulae do not depend on conditioning variables.

Example 1.1. Consider Frank's copula with density

$$c(u,v;\theta) = \theta\eta \frac{e^{-\theta(u+v)}}{(\eta - (1-e^{-\theta u})(1-e^{-\theta v}))^2} \tag{1.2}$$

where $\theta > 0$, $\eta = 1 - e^{-\theta}$. It is shown in Ref. 12 that this copula can be extended to the multivariate case. The trivariate copula density belonging to Frank's family is:

$$\begin{aligned} c_{123}(u_1,u_2,u_3;\theta) &= \theta^2\eta^2 e^{-\theta(u_1+u_2+u_3)} \\ &\quad \times \frac{\eta^2 + (1-e^{-\theta u_1})(1-e^{-\theta u_2})(1-e^{-\theta u_3})}{(\eta^2 - (1-e^{-\theta u_1})(1-e^{-\theta u_2})(1-e^{-\theta u_3}))^3}. \end{aligned} \tag{1.3}$$

All three bivariate margins of (1.3) are Frank's with the same parameter θ. If we consider a D-vine with c_{12} and c_{13} of the form (1.2), then to get a vine copula representation of the density (1.3) we must only find the conditional copula $c_{13|2}$, which is:

$$c_{13|2}(u,v;\theta,u_2) = \frac{2uv + (u - 3uv + v)e^{-\theta u_2} + (1-u)(1-v)e^{-2\theta u_2}}{(1 + (1-u)(1-v)(e^{-\theta u_2} - 1))^3}.$$

The conditional copula does not belong to Frank's family and it depends on the conditioning variable.

1.3.3 *Estimation*

When estimating a vine copula from data it is usually assumed that conditional copulae do not depend on conditioning variables.[a] Moreover, the type of a vine structure is fixed and only few families of bivariate copulae are taken into consideration.

The estimation of copula parameters using the maximum likelihood principle, for the vine copula or pair-copula construction (PCC),[2] is performed

[a]To our knowledge, it is not known how severe this assumption is. For more information, see Ref. 10.

sequentially starting from the first tree. This landmark advance in associating bivariate copulae to a vine and estimating copula parameters from data demonstrated the superiority of vines and opened up large areas of application in mathematical finance, risk analysis and uncertainty modeling in engineering[1] (for more information about PCC, see Chapter 3, and Chapters 13, 15 and 16 for examples of applications).

The assumption of constant conditional copulae and consideration of only a few types of bivariate families in fitting a vine to data cause a phenomenon where some types of vines fit the data better than the others. To find the best vine structure, we would in principle have to estimate all possible vines. In dimensions higher than seven or eight, this is infeasible as the number of vines grows rapidly with dimension (see Ref. 18 and Chapter 9). Moreover, because of sequential estimation in PCC, estimates for parameters of conditional copulae in higher-order trees are less reliable. For higher-dimensional cases, some simplifying assumptions for fitting vines to data will have to be made. Some ideas on this subject are based on optimal truncations of a vine (see Chapter 11). Searching for the best vine model can also be approached from a Bayesian perspective (see Chapter 12).

If joint data are not available, there exist protocols to elicit copula parameters from experts.[19]

1.3.4 *Properties and applications*

Some properties of vines are already well established (see Refs. 3, 4 and 13 and Chapter 3). In this volume, new properties of vine distributions are presented. In Chapter 8, the similarities and differences in dependence for different regular vines on n variables are studied. It is shown that for $n \leq 4$, only two types of vines are available, namely C-vines and D-vines. In higher dimensions, other vine structures appear. Distributions corresponding to different vines are compared from the perspective of marginal dependence. It is shown that under some conditions the bivariate marginal dependence from the C-vine is the highest of all vines on five nodes.

Tail dependence of vine copula is of great interest. It is shown in Chapter 9 that vine copulae have flexible asymmetry in the joint upper and lower tails. This flexibility can be achieved by appropriate choice of bivariate copulae. Figure 1.10 shows contours of bivariate densities for (X_1, X_3) with standard normal margins and a copula which is a bivariate margin of a D-vine with copulae c_{12}, c_{23} and $c_{13|2}$. Rank correlations on the D-vine are $r_{12} = 0.5$, $r_{23} = 0.6$ and $r_{13|2} = 0.7$ for all cases. They are, however, realized

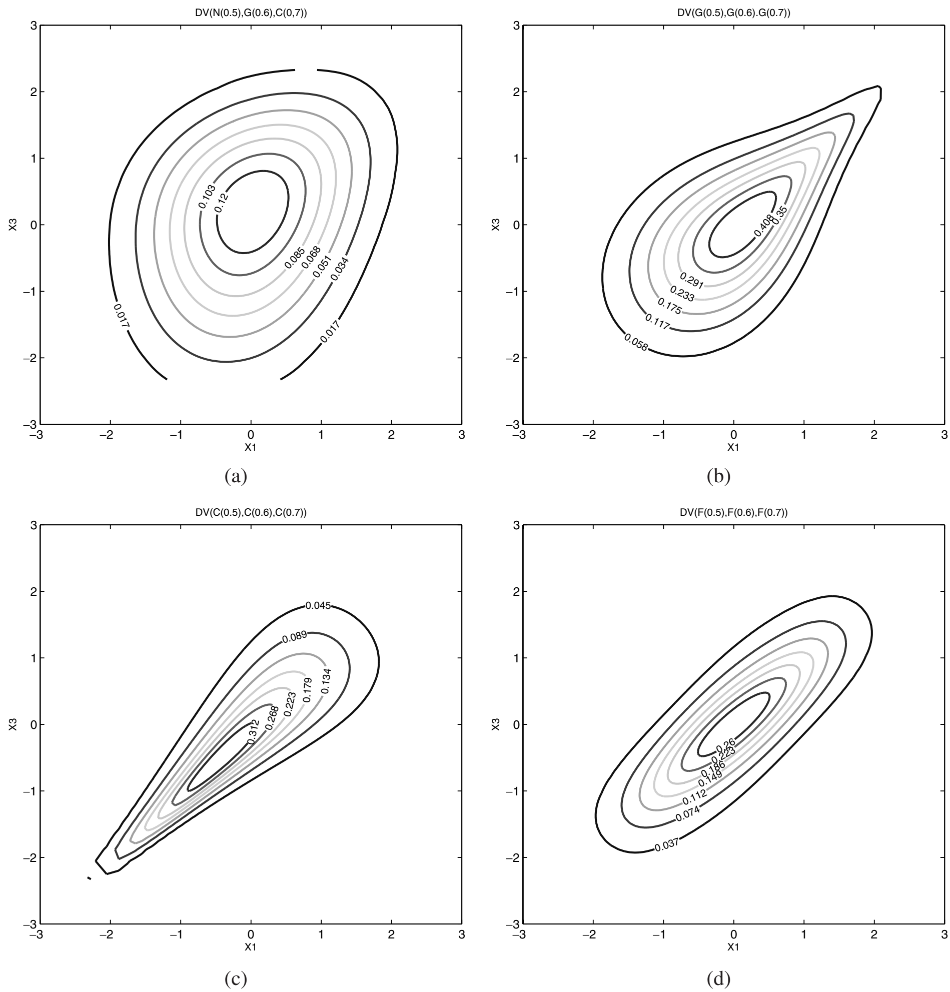

Figure 1.10. Contours of densities for (X_1, X_3) with standard normal margins and copula obtained from the D-vine DV(1,2,3) with Spearman correlations $r_{12} = 0.5$, $r_{23} = 0.6$ and $r_{13|2} = 0.7$ realized by copulae (a) Normal, Gumbel, Clayton (b) Gumbel, Gumbel, Gumbel (c) Clayton, Clayton, Clayton and (d) Frank, Frank, Frank.

by different bivariate copulae. A variety of distributions can be obtained with different choices of bivariate copulae.

Copulae are naturally used for continuous variables. They can also be useful in simulating models for high-dimensional count variables. The normal copulae were utilized for the construction of multivariate discrete distributions with specified correlation structure. However, due to the positive definiteness constraint, it is challenging to determine the appropriate normal copula parameters for a specified target correlation. Parameters of the normal copula can, however, be reparameterized into an algebraically independent set of partial correlations assigned to the edges of a vine. This idea allows efficient sampling of high-dimensional count variables and is presented in Chapter 4.

Vines are undirected graphical structures representing joint distributions. In Chapter 14, a directed graphical model, Non-Parametric Bayesian Belief Net (NPBBN), is presented and compared with vines. The most important difference between NPBBNs and regular vines appears to be in the conditional independencies that they can represent. The choice between representing a multivariate distribution using a regular vine or a NPBBN depends on many factors, some of which are discussed in Chapter 14.

1.4 Outline

The focus of this book is on vine copulae or PCCs. However, in Chapter 2, different multivariate copula classes and construction schemes of multivariate models are also reviewed. Popular multivariate copulae belonging to the elliptical family as well as Archimedean copulae and their generalizations are presented.

In Chapter 3, an introduction to the main idea of vines as graphical models is presented. This chapter traces the early history of vines and presents the motivation for their construction. Important properties and applications of vines are included.

In Chapter 4, vine copulae are used to sample multivariate count variables with target correlation structure.

Chapter 6 surveys the asymptotic theory of estimation of a copula from a frequentistic perspective and presents the problems involved in frequentistic model selection among several candidate copulae using the Maximum Pseudo Likelihood Estimator (MPLE). Frequentistic copula model selection has recently been addressed through the development of the Copula Information Criterion (CIC) — a model selection formula that extends the Akaike Information Criterion (AIC), based on maximum likelihood, to the MPLE.

Chapters 5 and 8 study tail dependence of various multivariate distributions. In Chapter 8, multivariate tail dependence functions are introduced and applied to vine copulae.

Chapter 9 explores how many different vines and regular vines are available. It is shown that the number of possible vines grows rapidly with dimension. A few algorithms for generating regular vines are presented in this chapter. It also contains a catalogue of regular vines up to dimension 9.

Chapter 10 shows an algorithm to generate all regular vines. This algorithm is used to obtain some general results about the number of equivalence classes for regular vines.

In Chapter 7, different types of five-dimensional vines are studied. Six equivalence classes for five-dimensional vines have been found. They are compared from the perspective of bivariate marginal dependence obtained when the bivariate copulae at each level were assumed to be the same. An interesting pattern of dependence emerges from this study, which may be helpful in the use of vine copulae for modeling multivariate data.

For high-dimensional problems, simplifying assumptions have to be made to reduce the complexity and computational burden involved in fitting vine copulae. Chapter 11 explores truncations of vines and proposes a heuristic search algorithm for the 'best vine' for the target correlation structure.

Chapter 12 reviews available MCMC estimation and model selection algorithms as well as their possible extensions for D-vine pair-copula constructions based on bivariate t-copulae. Theory presented in this chapter is then applied in Chapter 13 to Australian electricity load data.

Chapter 14 compares regular vines with the directed graphical models that represent joint distribution, called Non-Parametric Bayesian Belief Nets.

Chapter 15 compares three constructions for modeling higher-dimensional dependence: the Student copula, the partially nested Archimedean construction and the pair-copula construction. It is shown through two applications that the PCC provides a better fit to financial data than the two other structures.

In Chapter 16, the dependence structure of multivariate financial returns is modeled with a time-varying D-vine copula. Two different data sets, one with six exchange rates and another with five Asian equity indices were used for the analysis. The D-vine structure allows us to model the symmetric dependence of exchange rates and the asymmetric one of Asian equities. For both cases, the dependence structure was found to vary in time.

This book concludes with a short summary and a few future research directions.

All chapters are self-contained and provided with their own set of references. However, the notation in all chapters has been unified as much as possible. The general notation for this book is presented in the next section.

1.5 Glossary and Notation

Random vectors, distributions, densities, copulae

$\mathbf{X} = (X_1, \ldots, X_n)$ n-dimensional random vector;
$F_{\mathbf{X}} = F_{1,2,\ldots,n}$ cumulative distribution function (cdf) of $\mathbf{X}$,
$F_{\mathbf{X}}(\mathbf{x}) = F_{1,2,\ldots,n}(x_1, \ldots, x_n)$ value at $\mathbf{x} = (x_1, \ldots, x_n)$;
$f_{\mathbf{X}} = f_{1,2,\ldots,n}$ probability density function (pdf) of $\mathbf{X}$;
$\bar{F} = \bar{F}_{\mathbf{X}}$ the survival function of $\mathbf{X}$;
$F_{X_i} = F_i$ and $f_{X_i} = f_i$ marginal cdfs and pdfs, respectively;
F_S marginal distribution, where $S \subset \{1, \ldots, n\}$,
$F_{X_i|X_1,\ldots,X_k} = F_{i|1\cdots k}$ conditional distribution of X_i given $X_1, \ldots, X_k$; its value $F_{i|1\cdots k}(x_i|x_1, \ldots, x_k)$;
$C_{\mathbf{X}} = C_{1,2,\ldots,n}$ copula for $\mathbf{X}$ with density $c_{\mathbf{X}}$ where
$F_{\mathbf{X}} = C_{\mathbf{X}}(F_1, \ldots, F_n)$ and $f_{\mathbf{X}} = f_1 \cdots f_n \cdot c_{\mathbf{X}}(F_1, \ldots, F_n)$;
$C_{i_1,i_2|S}$ $(\cdot|x_k : k \in S)$ bivariate conditional copula of $F_{i_1|S}(\cdot|x_k : k \in S)$ and $F_{i_2|S}(\cdot|x_k : k \in S)$ where $F_{i_1|S}$ and $F_{i_2|S}$ univariate conditional cdfs and $i_j \notin S$ for $j = 1, 2$.

Correlations

$\rho(X_i, X_j) = \rho_{ij}$ product moment (pearson) correlation of X_i, X_j
$r_{X_i,X_j} = r_{ij} = \rho(F_i(X_i), F_j(X_j))$ Spearman rank correlation of X_i, X_j.

Graphs

$G = (N, E)$ graph, N vertex set, E edge set;
$T = (N, E)$ tree;
$\mathcal{V} = \mathcal{V}(n) = (T_1, \ldots, T_{n-1})$ vine on n elements.

Information

$I(f, g)$ or $KL(f, g)$ information (Kullback–Leibler) divergence between f and g;
$MI(f)$ mutual information of f.

Time series

$AR(m)$ autoregressive process of order m.

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